Another approach to the equivalence of measure-many one-way quantum finite automata and its application

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In memory of my grandparents

Abstract

In this paper, we present a much simpler, direct and elegant approach to the equivalence problem of *measure many one-way quantum finite automata* (MM-1QFAs). The approach is essentially generalized from the work of Carlyle [J. Math. Anal. Appl. 7 (1963) 167-175]. Namely, we reduce the equivalence problem of MM-1QFAs to that of two (initial) vectors.

As an application of the approach, we utilize it to address the equivalence problem of *Enhanced one-way quantum finite automata* (E-1QFAs) introduced by Nayak [Proceedings of the 40th Annual IEEE Symposium on Foundations of Computer Science, 1999, pp. 369-376]. We prove that two E-1QFAs $\mathcal{A}_1$ and $\mathcal{A}_2$ over Σ are equivalence if and only if they are $n_1^2 + n_2^2 - 1$-equivalent where n_1 and n_2 are the numbers of states in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.

Keywords: quantum finite automata, measure-many one-way quantum finite automata, enhanced one-way quantum finite automata, equivalence

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1. Introduction

The theory of quantum computing is unquestionably one of the hottest and front research fields in the theory of computing [1–3]. There exist a few works developed quantum computation model, such as *quantum Turing machines* [5, 6], *Quantum circuits* [7, 8], and the quantum generalizations of *finite automata*, i.e.,

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Quantum finite automata (QFAs) [9–16, 22]. In particular, the study of QFAs provides a good insight into the nature of quantum computation, since QFAs can be viewed as the simplest theoretical model based on quantum mechanism.

The so-called *measure-many one-way quantum finite automata* (MM-1QFAs), introduced in [10], is a kind of QFA model whose tape head is subjected to moving one cell to the right at each computation step, and measurement is performed after every computation step. There exist a few works dealt with the language recognized ability of MM-1QFAs, such as [10, 11, 14, 17–21]. Incidentally, the so-called *enhanced one-way quantum finite automata* (E-1QFAs) introduced by Nayak [22] can be viewed as a generalization of MM-1QFAs.

Just as the equivalence problem of the classical finite automata [23–25, 34, 35], the concept of "equivalence" gives us a classification of the elements of the set of MM-1QFAs over the same alphabet. On the equivalence issue of MM-1QFAs, Li and Qiu [26] have shown, with the help of the so-called *1qfa with control language* [11], that two MM-1QFAs $\mathcal{A}_1$ and $\mathcal{A}_2$ over the same alphabet are equivalent if and only if they are $3n_1^2+3n_2^2-1$-equivalent where n_1 and n_2 are the numbers of states in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, and factor 3 is the numbers of states in the minimal DFA [23–25] recognized the regular language $g^*a\{a,g,r\}^*$. Incidentally, there exist some works dealt with the equivalence issue with respect to other quantum finite automata [27–30]. However, the equivalence problem of E-1QFAs is still open thus far. A more comprehensive survey on this subject is [31] by Gruska.

We note that the method to the equivalence problem of MM-1QFAs, attributed to Li and Qiu [26], is roundabout and somewhat complicated. Therefore, the first aim of this paper is to present a much simpler, direct and elegant approach to the equivalence problem of MM-1QFAs. We summarize our motivations as follows. (1) As we know, the mathematical method is the essence of mathematics. The mathematician usually investigates the same problem with different mathematical methods and different concepts to fully understand it. This method can be followed; (2) It is an interesting work of its own to find a more general method to address the equivalence problem for MM-1QFAs;

(3) We want to know whether the upper-bound $3n_1^2 + 3n_2^2 - 1$ can be further improved. Such considerations lead us to transform the word function of MM-1QFAs defined in a "cumulation" manner (described in the sequel) to another version which is in a "non-cumulation" manner. Then, we improve the previous upper-bound to $n_1^2 + n_2^2 - 1$ by showing the following

Theorem 1. *Let $\mathcal{A}_i = (Q_i, \{U_i(\sigma)\}_{\sigma\in\Sigma\cup\{\$\}}, |\pi_i\rangle, \mathcal{O}_i)$, $i = 1, 2$, be two MM-1QFAs over Σ. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent iff they are $(n_1^2 + n_2^2 - 1)$-equivalent, where n_1 and n_2 are the numbers of states in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.*

As mentioned earlier, the E-1QFA model [22] can be seen as a finite memory version of the mixed state MM-1QFA. Thus, the approach to the equivalence problem of MM-1QFAs also can be applied to that of E-1QFAs. Therefore, as our second aim, we utilize the above approach to solve the equivalence problem of E-1QFAs, which remains open so far, by showing the following

Theorem 2. *Let $\mathcal{A}_i = (Q_i, Q_{acc,i}, Q_{rej,i}, \{\mathcal{U}_\sigma^{(i)}\}_{\sigma\in\Sigma\cup\{\#,\$\}}, \rho_i, \mathcal{O}_i)$, $i = 1, 2$, be two E-1QFAs over Σ. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent iff they are $(n_1^2+n_2^2-1)$-equivalent where n_1 and n_2 are the numbers of states in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.*

The remainder of the paper is organized in the following way: Section 2 is the preliminary part where basic concepts and notations used in the sequel are reviewed. Section 3 and Section 4 are devoted to the proofs of Theorem 1 and Theorem 2, respectively. Section 5 is the concluding section.

2. Preliminaries

For convenience, we briefly review some basic notions needed in the sequel. To a more exhaustive illustration about linear algebra, we refer to [32]. Also, we refer to [1–3] for a through treatment on the quantum theory.

2.1. Some notation on Linear algebra

Let $\mathbb{C}$ denote the field of complex number, M a complex matrix, i.e., $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ with $a_{ij} \in \mathbb{C}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Some times, we use $(a_{ij})_{m\times n}$ to denote M. In particular, $1 \times n$ (resp. $n \times 1$) complex matrices are called n dimensional row vectors (resp. column vectors). If $m = n$, then M is called a *complex square matrix* of order n (or m), and sometimes M is called a n-order (or m-order) complex matrix. Let $M = (a_{ij})_{m\times n}$ be a $m \times n$ complex matrix, then the transpose of M is denoted as M', i.e., $M' = (a_{ji})_{n\times m}$, and the conjugate-transpose of M is denoted as $M^\dagger$. In this paper, the set of all n-order complex matrices will be denoted as $\mathbb{M}_n(\mathbb{C})$. For any $H \in \mathbb{M}_n(\mathbb{C})$, H is said to be *Hermitian* if $H^\dagger = H$, and is said to be *Unitary* if $H^\dagger H = HH^\dagger = I_n$ where I_n denotes the n-order *identity* matrix. Suppose that A and B are m and n-order complex matrix, respectively, we define the *"diagonal sum"* of A and B to be

$$A \oplus B \triangleq \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Therefore, $A \oplus B$ is a $(m+n)$-order complex matrix.

Let $A = (a_{ij})$ be an $n \times n$ matrix over $\mathbb{C}$, let $\mathrm{Tr}(A)$ denote the *trace* of A, i.e., $\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$. It is well known that

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA), \quad \text{and} \quad \mathrm{Tr}(\lambda_1 A + \lambda_2 B) = \lambda_1 \mathrm{Tr}(A) + \lambda_2 \mathrm{Tr}(B)$$

where $\lambda_i \in \mathbb{C}$.

Let V be a finite dimensional vector space over $\mathbb{C}$, and $\mathcal{B} = \{\eta_1, \eta_2, \cdots, \eta_n\}$ a basis for V over $\mathbb{C}$. This means that for any vector $\alpha \in V$, it has a unique expression as a linear combination

$$\alpha = c_1\eta_1 + c_2\eta_2 + \cdots + c_n\eta_n$$

where $c_i \in \mathbb{C}$. The dimension of V, denoted by $\dim V$, is defined to be the cardinal number of $\mathcal{B}$. Let $\mathrm{span}\{\mathcal{B}\}$ denote the vector space generated by the vectors in $\mathcal{B}$. Then, as a matter of fact, $V=$span$\{\mathcal{B}\}$. Furthermore, $\mathbb{M}_n(\mathbb{C})$ is a vector space over $\mathbb{C}$ with the dimension n^2.

2.2. Some notation on Quantum mechanics

In quantum theory, for any isolated physical system, it is associated with a (finite-dimensional) *Hilbert space*, denoted as $\mathcal{H}$, which is called the state space of the system. In *Dirac* notation, the row vector (resp. column vector) φ is denoted as $\langle\varphi|$ (resp. $|\varphi\rangle$). Incidentally, $\langle\varphi|$ is the conjugate-transpose of $|\varphi\rangle$, i.e., $\langle\varphi| = |\varphi\rangle^\dagger$. The inner product of two vectors $|\varphi\rangle$ and $|\psi\rangle$ is denoted as $\langle\varphi|\psi\rangle$. The norm (or length) of the vector $|\varphi\rangle$, denoted by $\||\varphi\rangle\|$, is defined as $\||\varphi\rangle\| = \sqrt{\langle\varphi|\varphi\rangle}$. A vector $|\varphi\rangle$ (resp. $\langle\varphi|$) is said to be unit if $\||\varphi\rangle\| = 1$ (resp. $\|\langle\varphi|\| = 1$).

Suppose that $Q = \{q_1, q_2, \cdots, q_m\}$ is the basic state set of a quantum system. Then the corresponding Hilbert space is $\mathcal{H}_m = \text{span}\{|q_i\rangle \,|\, q_i \in Q, 1 \le i \le m\}$ where $|q_i\rangle = (0, \cdots, 0, 1, 0, \cdots, 0)'$ is a m dimensional column vector having only 1 at the $(i, 1)$ entry, together with the inner product $\langle\cdot|\cdot\rangle$, defined to be $\langle\alpha|\beta\rangle = \sum_{i=1}^m x_i^* y_i$ where λ^* stands for the conjugate of λ for each complex number $\lambda \in \mathbb{C}$, $|\alpha\rangle = (x_1, x_2, \cdots, x_m)'$ and $|\beta\rangle = (y_1, y_2, \cdots, y_m)'$ are two vectors in $\mathcal{H}_m$. At any time, the state of this system is a *superposition* of $|q_i\rangle$, $1 \le i \le m$, and can be represented by a unit vector $|\phi\rangle = \sum_{i=1}^m c_i|q_i\rangle$ with $c_i \in \mathbb{C}$ such that $\sum_{i=1}^m |c_i|^2 = 1$. One can perform a *measure* on $\mathcal{H}_m$ to extract some information about the system. A measurement can be described by an *observable*, i.e., a Hermitian matrix $\mathcal{O} = \lambda_1 P_1 + \cdots + \lambda_s P_s$ where λ_i is its eigenvalue and P_i is the projector onto the eigenspace corresponding to λ_i.

The above mathematical descriptions of quantum system are based on "pure state". We need some descriptions based on "mixed states". In mixed states picture, the states of quantum device are represented by density operator $\rho \in \mathcal{L}(\mathcal{H})$, i.e., ρ is self-adjoint, $\rho \ge 0$ (semi-positive definite) and $\text{Tr}(\rho) = 1$. The evolution of a closed quantum system is characterized by a unitary operation U which maps ρ to $U\rho U^\dagger$. However, a general quantum operation $\mathcal{U}$ from $\mathcal{L}(\mathcal{H}_1)$ to $\mathcal{L}(\mathcal{H}_2)$ is a trace-preserving completely positive mapping [1–3] with the form

$\mathcal{U}(\rho) = \sum_i M_i \rho M_i^\dagger$ for any $\rho \in \mathcal{L}(\mathcal{H}_1)$, where $\{M_i\}$ are Kraus operators of $\mathcal{U}$ satisfying $\sum_i M_i^\dagger M_i = I_{\dim \mathcal{H}_1}$. Let $\mathcal{H} = P_1 \oplus P_2 \oplus \cdots \oplus P_k$ be a decomposition. Then, for any $\rho \in \mathcal{L}(\mathcal{H})$, $\mathrm{Tr}(P_j \rho)$ (equivalent to $\mathrm{Tr}(P_j \rho P_j^\dagger)$) is the probability that the property P_j is observed.

2.3. On relevant definitions of MM-1QFAs

For any finite set S, $|S|$ denotes the cardinality of S. Throughout this paper, Σ denotes the non-empty finite alphabet. A *word* over the alphabet Σ is a finite sequence of symbols chosen from Σ. Let Σ^* denote the set of all words over Σ. For any word $\omega \in \Sigma^*$, $|\omega|$ denotes the length of ω. Let Σ^n denote the set of all words of length n over Σ where n is a non-negative integer. Then Σ^* can be represented as $\Sigma^* = \epsilon \cup \Sigma \cup \Sigma^2 \cup \cdots$ where ϵ denotes the empty word.

For a fixed alphabet Σ, let $M(x_i)$, where $x_i \in \Sigma$, be complex square matrices indexed by x_i. For convenience, we define the formal product $\prod_{i=n}^{1} M(x_i)$ by

$$\prod_{i=n}^{1} M(x_i) \quad \triangleq \quad M(x_n)M(x_{n-1}) \cdots M(x_1).$$

Now, we state the definition of MM-1QFA as follows.

Definition 1. Formally, an MM-1QFA with m states on the alphabet Σ is a quadruple tuple

$$\mathcal{A} \quad = \quad (Q, \{U(\sigma)\}_{\sigma \in \Sigma \cup \{\$\}}, |\pi\rangle, \mathcal{O})$$

where $Q = \{q_1, q_2, \cdots, q_m\}$ is the basic state set, $|\pi\rangle$ is the initial state vector with $\||\pi\rangle\| = 1$, $\$ \notin \Sigma$ is an end-mark, for each $\sigma \in \Sigma \cup \{\$\}$, $U(\sigma) \in \mathbb{M}_m(\mathbb{C})$ is an unitary matrix, and $\mathcal{O}$ is an observable with results in $\{a, r, g\}$, completely described by the projectors $P(a)$, $P(r)$ and $P(g)$.

The projectors $P(a)$, $P(g)$ and $P(r)$ are given by

$$P(a) = \sum_{q \in Q_{acc}} |q\rangle\langle q|, \qquad P(g) = \sum_{q \in Q_{non}} |q\rangle\langle q|, \qquad P(r) = \sum_{q \in Q_{rej}} |q\rangle\langle q|$$

where $Q_{non} = Q \backslash (Q_{acc} \cup Q_{rej})$ is the set of *non-halting* states, $Q_{acc} \subseteq Q$ and $Q_{rej} \subseteq Q$ (with $Q_{acc} \cap Q_{rej} = \emptyset$) are the sets of accepting states and rejecting states, respectively, and $|q\rangle\langle q|$ denotes the matrix product of column vector $|q\rangle$ and row vector $\langle q|$.

Fed with $x_1x_2\cdots x_n\$$ where $x_1x_2\cdots x_n\in\Sigma^*$, $\mathcal{A}$ computes as follows: starting from $|\pi\rangle$, $U(x_1)$ is applied and a measurement of $\mathcal{O}$ is performed reaching a new current state. If the measurement result is 'g', then $U(x_2)$ is applied and a new measurement of $\mathcal{O}$ is performed. This process continues as far as measurements yields the result 'g'. As far as the result of measurement is 'a', the computation stops and the word is accepted. If the measurement result is 'r', then the computation stops and the word is rejected. Therefore, $\mathcal{A}$ induces a word function $p_{\mathcal{A}}:\Sigma^*\$\to[0,1]$ in a "cumulation" manner, i.e.,

$$p_{\mathcal{A}}(x_1x_2\cdots x_n\$) \quad=\quad \sum_{k=1}^{n+1}\left\|P(a)U(x_k)\left(\prod_{i=k-1}^{1}\big(P(g)U(x_i)\big)\right)|\pi\rangle\right\|^2 \tag{1}$$

where x_{n+1} denotes \$. By $\prod_{i=0}^{1}\big(P(g)U(x_i)\big)$ we mean that

$$\prod_{i=0}^{1}\big(P(g)U(x_i)\big) \quad=\quad I_m$$

i.e., the m-order ($m=|Q|$) identity matrix. Further, the probability of $\mathcal{A}$ accepting the word $x_1x_2\cdots x_n$ is defined as

$$\mathcal{P}_{\mathcal{A}}(x_1x_2\cdots x_n) \quad=\quad p_{\mathcal{A}}(x_1x_2\cdots x_n\$). \tag{2}$$

Definition 2. Two MM-1QFAs $\mathcal{A}_1$ and $\mathcal{A}_2$ over Σ are said to be equivalent (resp. t-equivalent) if $\mathcal{P}_{\mathcal{A}_1}(\omega)=\mathcal{P}_{\mathcal{A}_2}(\omega)$ for all $\omega\in\Sigma^*$ (resp. for all $\omega\in\Sigma^*$ with $|\omega|\leq t$).

The probability $\mathcal{P}_{\mathcal{A}}(\omega)$ of $\mathcal{A}$ accepting the word ω given in terms of Eq. (2) is somewhat complicated. Now, we define another "probability function" of $\mathcal{A}$ 'accepting' the word ω as follows.

$$\mathcal{F}_{\mathcal{A}}(\omega) \quad=\quad \begin{cases}\mathcal{P}_{\mathcal{A}}(x_1x_2\cdots x_n)-\mathcal{P}_{\mathcal{A}}(x_1x_2\cdots x_{n-1}), & \omega=x_1x_2\cdots x_n;\\ \mathcal{P}_{\mathcal{A}}(\epsilon), & \omega=\epsilon.\end{cases} \tag{3}$$

Remark 1. Note that, if $n = 1$ in Eq. (3), then $x_1 x_2 \cdots x_0$ denotes the empty word ϵ. More specifically, we define $\mathcal{F}_{\mathcal{A}}(x)$ to be the value: $\mathcal{P}_{\mathcal{A}}(x) - \mathcal{P}_{\mathcal{A}}(\epsilon)$ for any $x \in \Sigma$.

For readability, we introduce the concept of "β-equivalence" for MM-1QFAs in terms of Eq. (3) as follows.

Definition 3. Two MM-1QFAs $\mathcal{A}_1$ and $\mathcal{A}_2$ over the same input alphabet Σ are said to be β-equivalent (resp. t-β-equivalent) if $\mathcal{F}_{\mathcal{A}_1}(\omega) = \mathcal{F}_{\mathcal{A}_2}(\omega)$ for all $\omega \in \Sigma^*$ (resp. for all $\omega \in \Sigma^*$ with $|\omega| \leq t$).

The following Theorem is the basis that allowed us to present a much simpler approach to the equivalence problem of MM-1QFAs.

Theorem 3. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two MM-1QFAs over Σ. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent iff they are β-equivalent.*

Proof. We show first the "only if" part. Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent, then we have

$$\mathcal{P}_{\mathcal{A}_1}(\omega) \quad = \quad \mathcal{P}_{\mathcal{A}_2}(\omega) \qquad (\forall\, \omega \in \Sigma^*). \tag{4}$$

We assert that $\mathcal{F}_{\mathcal{A}_1}(\omega) = \mathcal{F}_{\mathcal{A}_2}(\omega)$ for all $\omega \in \Sigma^*$. By Eq. (3) and Eq. (4), the assertion is obvious when $\omega = \epsilon$; For the case when $\omega = x_1 x_2 \cdots x_n$ with $n \geq 1$, by Eq. (4) we have

$$\mathcal{P}_{\mathcal{A}_1}(x_1 \cdots x_n) - \mathcal{P}_{\mathcal{A}_1}(x_1 \cdots x_{n-1}) \quad = \quad \mathcal{P}_{\mathcal{A}_2}(x_1 \cdots x_n) - \mathcal{P}_{\mathcal{A}_2}(x_1 \cdots x_{n-1})$$

i.e., $\mathcal{F}_{\mathcal{A}_1}(x_1 \cdots x_n) = \mathcal{F}_{\mathcal{A}_2}(x_1 \cdots x_n)$. Thus the assertion holds for all $\omega \in \Sigma^*$.

We show next the "if" part of the Theorem. By hypothesis

$$\mathcal{F}_{\mathcal{A}_1}(\omega) \quad = \quad \mathcal{F}_{\mathcal{A}_2}(\omega) \qquad (\forall\, \omega \in \Sigma^*) \tag{5}$$

Also, it is clear that $\mathcal{P}_{\mathcal{A}_1}(\omega) = \mathcal{P}_{\mathcal{A}_2}(\omega)$ when $\omega = \epsilon$. Assume that $\omega = x_1 x_2 \cdots x_n$ with $n \geq 1$. For simplicity, denote

$$a_n \quad = \quad \mathcal{P}_{\mathcal{A}_1}(x_1 \cdots x_n)$$

and

$$b_n \quad = \quad \mathcal{P}_{\mathcal{A}_2}(x_1 \cdots x_n)$$

for all $n \geq 1$. Setting $a_0 = \mathcal{P}_{\mathcal{A}_1}(\epsilon)$ and $b_0 = \mathcal{P}_{\mathcal{A}_2}(\epsilon)$, then by Eq. (3), we find that

$$\mathcal{F}_{\mathcal{A}_1}(x_1 \cdots x_n) = a_n - a_{n-1} \qquad \text{and} \quad \mathcal{F}_{\mathcal{A}_2}(x_1 \cdots x_n) = b_n - b_{n-1}.$$

Thus,

$$\begin{aligned}
\mathcal{P}_{\mathcal{A}_1}(x_1 \cdots x_n) &= a_0 + \sum_{k=1}^{n}(a_k - a_{k-1}) \\
&= \mathcal{F}_{\mathcal{A}_1}(\epsilon) + \sum_{k=1}^{n} \mathcal{F}_{\mathcal{A}_1}(x_1 \cdots x_k) \\
&= \mathcal{F}_{\mathcal{A}_2}(\epsilon) + \sum_{k=1}^{n} \mathcal{F}_{\mathcal{A}_2}(x_1 \cdots x_k) \qquad \text{(by Eq. (5))} \\
&= b_0 + \sum_{k=1}^{n}(b_k - b_{k-1}) = \mathcal{P}_{\mathcal{A}_2}(x_1 \cdots x_n).
\end{aligned}$$

Theorem 3 follows. ■

Remark 2. In fact, it is clear that the proof of Theorem 3 can be extended to prove that two MM-1QFAs $\mathcal{A}_1$ and $\mathcal{A}_2$ are t-equivalent if and only if they are t-β-equivalent.

For convenience, we expand Eq. (3) as follows. Note that, if $\omega = x_1 x_2 \cdots x_n$, then we have

$$\begin{aligned}
\mathcal{F}_{\mathcal{A}}(\omega) &= \mathcal{P}_{\mathcal{A}}(x_1 x_2 \cdots x_n) - \mathcal{P}_{\mathcal{A}}(x_1 x_2 \cdots x_{n-1}) \\
&= \langle \pi | \left(\prod_{i=n-1}^{1} (P(g)U(x_i)) \right)^{\dagger} U(x_n)^{\dagger} P(a)^{\dagger} P(a) U(x_n) \left(\prod_{i=n-1}^{1} (P(g)U(x_i)) \right) |\pi\rangle \\
&\quad + \langle \pi | \left(\prod_{i=n}^{1} (P(g)U(x_i)) \right)^{\dagger} U(\$)^{\dagger} P(a)^{\dagger} P(a) U(\$) \left(\prod_{i=n}^{1} (P(g)U(x_i)) \right) |\pi\rangle \\
&\quad - \langle \pi | \left(\prod_{i=n-1}^{1} (P(g)U(x_i)) \right)^{\dagger} U(\$)^{\dagger} P(a)^{\dagger} P(a) U(\$) \left(\prod_{i=n-1}^{1} (P(g)U(x_i)) \right) |\pi\rangle
\end{aligned}$$

Setting $A(\sigma) = P(g)U(\sigma)$ for each $\sigma \in \Sigma$ and noting that $P(a)^2 = P(a)$, $P(a)^\dagger = P(a)$, we find that

$$\mathcal{F}_{\mathcal{A}}(\omega) = \langle \pi | \, \eta_{\mathcal{A}}(\omega) \, | \pi \rangle \tag{6}$$

where

$$\eta_{\mathcal{A}}(\omega) = \begin{cases} \Big(\prod_{i=n-1}^{1} A(x_i) \Big)^\dagger \delta_{\mathcal{A}}(x_n) \Big(\prod_{i=n-1}^{1} A(x_i) \Big), & \omega = x_1 x_2 \cdots x_n \in \Sigma^n; \\ \\ U(\$)^\dagger P(a) U(\$), & \omega = \epsilon. \end{cases}$$

and $\delta_{\mathcal{A}}(x_n)$ is given by

$$\delta_{\mathcal{A}}(x_n) = U(x_n)^\dagger P(a) U(x_n) + A(x_n)^\dagger U(\$)^\dagger P(a) U(\$) A(x_n) - U(\$)^\dagger P(a) U(\$).$$

We further introduce the following auxiliary definitions needed in the sequel.

Definition 4. Let $\mathcal{A}_i = (Q_i, \{U_i(\sigma)\}_{\sigma \in \Sigma \cup \{\$\}}, |\pi_i\rangle, \mathcal{O}_i)$, $i = 1, 2$, be two MM-1QFAs over the alphabet Σ, where $\mathcal{O}_1 = \{P_1(a), P_1(g), P_1(r)\}$ and $\mathcal{O}_2 = \{P_2(a), P_2(g), P_2(r)\}$. The *diagonal sum* of $\mathcal{A}_1$ and $\mathcal{A}_2$, denoted by $\mathcal{A}_1 \oplus \mathcal{A}_2$, is an MM-1QFA, defined to be

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 = (Q, \{U(\sigma)\}_{\sigma \in \Sigma \cup \{\$\}}, |\vartheta\rangle, \mathcal{O})$$

where $Q = Q_1 \cup Q_2$ with $Q_1 \cap Q_2 = \emptyset$, $U(\sigma) = U_1(\sigma) \oplus U_2(\sigma)$ for each $\sigma \in \Sigma \cup \{\$\}$, $|\vartheta\rangle \in \mathcal{H}_{|Q_1|+|Q_2|}$ is an arbitrary unit vector and $\mathcal{O} = \{P_1(a) \oplus P_2(a), P_1(g) \oplus P_2(g), P_1(r) \oplus P_2(r)\}$.

It should be noted that the initial vector $|\vartheta\rangle$ of $\mathcal{A}$ is arbitrary. Of particular importance are the following two vectors

$$|\varphi\rangle = \begin{pmatrix} |\pi_1\rangle \\ 0 \end{pmatrix}, \qquad |\psi\rangle = \begin{pmatrix} 0 \\ |\pi_2\rangle \end{pmatrix}. \tag{7}$$

With respect to the above vectors, we introduce the following technical definition.

Definition 5. Let $\mathcal{A}_i = (Q_i, \{U_i(\sigma)\}_{\sigma\in\Sigma\cup\{\$\}}, |\pi_i\rangle, \mathcal{O}_i)$, $i = 1, 2$, be two MM-1QFAs over Σ. Let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Then, the vectors $|\varphi\rangle$ and $|\psi\rangle$, defined in Eqs. (7), are said to be equivalent with respect to $\mathcal{A}$ (resp. t-equivalent with respect to $\mathcal{A}$), if

$$\langle\varphi|\,\eta_{\mathcal{A}}(\omega)\,|\varphi\rangle \quad = \quad \langle\psi|\,\eta_{\mathcal{A}}(\omega)|\psi\rangle \tag{8}$$

for all $\omega \in \Sigma^*$ (resp. for all $\omega \in \Sigma^*$ with $|\omega| \leq t$).

Remark 3. In fact, the left side of Eq. (8) is $\mathcal{F}_{\mathcal{A}_1}(\omega)$, and the right side of Eq. (8) is $\mathcal{F}_{\mathcal{A}_2}(\omega)$. To see this, one can verify without difficulty that

$$\eta_{\mathcal{A}}(\omega) \quad = \quad \begin{pmatrix} \eta_{\mathcal{A}_1}(\omega) & 0 \\ 0 & \eta_{\mathcal{A}_2}(\omega) \end{pmatrix} \tag{9}$$

for all $\omega \in \Sigma^*$. Hence, it is clear that

$$\langle\varphi|\,\eta_{\mathcal{A}}(\omega)\,|\varphi\rangle = \langle\pi_1|\,\eta_{\mathcal{A}_1}(\omega)\,|\pi_1\rangle = \mathcal{F}_{\mathcal{A}_1}(\omega)$$

and

$$\langle\psi|\,\eta_{\mathcal{A}}(\omega)\,|\psi\rangle = \langle\pi_2|\,\eta_{\mathcal{A}_2}(\omega)\,|\pi_2\rangle = \mathcal{F}_{\mathcal{A}_2}(\omega).$$

Let $\mathcal{A} = (Q, \{U(\sigma)\}_{\sigma\in\Sigma\cup\{\$\}}, |\pi\rangle, \mathcal{O})$ be an MM-1QFA. Suppose that $\omega = x_1x_2\cdots x_n \in \Sigma^*$ and $y \in \Sigma$ are arbitrary. It should be noted that

$$\begin{aligned}
\eta_{\mathcal{A}}(y\omega) \quad &= \quad \left[\left(\prod_{i=n-1}^{1} A(x_i)\right) A(y)\right]^{\dagger} \delta_{\mathcal{A}}(x_n) \left[\left(\prod_{i=n-1}^{1} A(x_i)\right) A(y)\right] \\
&= \quad A(y)^{\dagger} \left[\left(\prod_{i=n-1}^{1} A(x_i)\right)^{\dagger} \delta_{\mathcal{A}}(x_n) \left(\prod_{i=n-1}^{1} A(x_i)\right)\right] A(y) \\
&= \quad A(y)^{\dagger}\, \eta_{\mathcal{A}}(\omega)\, A(y)
\end{aligned} \tag{10}$$

Remark 4. Eq. (10) pays a key role in the proof of Lemma 5 in the sequel, and is inspired by the proof of Lemma **8** in [27] attributed to Li and Qiu, and by the proof of Theorem **1** in [4] attributed to Carlyle.

2.4. On relevant definitions of E-1QFAs

As mentioned earlier, an E-1QFA is a theoretical model for a quantum computer with finite workspace [22] which can be seen as a generalization of MM-1QFA. In what follows, we first state the definition of E-1QFA as follows.

Definition 6 (*modification of [22]*)**.** An E-1QFA defined on the alphabet Σ is a sextuple

$$\mathcal{A} \quad = \quad (Q, Q_{acc}, Q_{rej}, \{\mathcal{U}_\sigma\}_{\sigma\in\Sigma\cup\{\#,\$\}}, \rho, \mathcal{O})$$

where Q is a finite set of states, and $Q_{acc} \subseteq Q$, $Q_{rej} \subseteq Q$ are the accepting and rejecting states sets, respectively; For each symbol $\sigma \in \Sigma \cup \{\#, \$\}$ where $\#$ and $\$$ are, respectively, the left and right end-marker, $\mathcal{A}$ has a corresponding "superoperator" [1] $\mathcal{U}_\sigma$; The density matrix $\rho = |q_0\rangle\langle q_0|$ ($q_0 \in Q$) is the initial state of $\mathcal{A}$, and $\mathcal{O} = \{P_a, P_g, P_r\}$ where P_a, P_g and P_r are the orthogonal projection onto $\text{span}\{|q\rangle | q \in Q_{acc}\}$, $\text{span}\{|q\rangle | q \in Q\backslash(Q_{acc} \cup Q_{rej})\}$ and $\text{span}\{|q\rangle | q \in Q_{rej}\}$, respectively.

The computing procedure of an E-1QFA is similar to that of an MM-1QFA. For more details, we refer to [22] (cf. [22], section 3.2). Therefore, for a word $\omega = x_1x_2\cdots x_n \in \Sigma^*$, an E-1QFA $\mathcal{A}$ induces a word function as follows

$$p_{\mathcal{A}}(\#\omega\$) \quad = \quad \text{Tr}\left(\sum_{k=0}^{n+1}(P_a \circ \mathcal{U}_{x_k}) \circ \left[\prod_{i=k-1}^{0}(P_g \circ \mathcal{U}_{x_i})\right](\rho)\right) \tag{11}$$

where x_0 = '#', x_{n+1} = '\$'. The probability of $\mathcal{A}$ accepting ω thus can be defined as

$$\mathcal{P}_{\mathcal{A}}(\omega) \quad = \quad p_{\mathcal{A}}(\#\omega\$). \tag{12}$$

[1]Here, the "superoperator" [22] is given by a composition of a finite sequence of unitary transformations and orthogonal measurements on the space $\mathbb{C}^Q$ (i.e., $\mathcal{H}_Q$, see subsection 2.2). However, if we allow any POVM measurements instead of orthogonal measurements, then the set of "superoperators" consists of all possible quantum operations (superoperators) [33].

In Eq. (11), the formal product $\prod_{i=m}^{0} \mathcal{U}_i$ is given by

$$\prod_{i=m}^{0} \mathcal{U}_i = \mathcal{U}_m \circ \mathcal{U}_{m-1} \circ \cdots \circ \mathcal{U}_0.$$

By $\prod_{i=-1}^{0} (P_g \circ U_{x_i})$ we mean $\mathcal{I}$, i.e. the identity superoperator from $\mathcal{L}(\mathcal{H}_Q)$ to $\mathcal{L}(\mathcal{H}_Q)$. The term $P_g \circ \mathcal{U}$ is defined by the following rule

$$\begin{aligned} P_g \circ \mathcal{U}(\rho') &= P_g \left(\sum_i M_i \rho' M_i^\dagger \right) P_g^\dagger \\ &= \sum_i \left[(P_g M_i) \rho' (P_g M_i)^\dagger \right] \end{aligned}$$

for any $\rho' \in \mathcal{L}(\mathcal{H}_Q)$, where $\{M_i\}$ are Kraus operators of $\mathcal{U}$. Also, $P_a \circ \mathcal{U}$ is defined similarly.

Definition 7. Two E-1QFAs $\mathcal{A}_1$ and $\mathcal{A}_2$ over the same alphabet Σ are said to be equivalent (resp. t-equivalent), if $\mathcal{P}_{\mathcal{A}_1}(\omega) = \mathcal{P}_{\mathcal{A}_2}(\omega)$ for all $\omega \in \Sigma^*$ (resp. for all $\omega \in \Sigma^*$ with $|\omega| \leq t$).

Similarly, the probability $\mathcal{P}_{\mathcal{A}}(\omega)$ of $\mathcal{A}$ accepting ω given by Eq. (12) is in a "cumulation" manner. We can define another version which is in a "non-cumulation" manner as follows

$$\mathcal{F}_{\mathcal{A}}(\omega) = \begin{cases} \mathcal{P}_{\mathcal{A}}(x_1 x_2 \cdots x_n) - \mathcal{P}_{\mathcal{A}}(x_1 x_2 \cdots x_{n-1}), & \omega = x_1 x_2 \cdots x_n; \\ \mathcal{P}_{\mathcal{A}}(\epsilon), & \omega = \epsilon. \end{cases} \tag{13}$$

Similar to the case of MM-1QFAs, we define the concept of "β-equivalence" for E-1QFAs in terms of Eq. (13) as follows.

Definition 8. Two E-1QFAs $\mathcal{A}_1$ and $\mathcal{A}_2$ over the same alphabet Σ are said to be β-equivalent (resp. t-β-equivalent) if $\mathcal{F}_{\mathcal{A}_1}(\omega) = \mathcal{F}_{\mathcal{A}_2}(\omega)$ for all $\omega \in \Sigma^*$ (resp. for all $\omega \in \Sigma^*$ with $|\omega| \leq t$).

The following Theorem allows us to apply the approach to the equivalence problem of MM-1QFAs to that of E-1QFAs.

Theorem 4. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two E-1QFAs over the same alphabet Σ. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent (resp. t-equivalent) iff they are β-equivalent (resp. t-β-equivalent).*

Proof. The proof is similar to that of Theorem 3, and the detail is omitted. ■

Note that, if $\omega = x_1x_2\cdots x_n$ with $n \geq 1$, then $\mathcal{F}_{\mathcal{A}}(\omega)$ can be reduced as follows

$$\mathcal{F}_{\mathcal{A}}(\omega) = \mathrm{Tr}\left(\Big(P_a \circ \mathcal{U}_{x_n} + (P_a \circ \mathcal{U}_{\$}) \circ (P_g \circ \mathcal{U}_{x_n}) - P_a \circ \mathcal{U}_{\$}\Big) \circ \prod_{i=n-1}^{0}\Big(P_g \circ \mathcal{U}_{x_i}\Big)(\rho)\right) \tag{14}$$

We could rewrite Eq. (14) as

$$\mathcal{F}_{\mathcal{A}}(\omega) \quad = \quad \mathrm{Tr}\left((P_a \circ \mathcal{U}_{x_n} + (P_a \circ \mathcal{U}_{\$}) \circ (P_g \circ \mathcal{U}_{x_n}) - P_a \circ \mathcal{U}_{\$})(\rho')\right)$$

where

$$\begin{aligned}
\rho' &= \prod_{i=n-1}^{0}\Big(P_g \circ \mathcal{U}_{x_i}\Big)(\rho) \\
&= \sum_{i_{x_{n-1}}}(P_gM_{i_{x_{n-1}}})\left[\cdots\left[\sum_{i_{x_0}}(P_gM_{i_{x_0}})|q_0\rangle\langle q_0|(P_gM_{i_{x_0}})^\dagger\right]\cdots\right](P_gM_{i_{x_{n-1}}})^\dagger \\
&= \sum_{i_{x_{n-1}}}\cdots\sum_{i_{x_0}}\Big[(P_gM_{i_{x_{n-1}}})\cdots(P_gM_{i_{x_0}})|q_0\rangle\langle q_0|(P_gM_{i_{x_0}})^\dagger\cdots(P_gM_{i_{x_{n-1}}})^\dagger\Big].
\end{aligned}$$

Setting $P_aM_j = A_j$ and $P_gM_j = B_j$ for all M_j, then a simple calculation leads to the following

$$\begin{aligned}
\mathrm{Tr}(P_a \circ \mathcal{U}_{x_n}(\rho')) &= \mathrm{Tr}\left(\sum_{i_{x_n}}\sum_{i_{x_{n-1}}}\cdots\sum_{i_{x_0}}A_{i_{x_n}}B_{i_{x_{n-1}}}\cdots B_{i_{x_0}}|q_0\rangle\langle q_0|B^\dagger_{i_{x_0}}\cdots B^\dagger_{i_{x_{n-1}}}A^\dagger_{i_{x_n}}\right) \\
&\qquad \text{(by the commutative law of Tr, we have)} \\
&= \mathrm{Tr}\left(\langle q_0|\left[\sum_{i_{x_0}}\cdots\sum_{i_{x_{n-1}}}\sum_{i_{x_n}}B^\dagger_{i_{x_0}}\cdots B^\dagger_{i_{x_{n-1}}}A^\dagger_{i_{x_n}}A_{i_{x_n}}B_{i_{x_{n-1}}}\cdots B_{i_{x_0}}\right]|q_0\rangle\right) \\
&= \langle q_0|\left[\sum_{i_{x_0}}\cdots\sum_{i_{x_{n-1}}}B^\dagger_{i_{x_0}}\cdots B^\dagger_{i_{x_{n-1}}}\left(\sum_{i_{x_n}}A^\dagger_{i_{x_n}}A_{i_{x_n}}\right)B_{i_{x_{n-1}}}\cdots B_{i_{x_0}}\right]|q_0\rangle;
\end{aligned}$$

$$\mathrm{Tr}(P_a \circ \mathcal{U}_\$ \circ P_g \circ \mathcal{U}_{x_n}(\rho')) \;=$$

$$\langle q_0| \left[\sum_{i_{x_0}} \cdots \sum_{i_{x_{n-1}}} B^\dagger_{i_{x_0}} \cdots B^\dagger_{i_{x_{n-1}}} \left(\sum_{i_{x_n}} \sum_{i_{x_{n+1}}} B^\dagger_{i_{x_n}} A^\dagger_{i_{x_{n+1}}} A_{i_{x_{n+1}}} B_{i_{x_n}} \right) B_{i_{x_{n-1}}} \cdots B_{i_{x_0}} \right] |q_0\rangle;$$

and

$$\mathrm{Tr}(P_a \circ \mathcal{U}_\$(\rho')) \;=\; \langle q_0| \left[\sum_{i_{x_0}} \cdots \sum_{i_{x_{n-1}}} B^\dagger_{i_{x_0}} \cdots B^\dagger_{i_{x_{n-1}}} \left(\sum_{i_{x_{n+1}}} A^\dagger_{i_{x_{n+1}}} A_{i_{x_{n+1}}} \right) B_{i_{x_{n-1}}} \cdots B_{i_{x_0}} \right] |q_0\rangle.$$

It is easy to verify that

$$\begin{aligned}
\mathcal{F}_\mathcal{A}(\omega) &= \mathrm{Tr}\left(P_a \circ \mathcal{U}_{x_n}(\rho')\right) + \mathrm{Tr}\left((P_a \circ \mathcal{U}_\$) \circ (P_g \circ \mathcal{U}_{x_n})(\rho')\right) - \mathrm{Tr}\left(P_a \circ \mathcal{U}_\$(\rho')\right) \\
&= \langle q_0| \left[\sum_{i_{x_0}} \cdots \sum_{i_{x_{n-1}}} B^\dagger_{i_{x_0}} \cdots B^\dagger_{i_{x_{n-1}}} \xi_\mathcal{A}(x_n)\; B_{i_{x_{n-1}}} \cdots B_{i_{x_0}} \right] |q_0\rangle
\end{aligned}$$

where $\xi_\mathcal{A}(x_n)$ is given by

$$\xi_\mathcal{A}(x_n) \;=\; \sum_{i_{x_n}} A^\dagger_{i_{x_n}} A_{i_{x_n}} + \sum_{i_{x_n}} \sum_{i_{x_{n+1}}} B^\dagger_{i_{x_n}} A^\dagger_{i_{x_{n+1}}} A_{i_{x_{n+1}}} B_{i_{x_n}} - \sum_{i_{x_{n+1}}} A^\dagger_{i_{x_{n+1}}} A_{i_{x_{n+1}}}.$$

Since an E-1QFA has a left end-marker '#' which is different from an MM-1QFA, the approach to the equivalence problem of MM-1QFAs may not be applied directly to that of E-1QFAs. We need a more careful pre-treatment. Thus, denote

$$\vartheta_\mathcal{A}(\omega) \;=\; \sum_{i_{x_1}} \cdots \sum_{i_{x_{n-1}}} B^\dagger_{i_{x_1}} \cdots B^\dagger_{i_{x_{n-1}}} \xi_\mathcal{A}(x_n) B_{i_{x_{n-1}}} \cdots B_{i_{x_1}}$$

and

$$\begin{aligned}
\theta_\mathcal{A}(\omega) &= \sum_{i_{x_0}} B^\dagger_{i_{x_0}} \left(\sum_{i_{x_1}} \cdots \sum_{i_{x_{n-1}}} B^\dagger_{i_{x_1}} \cdots B^\dagger_{i_{x_{n-1}}} \xi_\mathcal{A}(x_n) B_{i_{x_{n-1}}} \cdots B_{i_{x_1}} \right) B_{i_{x_0}} \\
&= \sum_{i_{x_0}} B^\dagger_{i_{x_0}} \vartheta_\mathcal{A}(\omega) B_{i_{x_0}} \qquad (15)
\end{aligned}$$

for any $\omega = x_1 x_2 \cdots x_n \in \Sigma^*$.

The following technical definition of "*diagonal sum*" of E-1QFAs will pay the same role as the definition of "*diagonal sum*" of MM-1QFAs.

Definition 9. Let $\mathcal{A}_i = (Q_i, Q_{acc,i}, Q_{rej,i}, \{\mathcal{U}_\sigma^{(i)}\}_{\sigma\in\Sigma\cup\{\#,\$\}}, \rho_i, \mathcal{O}_i)$, $i = 1, 2$, be two E-1QFAs over Σ where $\mathcal{O}_i = \{P_a^{(i)}, P_g^{(i)}, P_r^{(i)}\}$, and $\rho_i = |q_0^{(i)}\rangle\langle q_0^{(i)}|$. The *diagonal sum* of $\mathcal{A}_1$ and $\mathcal{A}_2$, denoted as $\mathcal{A}_1 \oplus \mathcal{A}_2$, is defined to be

$$\mathcal{A} \quad \triangleq \quad \mathcal{A}_1 \oplus \mathcal{A}_2 = (Q, Q_{acc}, Q_{rej}, \{\mathcal{U}_\sigma\}_{\sigma\in\Sigma\cup\{\#,\$\}}, \varrho, \mathcal{O})$$

where $Q = Q_1 \cup Q_2$ with $Q_1 \cap Q_2 = \emptyset$, $\mathcal{U}_\sigma = \mathcal{U}_\sigma^{(1)} \oplus \mathcal{U}_\sigma^{(2)}$ [2], $\varrho \in \mathcal{L}(\mathcal{H}_{Q_1\cup Q_2})$ is an arbitrary density matrix, and $\mathcal{O} = \{P_a^{(1)} \oplus P_a^{(2)}, P_g^{(1)} \oplus P_g^{(2)}, P_r^{(1)} \oplus P_r^{(2)}\}$.

Also, as the case of MM-1QFA, the initial state ϱ of $\mathcal{A}$ is arbitrary. Of particular importance are the following

$$\varphi = \begin{pmatrix} \rho_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \psi = \begin{pmatrix} 0 & 0 \\ 0 & \rho_2 \end{pmatrix}. \tag{16}$$

Similarly, we introduce the following definition.

Definition 10. Let $\mathcal{A}_i = (Q_i, Q_{acc,i}, Q_{rej,i}, \{\mathcal{U}_\sigma^{(i)}\}_{\sigma\in\Sigma\cup\{\#,\$\}}, \rho_i, \mathcal{O}_i)$, $i = 1, 2$, be two E-1QFAs over Σ where $\mathcal{O}_i = \{P_a^{(i)}, P_g^{(i)}, P_r^{(i)}\}$, and $\rho_i = |q_0^{(i)}\rangle\langle q_0^{(i)}|$. Let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Then the density matrices φ and ψ, defined in Eqs. (16), are said to be equivalent with respect to $\mathcal{A}$ (resp. t-equivalent with respect to $\mathcal{A}$), if

$$(\langle q_0^{(1)}|, \mathbf{0})\ \theta_{\mathcal{A}}(\omega) \begin{pmatrix} |q_0^{(1)}\rangle \\ \mathbf{0} \end{pmatrix} \quad = \quad (\mathbf{0}, \langle q_0^{(2)}|)\ \theta_{\mathcal{A}}(\omega) \begin{pmatrix} \mathbf{0} \\ |q_0^{(2)}\rangle \end{pmatrix} \tag{17}$$

for all $\omega \in \Sigma^*$ (resp. for all $\omega \in \Sigma^*$ with $|\omega| \leq t$).

Remark 5. Also, It is easy to find that

$$\theta_{\mathcal{A}}(\omega) \quad = \quad \begin{pmatrix} \theta_{\mathcal{A}_1}(\omega) & 0 \\ 0 & \theta_{\mathcal{A}_2}(\omega) \end{pmatrix} \tag{18}$$

[2]Here, if $\mathcal{U}_\sigma^{(1)}$ and $\mathcal{U}_\sigma^{(2)}$ are given by the operators sets $\{E_i\}$ and $\{Z_j\}$, respectively, then $\mathcal{U}_\sigma$ can be defined to be given by the operators set $\{M_i\} \triangleq \{E_i \oplus Z_i\}$. It is not hard to see that $\sum_i M_i^\dagger M_i = \begin{pmatrix} \sum_i E_i^\dagger E_i & 0 \\ 0 & \sum_i Z_i^\dagger Z_i \end{pmatrix}$ and $\mathcal{U}_\sigma(\rho) = \begin{pmatrix} \sum_i E_i \rho_1 E_i^\dagger & 0 \\ 0 & \sum_i Z_i \rho_2 Z_i^\dagger \end{pmatrix} = \begin{pmatrix} \mathcal{U}_\sigma^{(1)}(\rho_1) & 0 \\ 0 & \mathcal{U}_\sigma^{(2)}(\rho_2) \end{pmatrix}$ for any $\rho = \rho_1 \oplus \rho_2$.

for all $\omega \in \Sigma^*$. Thus,

$$(\langle q_0^{(1)}|, \mathbf{0})\ \theta_{\mathcal{A}}(\omega) \begin{pmatrix} |q_0^{(1)}\rangle \\ \mathbf{0} \end{pmatrix} = \langle q_0^{(1)}|\theta_{\mathcal{A}_1}(\omega)|q_0^{(1)}\rangle = \mathcal{F}_{\mathcal{A}_1}(\omega)$$

and

$$(\mathbf{0}, \langle q_0^{(2)}|)\ \theta_{\mathcal{A}}(\omega) \begin{pmatrix} \mathbf{0} \\ |q_0^{(2)}\rangle \end{pmatrix} = \langle q_0^{(2)}|\theta_{\mathcal{A}_2}(\omega)|q_0^{(2)}\rangle = \mathcal{F}_{\mathcal{A}_2}(\omega).$$

Namely, the left side of Eq. (17) is $\mathcal{F}_{\mathcal{A}_1}(\omega)$, and the right side of Eq. (17) is $\mathcal{F}_{\mathcal{A}_2}(\omega)$.

In the following, we derive a relation which is similar to Eq. (10). Let $\mathcal{A}_i = (Q_i, Q_{acc,i}, Q_{rej,i}, \{\mathcal{U}_\sigma^{(i)}\}_{\sigma\in\Sigma\cup\{\#,\$\}}, \rho_i, \mathcal{O}_i)$, $i = 1, 2$, be two E-1QFAs, and $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Suppose that $\omega = x_1x_2\cdots x_n \in \Sigma^*$ and $y \in \Sigma$ are arbitrary. Then, it is clear that

$$\begin{aligned}
\vartheta_{\mathcal{A}}(y\omega) &= \sum_{i_y}\sum_{i_{x_1}}\cdots\sum_{i_{x_{n-1}}} B_{i_y}^{\dagger}B_{i_{x_1}}^{\dagger}\cdots B_{i_{x_{n-1}}}^{\dagger}\xi_{\mathcal{A}}(x_n)B_{i_{x_{n-1}}}\cdots B_{i_{x_1}}B_{i_y} \\
&= \sum_{i_y} B_{i_y}^{\dagger}\left[\sum_{i_{x_1}}\cdots\sum_{i_{x_{n-1}}} B_{i_{x_1}}^{\dagger}\cdots B_{i_{x_{n-1}}}^{\dagger}\xi_{\mathcal{A}}(x_n)B_{i_{x_{n-1}}}\cdots B_{i_{x_1}}\right]B_{i_y} \\
&= \sum_{i_y} B_{i_y}^{\dagger}\vartheta_{\mathcal{A}}(\omega)B_{i_y}. \qquad (19)
\end{aligned}$$

Remark 6. Just as the relation: Eq. (10), will play an important role in the proof of Lemma 5, this relation, i.e., Eq. (19), will play a similar role in the proof of Lemma 8.

3. Proof of Theorem 1

In this section, we present our approach to the equivalence problem of MM-1QFAs. Let us first introduce some convenient notation.

For each $i \geq 0$, let $H_{\mathcal{A}}(i)$ denote the set $\{\eta_{\mathcal{A}}(\omega)|\omega \in \Sigma^*, |\omega| \leq i\}$ where $H_{\mathcal{A}}(0) = \{U(\$)^{\dagger}P(a)U(\$)\}$, and $\mathcal{V}_{\mathcal{A}}(i)$ the vector space spanned by $H_{\mathcal{A}}(i)$, i.e., $\mathcal{V}_{\mathcal{A}}(i)$ =span$\{H_{\mathcal{A}}(i)\}$. Then it is clear that $\mathcal{V}_{\mathcal{A}}(i) \subseteq \mathcal{V}_{\mathcal{A}}(i+1)$ since $H_{\mathcal{A}}(i) \subseteq H_{\mathcal{A}}(i+1)$. We prove

Lemma 5. *Let* $\mathcal{A} = (Q, \{U(\sigma)\}_{\sigma\in\Sigma\cup\{\$\}}, |\pi\rangle, \mathcal{O})$ *be an MM-1QFA. Then there exists an integer* $l < |Q|^2$*, such that* $\mathcal{V}_{\mathcal{A}}(l) = \mathcal{V}_{\mathcal{A}}(l+j)$ *for all* $j \geq 1$*.*

Proof. We show first that there exists an integer $l < |Q|^2$ such that $\mathcal{V}_{\mathcal{A}}(l) = \mathcal{V}_{\mathcal{A}}(l+1)$. Suppose there exists no such an integer, then for all $i \geq 0$ we find that $\mathcal{V}_{\mathcal{A}}(i) \neq \mathcal{V}_{\mathcal{A}}(i+1)$. This gives

$$\mathcal{V}_{\mathcal{A}}(0) \subset \mathcal{V}_{\mathcal{A}}(1) \subset \cdots \subseteq \mathbb{M}_{|Q|}(\mathbb{C}).$$

Since $\dim\mathbb{M}_{|Q|}(\mathbb{C}) = |Q|^2$ and $\dim\mathcal{V}_{\mathcal{A}}(0) \geq 1$, we have $\dim\mathcal{V}_{\mathcal{A}}(|Q|^2) \geq |Q|^2+1$ which contradicts the fact that $\mathcal{V}_{\mathcal{A}}(|Q|^2) \subseteq \mathbb{M}_{|Q|}(\mathbb{C})$.

We show next that $\mathcal{V}_{\mathcal{A}}(l) = \mathcal{V}_{\mathcal{A}}(l+j)$ for all $j \geq 1$ by induction on j. For $j = 1$, we have shown in the above. Assume it is true for $j < m$ ($m > 1$) and consider the case $j = m$. Note that $H_{\mathcal{A}}(l+m) = H_{\mathcal{A}}(l+(m-1)) \cup \{\eta_{\mathcal{A}}(\omega)|\omega \in \Sigma^*, |\omega| = l+m\}$ and $\mathcal{V}_{\mathcal{A}}(l+m) = \mathrm{span}\{H_{\mathcal{A}}(l+m)\}$. Thus, for all $\eta \in \mathcal{V}_{\mathcal{A}}(l+m)$, η can be written as

$$\eta \quad = \quad \sum_{i_1} a_{i_1}\eta_{\mathcal{A}}(\omega_{i_1}) + \sum_{i_2} a_{i_2}\eta_{\mathcal{A}}(\omega_{i_2})$$

where $\eta_{\mathcal{A}}(\omega_{i_1}) \in H_{\mathcal{A}}(l+(m-1))$ and $\eta_{\mathcal{A}}(\omega_{i_2}) \in \{\eta_{\mathcal{A}}(\omega)|\omega \in \Sigma^*, |\omega| = l+m\}$. Clearly, $\sum_{i_1} a_{i_1}\eta_{\mathcal{A}}(\omega_{i_1}) \in \mathcal{V}_{\mathcal{A}}(l+(m-1))$. We assert that $\sum_{i_2} a_{i_2}\eta_{\mathcal{A}}(\omega_{i_2}) \in \mathcal{V}_{\mathcal{A}}(l+(m-1))$. To see this it suffices to prove that, for each $\eta_{\mathcal{A}}(\omega_{i_2}) \in \{\eta_{\mathcal{A}}(\omega)|\omega \in \Sigma^*, |\omega| = l+m\}$, it can be expressed as $\eta_{\mathcal{A}}(\omega_{i_2}) = \sum_z b_z\eta_{\mathcal{A}}(\omega_z)$ with $\eta_{\mathcal{A}}(\omega_z) \in H_{\mathcal{A}}(l+(m-1))$ and $b_z \in \mathbb{C}$. This can be deduced as follows.

Note that ω_{i_2} can be written as $\omega_{i_2} = y_{i_2}\omega'_{i_2}$ with $y_{i_2} \in \Sigma$ and $|\omega'_{i_2}| = l+(m-1) < l+m$. By induction hypothesis, $\eta_{\mathcal{A}}(\omega'_{i_2}) \in \mathcal{V}_{\mathcal{A}}(l) = \mathcal{V}_{\mathcal{A}}(l+(m-1))$. Thus,

$$\eta_{\mathcal{A}}(\omega'_{i_2}) \quad = \quad \sum_k c_k\eta_{\mathcal{A}}(\omega'_{i_2,k}) \qquad (\omega'_{i_2,k} \in \Sigma^*,\ |\omega'_{i_2,k}| \leq l \text{ and } c_k \in \mathbb{C}). \tag{20}$$

It follows that

$$\begin{aligned}\eta_{\mathcal{A}}(\omega_{i_2}) \quad &= \quad \eta_{\mathcal{A}}(y_{i_2}\omega'_{i_2}) \\ &= \quad A(y_{i_2})^\dagger\eta_{\mathcal{A}}(\omega'_{i_2})A(y_{i_2}) \qquad \text{(by Eq. (10))}\end{aligned}$$

$$
\begin{aligned}
&= A(y_{i_2})^\dagger \left(\sum_k c_k \eta_{\mathcal{A}}(\omega'_{i_2,k}) \right) A(y_{i_2}) \qquad \text{(by Eq. (20))} \\
&= \sum_k c_k (A(y_{i_2})^\dagger \eta_{\mathcal{A}}(\omega'_{i_2,k}) A(y_{i_2})) \\
&= \sum_k c_k \eta_{\mathcal{A}}(y_{i_2}\omega'_{i_2,k}) \qquad \text{(by Eq. (10))}
\end{aligned}
$$

which means that $\eta_{\mathcal{A}}(\omega_{i_2}) \in \mathcal{V}_{\mathcal{A}}(l+1)$. Hence, the asserted result holds. ■

Remark 7. Further, it should be noted that, if $\mathcal{A}_i = (Q_i, \{U_i(\sigma)\}_{\sigma \in \Sigma \cup \{\$\}}, |\pi_i\rangle, \mathcal{O}_i)$, $i = 1, 2$, are two MM-1QFAs over Σ, and $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ is the diagonal sum of $\mathcal{A}_1$ and $\mathcal{A}_2$, then $\dim \mathcal{V}_{\mathcal{A}}(i) \le n_1^2 + n_2^2$ for all $i \ge 0$, where $n_1 = |Q_1|$ and $n_2 = |Q_2|$. To see this, let

$$
\mathcal{B} \;=\; \{\, E_{ij} \,|\, 1 \le i, j \le n_1 \,\} \cup \{\, E_{ij} \,|\, n_1 + 1 \le i, j \le n_1 + n_2 \,\}
$$

where the elements in $\mathcal{B}$ are (n_1+n_2)-order matrices having only 1 at the (i,j) entry and 0's elsewhere. Since, for all $\omega \in \Sigma^*$, $\eta_{\mathcal{A}}(\omega)$ are of the form

$$
\eta_{\mathcal{A}}(\omega) \;=\; \begin{pmatrix} \eta_{\mathcal{A}_1}(\omega) & 0 \\ 0 & \eta_{\mathcal{A}_2}(\omega) \end{pmatrix}
$$

where $\eta_{\mathcal{A}_1}(\omega)$ and $\eta_{\mathcal{A}_2}(\omega)$ are n_1-order and n_2-order complex matrices, respectively, one can easy verify that

$$
\mathcal{V}_{\mathcal{A}}(i) \;\subseteq\; \mathrm{span}\{\mathcal{B}\} \qquad (\ \forall i \ge 0\ )
$$

This implies $\dim \mathcal{V}_{\mathcal{A}}(i) \le n_1^2 + n_2^2$ for all $i \ge 0$. Hence, by replacing $\mathbb{M}_{|Q|}(\mathbb{C})$ with $\mathrm{span}\{\mathcal{B}\}$ in the proof of Lemma 5, we have $l < n_1^2 + n_2^2$. The above remark shows the following

Corollary 6. *Let $\mathcal{A}_i = (Q_i, \{U_i(\sigma)\}_{\sigma \in \Sigma \cup \{\$\}}, |\pi_i\rangle, \mathcal{O}_i)$, $i = 1, 2$, be two MM-1QFAs over Σ, and $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Then there exists an integer $l < n_1^2 + n_2^2$ where $n_1 = |Q_1|$ and $n_2 = |Q_2|$, such that $\mathcal{V}_{\mathcal{A}}(l) = \mathcal{V}_{\mathcal{A}}(l+j)$ for all $j \ge 1$.* ■

By virtue of Corollary 6, we prove the following

Theorem 7. *Let $\mathcal{A}_i = (Q_i, \{U_i(\sigma)\}_{\sigma\in\Sigma\cup\{\$\}}, |\pi_i\rangle, \mathcal{O}_i)$, $i = 1, 2$, be two MM-1QFAs over Σ, and $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Then the unit vectors $|\varphi\rangle$ and $|\psi\rangle$, defined in Eqs. (7), are equivalent with respect to $\mathcal{A}$ iff they are $n_1^2 + n_2^2 - 1$-equivalent with respect to $\mathcal{A}$, where n_1 and n_2 are the numbers of states in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.*

Proof. The "only if" part is obvious, we show the "if" part. Suppose that $|\varphi\rangle$ and $|\psi\rangle$ are $n_1^2 + n_2^2 - 1$-equivalent (with respect to $\mathcal{A}$), then for all $\omega = x_1x_2\cdots x_n \in \Sigma^*$ with $|\omega| < n_1^2 + n_2^2 - 1$, Eq. (8) holds. Namely,

$$\langle\varphi|\eta_{\mathcal{A}}(\omega)|\varphi\rangle \quad = \quad \langle\psi|\eta_{\mathcal{A}}(\omega)|\psi\rangle \qquad (\forall\, \eta_{\mathcal{A}}(\omega) \in H_{\mathcal{A}}(n_1^2 + n_2^2 - 1)) \tag{21}$$

By Corollary 6, for all $\omega \in \Sigma^*$, $\eta_{\mathcal{A}}(\omega) \in \mathcal{V}_{\mathcal{A}}(n_1^2+n_2^2-1) = \mathrm{span}\big\{H_{\mathcal{A}}(n_1^2+n_2^2-1)\big\}$. Hence,

$$\eta_{\mathcal{A}}(\omega) \quad = \quad \sum_i a_i\, \eta_{\mathcal{A}}(\omega_i) \qquad (\eta_{\mathcal{A}}(\omega_i) \in H_{\mathcal{A}}\big(n_1^2 + n_2^2 - 1\big)) \tag{22}$$

where $a_i \in \mathbb{C}$. It follows that

$$\begin{aligned}
\langle\varphi|\,\eta_{\mathcal{A}}(\omega)\,|\varphi\rangle \quad &= \quad \langle\varphi| \left(\sum_i a_i\, \eta_{\mathcal{A}}(\omega_i)\right) |\varphi\rangle \qquad \text{(by Eq. (22))} \\
&= \quad \sum_i a_i \Big(\langle\varphi|\,\eta_{\mathcal{A}}(\omega_i)\,|\varphi\rangle\Big) \qquad (\eta_{\mathcal{A}}(\omega_i) \in H_{\mathcal{A}}\big(n_1^2 + n_2^2 - 1\big)) \\
&= \quad \sum_i a_i \Big(\langle\psi|\,\eta_{\mathcal{A}}(\omega_i)\,|\psi\rangle\Big) \qquad \text{(by Eq. (21))} \\
&= \quad \langle\psi|\,\eta_{\mathcal{A}}(\omega)\,|\psi\rangle.
\end{aligned}$$

This means that Eq. (8) holds for all $\omega \in \Sigma^*$. Thus $|\varphi\rangle$ and $|\psi\rangle$ are equivalent with respect to $\mathcal{A}$. ■

Now, we can present the proof of Theorem 1 as follows.

Proof of Theorem 1. By Theorem 3, we only need to show that $\mathcal{A}_1$ and $\mathcal{A}_2$ are β-equivalent if and only if they are $(n_1^2 + n_2^2 - 1)$-β-equivalent.

Since it is obvious that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are β-equivalent then they are $(n_1^2 + n_2^2 - 1)$-β-equivalent, we only need to show that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are $(n_1^2+n_2^2-1)$-

β-equivalent, then they are β-equivalent. Let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. By Remark 3,

$$\mathcal{F}_{\mathcal{A}_1}(\omega) \quad = \quad \langle\varphi|\, \eta_{\mathcal{A}}(\omega)\, |\varphi\rangle \tag{23}$$

and

$$\mathcal{F}_{\mathcal{A}_2}(\omega) \quad = \quad \langle\psi|\, \eta_{\mathcal{A}}(\omega)\, |\psi\rangle \tag{24}$$

for all $\omega \in \Sigma^*$, where $|\varphi\rangle$ and $|\psi\rangle$ are defined in Eqs. (7).

Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are $(n_1^2 + n_2^2 - 1)$-β-equivalent. Then, we have

$$\mathcal{F}_{\mathcal{A}_1}(\omega) \quad = \quad \mathcal{F}_{\mathcal{A}_2}(\omega) \tag{25}$$

for all $\omega \in \Sigma^*$ with $|\omega| < n_2^2 + n_2^2 - 1$. It follows from Eq. (23), Eq. (24) and Eq. (25) that

$$\langle\varphi|\, \eta_{\mathcal{A}}(\omega)\, |\varphi\rangle \quad = \quad \langle\psi|\, \eta_{\mathcal{A}}(\omega)\, |\psi\rangle \qquad (|\omega| < n_1^2 + n_2^2 - 1)$$

This implies that $|\varphi\rangle$ and $|\psi\rangle$ are $n_1^2 + n_2^2 - 1$-equivalent with respect to $\mathcal{A}$. Thus, by Theorem 7, $|\varphi\rangle$ and $|\psi\rangle$ are equivalent with respect to $\mathcal{A}$. This implies that $\langle\varphi|\, \eta_{\mathcal{A}}(\omega)\, |\varphi\rangle = \langle\psi|\, \eta_{\mathcal{A}}(\omega)\, |\psi\rangle$ for all $\omega \in \Sigma^*$, i.e., $\mathcal{F}_{\mathcal{A}_1}(\omega) = \mathcal{F}_{\mathcal{A}_2}(\omega)$ for all $\omega \in \Sigma^*$. Hence, $\mathcal{A}_1$ and $\mathcal{A}_2$ are β-equivalent. ■

Someone may argue that the improvement from $3n_1^2 + 3n_2^2 - 1$ to $n_1^2 + n_2^2 - 1$ is not essential, since they are both quadratic. We conjecture that the upper-bound $n_1^2 + n_2^2 - 1$ can not be further improved to linear bound. However, we have no ability to prove it.

4. Proof of Theorem 2

In this section, we investigate the equivalence problem of E-1QFAs. For convenience, we will use the following notations.

For any $i \geq 0$, we let $H_{\mathcal{A}}(i)$ denote the set $\{\theta_{\mathcal{A}}(\omega) | \omega \in \Sigma^*, |\omega| \leq i\}$, $V_{\mathcal{A}}(i)$ the vector space spanned by $H_{\mathcal{A}}(i)$, $K_{\mathcal{A}}(i)$ the set $\{\vartheta_{\mathcal{A}}(\omega) | \omega \in \Sigma^*, |\omega| \leq i\}$, and $\mathcal{S}_{\mathcal{A}}(i)$ the vector space spanned by $K_{\mathcal{A}}(i)$. Also, the following relations are obvious

$$H_{\mathcal{A}}(i) \subseteq H_{\mathcal{A}}(i+1), \qquad V_{\mathcal{A}}(i) \subseteq V_{\mathcal{A}}(i+1)$$
$$K_{\mathcal{A}}(i) \subseteq K_{\mathcal{A}}(i+1), \qquad \mathcal{S}_{\mathcal{A}}(i) \subseteq \mathcal{S}_{\mathcal{A}}(i+1).$$

Lemma 8. *Let* $\mathcal{A}_i = (Q_i, Q_{acc,i}, Q_{rej,i}, \{\mathcal{U}_\sigma^{(i)}\}_{\sigma\in\Sigma\cup\{\#,\$\}}, \rho_i, \mathcal{O}_i)$*,* $i = 1,2$*, be two E-1QFAs over* Σ*, and* $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$*. Then, there exists an integer* $l < n_1^2 + n_2^2$*, where* $n_1 = |Q_1|$ *and* $n_2 = |Q_2|$*, such that* $\mathcal{S}_\mathcal{A}(l) = \mathcal{S}_\mathcal{A}(l+j)$ *for all* $j \geq 1$.

Proof. The proof of this lemma is similar to that of Lemma 5. First, we remark that, if $\mathcal{A}_i$, $i = 1,2$, are two E-1QFAs over Σ and $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, then

$$\vartheta_\mathcal{A}(\omega) \quad = \quad \begin{pmatrix} \vartheta_{\mathcal{A}_1}(\omega) & 0 \\ 0 & \vartheta_{\mathcal{A}_2}(\omega) \end{pmatrix}$$

for all $\omega \in \Sigma^*$. Hence, by the argument similar to Remark 7 we find that $\dim\mathcal{S}_\mathcal{A}(i) \leq n_1^2 + n_2^2$ for all $i \geq 0$.

Then by using the same argument that we have just used in the proof of Lemma 5, we see that there exists an integer $l < n_1^2 + n_2^2$ such that $\mathcal{S}_\mathcal{A}(l) = \mathcal{S}_\mathcal{A}(l+1)$.

Next, we show that $\mathcal{S}_\mathcal{A}(l) = \mathcal{S}_\mathcal{A}(l+j)$ for all $j \geq 1$ by induction on j. For $j = 1$, we have done. Assume it is true for $j < m$ $(m > 1)$ and consider the case $j = m$. Since $\mathcal{S}_\mathcal{A}(l+m) = \text{span}\{K_\mathcal{A}(l+m)\}$ and $K_\mathcal{A}(l+m) = K_\mathcal{A}(l+(m-1)) \cup \{\vartheta_\mathcal{A}(\omega) | \omega \in \Sigma^*, |\omega| = l+m\}$, thus, for any $\vartheta \in \mathcal{S}_\mathcal{A}(l+m)$, ϑ can be written as

$$\vartheta \quad = \quad \sum_{i_1} a_{i_1}\, \vartheta_\mathcal{A}(\omega_{i_1}) + \sum_{i_2} a_{i_2}\, \vartheta_\mathcal{A}(\omega_{i_2})$$

where $\vartheta_\mathcal{A}(\omega_{i_1}) \in K_\mathcal{A}(l+(m-1))$ and $\vartheta_\mathcal{A}(\omega_{i_2}) \in \{\vartheta_\mathcal{A}(\omega) | \omega \in \Sigma^*, |\omega| = l+m\}$. We must to show that $\vartheta \in \mathcal{S}_\mathcal{A}(l+(m-1))$. For this, we only need to prove that

$$\sum_{i_2} a_{i_2}\, \vartheta_\mathcal{A}(\omega_{i_2}) \quad \in \quad \mathcal{S}_\mathcal{A}(l+(m-1)). \tag{26}$$

Note that $|\omega_{i_2}| = l+m$. Assume that $\omega_{i_2} = yx_1x_2\cdots x_{l+(m-1)}$, then, we get

$$\begin{aligned} \vartheta_\mathcal{A}(\omega_{i_2}) \quad &= \quad \sum_{i_y} B_{i_y}^\dagger \vartheta_\mathcal{A}(x_1x_2\cdots x_{l+(m-1)}) B_{i_y} \qquad \text{(by Eq. (19))} \\ &\qquad \text{(by induction hypothesis, we have)} \\ &= \quad \sum_{i_y} B_{i_y}^\dagger \left(\sum_z a_z \vartheta_\mathcal{A}(\omega_z)\right) B_{i_y} \qquad (\vartheta_\mathcal{A}(\omega_z) \in K_\mathcal{A}(l)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_z a_z \left(\sum_{i_y} B_{i_y}^{\dagger} \vartheta_{\mathcal{A}}(\omega_z) B_{i_y} \right) \\
&= \sum_z a_z \, \vartheta_{\mathcal{A}}(y\omega_z) \qquad \text{(by Eq. (19))}
\end{aligned}
$$

with $|y\omega_z| \leq l+1$ and $a_z \in \mathbb{C}$, as required. ■

Now, we can prove the following

Lemma 9. *Let $\mathcal{A}_i = (Q_i, Q_{acc,i}, Q_{rej,i}, \{\mathcal{U}_\sigma^{(i)}\}_{\sigma\in\Sigma\cup\{\#,\$\}}, \rho_i, \mathcal{O}_i)$ be two E-1QFAs over Σ, and $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Then, $V_{\mathcal{A}}(n_1^2+n_2^2-1) = V_{\mathcal{A}}((n_1^2+n_2^2-1)+j)$ for all $j \geq 1$.*

Proof. For any $\omega \in \Sigma^*$ with $|\omega| = (n_1^2+n_2^2-1)+j$, we have

$$
\begin{aligned}
\theta_{\mathcal{A}}(\omega) &= \sum_{i_{x_0}} B_{i_{x_0}}^{\dagger} \vartheta_{\mathcal{A}}(\omega) B_{i_{x_0}} \qquad \text{(by Eq. (15))} \\
&\quad \text{(by Lemma 8, we have)} \\
&= \sum_{i_{x_0}} B_{i_{x_0}}^{\dagger} \left(\sum_z a_z \, \vartheta_{\mathcal{A}}(\omega_z) \right) B_{i_{x_0}} \qquad (\ \vartheta_{\mathcal{A}}(\omega_z) \in K_{\mathcal{A}}(n_1^2+n_2^2-1)\ ) \\
&= \sum_z a_z \left(\sum_{i_{x_0}} B_{i_{x_0}}^{\dagger} \vartheta_{\mathcal{A}}(\omega_z) B_{i_{x_0}} \right) \\
&= \sum_z a_z \, \theta_{\mathcal{A}}(\omega_z) \qquad \text{(by Eq. (15))}
\end{aligned}
$$

where $|\omega_z| \leq n_1^2+n_2^2-1$ and $a_z \in \mathbb{C}$. Hence, $V_{\mathcal{A}}((n_1^2+n_2^2-1)+j) = V_{\mathcal{A}}(n_1^2+n_2^2-1)$. The above argument holds for all $j \geq 1$. The lemma follows. ■

Remark 8. It should be noted that we achieve the proof of Lemma 9 by dint of Lemma 8. The reason for this is that an E-1QFA has the left end-mark '#', which prevents us from achieving the proof directly. This is also the reason for why the formula $\theta_{\mathcal{A}}(\omega)$ is given in the form of Eq. (15).

The proof of the following theorem and the proof of Theorem 2 are similar to the proof of Theorem 7 and the proof of Theorem 1, respectively. Since our presentation here is self-contained, we present the proofs in detail.

Theorem 10. *Let $\mathcal{A}_i = (Q_i, Q_{acc,i}, Q_{rej,i}, \{\mathcal{U}_\sigma^{(i)}\}_{\sigma\in\Sigma\cup\{\#,\$\}}, \rho_i, \mathcal{O}_i)$, $i = 1, 2$, be two E-1QFAs over Σ, and $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Then the density matrices φ and ψ, defined in Eqs. (16), are equivalent with respect to $\mathcal{A}$ iff they are $n_1^2+n_2^2-1$-equivalent with respect to $\mathcal{A}$, where n_1 and n_2 are the numbers of states in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.*

Proof. The "only if" part of the theorem is trivial, we only need to show the "if" part. Assume that φ and ψ are $n_1^2+n_2^2-1$-equivalent. Then, for all $\omega \in \Sigma^*$ with $|\omega| \le n_1^2+n_2^2-1$, Eq. (17) holds. Namely

$$(\langle q_0^{(1)}|, \mathbf{0})\theta_{\mathcal{A}}(\omega)\begin{pmatrix} |q_0^{(1)}\rangle \\ \mathbf{0}\end{pmatrix} = (\mathbf{0}, \langle q_0^{(2)}|)\theta_{\mathcal{A}}(\omega)\begin{pmatrix} \mathbf{0} \\ |q_0^{(2)}\rangle\end{pmatrix} \tag{27}$$

for all $\theta_{\mathcal{A}}(\omega) \in V_{\mathcal{A}}(n_1^2+n_2^2-1)$.

By Lemma 9, for all $\omega \in \Sigma^*$, we have

$$\theta_{\mathcal{A}}(\omega) = \sum_i a_i\theta_{\mathcal{A}}(\omega_i) \qquad (\theta_{\mathcal{A}}(\omega_i) \in H_{\mathcal{A}}(n_1^2+n_2^2-1), a_i \in \mathbb{C}) \tag{28}$$

Thus

$$\begin{aligned}
(\langle q_0^{(1)}|, \mathbf{0})\theta_{\mathcal{A}}(\omega)\begin{pmatrix} |q_0^{(1)}\rangle \\ \mathbf{0}\end{pmatrix} &= (\langle q_0^{(1)}|, \mathbf{0})\left(\sum_i a_i\theta_{\mathcal{A}}(\omega_i)\right)\begin{pmatrix} |q_0^{(1)}\rangle \\ \mathbf{0}\end{pmatrix} && \text{(by Eq. (28))}\\
&= \sum_i a_i\left((\langle q_0^{(1)}|, \mathbf{0})\theta_{\mathcal{A}}(\omega_i)\begin{pmatrix} |q_0^{(1)}\rangle \\ \mathbf{0}\end{pmatrix}\right)\\
&= \sum_i a_i\left((\mathbf{0}, \langle q_0^{(2)}|)\theta_{\mathcal{A}}(\omega_i)\begin{pmatrix} \mathbf{0} \\ |q_0^{(2)}\rangle\end{pmatrix}\right) && \text{(by Eq. (27))}\\
&= (\mathbf{0}, \langle q_0^{(2)}|)\left(\sum_i a_i\theta_{\mathcal{A}}(\omega_i)\right)\begin{pmatrix} \mathbf{0} \\ |q_0^{(2)}\rangle\end{pmatrix}\\
&= (\mathbf{0}, \langle q_0^{(2)}|)\theta_{\mathcal{A}}(\omega)\begin{pmatrix} \mathbf{0} \\ |q_0^{(2)}\rangle\end{pmatrix} && \text{(by Eq. (28))}
\end{aligned}$$

This implies that Eq. (17) holds for all $\omega \in \Sigma^*$. Thus, by Definition 10, φ and ψ are equivalent with respect to $\mathcal{A}$. ■

Finally, we present the proof of Theorem 2 as follows.

Proof of Theorem 2. By Theorem 4, we only need to show that $\mathcal{A}_1$ and $\mathcal{A}_2$ are β-equivalent if and only if they are $(n_1^2+n_2^2-1)$-β-equivalent.

Also, it is clear that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are β-equivalent then they are $(n_1^2+n_2^2-1)$-β-equivalent. Let $\mathcal{A}=\mathcal{A}_1\oplus\mathcal{A}_2$. Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are $(n_1^2+n_2^2-1)$-β-equivalent. Then for all $\omega\in\Sigma^*$ with $|\omega|\leq n_1^2+n_2^2-1$, we have

$$\mathcal{F}_{\mathcal{A}_1}(\omega) \quad = \quad \mathcal{F}_{\mathcal{A}_2}(\omega) \qquad (\ |\omega|\leq n_1^2+n_2^2-1\ ) \tag{29}$$

By Remark 5,

$$\mathcal{F}_{\mathcal{A}_1}(\omega) \quad = \quad (\langle q_0^{(1)}|, \mathbf{0})\ \theta_{\mathcal{A}}(\omega) \begin{pmatrix} |q_0^{(1)}\rangle \\ \mathbf{0} \end{pmatrix} \tag{30}$$

$$\mathcal{F}_{\mathcal{A}_2}(\omega) \quad = \quad (\mathbf{0}, \langle q_0^{(2)}|)\ \theta_{\mathcal{A}}(\omega) \begin{pmatrix} \mathbf{0} \\ |q_0^{(2)}\rangle \end{pmatrix} \tag{31}$$

Eqs. (29), (30) and (31) imply that φ and ψ are $n_1^2+n_2^2-1$-equivalent with respect to $\mathcal{A}$. By Theorem 10, φ and ψ are equivalent with respect to $\mathcal{A}$, which means that $\mathcal{F}_{\mathcal{A}_1}(\omega)=\mathcal{F}_{\mathcal{A}_2}(\omega)$ for all $\omega\in\Sigma^*$. i.e., $\mathcal{A}_1$ and $\mathcal{A}_2$ are β-equivalent. Theorem 2 follows. ■

5. Conclusions

In this paper, it has shown that two MM-1QFAs $\mathcal{A}_1$ and $\mathcal{A}_2$ over the same alphabet Σ are equivalent if and only if they are $(n_1^2+n_2^2-1)$-equivalent. Our result indicates that the upper-bound for the equivalence problem of MM-1QFAs is irrelevant to the numbers of states in the minimal DFA recognized the regular language $g^*a\{a,r,g\}^*$. The approach used in this paper is similar to the work of Carlyle [4]. Also, comparing with [26], the reader may find that the approach used in this paper is much simpler, direct and elegant.

As an application of the approach, we utilize it to address the equivalence problem of E-1QFAs which has not been answered previously by showing Theorem 2.

As mentioned earlier, from the algebraic point of view, the concept of "equivalence" provides us a classification of the elements of the set of MM-1QFAs over the same alphabet. Let $\mathcal{A}$ be an MM-1QFA over Σ, and let $\widetilde{\mathcal{A}}$ denote the set of MM-1QFAs over Σ which is equivalent to $\mathcal{A}$. Then, a natural question to be asked is whether there exists an MM-1QFA $\mathcal{A}' \in \widetilde{\mathcal{A}}$ with least (minimal) numbers of basic states? If such an element exists, then how to construct it? It is our future work to consider these interesting and more challenging problems.

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