Comment on "Hamiltonian splitting for the Vlasov-Maxwell equations"

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Abstract

The paper [1] by Crouseilles, Einkemmer, and Faou used an incorrect Poisson bracket for the Vlasov-Maxwell equations. If the correct Poisson bracket is used, the solution of one of the subsystems cannot be computed exactly in general. As a result, one cannot construct a symplectic scheme for the Vlasov-Maxwell equations using the splitting Hamiltonian method proposed in Ref. [1].

Keywords: splitting method, Vlasov-Maxwell system, symplectic integrator, Poisson bracket

In a recent paper [1] by Crouseilles, Einkemmer, and Faou, a new symplectic splitting method for the Vlasov-Maxwell equations is proposed. In comparison with previous splitting methods, the exciting new feature of the proposed method is that it is designed to preserve the symplectic structure of the Vlasov-Maxwell system, and thus enjoys the benefits of symplectic integration, such as the global bound on energy error and long-term accuracy and fidelity.

Crouseilles, Einkemmer, and Faou developed an innovative technique to achieve this goal. A non-canonical Poisson bracket for the Vlasov-Maxwell system as an infinite-dimensional Hamiltonian system is employed. The system is split into three subsystems by splitting the Hamiltonian functional

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into three parts. It turns out that the solution for each subsystem can be computed exactly and therefore preserves exactly the symplectic structure corresponding to the Poisson bracket. As a consequence, the combined algorithm according to the splitting scheme preserves the symplectic structure as well. In addition, higher order methods can be constructed using various familiar composition methods.

The Poisson bracket adopted by Ref. [1] is the bracket discovered by Morrison in 1980 [2],

$$\begin{aligned}
[F,G](E,B,f) = &\int f\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\}_{xv} dxdv \\
&+\int\left[\frac{\delta F}{\delta E}\cdot\left(\nabla\times\frac{\delta G}{\delta B}\right)-\frac{\delta G}{\delta E}\cdot\left(\nabla\times\frac{\delta F}{\delta B}\right)\right]dx \\
&+\int\left(\frac{\delta F}{\delta E}\cdot\frac{\partial f}{\partial v}\frac{\delta G}{\delta f}-\frac{\delta G}{\delta E}\cdot\frac{\partial f}{\partial v}\frac{\delta F}{\delta f}\right)dxdv \\
&+\int\left[\frac{\delta F}{\delta B}\cdot\left(\frac{\partial f}{\partial v}\times v\right)\frac{\delta G}{\delta f}-\frac{\delta G}{\delta B}\cdot\left(\frac{\partial f}{\partial v}\times v\right)\frac{\delta F}{\delta f}\right]dxdv, \qquad (1)
\end{aligned}$$

for functionals F and G of E, B, and f. Here $\{h,g\}_{xv}$ is the canonical Poisson bracket in the (x,v) space. The Hamiltonian for the system is

$$H(f,E,B)=\frac{1}{2}\int v^2 f dxdv+\frac{1}{2}\int\left(E^2+B^2\right)dx. \qquad (2)$$

This Hamiltonian can be split into three parts [1] as follows,

$$H = H_f + H_E + H_B, \qquad (3)$$

$$H_f = \frac{1}{2}\int v^2 f dxdv, \qquad (4)$$

$$H_E = \frac{1}{2}\int E^2 dx, \qquad (5)$$

$$H_B = \frac{1}{2}\int B^2 dx. \qquad (6)$$

The scheme developed in Ref. [1] is based on the observation that solutions of the subsystems corresponding to H_f, H_E, and H_B can all be computed exactly.

Unfortunately, this Poisson bracket (1) is known to be incorrect, because it does not satisfy the Jacobi identity. This error had been discovered and corrected [3, 4] shortly after its publication [2]. The correct Poisson bracket [3, 4] is

$$
\begin{aligned}
[F, G](E, B, f) = & \int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} dx dv \\
& + \int \left[\frac{\delta F}{\delta E} \cdot \left(\nabla \times \frac{\delta G}{\delta B} \right) - \frac{\delta G}{\delta E} \cdot \left(\nabla \times \frac{\delta F}{\delta B} \right) \right] dx \\
& + \int \left(\frac{\delta F}{\delta E} \cdot \frac{\partial f}{\partial v} \frac{\delta G}{\delta f} - \frac{\delta G}{\delta E} \cdot \frac{\partial f}{\partial v} \frac{\delta F}{\delta f} \right) dx dv \\
& + \int f B \left(\frac{\partial}{\partial v} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial v} \frac{\delta G}{\delta f} \right) dx dv. \qquad (7)
\end{aligned}
$$

Following Ref. [5], we will call this bracket the Morrison-Marsden-Weinstein (MMW) bracket. Integrating the third term on the right-hand side of Eq. (7) and considering the fact that

$$
\frac{\partial}{\partial v} \left(\frac{\delta F}{\delta E} \right) = \frac{\partial}{\partial v} \left(\frac{\delta G}{\delta E} \right) = 0, \qquad (8)
$$

we can recast the MMW bracket as

$$
\begin{aligned}
[F, G](E, B, f) = & \int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} dx dv \\
& + \int \left[\frac{\delta F}{\delta E} \cdot \left(\nabla \times \frac{\delta G}{\delta B} \right) - \frac{\delta G}{\delta E} \cdot \left(\nabla \times \frac{\delta F}{\delta B} \right) \right] dx \\
& + \int f \left(\frac{\delta G}{\delta E} \cdot \frac{\partial}{\partial v} \frac{\delta F}{\delta f} - \frac{\delta F}{\delta E} \cdot \frac{\partial}{\partial v} \frac{\delta G}{\delta f} \right) dx dv \\
& + \int f B \left(\frac{\partial}{\partial v} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial v} \frac{\delta G}{\delta f} \right) dx dv. \qquad (9)
\end{aligned}
$$

In Ref. [6], the MMW bracket in the form of Eq. (7) is used, and in Refs. [7, 8], the equivalent form of Eq. (9) is used. We emphasize that in order for the MMW bracket in the form of Eq. (7) or Eq. (9) to satisfy the Jacobi identity, some constraints in terms of B and/or E are necessary. Marsden and Weinstein [4] restricted the solution space to be $Mv = \{(f, E, B) \mid \nabla \cdot B = 0, \nabla \cdot E = \int f dv\}$. Morrison [7] pointed out that it is only necessary to require $\nabla \cdot B = 0$ for the MMW bracket in the form of Eq. (7) or Eq. (9) to satisfy the Jacobi identity. Of course, a solution of the Vlasov-Maxwell equations is always in Mv, if it is initially in Mv.

If we use this correct Poisson bracket instead, in the form of either Eq. (7) or Eq. (9), and apply the same splitting scheme as in Eq. (3), it is very disappointing to find out that the solution for the subsystem corresponding to H_f can not be computed exactly, whereas the solutions for the subsystems corresponding to H_E and H_B can. Specifically, the subsystem associated with H_f is

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + (v \times B) \cdot \frac{\partial f}{\partial v} = 0, \tag{10}$$

$$\frac{\partial E}{\partial t} = -\int v f dv, \tag{11}$$

$$\frac{\partial B}{\partial t} = 0. \tag{12}$$

This subsystem is more complicated than its counterpart obtained using the incorrect Poisson bracket (1). Unless the magnetic field B is uniform in space or vanishes, its solution cannot be computed exactly. As a result, for systems with general magnetic field, one cannot construct a symplectic scheme for the Vlasov-Maxwell equations using the splitting Hamiltonian method proposed in Ref. [1].

If a symplectic integration method to a desired order for Eqs. (10)-(12) can be found, then we can apply this splitting method to obtain a symplectic scheme. However, such a symplectic method for Eqs. (10)-(12) is not available yet. Further investigation is needed.

As a final note, it is necessary to mention the following bracket proposed by Chandre et al. [8] to remove the $\nabla \cdot B = 0$ constraint for the MMW bracket for the Vlasov-Maxwell equations,

$$\begin{aligned}[F, G](E, B, f) = &\int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} dx dv \\ &+ \int \left[\frac{\delta F}{\delta E} \cdot \left(\nabla \times \frac{\delta G}{\delta B} \right) - \frac{\delta G}{\delta E} \cdot \left(\nabla \times \frac{\delta F}{\delta B} \right) \right] dx \\ &+ \int f \left(\frac{\delta G}{\delta E} \cdot \frac{\partial}{\partial v} \frac{\delta F}{\delta f} - \frac{\delta F}{\delta E} \cdot \frac{\partial}{\partial v} \frac{\delta G}{\delta f} \right) dx dv \\ &+ \int f (B - \nabla \triangle^{-1} \nabla \cdot B) \left(\frac{\partial}{\partial v} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial v} \frac{\delta G}{\delta f} \right) dx dv. \end{aligned} \tag{13}$$

Here, $B - \nabla \triangle^{-1} \nabla \cdot B$ is the projection of B that "removes" the non-divergence-free part of B. It is straightforward to verify that the Jacobi identity is unconditionally satisfied [8]. For this bracket given by Eq. (13), the splitting Hamiltonian method proposed in Ref. [1] generates identical subsystems as for the MMW bracket given by Eq. (7) or Eq. (9), when the $\nabla \cdot B = 0$ constraint is satisfied initially. Therefore, the splitting Hamiltonian method proposed in Ref. [1] is not valid for the bracket given by Eq. (13) as well.

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