Decidability of Order-Based Modal Logics[☆]

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Abstract

Decidability of the validity problem is established for a family of many-valued modal logics, notably Gödel modal logics, where propositional connectives are evaluated according to the order of values in a complete sublattice of the real unit interval $[0, 1]$, and box and diamond modalities are evaluated as infima and suprema over (many-valued) Kripke frames. If the sublattice is infinite and the language is sufficiently expressive, then the standard semantics for such a logic lacks the finite model property. It is shown here, however, that, given certain regularity conditions, the finite model property holds for a new semantics for the logic, providing a basis for establishing decidability and PSPACE-completeness. Similar results are also established for S5 logics that coincide with one-variable fragments of first-order many-valued logics. In particular, a first proof is given of

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the decidability and co-NP-completeness of validity in the one-variable fragment of first-order Gödel logic.

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1. Introduction

Many-valued modal logics extend the Kripke frame setting of classical modal logic with a many-valued semantics at each world and a many-valued or crisp (Boolean-valued) accessibility relation to model modal notions such as necessity, belief, and spatio-temporal relations in the presence of uncertainty, possibility, or vagueness. Applications include modelling fuzzy belief [17,22], spatial reasoning with vague predicates [33], many-valued tense logics [12], and fuzzy similarity measures [18]. Fuzzy description logics may also be interpreted, analogously to the classical case, as many-valued multi-modal logics (see, e.g., [5,21,35]).

Quite general approaches to many-valued modal logics, focussing largely on decidability and axiomatization issues for finite-valued modal logics, are described in [6,15,16,30]. For modal logics based on an infinite-valued semantics, typically over the real unit interval $[0,1]$, two core families can be identified. Many-valued modal logics of "magnitude" are based on a semantics related to Łukasiewicz infinite-valued logic with connectives interpreted by continuous functions over real numbers [13,19,23]. Typical many-valued modal logics of the second family are based instead on the semantics of infinite-valued Gödel logics [9,10,19,27]. The standard infinite-valued Gödel logic (also known as Gödel-Dummett logic) interprets truth values as elements of $[0,1]$, conjunction and disjunction as minimum and maximum, respectively, and implication $x \to y$ as y for $x > y$ and 1 otherwise. Modal operators $\Box$ and $\Diamond$ (not inter-definable in this setting) are interpreted as infima and suprema of values at accessible worlds. More generally, "order-based" modal logics may be defined over a complete sublattice of $[0,1]$ with additional operations depending only on the order.

Propositional Gödel logic has been studied intensively both as a fundamental "t-norm based" fuzzy logic [19,28] and as an intermediate (or superintuitionistic) logic, obtained as an extension of an axiomatization of propositional intuitionistic logic with the prelinearity axiom schema $(\varphi \to \psi) \lor (\psi \to \varphi)$. The many-valued modal logics considered in this paper diverge considerably, however, from the modal intermediate logics investigated in [36] (and elsewhere), which use two accessibility relations for Kripke models, one for the modal operators and one for the intuitionistic connectives. We remark also that, unlike the operations added to infinite-valued logics in [11, 20], which represent truth stressers such as "very true" or "classically true", the modalities considered here cannot be interpreted simply as unary connectives on the real unit interval $[0, 1]$.

The first main contribution of this paper is to establish PSPACE-completeness results (matching the complexity of the classical modal logic K [25]) for the validity problem of Gödel modal logics and other order-based modal logics defined over complete sublattices of $[0, 1]$ satisfying certain local regularity conditions (e.g., sublattices order-isomorphic to the positive integers with an added top element and the negative integers with an added bottom element). The finite model property typically fails even for the box and diamond fragments of these logics. Decidability and PSPACE-completeness of the validity problem for these fragments of Gödel modal logics over $[0, 1]$ was established in [27] using analytic Gentzen-style proof systems, but this methodology does not seem to extend easily to the full logics. Here, alternative Kripke semantics are provided for order-based modal logics that not only have the same valid formulas as the original semantics, but also admit the finite model property. The key idea of this new semantics is to restrict evaluations of modal formulas at a world to a particular set of truth values.

The second main contribution of the paper is to establish co-NP-completeness results for the validity problem of crisp order-based "$\mathsf{S5}$" logics: order-based modal logics where accessibility is an equivalence relation. Such logics may be interpreted also as one-variable fragments of first-order many-valued logics. In particular, the open decidability problem for validity in the one-variable fragment of first-order Gödel logic (see, e.g., [19, Chapter 9, Problem 13])) is answered positively and shown to be co-NP-complete. This result matches the complexity of the one-variable fragments of classical first-order logic (equivalently, $\mathsf{S5}$) and first-order Łukasiewicz logic (see [19]), and contrasts with the co-NEXPTIME-completeness of the one-variable fragment of first-order intuitionistic logic (equivalently, the intuitionistic modal logic MIPC) [26].

2. Order-Based Modal Logics

We consider "order-based" modal logics where propositional connectives are interpreted at individual worlds in an algebra consisting of a complete sublattice of $\langle [0, 1], \wedge, \vee, 0, 1 \rangle$ with operations defined based only on the order. Modalities $\Box$ and $\Diamond$ are defined using infima and suprema, respectively, according to either a (crisp, i.e., Boolean-valued) binary relation on the set of worlds or a binary mapping (many-valued relation) from worlds to values of the algebra. For convenience, we consider only finite algebraic languages, noting that to decide the validity of a formula we may in any case restrict to the language containing only operation symbols occurring in that formula.

We reserve the symbols $\Rightarrow$, $\&$, $\sim$, and $\approx$ to denote implication, conjunction, negation, and equality, respectively, in classical first-order logic. We also recall an appropriate notion of first-order definability of operations for algebraic structures. Let $\mathcal{L}$ be an algebraic language, $\mathbf{A}$ an algebra for $\mathcal{L}$, and $\mathcal{L}'$ a sublanguage of $\mathcal{L}$. An operation $f\colon A^n \to A$ is *defined* in $\mathbf{A}$ by a first-order $\mathcal{L}'$-formula $F(x_1, \ldots, x_n, y)$ with free variables $x_1, \ldots, x_n, y$ if for all $a_1, \ldots, a_n, b \in A$,

$$\mathbf{A} \models F(a_1, \ldots, a_n, b) \quad \Leftrightarrow \quad f(a_1, \ldots, a_n) = b.$$

2.1. Order-Based Algebras

Let $\mathcal{L}$ be a finite algebraic language that includes the binary operation symbols $\wedge$ and $\vee$ and constant symbols $\bar{0}$ and $\bar{1}$ (to be interpreted by the usual lattice operations), and denote the finite set of constants (nullary operation symbols) of this language by $\mathrm{C}_{\mathcal{L}}$. An algebra $\mathbf{A}$ for $\mathcal{L}$ will be called *order-based* if it satisfies the following conditions:

(1) $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, 0, 1\rangle$ is a complete sublattice of $\langle [0,1], \min, \max, 0, 1\rangle$; i.e., $\{0,1\} \subseteq A \subseteq [0,1]$ and for all $B \subseteq A$, $\bigwedge^{[0,1]} B$ and $\bigvee^{[0,1]} B$ belong to A.

(2) For each operation symbol $\star$ of $\mathcal{L}$, the operation $\star^{\mathbf{A}}$ is definable in $\mathbf{A}$ by a quantifier-free first-order formula in the algebraic language consisting of $\wedge$, $\vee$, and constants from $\mathrm{C}_{\mathcal{L}}$.

We also let $\mathrm{C}_{\mathcal{L}}^{\mathbf{A}}$ denote the finite set of constant operations $\{c^{\mathbf{A}} : c \in \mathrm{C}_{\mathcal{L}}\}$ and define $R(\mathbf{A})$ and $L(\mathbf{A})$ to be the sets of *right* and *left accumulation points*, respectively, of $\mathbf{A}$ in the usual topology inherited from $[0,1]$; that is,

$$\begin{array}{lcl} a \in R(\mathbf{A}) & \Leftrightarrow & \text{there is a } c \in A \text{ such that } a <^{\mathbf{A}} c \text{ and for all such } c, \\ & & \text{there is an } e \in A \text{ such that } a <^{\mathbf{A}} e <^{\mathbf{A}} c. \\ b \in L(\mathbf{A}) & \Leftrightarrow & \text{there is a } d \in A \text{ such that } d <^{\mathbf{A}} b, \text{ and for all such } d, \\ & & \text{there is an } f \in A \text{ such that } d <^{\mathbf{A}} f <^{\mathbf{A}} b. \end{array}$$

Note that, because $\mathbf{A}$ is a chain, an implication operation $\to^{\mathbf{A}}$ may always be introduced as the residual of $\wedge^{\mathbf{A}}$:

$$a \to^{\mathbf{A}} b \;=\; \bigvee\nolimits^{\mathbf{A}} \{c \in A : c \wedge^{\mathbf{A}} a \leq^{\mathbf{A}} b\} \;=\; \begin{cases} 1 & \text{if } a \leq^{\mathbf{A}} b \\ b & \text{otherwise.} \end{cases}$$

Let $s \leq t$ stand for $s \wedge t \approx s$ and let $s < t$ stand for $(s \leq t)$ & $\sim(s \approx t)$. Then the implication operation $\rightarrow^{\mathbf{A}}$ is definable in $\mathbf{A}$ by the quantifier-free first-order formula

$$F^{\rightarrow}(x, y, z) \;=\; ((x \leq y) \Rightarrow (z \approx \bar{1}))\ \&\ ((y < x) \Rightarrow (z \approx y)).$$

That is, for all $a, b, c \in A$,

$$\mathbf{A} \models F^{\rightarrow}(a, b, c) \quad \Leftrightarrow \quad a \rightarrow^{\mathbf{A}} b = c.$$

In this paper, the connective $\rightarrow$ will always be interpreted by $\rightarrow^{\mathbf{A}}$ in $\mathbf{A}$. We will also make use of the negation connective $\neg\varphi := \varphi \rightarrow \bar{0}$, which is interpreted by the unary operation

$$\neg^{\mathbf{A}} a \;=\; \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Examples of other useful operations (see, e.g., [1]) covered by the order-based approach are the globalization and Nabla operators

$$\Delta^{\mathbf{A}} a \;=\; \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \nabla^{\mathbf{A}} a \;=\; \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{otherwise,} \end{cases}$$

definable in $\mathbf{A}$ (noting also that $\nabla^{\mathbf{A}} a = \neg^{\mathbf{A}}\neg^{\mathbf{A}} a$), by

$$\begin{aligned} F^{\Delta}(x, y) \;&=\; ((x \approx \bar{1}) \Rightarrow (y \approx \bar{1}))\ \&\ ((x < \bar{1}) \Rightarrow (y \approx \bar{0})) \\ F^{\nabla}(x, y) \;&=\; ((x \approx \bar{0}) \Rightarrow (y \approx \bar{0}))\ \&\ ((\bar{0} < x) \Rightarrow (y \approx \bar{1})), \end{aligned}$$

and the dual-implication connective (the residual of $\vee^{\mathbf{A}}$)

$$a \leftarrow^{\mathbf{A}} b \;=\; \bigwedge\nolimits^{\mathbf{A}} \{c \in A : b \leq^{\mathbf{A}} a \vee^{\mathbf{A}} c\} \;=\; \begin{cases} 0 & \text{if } b \leq^{\mathbf{A}} a \\ b & \text{otherwise,} \end{cases}$$

definable in $\mathbf{A}$ by

$$F^{\leftarrow}(x, y, z) \;=\; ((y \leq x) \Rightarrow (z \approx \bar{0}))\ \&\ ((x < y) \Rightarrow (z \approx y)).$$

For the remainder of this work, we will omit the superscript $\mathbf{A}$ when the algebra or order is clear from the context.

2.2. Many-Valued Kripke Semantics

Let us fix a finite language $\mathcal{L}$ including the operational symbols $\bar{1}$, $\bar{0}$, $\wedge$, $\vee$, and $\to$, and an order-based algebra $\mathbf{A}$ for $\mathcal{L}$. We define order-based modal logics $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ and $\mathsf{K}(\mathbf{A})$ based on standard (crisp) Kripke frames and Kripke frames with an accessibility relation taking values in A, respectively.

An $\mathbf{A}$-*frame* is a pair $\mathfrak{F} = \langle W, R \rangle$ such that W is a non-empty set of *worlds* and $R \colon W \times W \to A$ is an $\mathbf{A}$-*accessibility relation* on W. If $Rxy \in \{0, 1\}$ for all $x, y \in W$, then R is *crisp* and $\mathfrak{F}$ is called a *crisp* $\mathbf{A}$-*frame*. In this case, we often write $R \subseteq W \times W$ and Rxy to mean $Rxy = 1$.

Now let Fm be the set of formulas, denoted by $\varphi, \psi, \chi \ldots$, of the language $\mathcal{L}$ with additional unary operation symbols (modal connectives) $\Box$ and $\Diamond$, defined inductively over a countably infinite set Var of propositional variables, denoted by $p, q, \ldots$. We call formulas of the form $\Box\varphi$ and $\Diamond\varphi$ *box-formulas* and *diamond-formulas*, respectively. *Subformulas* are defined as usual, and the *length* of a formula φ, denoted by $\ell(\varphi)$, is the total number of occurrences of subformulas in φ. We also let $\mathrm{Var}(\varphi)$ denote the set of variables occurring in the formula φ.

A $\mathsf{K}(\mathbf{A})$-*model* is a triple $\mathfrak{M} = \langle W, R, V \rangle$ such that $\langle W, R \rangle$ is an $\mathbf{A}$-frame and $V \colon \mathrm{Var} \times W \to A$ is a mapping, called a *valuation*, that is extended to $V \colon \mathrm{Fm} \times W \to A$ by

$$V(\star(\varphi_1, \ldots, \varphi_n), x) \quad = \quad \star(V(\varphi_1, x), \ldots, V(\varphi_n, x))$$

for each n-ary operation symbol $\star$ of $\mathcal{L}$, and

$$\begin{aligned} V(\Box\varphi, x) &= \bigwedge\{Rxy \to V(\varphi, y) : y \in W\} \\ V(\Diamond\varphi, x) &= \bigvee\{Rxy \wedge V(\varphi, y) : y \in W\}. \end{aligned}$$

A $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-*model* satisfies the extra condition that $\langle W, R \rangle$ is a crisp $\mathbf{A}$- frame. In this case, the conditions for $\Box$ and $\Diamond$ simplify to

$$\begin{aligned} V(\Box\varphi, x) &= \bigwedge\{V(\varphi, y) : Rxy\} \\ V(\Diamond\varphi, x) &= \bigvee\{V(\varphi, y) : Rxy\}. \end{aligned}$$

A formula $\varphi \in \mathrm{Fm}$ will be called *valid* in a $\mathsf{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V \rangle$ if $V(\varphi, x) = 1$ for all $x \in W$. If φ is valid in all L-models for some logic L, then φ is said to be L-*valid*, written $\models_{\mathsf{L}} \varphi$.

We now introduce some useful notation and terminology. A subset $\Sigma \subseteq \mathrm{Fm}$ will be called a *fragment* if it contains all constants in $\mathrm{C}_{\mathcal{L}}$ and is closed with respect to taking subformulas. For a formula $\varphi \in \mathrm{Fm}$, we let $\Sigma(\varphi)$ be the smallest (always finite) fragment containing φ. Also, for any $\mathsf{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$, subset $X \subseteq W$, and fragment $\Sigma \subseteq \mathrm{Fm}$, we let

$$V[\Sigma, X] = \{V(\varphi, x) : \varphi \in \Sigma \text{ and } x \in X\}.$$

We shorten $V[\Sigma, \{x\}]$ to $V[\Sigma, x]$. For $\Sigma \subseteq \mathrm{Fm}$, we let $\Sigma_\Box$ and $\Sigma_\Diamond$ be the sets of all box-formulas in Σ and diamond-formulas in Σ, respectively.

Given a linearly ordered set $\langle P, \leq\rangle$ and $C \subseteq P$, a map $h\colon P \to P$ will be called a *C-order embedding* if it is an order-preserving embedding (i.e., $a \leq b$ if and only if $h(a) \leq h(b)$ for all $a, b \in P$) satisfying $h(c) = c$ for all $c \in C$. We will call an order embedding $h\colon P \to P$ *inflationary* or *deflationary* if for all $a \in P$, $a \leq h(a)$, or for all $a \in P$, $a \geq h(a)$, respectively. h will be called *B-complete* for $B \subseteq P$ if whenever $\bigvee D \in B$ or $\bigwedge D \in B$ for some $D \subseteq P$, respectively,

$$h(\bigvee D) = \bigvee h[D] \quad \text{or} \quad h(\bigwedge D) = \bigwedge h[D].$$

The following lemma establishes the critical property of order-based modal logics for our purposes. Namely, it is only the relative order of the values taken by variables and the accessibility relation between worlds that plays a role in determining the values of formulas and checking validity.

Lemma 1. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a $\mathsf{K}(\mathbf{A})$-model and $\Sigma \subseteq \mathrm{Fm}$ a fragment, and let $h\colon A \to A$ be a $V[\Sigma_\Box \cup \Sigma_\Diamond, W]$-complete $\mathrm{C}_\mathcal{L}$-order embedding. Consider the $\mathsf{K}(\mathbf{A})$-model $\widehat{\mathfrak{M}} = \langle W, \widehat{R}, \widehat{V}\rangle$ with $\widehat{R}xy = h(Rxy)$ and $\widehat{V}(p, x) = h(V(p, x))$ for all $p \in \mathrm{Var}$ and $x, y \in W$. Then for all $\varphi \in \Sigma$ and $x \in W$:*

$$\widehat{V}(\varphi, x) = h(V(\varphi, x)).$$

Proof. We proceed by induction on $\ell(\varphi)$. The case $\varphi \in \mathrm{Var} \cup \mathrm{C}_\mathcal{L}$ follows from the definition of $\widehat{V}$. For the induction step, suppose that $\varphi = \star(\varphi_1, \ldots, \varphi_n)$ for some operation symbol $\star$ of $\mathcal{L}$ and $\varphi_1, \ldots, \varphi_n \in \Sigma$. Recall that $\star$ is definable in $\mathbf{A}$ by some quantifier-free first-order formula $F^\star(x_1, \ldots, x_n, y)$ in the first-order language with $\wedge$, $\vee$, and constants from $\mathrm{C}_\mathcal{L}$, i.e.

$$\star(a_1, \ldots, a_n) = b \quad \Leftrightarrow \quad \mathbf{A} \models F^\star(a_1, \ldots, a_n, b).$$

Because $F^\star(x_1, \ldots, x_n, y)$ is quantifier-free and h preserves $\wedge$, $\vee$, and $\mathrm{C}_\mathcal{L}$,

$$\mathbf{A} \models F^\star(a_1, \ldots, a_n, b) \quad \Leftrightarrow \quad \mathbf{A} \models F^\star(h(a_1), \ldots, h(a_n), h(b)).$$

So we may also conclude

$$\star(h(a_1),\ldots,h(a_n)) = h(\star(a_1,\ldots,a_n)).$$

Hence for all $x \in W$, using the induction hypothesis for the step from (1) to (2):

$$\begin{aligned}
\widehat{V}(\star(\varphi_1,\ldots,\varphi_n),x) &= \star(\widehat{V}(\varphi_1,x),\ldots,\widehat{V}(\varphi_n,x)) &(1)\\
&= \star(h(V(\varphi_1,x)),\ldots,h(V(\varphi_n,x))) &(2)\\
&= h(\star(V(\varphi_1,x),\ldots,V(\varphi_n,x))) &(3)\\
&= h(V(\star(\varphi_1,\ldots,\varphi_n),x)). &(4)
\end{aligned}$$

If $\varphi = \Diamond\psi$ for some $\psi \in \Sigma$, then we obtain for all $x \in W$:

$$\begin{aligned}
\widehat{V}(\Diamond\psi,x) &= \bigvee\{\widehat{R}xy \wedge \widehat{V}(\psi,y) : y \in W\} &(5)\\
&= \bigvee\{h(Rxy) \wedge h(V(\psi,y)) : y \in W\} &(6)\\
&= \bigvee\{h(Rxy \wedge V(\psi,y)) : y \in W\} &(7)\\
&= h(\bigvee\{Rxy \wedge V(\psi,y) : y \in W\}) &(8)\\
&= h(V(\Diamond\psi,x)). &(9)
\end{aligned}$$

(5) to (6) follows from the definition of $\widehat{R}$ and the induction hypothesis, (6) to (7) follows because h is an order embedding, and (7) to (8) follows because h is $V[\Sigma_\Box \cup \Sigma_\Diamond, W]$-complete and $\bigvee\{Rxy \wedge V(\psi,y) : y \in W\} = V(\Diamond\psi,x) \in V[\Sigma_\Diamond, W]$. The case $\varphi = \Box\psi$ is very similar. $\Box$

We now consider many-valued analogues of some useful notions and results from classical modal logic (see, e.g., [4]). For an $\mathbf{A}$-frame $\langle W, R\rangle$, we define the crisp relation R^+ as follows:

$$R^+ = \{(x,y) \in W^2 : Rxy > 0\}, \quad R^+[x] = \{y \in W : R^+xy\} \text{ for } x \in W.$$

Let $\mathfrak{M} = \langle W, R, V\rangle$ be a $\mathsf{K}(\mathbf{A})$-model. We call $\mathfrak{M}' = \langle W', R', V'\rangle$ a $\mathsf{K}(\mathbf{A})$-*submodel* of $\mathfrak{M}$, written $\mathfrak{M}' \subseteq \mathfrak{M}$, if $W' \subseteq W$ and R' and V' are the restrictions to W' of R and V, respectively. In particular, given $x \in W$, the $\mathsf{K}(\mathbf{A})$-*submodel of* $\mathfrak{M}$ *generated by* x is the smallest $\mathsf{K}(\mathbf{A})$-submodel $\mathfrak{M}' = \langle W', R', V'\rangle$ of $\mathfrak{M}$ such that $x \in W'$ and for all $y \in W'$, whenever $z \in R^+[y]$, also $z \in W'$.

Lemma 2. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a $\mathsf{K}(\mathbf{A})$-model and $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ a generated $\mathsf{K}(\mathbf{A})$-submodel of $\mathfrak{M}$. Then $\widehat{V}(\varphi,x) = V(\varphi,x)$ for all $x \in \widehat{W}$ and $\varphi \in \mathrm{Fm}$.*

Proof. We proceed by induction on $\ell(\varphi)$. The base case is trivial for any submodel of $\mathfrak{M}$, so also for $\widehat{\mathfrak{M}}$. For the induction step, the case where $\varphi = \star(\varphi_1, \dots, \varphi_n)$ for some operation symbol $\star$ follows immediately using the induction hypothesis.

Suppose now that $\varphi = \Box\psi$. Fix $x \in \widehat{W}$ and note that for any $y \in W \setminus \widehat{W}$, we have $Rxy = 0$. Observe also that $0 \to a = 1$ for all $a \in A$. Hence, excluding all worlds $y \in W$ such that $Rxy = 0$ does not change the value of $\bigwedge\{Rxy \to V(\psi, y) : y \in W\}$. So, using the induction hypothesis,

$$\begin{aligned} V(\Box\psi, x) &= \bigwedge\{Rxy \to V(\psi, y) : y \in \widehat{W}\} \\ &= \bigwedge\{\widehat{R}xy \to \widehat{V}(\psi, y) : y \in \widehat{W}\} \\ &= \widehat{V}(\Box\psi, x). \end{aligned}$$

The case where $\varphi = \Diamond\psi$ is very similar. $\square$

Following the usual terminology of modal logic, a *tree* is defined as a relational structure $\langle T, S\rangle$ such that (i) $S \subseteq T^2$ is irreflexive, (ii) there exists a unique *root* $x_0 \in T$ satisfying S^*x_0x for all $x \in T$ where S^* is the reflexive transitive closure of S, (iii) for each $x \in T \setminus \{x_0\}$, there is a unique $x' \in T$ such that $Sx'x$. A tree $\langle T, S\rangle$ has *height* $m \in \mathbb{N}$ if $m = \max\{|\{y \in T : S^*yx\}| : x \in T\}$. A $\mathsf{K}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V\rangle$ is called a $\mathsf{K}(\mathbf{A})$-*tree-model* if $\langle W, R^+\rangle$ is a tree, and has finite height $\mathrm{hg}(\mathfrak{M}) = m$ if $\langle W, R^+\rangle$ has height m.

Lemma 3. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a $\mathsf{K}(\mathbf{A})$-model, $x_0 \in W$, and $k \in \mathbb{N}$. Then there exists a $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ with root $\widehat{x}_0$ and $\mathrm{hg}(\widehat{\mathfrak{M}}) \leq k$ such that $\widehat{V}(\varphi, \widehat{x}_0) = V(\varphi, x_0)$ for all $\varphi \in \mathrm{Fm}$ with $\ell(\varphi) \leq k$. Moreover, if $\mathfrak{M}$ is a $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-model, then so is $\widehat{\mathfrak{M}}$.*

Proof. Consider the $\mathsf{K}(\mathbf{A})$-model $\mathfrak{M}' = \langle W', R', V'\rangle$ obtained by "unravelling" $\mathfrak{M}$ at the world x_0; i.e., for all $n \in \mathbb{N}$ (noting that $0 \in \mathbb{N}$),

$$\begin{aligned} W' &= \bigcup_{n \in \mathbb{N}} \{(x_0, \dots, x_n) \in W^{n+1} : R^+x_ix_{i+1} \text{ for } i < n\} \\ R'yz &= \begin{cases} Rx_nx_{n+1} & \text{if } y = (x_0, \dots, x_n),\ z = (x_0, \dots, x_{n+1}) \\ 0 & \text{otherwise} \end{cases} \\ V'(p, (x_0, \dots, x_n)) &= V(p, x_n). \end{aligned}$$

Clearly, $\mathfrak{M}'$ is a $\mathsf{K}(\mathbf{A})$-tree-model with root $\widehat{x}_0 = (x_0)$. Now let $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ be the $\mathsf{K}(\mathbf{A})$-tree-submodel of $\mathfrak{M}'$ defined by cutting $\mathfrak{M}'$ at depth k; i.e., let $\widehat{W} =$

$\{(x_0,\dots,x_n)\in W' : n\le k\}$ and let $\widehat{R}$ and $\widehat{V}$ be the restrictions of R' and V' to $\widehat{W}\times\widehat{W}$ and $\mathrm{Var}\times\widehat{W}$, respectively. A straightforward induction on $\ell(\varphi)$ shows that for all $\varphi\in\mathrm{Fm}$ and $n\in\mathbb{N}$ such that $\ell(\varphi)\le k-n$, $\widehat{V}(\varphi,(x_0,\dots,x_n))=V(\varphi,x_n)$. In particular, $\widehat{V}(\varphi,\widehat{x}_0)=V(\varphi,x_0)$ for all $\varphi\in\mathrm{Fm}$ with $\ell(\varphi)\le k$. $\square$

2.3. *Gödel Modal Logics*

The "Gödel modal logics" GK and GK^{C} studied in [9, 10, 27] are $\mathsf{K}(\mathbf{G})$ and $\mathsf{K}(\mathbf{G})^{\mathsf{C}}$, respectively, defined with respect to the infinite-valued Gödel algebra

$$\mathbf{G}=\langle[0,1],\wedge,\vee,\to,0,1\rangle.$$

Axiomatizations of the box and diamond fragments of GK are obtained in [9] as extensions of an axiomatization of Gödel logic (intuitionistic logic plus the prelinearity axiom schema $(\varphi\to\psi)\vee(\psi\to\varphi)$) with, respectively,

$$\begin{array}{l}\neg\neg\Box\varphi\to\Box\neg\neg\varphi\\ \Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi)\\ \dfrac{\varphi}{\Box\varphi}\end{array}\qquad\text{and}\qquad\begin{array}{l}\Diamond(\varphi\vee\psi)\to(\Diamond\varphi\vee\Diamond\psi)\\ \Diamond\neg\neg\varphi\to\neg\neg\Diamond\varphi\\ \neg\Diamond\bar{0}\\ \dfrac{\varphi\to\psi}{\Diamond\varphi\to\Diamond\psi}.\end{array}$$

An axiomatization of the full logic GK is obtained in [10] by extending the union of these axiomatizations with the Fischer Servi axioms (see [34])

$$\begin{array}{l}\Diamond(\varphi\to\psi)\to(\Box\varphi\to\Diamond\psi)\\ (\Diamond\varphi\to\Box\psi)\to\Box(\varphi\to\psi).\end{array}$$

It is also shown in [10] that GK coincides with the extension of the intuitionistic modal logic IK (see [34]) with the prelinearity axiom schema $(\varphi\to\psi)\vee(\psi\to\varphi)$.

No axiomatization has yet been found for the full logic GK^{C}. However, the box fragment of GK^{C} coincides with the box fragment of GK [9], and the diamond fragment of GK^{C} is axiomatized in [27] as an extension of the diamond fragment of GK with

$$\frac{\varphi\vee(\psi_1\to\psi_2)}{\Diamond\varphi\vee(\Diamond\psi_1\to\Diamond\psi_2)}.$$

More generally, we may consider the family of *Gödel modal logics* $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ where $\mathbf{A}$ is a complete subalgebra of $\mathbf{G}$: in particular, when $\mathbf{A}$ is $\mathbf{G}_\downarrow=\langle G_\downarrow,\wedge,\vee,\to,0,1\rangle$ or $\mathbf{G}_\uparrow=\langle G_\downarrow,\wedge,\vee,\to,0,1\rangle$ with

$$G_\downarrow=\{0\}\cup\{\tfrac{1}{n+1}\mid n\in\mathbb{N}\}\quad\text{and}\quad G_\uparrow=\{1-\tfrac{1}{n+1}\mid n\in\mathbb{N}\}\cup\{1\}.$$

Clearly, order-based algebras with universes $G_\downarrow$ and $G_\uparrow$ are isomorphic to algebras with universes $\{-n : n \in \mathbb{N}\} \cup \{-\infty\}$ and $\mathbb{N} \cup \{\infty\}$, respectively.

It is not hard to show (see below) that for finite $\mathbf{A}$, the sets of valid formulas of $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ depend only on the cardinality of A and are decidable. Note, moreover, that although all infinite subalgebras of $\mathbf{G}$ produce the same set of valid propositional formulas [14], there are countably infinitely many different infinite-valued first-order Gödel logics (considered as sets of valid formulas) [2]. This result holds also for Gödel modal logics.

Proposition 4. *There are countably infinitely many different logics $\mathsf{K}(\mathbf{A})$ (considered as sets of valid formulas), where $\mathbf{A}$ is an infinite subalgebra of $\mathbf{G}$. Moreover, the same is true for $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$.*

Proof. By the mentioned result of [2], there are at most countably many such logics. Just note that for each infinite subalgebra $\mathbf{A}$ of $\mathbf{G}$, the modal logic $\mathsf{K}(\mathbf{A})$ corresponds to a specific fragment of the first-order logic over $\mathbf{A}$, determined by the same standard translation π as in the classical setting, where box- and diamond-formulas are translated as follows:

$$\pi(\Box\varphi) = (\forall y)(Rxy \to \pi(\varphi)(y)) \quad \text{and} \quad \pi(\Diamond\varphi) = (\exists y)(Rxy \wedge \pi(\varphi)(y)).$$

To obtain the fragment in the crisp case, we may use the usual "crispification" of the relation symbol R by prefixing it with $\neg\neg$.

To show that there are infinitely many such logics, let us fix, for each $n \in \mathbb{Z}^+$, a complete subalgebra $\mathbf{A}_n$ of $\mathbf{G}$ with exactly n right accumulation points (i.e., $|R(\mathbf{A})| = n$). We then prove that for all distinct $n, m \in \mathbb{Z}^+$, the logics $\mathsf{K}(\mathbf{A}_n)$ and $\mathsf{K}(\mathbf{A}_m)$ are mutually distinct, and so are $\mathsf{K}^{\mathsf{c}}(\mathbf{A}_n)$ and $\mathsf{K}^{\mathsf{c}}(\mathbf{A}_m)$. For this, we define

$$\varphi(p, q) = (\Box(q \to p) \wedge (q \to \Box q) \wedge \Box((p \to q) \to q)) \to ((\Box p \to q) \to q),$$

which detects right accumulation points, and for each $n \in \mathbb{Z}^+$, let

$$\varphi_n(p_1, q_1, \ldots, p_n, q_n) = \bigwedge_{i=1}^{n-1} ((q_{i+1} \to q_i) \to q_i) \to \bigvee_{i=1}^{n} \varphi(p_i, q_i).$$

We leave the reader to show that for each $n \in \mathbb{Z}^+$, the formula φ_n is $\mathsf{K}(\mathbf{A})$-valid if and only if $|R(\mathbf{A})| < n$. $\Box$

The logics $\mathsf{K}(\mathbf{G})$, $\mathsf{K}(\mathbf{G}_\uparrow)$, and $\mathsf{K}(\mathbf{G}_\downarrow)$ and their crisp counterparts are all distinct. The formula $\Box\neg\neg p \to \neg\neg\Box p$ is valid in the logics based on $\mathbf{G}_\uparrow$, but not in those based on $\mathbf{G}$ or $\mathbf{G}_{\downarrow}$. To see this, note that 0 is an accumulation point in $[0,1]$ and $G_{\downarrow}$ (but not in $G_{\uparrow}$); hence for these sets there is an infinite strictly descending sequence of values $(a_i)_{i\in I}$ with limit 0, giving $\neg\neg a_i = 1$ for each $i \in I$ and $\inf_{i\in I} \neg\neg a_i = 1$, while $\neg\neg \inf_{i\in I} a_i = \neg\neg 0 = 0$ (see the proof of Theorem 7). Similarly, $(\Diamond p \to \Diamond q) \to (\neg\Diamond q \vee \Diamond(p \to q))$ is valid in the logics based on $\mathbf{G}_{\downarrow}$ but not those based on $\mathbf{G}$. Moreover, the formula $\neg\neg\Diamond p \to \Diamond\neg\neg p$ is valid in any of the crisp logics, but not in their non-crisp versions.

2.4. The Finite Model Property

Let us call an L-model for a logic L *countable* or *finite* if its set of worlds is countable or finite, respectively. We say that a logic L has the *finite model property* if validity in the logic coincides with validity in all finite L-models. Observe first that if the underlying algebra of an order-based modal logic is finite, then the logic has the finite model property.

Lemma 5. *If $\mathbf{A}$ is a finite order-based algebra, then $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ have the finite model property.*

Proof. By Lemma 3, it suffices to show that for any finite fragment $\Sigma \subseteq \mathrm{Fm}$ and $\mathsf{K}(\mathbf{A})$-tree-model $\mathfrak{M} = \langle W, R, V\rangle$ of finite height with root x, there is a finite $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle \subseteq \mathfrak{M}$ with root x such that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$. We prove this claim by induction on $\mathrm{hg}(\mathfrak{M})$. For the base case, $W = \{x\}$ and we let $\widehat{\mathfrak{M}} = \mathfrak{M}$.

For the induction step, consider for each $y \in R^{+}[x]$, the submodel $\mathfrak{M}_y = \langle W_y, R_y, V_y\rangle$ of $\mathfrak{M}$ generated by y. Each $\mathfrak{M}_y$ is a $\mathsf{K}(\mathbf{A})$-tree-model of finite height with root y and $\mathrm{hg}(\mathfrak{M}_y) < \mathrm{hg}(\mathfrak{M})$. Hence, by the induction hypothesis, for each $y \in R^{+}[x]$, there is a finite $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y\rangle \subseteq \mathfrak{M}_y \subseteq \mathfrak{M}$ with root $y \in \widehat{W}_y$ such that for all $\varphi \in \Sigma$, by Lemma 2, $\widehat{V}_y(\varphi, y) = V_y(\varphi, y) = V(\varphi, y)$.

Because $\mathbf{A}$ is finite, we can now choose for each $\varphi \in \Sigma_{\Box} \cup \Sigma_{\Diamond}$, a world y_φ such that $V(\varphi, x) = Rxy_\varphi \to \widehat{V}_y(\psi, y_\varphi)$ when $\varphi = \Box\psi$, and $V(\varphi, x) = Rxy_\varphi \wedge \widehat{V}_y(\psi, y_\varphi)$ when $\varphi = \Diamond\psi$. Define the finite set $Y = \{y_\varphi \in R^{+}[x] : \varphi \in \Sigma_{\Box} \cup \Sigma_{\Diamond}\}$. We let $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ where

$$\widehat{W} = \{x\} \cup \bigcup_{y\in Y} \widehat{W}_y,$$

and $\widehat{R}$ and $\widehat{V}$ are R and V, respectively, restricted to $\widehat{W}$. An easy induction on $\ell(\varphi)$ establishes that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$. $\Box$

We are also able to establish the finite model property when the underlying (infinite) algebra is $\mathbf{G}_{\uparrow}$.

Theorem 6. $\mathsf{K}(\mathbf{G}_{\uparrow})$ *and* $\mathsf{K}(\mathbf{G}_{\uparrow})^{\mathsf{C}}$ *have the finite model property.*

Proof. By Lemmas 3 and 5, it suffices to show that if $\varphi \in \mathrm{Fm}$ is not valid in some $\mathsf{K}(\mathbf{G}_{\uparrow})$-tree-model $\mathfrak{M}$ of finite height, then there is a finite subalgebra $\mathbf{B}$ of $\mathbf{G}_{\uparrow}$ and a $\mathsf{K}(\mathbf{B})$-model $\widehat{\mathfrak{M}}$ (that is crisp if $\mathfrak{M}$ is crisp) such that φ is not valid in $\widehat{\mathfrak{M}}$.

Suppose that $\beta = V(\varphi, x) < 1$ for some $\mathsf{K}(\mathbf{G}_{\uparrow})$-tree-model of finite height $\mathfrak{M} = \langle W, R, V\rangle$ with root x. Let $\mathbf{B}$ be the finite subalgebra of $\mathbf{G}_{\uparrow}$ with universe $(G_{\uparrow} \cap [0, \beta]) \cup \{1\}$ and consider $h\colon \mathbf{G}_{\uparrow} \to \mathbf{B}$ defined by

$$h(a) = \begin{cases} a & \text{if } a \le \beta \\ 1 & \text{otherwise.} \end{cases}$$

We define a $\mathsf{K}(\mathbf{B})$-model $\widehat{\mathfrak{M}} = \langle W, \widehat{R}, \widehat{V}\rangle$ (that is crisp if $\mathfrak{M}$ is crisp) as follows. Let $\widehat{R}yz = h(Ryz)$ for all $y, z \in W$ and $\widehat{V}(p, y) = h(V(p, y))$ for all $y \in W$ and $p \in \mathrm{Var}$. We prove that $\widehat{V}(\psi, y) = h(V(\psi, y))$ for all $y \in W$ and $\psi \in \mathrm{Fm}$ by induction on $\ell(\psi)$. The base case follows by definition (recalling that the only constants are $\bar{0}$ and $\bar{1}$). For the induction step, the propositional cases follow by observing that h is a Heyting algebra homomorphism (i.e., preserves the operations $\wedge$, $\vee$, $\to$, $\bar{0}$, and $\bar{1}$). The case of $\psi = \Box\chi$ is also straightforward. If $\psi = \Diamond\chi$, then

$$\begin{aligned}
\widehat{V}(\Diamond\chi, y) &= \bigvee\{\widehat{R}yz \wedge \widehat{V}(\chi, z) : z \in W\} && (10)\\
&= \bigvee\{h(Ryz) \wedge h(V(\chi, z)) : z \in W\} && (11)\\
&= \bigvee\{h(Ryz \wedge V(\chi, z)) : z \in W\} && (12)\\
&= h(\bigvee\{Ryz \wedge V(\chi, z) : z \in W\}) && (13)\\
&= h(V(\Diamond\chi, y)). && (14)
\end{aligned}$$

The step from (10) to (11) follows using the induction hypothesis and the step from (11) to (12) follows because h is a Heyting algebra homomorphism. For the step from (12) to (13), note that for $\bigvee\{Ryz \wedge V(\chi, z) : z \in W\} \le \beta$, the equality is immediate. Otherwise, $Ryz \wedge V(\chi, z) > \beta$ for some $z \in W$ and $h(Ryz \wedge V(\chi, z)) = 1$, so $h(\bigvee\{Ryz \wedge V(\chi, z) : z \in W\}) = 1 = \bigvee\{h(Ryz \wedge V(\chi, z)) : z \in W\}$.

Hence $\widehat{V}(\varphi, x) = h(V(\varphi, x)) = h(\beta) = \beta < 1$ as required. ☐

The finite model property does not hold, however, for Gödel modal logics with universe $[0,1]$ or $G_{\downarrow}$, or even $G_{\uparrow}$ if we add also the connective Δ to the language. The problem in these cases stems from the existence of accumulation points in the universe of truth values considered together with the non-continuous operation $\neg$ or Δ. If infinitely many worlds are accessible from a world x, then the value taken by a formula $\Box\varphi$ (or $\Diamond\varphi$) at x will be the infimum (supremum) of values calculated from values of φ at these worlds, but may not be the minimum (maximum). A formula may therefore not be valid in such a model, but valid in all finite models where infima (suprema) and minima (maxima) coincide.

Theorem 7. *Suppose that either (i) the universe of* $\mathbf{A}$ *is* $[0,1]$ *or* $G_{\downarrow}$*, or (ii) the universe of* $\mathbf{A}$ *is* $G_{\uparrow}$ *and the language contains* Δ*. Then neither* $\mathsf{K}(\mathbf{A})$ *nor* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ *has the finite model property.*

Proof. For (i), we follow [9] where it is shown that the following formula provides a counterexample to the finite model property of GK and GK^{C}:

$$\Box\neg\neg p \to \neg\neg\Box p.$$

Just observe that the formula is valid in all finite $\mathsf{K}(\mathbf{A})$-models, but not in the infinite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-model $\langle \mathbb{N}, R, V\rangle$ where $Rmn = 1$ for all $m, n \in \mathbb{N}$ and $V(p,n) = \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Hence neither $\mathsf{K}(\mathbf{A})$ nor $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ has the finite model property.

Similarly, for (ii), the formula

$$\Delta\Diamond p \to \Diamond\Delta p$$

is valid in all finite $\mathsf{K}(\mathbf{A})$-models, but not in the infinite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-model $\langle \mathbb{N}, R, V\rangle$ where $Rmn = 1$ for all $m, n \in \mathbb{N}$ and $V(p,n) = \frac{n}{n+1}$ for all $n \in \mathbb{N}$. $\Box$

Let us remark also that decidability and indeed PSPACE-completeness of validity in the box and diamond fragments of both GK and GK^{C} have been established in [27] using analytic Gentzen-style proof systems, but that decidability of validity in the full logics GK and GK^{C} has remained open.

3. A New Semantics for the Modal Operators

Consider again the failure of the finite model property for GK^{C} established in the proof of Theorem 7. For a GK^{C}-model to render $\Box\neg\neg p \to \neg\neg\Box p$ invalid at a world x, there must be values of p at worlds accessible to x that form an infinite descending sequence tending to but never reaching 0. This ensures that the infinite model falsifies the formula, but also that no particular world acts as a "witness" to the value of $\Box p$. Here, we redefine models to restrict the values at each world that can be taken by box-formulas and diamond-formulas. A formula such as $\Box p$ can then be "witnessed" at a world where the value of p is merely "sufficiently close" to the value of $\Box p$.

To ensure that these redefined models accept the same valid formulas as the original models, we restrict our attention to order-based algebras where the order satisfies a certain homogeneity property. Recall that $R(\mathbf{A})$ and $L(\mathbf{A})$ are the sets of right and left accumulation points, respectively, of an order-based algebra $\mathbf{A}$ in the usual topology inherited from $[0,1]$. Note also that by (a,b), $[a,b)$, etc. we denote here the intervals $(a,b) \cap A$, $[a,b) \cap A$, etc. in $\mathbf{A}$. We say that $\mathbf{A}$ is *locally right homogeneous* if for any $a \in R(\mathbf{A})$, there is a $c \in A$ such that $a < c$ and for any $e \in (a,c)$, there is a complete deflationary order embedding $h\colon [a,c) \to [a,e)$ such that $h(a) = a$. In this case, c is called a *witness* of right homogeneity at a. Similarly, $\mathbf{A}$ is said to be *locally left homogeneous* if for any $b \in L(\mathbf{A})$, there is a $d \in A$ such that $d < b$ and for any $f \in (d,b)$, there is a complete inflationary order embedding $h\colon (d,b] \to (f,b]$ such that $h(b) = b$. In this case, d is called a *witness* of left homogeneity at b. We will call $\mathbf{A}$ *locally homogeneous* if it is both locally right homogeneous and locally left homogeneous.

Observe that if $c \in A$ is a witness of right homogeneity at a, then any $e \in (a,c)$ will also be a witness of right homogeneity at a. Hence c can be chosen sufficiently close to a so that (a,c) is disjoint to any given finite subset of A. A similar observation holds for witnesses of left homogeneity.

Example 8. *Any finite $\mathbf{A}$ is trivially locally homogeneous. Also any $\mathbf{A}$ with $A = [0,1]$ is locally homogeneous: for $a \in R(\mathbf{A}) = [0,1)$, choose any $c > a$ to witness right homogeneity at a, and similarly for $b \in L(\mathbf{A}) = (0,1]$, choose any $d < b$ to witness left homogeneity at b. In the case of $A = G_{\downarrow}$, $L(\mathbf{A}) = \emptyset$, $R(\mathbf{A}) = \{0\}$, and any $c > 0$ witnesses right homogeneity at 0. Similarly, for $A = G_{\uparrow}$, $R(\mathbf{A}) = \emptyset$, $L(\mathbf{A}) = \{1\}$, and any $d < 1$ witnesses left homogeneity at 1. Moreover, infinitely many more non-isomorphic examples can be constructed using the fact that any ordered sum or lexicographical product of two locally homogeneous ordered sets is locally homogeneous.*

Let us assume for the remainder of this section that $\mathbf{A}$ is a locally homogeneous order-based algebra. An $\mathsf{FK}(\mathbf{A})$-*model* is a five-tuple $\mathfrak{M} = \langle W, R, V, T_{\Box}, T_{\Diamond} \rangle$ such that $\langle W, R, V \rangle$ is a $\mathsf{K}(\mathbf{A})$-model and $T_{\Box}\colon W \to \mathcal{P}(A)$ and $T_{\Diamond}\colon W \to \mathcal{P}(A)$ are functions satisfying for each $x \in W$:

(i) $\mathrm{C}^{\mathbf{A}}_{\mathcal{L}} \subseteq T_{\Box}(x) \cap T_{\Diamond}(x)$,

(ii) $T_{\Box}(x) = A \setminus \bigcup_{i \in I}(a_i, c_i)$ for some finite $I \subseteq \mathbb{N}$ (possibly empty), where $a_i \in R(\mathbf{A})$, c_i witnesses right homogeneity at a_i, and the intervals (a_i, c_i) are pairwise disjoint,

(iii) $T_{\Diamond}(x) = A \setminus \bigcup_{j \in J}(d_j, b_j)$ for some finite $J \subseteq \mathbb{N}$ (possibly empty), where $b_j \in L(\mathbf{A})$, d_j witnesses left homogeneity at b_j, and the intervals (d_j, b_j) are pairwise disjoint.

The valuation V is extended to the mapping $V \colon \mathrm{Fm} \times W$ inductively as follows:

$$V(\star(\varphi_1, \ldots, \varphi_n), x) \quad = \quad \star(V(\varphi_1, x), \ldots, V(\varphi_n, x))$$

for each n-ary operational symbol $\star$ of $\mathcal{L}$, and

$$\begin{aligned} V(\Box\varphi, x) &= \bigvee\{r \in T_{\Box}(x) : r \leq \bigwedge\{Rxy \to V(\varphi, y) : y \in W\}\} \\ V(\Diamond\varphi, x) &= \bigwedge\{r \in T_{\Diamond}(x) : r \geq \bigvee\{Rxy \wedge V(\varphi, y) : y \in W\}\}. \end{aligned}$$

As before, an $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$-*model* satisfies the extra condition that $\langle W, R\rangle$ is a crisp $\mathbf{A}$-frame, and the conditions for $\Box$ and $\Diamond$ simplify to

$$\begin{aligned} V(\Box\varphi, x) &= \bigvee\{r \in T_{\Box}(x) : r \leq \bigwedge\{V(\varphi, y) : Rxy\}\} \\ V(\Diamond\varphi, x) &= \bigwedge\{r \in T_{\Diamond}(x) : r \geq \bigvee\{V(\varphi, y) : Rxy\}\}. \end{aligned}$$

A formula $\varphi \in \mathrm{Fm}$ is *valid* in $\mathfrak{M}$ if $V(\varphi, x) = 1$ for all $x \in W$.

Example 9. *Note that when A is finite, $T_{\Box}(x) = T_{\Diamond}(x) = A$. For $A = [0,1]$, both $T_{\Box}(x)$ and $T_{\Diamond}(x)$ are obtained by removing finitely many arbitrary disjoint intervals (a, b) not containing constants. For $A = G_{\downarrow}$, the only possibilities are $T_{\Diamond}(x) = A$ and $T_{\Box}(x) = A$ or $T_{\Box}(x) = \{0, \frac{1}{n}, \frac{1}{n-1}, \ldots, 1\}$ for some $n \in \mathbb{Z}^{+}$ respecting $\mathrm{C}_{\mathcal{L}} \subseteq T_{\Box}(x)$. The case of $A = G_{\uparrow}$ is very similar.*

It is worth pointing out that in every $\mathsf{FK}(\mathbf{A})$-model $\mathfrak{M} = \langle W, R, V, T_{\Box}, T_{\Diamond}\rangle$ and for any $x \in W$, $T_{\Box}(x)$ and $T_{\Diamond}(x)$ will be complete subsets of A. Hence, the suprema and infima defining $V(\Box\varphi, x)$ and $V(\Diamond\varphi, x)$ will actually be maxima and minima, and always $V(\Box\varphi, x) \in T_{\Box}(x)$ and $V(\Diamond\varphi, x) \in T_{\Diamond}(x)$.

We now extend some previously introduced notions to $\mathsf{FK}(\mathbf{A})$-models. Given an $\mathsf{FK}(\mathbf{A})$- model $\mathfrak{M} = \langle W, R, V, T_{\Box}, T_{\Diamond}\rangle$, we call $\mathfrak{M}' = \langle W', R', V', T'_{\Box}, T'_{\Diamond}\rangle$ an

$\mathsf{FK}(\mathbf{A})$-*submodel* of $\mathfrak{M}$, written $\mathfrak{M}' \subseteq \mathfrak{M}$, if $W' \subseteq W$ and R', V', $T'_\Box$, and $T'_\Diamond$ are the restrictions to W' of R, V, $T_\Box$, and $T_\Diamond$, respectively. As before, given $x \in W$, the $\mathsf{FK}(\mathbf{A})$-*submodel of* $\mathfrak{M}$ *generated by* x is the smallest $\mathsf{FK}(\mathbf{A})$-submodel $\mathfrak{M}' = \langle W', R', V', T'_\Box, T'_\Diamond\rangle$ of $\mathfrak{M}$ satisfying $x \in W'$ and for all $y \in W'$, $z \in R^+[y]$ implies $z \in W'$. Lemmas 2 and 3 then extend to $\mathsf{FK}(\mathbf{A})$-models as follows with minimal changes in the proofs.

Lemma 10. *Let* $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ *be an* $\mathsf{FK}(\mathbf{A})$*-model.*

(a) *Let* $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond\rangle$ *be a generated* $\mathsf{FK}(\mathbf{A})$*-submodel of* $\mathfrak{M}$*. Then* $\widehat{V}(\varphi, x) = V(\varphi, x)$ *for all* $x \in \widehat{W}$*, and* $\varphi \in \mathrm{Fm}$.

(b) *Given any* $x \in W$ *and* $k \in \mathbb{N}$*, there exists an* $\mathsf{FK}(\mathbf{A})$*-tree-model* $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond\rangle$ *with root* $\widehat{x}$ *and* $\mathrm{hg}(\widehat{\mathfrak{M}}) \leq k$ *such that* $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x)$ *for all* $\varphi \in \mathrm{Fm}$ *with* $\ell(\varphi) \leq k$*, and if* $\mathfrak{M}$ *is an* $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$*-model, then so is* $\widehat{\mathfrak{M}}$*.*

Example 11. *There are very simple finite* $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$*-counter-models for the formula* $\Box\neg\neg p \to \neg\neg\Box p$ *when* $A = [0,1]$*. For* $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ *with* $W = \{a\}$*,* $Raa = 1$*,* $T_\Box(a) = T_\Diamond(a) = \mathrm{C}_\mathcal{L}$*, and* $0 < V(p, a) < \min(\mathrm{C}_\mathcal{L} \setminus \{0\})$*:*

$$\begin{aligned}
V(\Box\neg\neg p, a) &= \bigvee\{r \in \mathrm{C}_\mathcal{L} : r \leq \bigwedge\{V(\neg\neg p, y) : Ray\}\} \\
&= \bigvee\{r \in \mathrm{C}_\mathcal{L} : r \leq V(\neg\neg p, a)\} \\
&= \bigvee\{r \in \mathrm{C}_\mathcal{L} : r \leq 1\} \\
&= 1 \\
V(\neg\neg\Box p, a) &= \neg\neg\bigvee\{r \in \mathrm{C}_\mathcal{L} : r \leq \bigwedge\{V(p, y) : Ray\}\} \\
&= \neg\neg\bigvee\{r \in \mathrm{C}_\mathcal{L} : r \leq V(p, a)\} \\
&= \neg\neg 0 \\
&= 0 \\
V(\Box\neg\neg p \to \neg\neg\Box p, a) &= V(\Box\neg\neg p, a) \to V(\neg\neg\Box p, a) \\
&= 1 \to 0 \\
&= 0.
\end{aligned}$$

The same formula fails in a similar finite $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$*-model when* $A = G_\downarrow$*, and* $\Delta\Diamond p \to \Diamond\Delta p$ *fails in a similar* $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$*-model when* $A = G_\uparrow$*.*

Indeed, as shown below, given an $\mathsf{FK}(\mathbf{A})$-tree-model of finite height where $\varphi \in \mathrm{Fm}$ is not valid, we can always "prune" (i.e., remove branches from) the model in such a way that φ is still not valid in the resulting finite $\mathsf{FK}(\mathbf{A})$-tree-model. It then follows from part (b) of Lemma 10 that $\mathsf{FK}(\mathbf{A})$ and $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$ have the finite model property.

Lemma 12. *Let $\Sigma \subseteq \mathrm{Fm}$ be a finite fragment. Then for any $\mathsf{FK}(\mathbf{A})$-tree-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ of finite height with root x, there is a finite $\mathsf{FK}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond\rangle$ with $\langle \widehat{W}, \widehat{R}, \widehat{V}\rangle \subseteq \langle W, R, V\rangle$, root $x \in \widehat{W}$, and $|\widehat{W}| \leq |\Sigma|^{\mathrm{hg}(\mathfrak{M})}$ such that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$.*

Proof. We prove the lemma by induction on $\mathrm{hg}(\mathfrak{M})$. For the base case, $W = \{x\}$ and it suffices to define $\widehat{\mathfrak{M}} = \mathfrak{M}$.

For the induction step $\mathrm{hg}(\mathfrak{M}) = n + 1$, consider for each $y \in R^+[x]$, the submodel $\mathfrak{M}_y = \langle W_y, R_y, V_y, T_{\Box y}, T_{\Diamond y}\rangle$ of $\mathfrak{M}$ generated by y. Each $\mathfrak{M}_y$ is an $\mathsf{FK}(\mathbf{A})$-tree-model of finite height with root y and $\mathrm{hg}(\mathfrak{M}_y) \leq n$. Hence, by the induction hypothesis, for each $y \in R^+[x]$, there is a finite $\mathsf{FK}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y, \widehat{T}_{\Box y}, \widehat{T}_{\Diamond y}\rangle$ with $\langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y, \rangle \subseteq \langle W_y, R_y, V_y\rangle$ and root $y \in \widehat{W}_y$, such that $|\widehat{W}_y| \leq |\Sigma|^n$ and for all $\varphi \in \Sigma$, using Lemma 10(a), $\widehat{V}_y(\varphi, y) = V_y(\varphi, y) = V(\varphi, y)$.

We choose a finite number of appropriate $y \in R^+[x]$ in order to build our finite $\mathsf{FK}(\mathbf{A})$-submodel $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond\rangle$ of $\mathfrak{M}$ as the "union" of these $\widehat{\mathfrak{M}}_y$ connected by the root world $x \in \widehat{W}$. First we define $\widehat{T}_\Box(x)$ and $\widehat{T}_\Diamond(x)$.

Consider $T_\Box(x) = A \setminus \bigcup_{i\in I}(a_i, c_i)$ for some finite $I \subseteq \mathbb{N}$ (possibly empty), where for all $i \in I$, $a_i \in R(\mathbf{A})$, c_i witnesses right homogeneity at a_i, and the intervals (a_i, c_i) are pairwise disjoint. Consider also the finite (possibly empty) set $(V[\Sigma_\Box, x] \cap R(\mathbf{A})) \setminus \{a_i : i \in I\} = \{a_j : j \in J\}$ where $I \cap J = \emptyset$. For $j \in J$, choose a witness of right homogeneity c_j at a_j such that the intervals (a_i, c_i) are pairwise disjoint, for all $i \in I \cup J$, and $(V[\Sigma_\Box, x] \cup \mathrm{C}_{\mathcal{L}}) \cap (\bigcup_{i\in I\cup J}(a_i, c_i)) = \emptyset$. We define $\widehat{T}_\Box(x) = A \setminus \bigcup_{i\in I\cup J}(a_i, c_i)$, satisfying conditions (i) and (ii) of the definition of an $\mathsf{FK}(\mathbf{A})$-model by construction. Note also that $V[\Sigma_\Box, x] \cup \mathrm{C}_{\mathcal{L}} \subseteq \widehat{T}_\Box(x) \subseteq T_\Box(x)$.

Similarly, consider $T_\Diamond(x) = A \setminus \bigcup_{i\in I'}(d_i, b_i)$ for some finite $I' \subseteq \mathbb{N}$ (possibly empty), where for all $i \in I'$, $b_i \in L(\mathbf{A})$, d_i witnesses left homogeneity at b_i, and the intervals (d_i, b_i) are pairwise disjoint. Consider also the finite (possibly empty) set $(V[\Sigma_\Diamond, x] \cap L(\mathbf{A})) \setminus \{b_i : i \in I'\} = \{b_j : j \in J'\}$. For $j \in J'$, choose a witness of left homogeneity d_j at b_j such that the intervals (d_i, b_i) are pairwise disjoint for all $i \in I' \cup J'$, and $(V[\Sigma_\Diamond, x] \cup \mathrm{C}_\mathcal{L}) \cap (\bigcup_{i \in I' \cup J'}(d_i, b_i)) = \emptyset$. We define $\widehat{T}_\Diamond(x) = A \setminus \bigcup_{i \in I' \cup J'}(d_i, b_i)$, satisfying conditions (i) and (iii) of the definition of an $\mathsf{FK}(\mathbf{A})$-model by construction. Note also that $V[\Sigma_\Diamond, x] \cup \mathrm{C}_\mathcal{L} \subseteq \widehat{T}_\Diamond(x) \subseteq T_\Diamond(x)$.

Consider now $\varphi = \Box\psi \in \Sigma_\Box$ and let $a = V(\Box\psi, x) \in \widehat{T}_\Box(x)$. If $a \notin R(\mathbf{A})$, choose $y_\varphi \in R^+[x]$ such that $a = Rxy_\varphi \to V(\psi, y_\varphi)$. If $a \in R(\mathbf{A})$, there is an $i \in I \cup J$, such that $a = a_i$, and we choose $y_\varphi \in R^+[x]$ such that $Rxy_\varphi \to V(\psi, y_\varphi) \in [a_i, c_i)$. Similarly, for each $\varphi = \Diamond\psi \in \Sigma_\Diamond$, let $b = V(\Diamond\psi, x) \in T_\Diamond(x)$. If $b \notin L(\mathbf{A})$, choose $y_\varphi \in R^+[x]$ such that $b = Rxy_\varphi \wedge V(\psi, y_\varphi)$. If $b \in L(\mathbf{A})$, there is an $i \in I' \cup J'$, such that $b = b_i$ and we choose $y_\varphi \in R^+[x]$ such that $Rxy_\varphi \wedge V(\psi, y_\varphi) \in (d_i, b_i]$.

Now let $Y = \{y_\varphi \in R^+[x] : \varphi \in \Sigma_\Box \cup \Sigma_\Diamond\}$, noting that $|Y| \leq |\Sigma_\Box \cup \Sigma_\Diamond| < |\Sigma|$. We define $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond \rangle$ where

$$\widehat{W} = \{x\} \cup \bigcup_{y \in Y} \widehat{W}_y,$$

and $\widehat{R}$ and $\widehat{V}$ are R and V, respectively, restricted to $\widehat{W}$. $\widehat{T}_\Box(z)$ and $\widehat{T}_\Diamond(z)$ are defined as $\widehat{T}_{\Box y}(z)$ and $\widehat{T}_{\Diamond y}(z)$, respectively, if $z \in \widehat{W}_y$, for some $y \in Y$. $\widehat{T}_\Box(x)$ and $\widehat{T}_\Diamond(x)$ are defined as above.

Observe that $\langle \widehat{W}, \widehat{R}, \widehat{V} \rangle \subseteq \langle W, R, V \rangle$, $x \in \widehat{W}$ is the root of $\widehat{\mathfrak{M}}$, and $|\widehat{W}| \leq |Y||\Sigma|^n + 1 < |\Sigma||\Sigma|^n = |\Sigma|^{\mathrm{hg}(\mathfrak{M})}$. Moreover, for each $y \in Y$, $\widehat{\mathfrak{M}}_y$ is an $\mathsf{FK}(\mathbf{A})$-submodel of $\widehat{\mathfrak{M}}$ generated by y. Hence, by Lemma 10(a) and the induction hypothesis, for all $\varphi \in \Sigma$,

$$\widehat{V}(\varphi, y) = \widehat{V}_y(\varphi, y) = V_y(\varphi, y) = V(\varphi, y). \tag{15}$$

We show now that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$, proceeding by induction on $\ell(\varphi)$. The base case follows directly from the definition of $\widehat{V}$. For the inductive step, the non-modal cases follow directly using the induction hypothesis. For $\varphi = \Box\psi$, there are two cases. Suppose first that $V(\Box\psi, x) = a \notin R(\mathbf{A})$ and recall that

$$V(\Box\psi, x) = \bigvee\{r \in T_\Box(x) : r \leq \bigwedge\{Rxy \to V(\psi, y) : y \in W\}\} = a.$$

This implies that $Rxy \to V(\psi, y) \geq a$ for all $y \in Y \subseteq R^+[x]$. Hence, by (15), $\widehat{R}xy \to \widehat{V}(\psi, y) \geq a$ for all $y \in Y = \widehat{R}^+[x]$. Moreover, $\widehat{R}xy_\varphi \to \widehat{V}(\psi, y_\varphi) = a$ and hence, because $a \in V[\Sigma_\Box, x] \subseteq \widehat{T}_\Box(x)$,

$$\widehat{V}(\Box\psi, x) = \bigvee\{r \in \widehat{T}_\Box(x) : r \leq \bigwedge\{\widehat{R}xy \to \widehat{V}(\psi, y) : y \in \widehat{W}\}\} = a.$$

For the second case, suppose that $V(\Box\psi, x) = a \in R(\mathbf{A})$. Then $a = a_i$, for some $i \in I \cup J$, and we observe that

$$a_i = a \leq \bigwedge\{Rxy \to V(\psi, y) : y \in W\}.$$

By (15), we know that $\widehat{R}xy \to \widehat{V}(\psi, y) = Rxy \to V(\psi, y)$ for each $y \in \widehat{W}$, and because $\widehat{W} \subseteq W$, it follows that

$$a_i \leq \bigwedge\{Rxy \to V(\psi, y) : y \in W\} \leq \bigwedge\{\widehat{R}xy \to \widehat{V}(\psi, y) : y \in \widehat{W}\}.$$

By the choice of $y_\varphi \in \widehat{W}$,

$$\widehat{R}xy_\varphi \to \widehat{V}(\psi, y_\varphi) = Rxy_\varphi \to V(\psi, y_\varphi) < c_i.$$

Hence $a_i \leq \bigwedge\{\widehat{R}xy \to \widehat{V}(\psi, y) : y \in \widehat{W}\} < c_i$ and

$$\widehat{V}(\Box\psi, x) = \bigvee\{r \in \widehat{T}_\Box(x) : r \leq \bigwedge\{\widehat{R}xy \to \widehat{V}(\psi, y) : y \in \widehat{W}\}\} = a_i = a.$$

The case where $\varphi = \Diamond\psi$ is very similar. $\Box$

Corollary 13. $\mathsf{FK}(\mathbf{A})$ *and* $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$ *have the finite model property.*

4. Equivalence of the Semantics

Let us assume again that $\mathbf{A}$ is a locally homogeneous order-based algebra. We devote this section to establishing that a formula is valid in $\mathsf{K}(\mathbf{A})$ or $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ if and only if it is valid in $\mathsf{FK}(\mathbf{A})$ or $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$, respectively. Observe first that any $\mathsf{K}(\mathbf{A})$-model can be extended to an $\mathsf{FK}(\mathbf{A})$-model with the same valid formulas simply by defining $T_\Box$ and $T_\Diamond$ to be constantly A. Hence any $\mathsf{FK}(\mathbf{A})$-valid formula is also $\mathsf{K}(\mathbf{A})$-valid. We therefore turn our attention to the other (much harder) direction: proving that any $\mathsf{K}(\mathbf{A})$-valid formula is also $\mathsf{FK}(\mathbf{A})$-valid.

The main ingredient of the proof (see Lemma 16) is the construction of a $\mathsf{K}(\mathbf{A})$-tree- model taking the same values for formulas at its root as a given $\mathsf{FK}(\mathbf{A})$-tree-model. Note that the original $\mathsf{FK}(\mathbf{A})$-tree- model without the functions $T_\Box$ and $T_\Diamond$ cannot play this role in general; in $[0, 1]$, for example, the infimum or supremum required for calculating the value of a box-formula or diamond-formula at the root x might not be in the set $T_\Box(x)$ or $T_\Diamond(x)$. This problem is resolved by taking infinitely many copies of an inductively defined $\mathsf{K}(\mathbf{A})$-model in such a way that certain parts of the intervals in A missing in $T_\Box(x)$ or $T_\Diamond(x)$ are "squeezed" closer to either their lower or upper bounds. The obtained infima and suprema will then coincide with the next smaller or larger member of $T_\Box(x)$ and $T_\Diamond(x)$: that is, the required values of the formulas at x in the original $\mathsf{FK}(\mathbf{A})$-tree-model. The following example illustrates this idea for the relatively simple case where $\mathbf{A} = \mathbf{G}$.

Example 14. *Consider the* $\mathsf{FK}(\mathbf{G})^{\mathsf{C}}$*-tree-model* $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ *with* $W = \{x, y\}$, $R = \{(x, y)\}$, *and* $T_\Box(x) = [0, 1] \setminus (0.2, 0.8)$. *Note that* $0.2 \in R(\mathbf{G})$ *and that* 0.8 *witnesses right homogeneity at* 0.2. *Suppose that* $V(p, y) = 0.6$, *so that*

$$\begin{aligned} V(\Box p, x) &= \bigvee\{r \in T_\Box(x) : r \leq \bigwedge\{V(p, y) : Rxy\}\} \\ &= \bigvee\{r \in [0, 1] \setminus (0.2, 0.8) : r \leq 0.6\} \\ &= 0.2. \end{aligned}$$

For each $k \geq 2$, *we then consider* $\mathfrak{M}_k = \langle W_k, R_k, V_k\rangle$ *with* $W_k = \{y_k\}$, $R_k = \emptyset$, *and* $V_k(p, y_k) = h_k(V(p, y))$, *for some deflationary* $\{0, 1\}$*-order embedding* $h_k \colon [0, 1] \to [0, 1]$, *satisfying for each* $k \geq 2$,

$$h_k[[0.2, 0.8)] = [0.2, 0.2 + \tfrac{1}{k}).$$

Defining the $\mathsf{K}(\mathbf{G})^{\mathsf{C}}$*-tree-model* $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$, *with* $\widehat{W} = \{x\} \cup \{y_k : k \geq 2\}$, $\widehat{R} = \{(x, y_k) : k \geq 2\}$, *and* $\widehat{V}(p, y_k) = V_k(p, y_k)$, *we obtain (see Figure 1):*

$$\begin{aligned} \widehat{V}(\Box p, x) &= \bigwedge\{\widehat{V}(p, y_k) : \widehat{R}xy_k\} \\ &= 0.2 \\ &= V(\Box p, x). \end{aligned}$$

A central tool in the proof of Lemma 16 is the following result which allows the "squeezing" of $\mathsf{K}(\mathbf{A})$-models so that the values of formulas are arbitrarily close to certain points (as in Example 14). Intuitively, in the proof of Lemma 16, the set B below (in Lemma 15) will be the set of values at the root world x of all box-formulas and diamond-formulas in some fragment Σ. In (a) below, the values a and c will denote the endpoints of the removed interval and s will be the relevant value that we want to squeeze closer and closer towards a. The value t, the upper endpoint of the squeezed interval, will then be chosen in $A \setminus (B \cap L(\mathbf{A}))$ in order to ensure that all the suprema in B (relevant for determining the values of diamond-

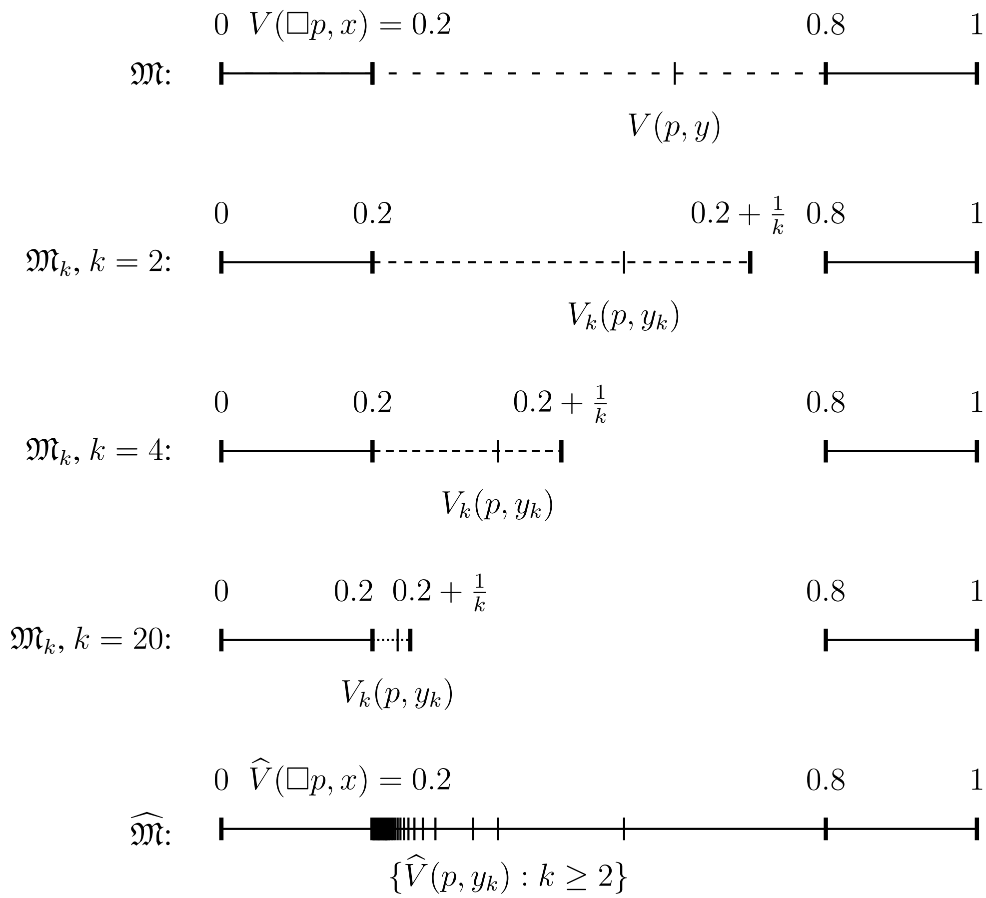

Figure 1: Squeezing models

formulas in Σ) are preserved by the squeezing. Note that $u \in [0,1]$ can be any value as close to a as needed (e.g., $u = a + \frac{1}{k}$ for any $k \in \mathbb{Z}^+$) so as to squeeze the interval $[a,t)$ into $[a,u)$ by the B-complete deflationary order embedding h, with the intention that $s \in [a,t)$ and $h(s) \in [a,u)$. For (b), the ideas are very similar.

Lemma 15. *Let $B \subseteq A$ be countable.*

(a) *Given $a \in R(\mathbf{A})$, some witness $c > a$ of right homogeneity at a, and an $s \in [a,c)$, there is a $t \in (s,c]$ such that $t \notin B \cap L(\mathbf{A})$. Moreover, for all $u \in (a,t]$, there is a B-complete deflationary order embedding $h\colon A \to A$ such that*

$$h[[a,t)] \subseteq [a,u), \quad \textit{and} \quad h|_{A\setminus(a,t)} = \mathrm{id}_A.$$

(b) *Given $b \in L(\mathbf{A})$, some witness $d < b$ of left homogeneity at b, and an $s \in (d,b]$, there is a $t \in [d,s)$ such that $t \notin B \cap R(\mathbf{A})$. Moreover, for all $u \in [t,b)$, there is a B-complete inflationary order embedding $h\colon A \to A$ such that*

$$h[(t,b]] \subseteq (u,b] \quad \textit{and} \quad h|_{A\setminus(t,b)} = \mathrm{id}_A.$$

Proof. For (a), let $B \subseteq A$ be countable and consider $a \in R(\mathbf{A})$, a witness c of right homogeneity at a, and $s \in [a,c)$. We first prove that there is a $t \in (s,c]$ which is either in $A \setminus L(\mathbf{A})$ or in $A \setminus B$. If $c \notin L(\mathbf{A})$, choose $t = c$. If $c \in L(\mathbf{A})$, then $[s,c]$ is infinite. Recall that A is a complete sublattice of $[0,1]$ and that every non-empty perfect set of real numbers (closed and containing no isolated points) is uncountable. Hence if $[s,c]$ is countable, there must be an isolated point $t \in (s,c]$ such that $t \notin L(\mathbf{A})$. If $[s,c]$ is uncountable, then there is a $t \in (s,c] \setminus B$, as B is countable. Either way, there is a $t \in (s,c]$ such that $t \notin B \cap L(\mathbf{A})$.

Now we define the embedding. Because $t \leq c$ also witnesses right homogeneity at a, for each $u \in (a,t]$, there is a complete deflationary order embedding $g\colon [a,t) \to [a,u)$ with $g(a) = a$. Define h as g on $[a,t)$ and as the identity on $A \setminus [a,t)$. Then all arbitrary meets and joins in A are preserved except in the case where t is a join of elements in $[a,t)$ and so $t \in L(\mathbf{A})$. But in this case $t \notin B$. Hence (a) holds. For (b), we use a very similar argument. $\square$

Lemma 16. *Let Σ be a finite fragment and let $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ be a finite $\mathsf{FK}(\mathbf{A})$-tree-model with root x. Then there is a countable $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ with root $\widehat{x}$ such that $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x)$ for all $\varphi \in \Sigma$. Moreover, if $\mathfrak{M}$ is crisp, then so is $\widehat{\mathfrak{M}}$.*

Proof. The lemma is proved by induction on $\mathrm{hg}(\mathfrak{M})$. The base case is immediate, fixing $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ with $\widehat{W} = W = \{x\}$, $\widehat{R} = R$, and $\widehat{V} = V$. For the induction step, given $y \in R^+[x]$, let $\mathfrak{M}_y = \langle W_y, R_y, V_y, T_{\Box y}, T_{\Diamond y} \rangle$ be the submodel of $\mathfrak{M}$ generated by y. Then $\mathfrak{M}_y$ is a finite $\mathsf{FK}(\mathbf{A})$-tree-model with root y, $\mathrm{hg}(\mathfrak{M}_y) < \mathrm{hg}(\mathfrak{M})$, and, by Lemma 10(a), $V_y(\varphi, z) = V(\varphi, z)$ for all $z \in W_y$ and $\varphi \in \mathrm{Fm}$. So, by the induction hypothesis, there is a countable $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y \rangle$ (crisp if $\mathfrak{M}$ is crisp) with root $\widehat{y}$ such that $\widehat{V}_y(\varphi, \widehat{y}) = V_y(\varphi, y) = V(\varphi, y)$ for all $\varphi \in \Sigma$.

For each $\varphi \in \Sigma_\Box \cup \Sigma_\Diamond$, we will choose a world $y_\varphi \in R^+[x]$ as described below and then, using Lemma 15, define for each $k \in \mathbb{Z}^+$ a copy of the $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}_{y_\varphi}$, denoted $\widehat{\mathfrak{M}}^k_\varphi$. Suppose that $\varphi = \Box\psi \in \Sigma_\Box$. Consider $T_\Box(x) = A \setminus \bigcup_{i \in I}(a_i, c_i)$ for some finite $I \subseteq \mathbb{N}$ (possibly empty), where for all $i \in I$, $a_i \in R(\mathbf{A})$, c_i witnesses right homogeneity at a_i, and the intervals (a_i, c_i) are pairwise disjoint. There are two cases.

(i) Suppose that $V(\Box\psi, x) = a_i$ for some $i \in I$. Recalling that

$$a_i = V(\Box\psi, x) = \bigvee\{r \in T_\Box(x) : r \leq \bigwedge\{Rxy \to V(\psi, y) : y \in W\}\},$$

there must be a world $y_\varphi \in R^+[x]$ such that

$$Rxy_\varphi \to V(\psi, y_\varphi) \in [a_i, c_i).$$

We fix $B = \widehat{V}_{y_\varphi}[\Sigma_\Box \cup \Sigma_\Diamond, \widehat{W}_{y_\varphi}]$, which is countable because $\widehat{W}_{y_\varphi}$ is countable and $\Sigma_\Box \cup \Sigma_\Diamond$ is finite. Using Lemma 15, for some t satisfying

$$a_i \leq s = Rxy_\varphi \to V(\psi, y_\varphi) < t \leq c_i,$$

there exists for each $k \in \mathbb{Z}^+$, a B-complete deflationary order embedding $h_k \colon A \to A$ mapping $[a_i, t)$ into $[a_i, a_i + \frac{1}{k})$, and $h_k|_{A \setminus (a_i, t)} = \mathrm{id}_A$. Clearly, this implies that for all $k \in \mathbb{Z}^+$, h_k is a $\widehat{V}_{y_\varphi}[\Sigma_\Box \cup \Sigma_\Diamond, \widehat{W}_{y_\varphi}]$-complete deflationary $\mathrm{C}_\mathcal{L}$-order embedding. We then define the copy $\widehat{\mathfrak{M}}^k_\varphi = \langle \widehat{W}^k_\varphi, \widehat{R}^k_\varphi, \widehat{V}^k_\varphi \rangle$ of $\widehat{\mathfrak{M}}_{y_\varphi}$ as follows:

- $\widehat{W}^k_\varphi$ is a copy of $\widehat{W}_{y_\varphi}$, denoting the copy of $\widehat{x}_{y_\varphi} \in \widehat{W}_{y_\varphi}$ by $\widehat{x}^k_\varphi$
- $\widehat{R}^k_\varphi \widehat{x}^k_\varphi \widehat{z}^k_\varphi = h_k(\widehat{R}_{y_\varphi} \widehat{x}_{y_\varphi} \widehat{z}_{y_\varphi})$ for $\widehat{x}_{y_\varphi}, \widehat{z}_{y_\varphi} \in \widehat{W}_{y_\varphi}$
- $\widehat{V}^k_\varphi(p, \widehat{x}^k_\varphi) = h_k(\widehat{V}_{y_\varphi}(p, \widehat{x}_{y_\varphi}))$ for $\widehat{x}_{y_\varphi} \in \widehat{W}_{y_\varphi}$.

Because h_k is a $\widehat{V}_{y_\varphi}[\Sigma_\Box \cup \Sigma_\Diamond, \widehat{W}_{y_\varphi}]$-complete deflationary $\mathrm{C}_{\mathcal{L}}$- order embedding, by Lemma 1, $\widehat{V}^k_\varphi(\chi, \widehat{y}^k_\varphi) = h_k(\widehat{V}_{y_\varphi}(\chi, \widehat{y}_\varphi))$ for all $\chi \in \Sigma$. By the induction hypothesis,

$$
\begin{array}{rrcl}
(\dagger) & h_k(Rxy_\varphi) \to \widehat{V}^k_\varphi(\psi, \widehat{y}^k_\varphi) & = & h_k(Rxy_\varphi) \to h_k(\widehat{V}_{y_\varphi}(\psi, \widehat{y}_\varphi)) \\
& & = & h_k(Rxy_\varphi \to \widehat{V}_{y_\varphi}(\psi, \widehat{y}_\varphi)) \\
& & = & h_k(Rxy_\varphi \to V(\psi, y_\varphi)) \\
& & = & h_k(s) \\
& & \in & [a_i, a_i + \frac{1}{k}).
\end{array}
$$

(ii) Suppose that $V(\Box\psi, x) \neq a_i$ for all $i \in I$. In this case, $V(\Box\psi, x) = \bigwedge\{Rxy \to V(\psi, y) : y \in W\}$ and, because W is finite, there is a $y_\varphi \in W$, such that, by the induction hypothesis,

$$V(\Box\psi, x) = Rxy_\varphi \to V(\psi, y_\varphi) = Rxy_\varphi \to \widehat{V}_{y_\varphi}(\psi, y_\varphi).$$

In this case, let h_k be the identity function on A and $\widehat{\mathfrak{M}}^k_\varphi = \langle \widehat{W}^k_\varphi, \widehat{R}^k_\varphi, \widehat{V}^k_\varphi \rangle = \widehat{\mathfrak{M}}_{y_\varphi}$.

Similarly, when $\varphi = \Diamond\psi \in \Sigma_\Diamond$, we obtain for each $k \in \mathbb{Z}^+$, a $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}}^k_\varphi$ as a copy of $\widehat{\mathfrak{M}}_{y_\varphi}$.

We now define the $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ by

$$
\begin{array}{rcl}
\widehat{W} & = & \{\widehat{x}\} \cup \bigcup\limits_{\varphi \in \Sigma_\Box \cup \Sigma_\Diamond} \bigcup\limits_{k \in \mathbb{Z}^+} \widehat{W}^k_\varphi \\
\widehat{R}wz & = & \begin{cases} \widehat{R}^k_\varphi wz & \text{if } w, z \in \widehat{W}^k_\varphi \text{ for some } \varphi \in \Sigma_\Box \cup \Sigma_\Diamond,\ k \in \mathbb{Z}^+ \\ h_k(Rxy_\varphi) & \text{if } w = \widehat{x},\ z = \widehat{y}^k_\varphi \in \widehat{W}^k_\varphi \text{ for } \varphi \in \Sigma_\Box \cup \Sigma_\Diamond,\ k \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases} \\
\widehat{V}(p, z) & = & \begin{cases} \widehat{V}^k_\varphi(p, z) & \text{if } z \in \widehat{W}^k_\varphi \text{ for some } \varphi \in \Sigma_\Box \cup \Sigma_\Diamond,\ k \in \mathbb{Z}^+ \\ V(p, x) & \text{if } z = \widehat{x}. \end{cases}
\end{array}
$$

If $\mathfrak{M}$ is crisp, then for all $\varphi \in \Sigma_\Box \cup \Sigma_\Diamond$, $\widehat{\mathfrak{M}}_{y_\varphi}$ is crisp and so also are $\widehat{\mathfrak{M}}^k_\varphi$ for all $k \in \mathbb{Z}^+$. Hence, by construction, $\widehat{\mathfrak{M}}$ is crisp. Moreover, as there are only finitely many different countable $\widehat{\mathfrak{M}}_{y_\varphi}$, and we only take countably many copies of each one, $\widehat{\mathfrak{M}}$ is also countable.

Observe now that for each $\widehat{y}^k_\varphi \in \widehat{R}^+[\widehat{x}]$, we have that $\widehat{\mathfrak{M}}^k_\varphi$ is the submodel of $\widehat{\mathfrak{M}}$ generated by $\widehat{y}^k_\varphi$. Hence, by Lemma 2, for all $\chi \in \Sigma$ and $\widehat{y}^k_\varphi \in \widehat{R}^+[\widehat{x}]$,

$$(\ddagger) \quad \widehat{V}(\chi, \widehat{y}^k_\varphi) = \widehat{V}^k_\varphi(\chi, \widehat{y}^k_\varphi) = h_k(\widehat{V}_{y_\varphi}(\chi, \widehat{y}_\varphi)) = h_k(V_{y_\varphi}(\chi, y_\varphi)) = h_k(V(\chi, y_\varphi)).$$

Finally, we prove that $\widehat{V}(\chi, \widehat{x}) = V(\chi, x)$ for all $\chi \in \Sigma$, proceeding by induction on $\ell(\chi)$. The base case follows directly from the definition of $\widehat{V}$. For the induction step, the cases for the non-modal connectives follow easily using the induction hypothesis. Let us just consider the case $\chi = \varphi = \Box\psi$ (a formula in $\Sigma_\Box$), the case $\chi = \Diamond\psi$ being very similar. There are two possibilities.

(i) Suppose that $V(\Box\psi, x) = a_i$ for some $i \in I$. Then for all $z \in W$, we have $Rxz \to V(\psi, z) \geq a_i$. Note that it is not possible for any $a \in A$ and h_k defined above that $h_k(a) < a_i \leq a$, as h_k is either the identity on $T_\Box(x)$ or is inflationary on A. So by construction, for all $\widehat{z} \in \widehat{W}$,

$$\widehat{R}\widehat{x}\widehat{z} \to \widehat{V}(\psi, \widehat{z}) \geq a_i.$$

Moreover, for $y_\varphi \in W$,

$$Rxy_\varphi \to V(\psi, y_\varphi) \in [a_i, c_i),$$

and by $(\dagger)$ and $(\ddagger)$,

$$\begin{array}{rcl}
a_i & \leq & \bigwedge\{\widehat{R}\widehat{x}\widehat{z} \to \widehat{V}(\psi, \widehat{z}) : \widehat{z} \in \widehat{W}\} \\
& \leq & \bigwedge\{\widehat{R}\widehat{x}\widehat{y}^k_\varphi \to \widehat{V}(\psi, \widehat{y}^k_\varphi) : k \in \mathbb{Z}^+\} \\
& = & \bigwedge\{h_k(Rxy_\varphi) \to \widehat{V}^k_\varphi(\psi, \widehat{y}^k_\varphi) : k \in \mathbb{Z}^+\} \\
& = & \bigwedge\{h_k(Rxy_\varphi \to V(\psi, y_\varphi)) : k \in \mathbb{Z}^+\} \\
& \leq & \bigwedge\{a_i + \frac{1}{k} : k \in \mathbb{Z}^+\} \\
& = & a_i.
\end{array}$$

So $\widehat{V}(\Box\psi, \widehat{x}) = \bigwedge\{\widehat{R}\widehat{x}\widehat{z} \to \widehat{V}(\psi, \widehat{z}) : \widehat{z} \in \widehat{W}\} = a_i = V(\Box\psi, x)$ as required.

(ii) Suppose that $V(\Box\psi, x) \neq a_i$ for all $i \in I$. Again, for all $z \in W$, we have that $Rxz \to V(\psi, z) \geq V(\Box\psi, x) \in T_\Box(x)$. As h_k is either the identity on $T_\Box(x)$ or is inflationary on A, by construction, for all $\widehat{z} \in \widehat{W}$,

$$\widehat{R}\widehat{x}\widehat{z} \to \widehat{V}(\psi, \widehat{z}) \geq V(\Box\psi, x).$$

Moreover, as in (ii) above, because W is finite, there is a $y_\varphi \in W$ such that

$$Rxy_\varphi \to V(\psi, y_\varphi) = V(\Box\psi, x).$$

Using (‡) and the fact that h_k is either the identity on $T_\Box(x)$ or inflationary on A,

$$\begin{array}{rcl}\widehat{V}(\Box\psi,\widehat{x}) & = & \bigwedge\{\widehat{R}\widehat{x}\widehat{z} \to \widehat{V}(\psi,\widehat{z}) : \widehat{z} \in \widehat{W}\} \\ & = & \bigwedge\{\widehat{R}\widehat{x}\widehat{y}^k_\varphi \to \widehat{V}(\psi,\widehat{y}^k_\varphi) : k \in \mathbb{Z}^+\} \\ & = & \bigwedge\{h_k(Rxy_\varphi) \to \widehat{V}^k_\varphi(\psi,\widehat{y}^k_\varphi) : k \in \mathbb{Z}^+\} \\ & = & \bigwedge\{h_k(Rxy_\varphi \to V(\psi,y_\varphi)) : k \in \mathbb{Z}^+\} \\ & = & Rxy_\varphi \to V(\psi,y_\varphi) \\ & = & V(\Box\psi,x).\end{array}$$

So $\widehat{V}(\Box\psi,\widehat{x}) = V(\Box\psi,x)$ as required. □

We obtain the following equivalence results.

Theorem 17.

(a) $\models_{\mathsf{K}(\mathbf{A})} \varphi$ *if and only if* $\models_{\mathsf{FK}(\mathbf{A})} \varphi$.

(b) $\models_{\mathsf{K}(\mathbf{A})^\mathsf{C}} \varphi$ *if and only if* $\models_{\mathsf{FK}(\mathbf{A})^\mathsf{C}} \varphi$.

Proof. For (a), the right-to-left direction is immediate using the fact that every $\mathsf{K}(\mathbf{A})$-tree-model can be extended to an $\mathsf{FK}(\mathbf{A})$-tree-model with the same valid formulas by setting $T_\Box$ and $T_\Diamond$ to be constantly A. Suppose now that $\not\models_{\mathsf{FK}(\mathbf{A})} \varphi$. By Lemmas 10 and 12, there is a finite $\mathsf{FK}(\mathbf{A})$-tree-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ with root x such that $V(\varphi,x) < 1$. By Lemma 16, we obtain a $\mathsf{K}(\mathbf{A})$-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}\rangle$ with root $\widehat{x}$ such that $\widehat{V}(\varphi,\widehat{x}) = V(\varphi,x) < 1$. So $\not\models_{\mathsf{K}(\mathbf{A})} \varphi$.

The proof of (b) is very similar, using the fact that Lemmas 10, 12, and 16 preserve crisp models. □

5. Decidability and Complexity

Let us assume again that $\mathbf{A}$ is a locally homogeneous order-based algebra. In this section, we will use the finite model property of $\mathsf{FK}(\mathbf{A})$ and $\mathsf{FK}(\mathbf{A})^\mathsf{C}$ to obtain decidability and complexity results for $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})^\mathsf{C}$ in various cases. We prove, in particular, that the Gödel modal logics GK and GK^C (i.e., where $\mathbf{A}$ is $\mathbf{G}$) are both PSPACE-complete and that the same is true for the cases where $\mathbf{A}$ is $\mathbf{G}_\downarrow$ or $\mathbf{G}_\uparrow$. These and other results in this section contrast with the fact that no first-order Gödel logic based on a countably infinite set of truth values is recursively axiomatizable [1].

For simplicity of exposition, we will assume that the only constants are $\bar{0}$ and $\bar{1}$. To explain the ideas involved in the proofs, consider $\varphi \in \mathrm{Fm}$ and $n = |\Sigma(\varphi)| = \ell(\varphi) + |\mathrm{C}_{\mathcal{L}}| = \ell(\varphi)+2$. To check that φ is not $\mathsf{K}(\mathbf{A})$-valid, it suffices, by Lemmas 10, 12, and 16, to find a finite $\mathsf{FK}(\mathbf{A})$-tree-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle$ of height $\leq \ell(\varphi)$ with root x and $|W| \leq |\Sigma(\varphi)|^{\ell(\varphi)} \leq n^n$ such that $V(\varphi, x) < 1$.

If $\mathbf{A}$ is infinite, then $T_\Box(x)$ and $T_\Diamond(x)$ may also be infinite, and hence $\mathfrak{M}$ may not be a computational object. We therefore introduce a modified version of $\mathfrak{M}$:

$$\mathfrak{M}^* = \langle W, R, V, \{\Phi(x)\}_{x\in W}, \{\Psi(x)\}_{x \in W}\rangle,$$

where for each $x \in W$, $\Phi(x) \subseteq A^2$ is the set of ordered pairs for which $T_\Box(x) = A \setminus \bigcup_{\langle r,s\rangle \in \Phi(x)} (r, s)$, and $\Psi(x) \subseteq A^2$ is the set of ordered pairs defining $T_\Diamond(x)$. Using the proof of Lemma 12 applied to a $\mathsf{K}(\mathbf{A})$-model, we may assume that $|\Phi(x)|$, $|\Psi(x)| \leq |\Sigma(\varphi)| = n$ for all $x \in W$, as the left endpoints of the intervals utilized in the proof to define $\widehat{T}_\Box(x)$ in the finite $\mathsf{FK}(\mathbf{A})$-tree-model belong to $V[\Sigma(\varphi)_\Box, x]$, and similarly for $\widehat{T}_\Diamond(x)$. Let us define inductively in $\mathfrak{M}^*$, for all $x \in W$ and $\psi \in \mathrm{Fm}$,

$$V(\Box\psi, x) = \begin{cases} r & \text{if } \bigwedge_{y\in W}(Rxy \to V(\psi, y)) \in (r, s) \\ & \text{for some } \langle r, s\rangle \in \Phi(x) \\ \bigwedge_{y\in W}(Rxy \to V(\psi, y)) & \text{otherwise,} \end{cases}$$

$$V(\Diamond\psi, x) = \begin{cases} s & \text{if } \bigvee_{y\in W}(Rxy \wedge V(\psi, y)) \in (r, s) \\ & \text{for some } \langle r, s\rangle \in \Psi(x) \\ \bigvee_{y\in W}(Rxy \wedge V(\psi, y)) & \text{otherwise.} \end{cases}$$

Then $\mathfrak{M}^*$ and $\mathfrak{M}$ assign the same values to a formula at any world. Moreover, for $\chi \in \Sigma(\varphi)$, the computation of $V(\chi, x)$ in $\mathfrak{M}^*$ involves only the set of values

$$N = V[\Sigma(\varphi), W] \cup \{Rxy : x, y \in W\} \cup \{r, s : \langle r, s\rangle \in \Phi(x) \cup \Psi(x),\ x \in W\}.$$

Note that $|N| \leq 4n^{2n} = e_n$. Hence, we may assume that R and V take values in the fixed set $A(e_n)$, where for $m \in \mathbb{Z}^+$,

$$A(m) = \{0, \tfrac{1}{m}, \dots, \tfrac{m-1}{m}, 1\}.$$

We can also assume that W is $W_n \subseteq \{0, 1, \dots, n^n\}$, yielding a finite structure

$$\mathfrak{M}^*(e_n) = \langle W_n, R, V, \{\Phi(i)\}_{i\in W_n}, \{\Psi(i)\}_{i \in W_n}\rangle,$$

where $\langle W_n, R^+\rangle$ is a tree with root 0 of height $\leq n$ and branching $\leq n$, and the sets $\Phi(i), \Psi(i)$, for $i \in W_n$, determine the endpoints of a family of disjoint open intervals in $A(e_n)$. We will call this kind of structure a (*crisp* if R is crisp) $\mathsf{FK}(e_n)$-*tree-model*. In order to recover the connection with the original $\mathsf{FK}(\mathbf{A})$-model, we introduce the following convenient notion.

A *finite system* is a triple $\mathbf{A}(m) = \langle A(m), \Phi, \Psi\rangle$ where $\Phi, \Psi \subseteq A(m)^2$. We call $\mathbf{A}(m)$ *consistent with* $\mathbf{A}$ if for some order-preserving embedding $h\colon A(m) \to A$, satisfying $h(0) = 0$ and $h(1) = 1$,

- $h(c)$ witnesses right homogeneity at $h(a) \in R(\mathbf{A})$ for all $\langle a, c\rangle \in \Phi$,
- $h(d)$ witnesses left homogeneity at $h(b) \in L(\mathbf{A})$ for all $\langle d, b\rangle \in \Psi$.

Then we obtain from the previous discussion:

Theorem 18. *The validity problems of $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ are decidable if the problem of consistency of finite systems $\mathbf{A}(m)$ with $\mathbf{A}$ is decidable. Moreover, the validity problems of $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ are co-NEXPTIME reducible (in the length of the formula) to the problem of consistency of finite systems $\mathbf{A}(m)$ with $\mathbf{A}$.*

Proof. As observed above, $\varphi \in \mathrm{Fm}$ with $n = \ell(\varphi)+2$ is not $\mathsf{K}(\mathbf{A})$-valid ($\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-valid) if and only if there is a (crisp) $\mathsf{FK}(e_n)$-tree-model of the form $\mathfrak{M}^*(e_n) = \langle W_n, R, V, \{\Phi(i)\}_{i\in W_n}, \{\Psi(i)\}_{i\in W_n}\rangle$ for which $V(\varphi, 0) < 1$ and the finite system $\mathbf{A}(e_n) = \langle A(e_n), \bigcup_{i\in W_n} \Phi(i), \bigcup_{i\in W_n} \Psi(i)\rangle$ is consistent with $\mathbf{A}$.

Choose non-deterministically $V\colon \mathrm{Var}(\varphi) \to A(e_n)$, $R\colon W_n^2 \to A(e_n)$, and $\Phi(i)$, $\Psi(i) \subseteq A(e_n)^2$ for all $i \in W_n$ to obtain $\mathfrak{M}^*(e_n)$, and compute $V(\varphi, 0)$ to verify $V(\varphi, 0) < 1$. This takes a number of steps bounded by a constant multiple of e_n. Then utilize an oracle to verify the consistency of $\mathbf{A}(e_n)$ with $\mathbf{A}$. $\square$

Example 19. *Any finite system $\mathbf{A}(m) = \langle A(m), \Phi, \Psi\rangle$ is consistent with $\mathbf{G}$. Also $\mathbf{A}(m)$ is consistent with $\mathbf{G}_\downarrow$ if and only if $\Psi = \emptyset$ and $\Phi = \{(0, \frac{k_1}{m}), \ldots, (0, \frac{k_l}{m})\}$ for some $l \in \mathbb{Z}^+$ and $k_1, \ldots, k_l \in \mathbb{N}$, or is $\emptyset$, and $\mathbf{A}(m)$ is consistent with $\mathbf{G}_\uparrow$ if and only if $\Phi = \emptyset$ and $\Psi = \{(\frac{k_1}{m}, 1), \ldots, (\frac{k_l}{m}, 1)\}$ for some $l \in \mathbb{Z}^+$ and $k_1, \ldots, k_l \in \mathbb{N}$, or is $\emptyset$. Hence in these cases the consistency problem is obviously decidable in linear time and space (null-space if the size of the input tape is not considered).*

Moreover, it is easy to verify inductively that any algebra $\mathbf{A}$ obtained from $\mathbf{G}$, $\mathbf{G}_\downarrow$, $\mathbf{G}_\uparrow$, and finite order-based algebras as a finite combination of ordered sums, lexicographical products, and fusion of consecutive points has a (PTIME) decidable consistency problem. In all of these cases, validity in $\mathsf{K}(\mathbf{A})$ and $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ is (co-NEXPTIME) decidable. This includes the case when A, as an ordered set, is isomorphic to an ordinal $\alpha + 1 < \omega^\omega$ or its reverse.

The algebras $\mathbf{G}$, $\mathbf{G}_{\downarrow}$, $\mathbf{G}_{\uparrow}$, and finite order-based algebras have the additional property that if the finite systems $\langle A(m), \Phi_i, \Psi_i\rangle$, for $i = 0, \dots, k$, are consistent with $\mathbf{A}$, then the same holds for $\langle A(m), \bigcup_{i\leq k} \Phi_i, \bigcup_{i\leq k} \Psi_i\rangle$. This will allow us to improve the decidability result in these cases to PSPACE- completeness. First, however, we need a result about $\mathsf{FK}(e_n)$-tree-models.

Lemma 20. *The following problem is PSPACE-reducible (in n) to the consistency of finite systems with $\mathbf{A}$:*

Given $\Sigma = \{\varphi_1, \dots, \varphi_k\} \subseteq \mathrm{Fm}$ (not necessarily distinct formulas) such that $k \leq n$ and $\ell(\varphi_j) \leq n$ for $j = 1, \dots, k$, and given intervals $I_1, \dots, I_k \subseteq A(e_n)$ (closed or open at their endpoints), determine if there exists a (crisp) $\mathsf{FK}(e_n)$-tree-model $\mathfrak{M}^* = \langle W_n, R, V, \{\Phi(i)\}_{i\in W_n}, \{\Psi(i)\}_{i\in W_n}\rangle$ with root 0 and height $\leq n$ such that $V(\varphi_j, 0) \in I_j$, for $j = 1, \dots, k$, and for $i \in W_n$, the system $\langle A(e_n), \Phi(i), \Psi(i)\rangle$ is consistent with $\mathbf{A}$.

Proof. As PSPACE = NPSPACE (see [32]), it suffices to give a non-deterministic polynomial space algorithm to produce the $\mathsf{FK}(e_n)$-tree-mode $\mathfrak{M}^*$. Because the full model may need exponential space to be displayed, our strategy is to search sequentially the branches of $\mathfrak{M}^*$, from the root down, so that all branches are built in the same polynomial space. This is the basic idea of Ladner's proof in [25] of the PSPACE complexity of the classical modal logic K. We do not try to optimize the space bound but show that $22n^5$ does the job.

Input. Each value in $A(e_n)$ may be represented by a binary word of length at most $\log e_n \leq 2n^2$, and the only information we need from the input, besides Σ, is the maximum (strictly smaller than 1) of $A(e_n)$ and the endpoints of the intervals I_j, indicating if they are included or not in the intervals. We consider also as part of the input a particular world $x \in W_n$, written in binary notation (length $\leq \log n^n \leq n^2$). At the initial stage, $x = 0$. With appropriate markings in the formulas, we may also assume that each φ_j appears decomposed in the form:

$$\varphi_j = \chi_j(p_1, \dots, p_l, \Box\psi_1^j, \dots, \Box\psi_{n_j}^j, \Diamond\theta_1^j, \dots, \Diamond\theta_{m_j}^j),$$

where $P = \{p_1, \dots, p_l\} \subseteq \mathrm{Var}$ and $\chi_j(p_1, \dots, p_l, q_1, \dots, q_{n_j}, s_1, \dots, s_{m_j})$ is a non-modal formula. Set:

$$S_\Box = \{\Box\psi_1^j, \dots, \Box\psi_{n_j}^j : j = 1, \dots, k\}, \quad S_\Diamond = \{\Diamond\theta_1^j, \dots, \Diamond\theta_{m_j}^j : j = 1, \dots, k\},$$
$$F_\Box = \{\psi_1^j, \dots, \psi_{n_j}^j : j = 1, \dots, k\}, \qquad F_\Diamond = \{\theta_1^j, \dots, \theta_{m_j}^j : j = 1, \dots, k\}.$$

Note that the input may be displayed in space at most $3n^2 + (1 + 2n)2n^2 \leq 9n^3$.

Step 1. Choose values $V(\rho, x) \in A(e_n)$, for all $\rho \in P \cup S_\Box \cup S_\Diamond$, and verify that $V(\varphi_j, x) \in I_j$ for each $j \leq k$.

Choose partial functions $\Phi(x) = \{\langle a, c_a\rangle : a \in G\} \subseteq V[S_\Box, x] \times A(e_n)$ and $\Psi(x) = \{\langle d_b, b\rangle : b \in H\} \subseteq A(e_n) \times V[S_\Diamond, x]$ and verify that the finite system $\langle A(e_n), \Phi(x), \Psi(x)\rangle$ is consistent with $\mathbf{A}$. Each $a \in G$ plays the role of a "right accumulation point" and c_a plays the role of a "witness of right homogeneity" at a; similarly, each $b \in H$ plays the role of a "left accumulation point" and d_b plays the role of a "witness of left homogeneity" at b. An oracle for the consistency problem must certify that this distribution can be realized in $\mathbf{A}$.

Choose also worlds $y_1, \ldots, y_m \in W_n$ for $m \leq n$ in the next level of the tree and values $Rxy_t \in A(e_n)$ for $t = 1, \ldots, m$.

Note that the space required to perform this step and store the data produced is at most $3n \cdot 2n^2 + n \cdot n^2 = 7n^3$. The values of the desired tree-model $\mathfrak{M}^*$ are guessed at the root. Hence, this model exists if and only if it is possible to find further (crisp, if necessary) $\mathsf{FK}(e_n)$-tree-models $\mathfrak{M}^*_t$ of height $\leq n-1$ with respective roots y_t, for $t = 1, \ldots, m$, such that for any $\rho \in F_\Box \cup F_\Diamond$,

1. $\bigwedge_{t=1}^{m} (Rxy_t \to V(\rho, y_t)) \in [V(\Box\rho, x), c_a)$ if $\rho \in F_\Box$ and $V(\Box\rho, x) = a \in G$,
2. $\bigwedge_{t=1}^{m} (Rxy_t \to V(\rho, y_t)) = V(\Box\rho, x)$ if $\rho \in F_\Box$ and $V(\Box\rho, x) \notin G$,
3. $\bigvee_{t=1}^{m} (Rxy_t \wedge V(\rho, y_t)) \in (d_b, V(\Diamond\rho, x)]$ if $\rho \in F_\Diamond$ and $V(\Diamond\rho, x) = b \in H$,
4. $\bigvee_{t=1}^{m} (Rxy_t \wedge V(\rho, y_t)) = V(\Diamond\rho, x)$ if $\rho \in F_\Diamond$ and $V(\Diamond\rho, x) \notin H$.

If $F^t_\Box$ ($F^t_\Diamond$) denotes the set of $\rho \in F_\Box$ ($\rho \in F_\Diamond$) for which the minimum (maximum) associated to ρ above is realized at y_t, then the situation $\rho \in F^t_\Diamond$, $V(\Diamond\rho, x) = b \in H$ and $Rxy_t \leq d_b$ does not arise and, similarly, the situation $\rho \in F^t_\Diamond$, and $Rxy_t < V(\Diamond\rho, x) \notin H$ is impossible. Moreover, the above conditions are equivalent to asking for all t and ρ:

1. $Rxy_t \to V(\rho, y_t) \geq V(\Box\rho, x)$ if $\rho \in F_\Box$
2. $Rxy_t \to V(\rho, y_t) \in [V(\Box\rho, x), c_a)$ if $\rho \in F^t_\Box$ and $V(\Box\rho, x) \in G$,
3. $Rxy_t \to V(\rho, y_t) = V(\Box\rho, x)$ if $\rho \in F^t_\Box$ and $V(\Box\rho, x) \notin G$,
4. $Rxy_t \wedge V(\rho, y_t) \leq V(\Diamond\rho, x)$ if $\rho \in F_\Diamond$
5. $Rxy_t \wedge V(\rho, y_t) \in (d_b, V(\Diamond\rho, x)]$ if $\rho \in F^t_\Diamond$ and $V(\Diamond\rho, x) \in H$,
6. $Rxy_t \wedge V(\rho, y_t) = V(\Diamond\rho, x)$ if $\rho \in F^t_\Diamond$ and $V(\Diamond\rho, x) \notin H$.

These conditions are equivalent, in turn, to asking that for each model $\mathfrak{M}_t^*$ and $\rho \in F_\Box \cup F_\Diamond$, the value $V(\rho, y_t)$ belongs to the interval $I_{\rho,t}$, fixed to be

1.	$[V(\Box\rho, x), Rxy_t)$	if $\rho \in F_\Box$ and $V(\Box\rho, x) < 1$,
	$[Rxy_t, 1]$	if $\rho \in F_\Box$ and $V(\Box\rho, x) = 1$,
2.	$[V(\Box\rho, x), c_a \wedge Rxy_t)$	if $\rho \in F_\Box^t$ and $V(\Box\rho, x) = a \in G$,
3.	$[V(\Box\rho, x), V(\Box\rho, x)]$	if $\rho \in F_\Box^t$, $V(\Box\rho, x) \notin G$, and $V(\Box\rho, x) < 1$,
	$[Rxy_t, 1]$	if $\rho \in F_\Box^t$, $V(\Box\rho, x) \notin G$, and $V(\Box\rho, x) = 1$,
4.	$[0, Rxy_t \to V(\Diamond\rho, x)]$	if $\rho \in F_\Diamond$,
5.	$(d_b, Rxy_t \to V(\Diamond\rho, x)]$	if $\rho \in F_\Diamond^t$, $V(\Diamond\rho, x) = b \in H$,
6.	$[V(\Diamond\rho, x), Rxy_t \to V(\Diamond\rho, x)]$	if $\rho \in F_\Diamond^t$, $V(\Diamond\rho, x) \notin H$.

But this amounts to the original problem: the existence of $\mathfrak{M}_t^*$ with root y_t satisfying the conditions of the lemma for the input $\Sigma' = F_\Box \cup F_\Diamond$ and intervals $I_{\rho,t}$, $\rho \in \Sigma'$. This justifies the next steps of the algorithm.

Step 2. Find coverings $F_\Box = \bigcup_{t\in(1,m]} F_\Box^t$ and $F_\Diamond = \bigcup_{t\in(1,m]} F_\Diamond^t$, verify that the situations $\rho \in F_\Diamond^t$, $V(\Diamond\rho, x) = b \in H$, and $Rxy_t \leq d_b$, or $\rho \in F_\Diamond^t$ and $Rxy_t < V(\Diamond\rho, x) \notin H$ do not arise, and compute for each t and $\rho \in F_\Box \cup F_\Diamond$ the interval $I_{\rho,t}$.

Note that computing and storing the data produced in this step requires space at most $2n \cdot n^2 + 2n^2 \cdot 2n^2 \leq 6n^4$.

Step 3. For $t = 1, \ldots, m$, return consecutively to Step 1 with input: $\Sigma' = F_\Box \cup F_\Diamond$, $\{I_{\rho,t} : \rho \in \Sigma'\}$, and $x = y_t$, traversing the resulting tree of worlds in pre-order; that is, the leftmost branch is exhausted before passing to the next unexplored sub-branch at the right.

Note that the cyclic repetition of Steps 1 and 2 (an exponential number of times), if successful at each stage, runs through a tree of height less than n, so the space needed to guess a branch of the tree is at most $22n^5$. The key point is that having verified successfully the existence of a branch we may utilize the same space for the next one, and thus the total space required is bounded by $22n^5$. Informally, returning to Step 1 with $t = 1$ starts a search for $\mathfrak{M}_1^*$, after finishing it successfully, we return to Step 1 with $t = 2$ and utilize the same space, bounded by $22n^4(n-1)$, to search for $\mathfrak{M}_2^*$, etc. Adding to this common space the space of the first cycle, we obtain $22n^5$. $\Box$

Theorem 21. *The validity problems for* $\mathsf{K}(\mathbf{A})$ *and* $\mathsf{K}(\mathbf{A})^\mathsf{C}$ *are PSPACE-complete for the algebras* $\mathbf{G}$, $\mathbf{G}_\downarrow$, *and* $\mathbf{G}_\uparrow$.

Proof. Lemma 20 applied to a formula φ and the interval $I = [0,1)$ yields a PSPACE algorithm in the length of φ to determine for these algebras, whether there is an $\mathsf{FK}(e_n)$-tree-model for which $V(\varphi, 0) < 1$ and $\langle A(e_n), \Phi(i), \Psi(i)\rangle$ is consistent with $\mathbf{A}$, for each $i \in W_n \subseteq \{0, 1, \ldots, n^n\}$. The latter condition is equivalent to consistency with $\mathbf{A}$ of $\langle A(e_n), \bigcup_{i\in W_n} \Phi(i), \bigcup_{i\in W_n} \Psi(i)\rangle$. The existence of this model is equivalent, recalling the earlier discussion in this section, to the existence of a $\mathsf{K}(\mathbf{A})$-counter-model for φ. The lower bound follows from the fact that classical modal logic K is PSPACE-hard [25] and can be interpreted faithfully in $\mathsf{K}(\mathbf{A})$ or $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ by the double negation interpretation which adds $\neg\neg$ in front of any subformula of a formula. □

Note that the last theorem applies to any algebra for which the consistency problem is PSPACE decidable and the union of consistent finite systems is consistent. Examples of these algebras are finite algebras (trivially), the ordinals $\omega^n + 1$, $n \in \mathbb{N}^+$, and their reverse orders. We also expect that PSPACE-completeness holds for all finite combinations of $\mathbf{G}$, $\mathbf{G}_\downarrow$, $\mathbf{G}_\uparrow$, and finite algebras built via ordered sums, lexicographical products, and fusion of consecutive points, but will not prove this here.

To generalize the results in this section to languages with a finite set of constants $\mathrm{C}_\mathcal{L} = \{c_1 < \ldots < c_l\}$, utilize a set of values $A'(e_n)$ containing an isomorphic copy $\mathrm{C}'_\mathcal{L} = \{c'_1 < \ldots < c'_l\}$ of $\mathrm{C}_\mathcal{L}$ such that $|[c'_i, c'_{i+1}]_{A'(e_n)}| = |[c_i, c_{i+1}]_\mathbf{A}|$, if $|[c_i, c_{i+1}]_\mathbf{A}| < e_n$, and $|[c'_i, c'_{i+1}]_{A'(e_n)}| = e_n$, otherwise. This allows V and R to take values in any possible interval of consecutive constants. Moreover, $|A'(e_n)| \leq |\mathrm{C}_\mathcal{L}| e_n$ and all bounds are multiplied by a constant. Finite systems must have now the form $\langle A(m), \Phi, \Psi, \{c'\}_{c\in \mathrm{C}_\mathcal{L}}\rangle$ and the embeddings granting consistency must send c' to c.

6. Order-Based Crisp S5 Logics

As in the classical setting, further many-valued modal logics may be defined for a given order-based algebra $\mathbf{A}$ as logics of particular classes of $\mathsf{K}(\mathbf{A})$-models (see, e.g., [9, 10]). In this section, we restrict our attention to proving decidability and co-NP-completeness for crisp order-based "S5" logics that may be understood also as one-variable fragments of order-based first-order logics. In particular, we give a positive answer to the open decidability problem (and establish co-NP-completeness) for validity in the one-variable fragment of first-order Gödel logic (see, e.g., [19, Chapter 9, Problem 13]).

We define an $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-*model* to be a $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, V, R\rangle$ such that R is an equivalence relation. We call $\mathfrak{M}$ *universal* if $R = W \times W$ and in this case just write $\mathfrak{M} = \langle W, V\rangle$, noting that the clauses for $\Box$ and $\Diamond$ simplify to

$$\begin{array}{rcl} V(\Box\varphi, x) & = & \bigwedge\{V(\varphi, y) : y \in W\} \\ V(\Diamond\varphi, x) & = & \bigvee\{V(\varphi, y) : y \in W\}. \end{array}$$

The following lemma is an immediate corollary of Lemma 2 and the fact that the generated submodel of an $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model is universal.

Lemma 22. $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi$ *if and only if* φ *is valid in all universal* $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$*-models.*

It follows that each order-based modal logic $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ may be viewed as the one-variable fragment of a corresponding order-based first-order logic. Rather than define this first-order logic and then restrict to its one-variable fragment, let us simply note that the first-order translation of $\varphi \in \mathrm{Fm}$ is obtained by replacing each propositional variable p with the predicate $p(x)$, $\Box$ with $\forall x$, and $\Diamond$ with $\exists x$. In particular, $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ is the Gödel modal logic $\mathsf{GS5}^{\mathsf{C}}$ corresponding to the one-variable fragment of first-order Gödel logic (see, e.g., [1, 19]). $\mathsf{GS5}^{\mathsf{C}}$ is axiomatized in [10] as an extension of the intuitionistic modal logic MIPC studied in [7, 31] with the prelinearity axiom schema $(\varphi \to \psi) \lor (\psi \to \varphi)$ and $\Box(\Box\varphi \lor \psi) \to (\Box\varphi \lor \Box\psi)$. Let us also remark in passing that the logic $\mathsf{GS5}$ based on non-crisp frames may be axiomatized as MIPC extended with just prelinearity [10], and that decidability of the validity problem follows from the finite model property for the semantics with two accessibility relations [3].

The infinite $\mathsf{K}(\mathbf{A})$-model defined in the proof of Theorem 7 for the formula $\Box\neg\neg p \to \neg\neg\Box p$ is a universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model. Hence, if the universe of $\mathbf{A}$ is $[0, 1]$ or $G_{\downarrow}$, then $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ does not have the finite model property. Also, as in Theorem 6, the logic $\mathsf{S5}(\mathbf{G}_{\uparrow})^{\mathsf{C}}$ has the finite model property, but not if Δ is added to the language. We will prove decidability for these and other cases here using again a new equivalent semantics.

Let us assume once more that $\mathbf{A}$ is a locally homogeneous order-based algebra. We define an $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-*model* as an $\mathsf{FK}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond\rangle$ such that $\langle W, R, V\rangle$ is an $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model, and for all $x, y \in W$,

(i) $T_\Box(x) = T_\Box(y)$ and $T_\Diamond(x) = T_\Diamond(y)$ whenever Rxy,

(ii) $\{V(\Diamond p, x) : p \in \mathrm{Var}\} \subseteq T_\Box(x)$ and $\{V(\Box p, x) : p \in \mathrm{Var}\} \subseteq T_\Diamond(x)$.

We call $\mathfrak{M}$ *universal* if $R = W \times W$ and in this case write $\mathfrak{M} = \langle W, V, T_\Box, T_\Diamond \rangle$, where $T_\Box$ and $T_\Diamond$ may now be understood as fixed subsets of A, and the clauses for $\Box$ and $\Diamond$ simplify to

$$\begin{aligned} V(\Box\varphi, x) &= \bigvee\{r \in T_\Box : r \leq \bigwedge\{V(\varphi, y) : y \in W\}\} \\ V(\Diamond\varphi, x) &= \bigwedge\{r \in T_\Diamond : r \geq \bigvee\{V(\varphi, y) : y \in W\}\}. \end{aligned}$$

Note in particular that, by condition (i), in universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-models and $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-models, the truth values of box-formulas and diamond-formulas are independent of the world.

The new condition (ii) for $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-models reflects the fact that we deal here with universal models not tree models and must therefore take into account the values of diamond-formulas and box-formulas when fixing the values in $T_\Box$ and $T_\Diamond$, respectively. It is easily shown that (ii) extends inductively for universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$- models to the following condition on all diamond and box formulas:

Lemma 23. *For any universal* $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$*-model* $\mathfrak{M} = \langle W, V, T_\Box, T_\Diamond \rangle$ *and* $x \in W$*,*

$$\{\widehat{V}(\Diamond\varphi, x) : \varphi \in \mathrm{Fm}\} \subseteq T_\Box \quad \textit{and} \quad \{\widehat{V}(\Box\varphi, x) : \varphi \in \mathrm{Fm}\} \subseteq T_\Diamond.$$

We now show that $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-validity is equivalent to validity in finite universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$- models, following fairly closely the corresponding proofs from previous sections.

Lemma 24. *Let* $\Sigma \subseteq \mathrm{Fm}$ *be a finite fragment,* $\mathfrak{M} = \langle W, V \rangle$ *a universal* $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$*-model, and* $x \in W$*. Then there is a finite universal* $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$*-model* $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V}, \widehat{T_\Box}, \widehat{T_\Diamond} \rangle$ *with* $x \in \widehat{W} \subseteq W$ *and* $|\widehat{W}| \leq |\Sigma|$ *such that* $\widehat{V}(\varphi, y) = V(\varphi, y)$ *for all* $\varphi \in \Sigma$ *and* $y \in \widehat{W}$*.*

Proof. The proof is similar to the proof of Lemma 12. Let us fix a finite fragment $\Sigma \subseteq \mathrm{Fm}$, a universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, V \rangle$, and $x \in W$. Consider the finite (possibly empty) sets

$$V[\Sigma_\Box, x] \cap R(\mathbf{A}) = \{a_i : i \in I\} \quad \text{and} \quad V[\Sigma_\Diamond, x] \cap L(\mathbf{A}) = \{b_j : j \in J\},$$

noting that these sets are independent of the choice of the world $x \in W$. For each $i \in I$, choose a witness of right homogeneity c_i at a_i such that the intervals (a_i, c_i) are pairwise disjoint for all $i \in I$, and

$$(V[\Sigma_\Box, x] \cup \{V(\Diamond p, x) : p \in \mathrm{Var} \cap \Sigma\} \cup \mathrm{C}_{\mathcal{L}}) \cap (\bigcup_{i \in I} (a_i, c_i)) = \emptyset.$$

Similarly, for each $j \in J$, choose a witness of left homogeneity d_j at b_j such that the intervals (d_j, b_j) are pairwise disjoint for all $j \in J$, and

$$(V[\Sigma_\Diamond, x] \cup \{V(\Box p, x) : p \in \mathrm{Var} \cap \Sigma\} \cup \mathrm{C}_{\mathcal{L}}) \cap (\bigcup_{j \in J}(d_j, b_j)) = \emptyset.$$

We define

$$\widehat{T}_\Box = A \setminus \bigcup_{i \in I}(a_i, c_i) \quad \text{and} \quad \widehat{T}_\Diamond = A \setminus \bigcup_{j \in J}(d_j, b_j).$$

Now consider $\varphi = \Box\psi \in \Sigma_\Box$ and $a = V(\Box\psi, x) \in \widehat{T}_\Box$. If $a \notin R(\mathbf{A})$, then we choose $y_\varphi \in W$ such that $a = V(\psi, y_\varphi)$. If $a \in R(\mathbf{A})$, then there is an $i \in I$ such that $a = a_i$, and we choose $y_\varphi \in W$ such that $V(\psi, y_\varphi) \in [a_i, c_i)$. Suppose now that $\varphi = \Diamond\psi \in \Sigma_\Diamond$ and $b = V(\Diamond\psi, x) \in \widehat{T}_\Diamond$. If $b \notin L(\mathbf{A})$, then we choose $y_\varphi \in W$ such that $b = V(\psi, y_\varphi)$. If $b \in L(\mathbf{A})$, then there is a $j \in J$ such that $b = b_j$, and we choose $y_\varphi \in W$ such that $V(\psi, y_\varphi) \in (d_j, b_j]$.

Now let $\widehat{W} = \{x\} \cup \{y_\varphi \in W : \varphi \in \Sigma_\Box \cup \Sigma_\Diamond\}$, noting that $|\widehat{W}| \leq 1 + |\Sigma_\Box \cup \Sigma_\Diamond| \leq |\Sigma|$. Define for each $y \in \widehat{W}$ and $p \in \mathrm{Var}$:

$$\widehat{V}(p, y) = \begin{cases} V(p, y) & \text{if } p \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V}, \widehat{T}_\Box, \widehat{T}_\Diamond \rangle$ is a finite $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$- model satisfying $x \in \widehat{W} \subseteq W$ and $|\widehat{W}| \leq |\Sigma|$. It then follows by an easy induction on $\ell(\varphi)$ that $\widehat{V}(\varphi, y) = V(\varphi, y)$ for all $y \in \widehat{W}$ and $\varphi \in \Sigma$. $\Box$

Note that the number of intervals omitted from $\widehat{T}_\Box$ and $\widehat{T}_\Diamond$, defined in Lemma 24, is smaller than or equal to the cardinality of $\Sigma_\Box$ and $\Sigma_\Diamond$, respectively, for the given fragment Σ.

Lemma 25. *Let $\mathfrak{M} = \langle W, V, T_\Box, T_\Diamond \rangle$ be a finite universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-model. Then there is a universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V} \rangle$ with $W \subseteq \widehat{W}$ such that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \mathrm{Fm}$ and $x \in W$.*

Proof. Given a finite universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M}$, we construct our universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$- model $\widehat{\mathfrak{M}}$ directly by taking infinitely many copies of $\mathfrak{M}$.

Consider $T_\Box = A \setminus \bigcup_{i \in I}(a_i, c_i)$ and $T_\Diamond = A \setminus \bigcup_{j \in J}(d_j, b_j)$ for finite (possibly empty) sets I, J, where for each $i \in I$, right homogeneity at $a_i \in R(\mathbf{A})$ is witnessed by c_i such that the intervals (a_i, c_i) are pairwise disjoint, and, similarly, for each $j \in J$, left homogeneity at $b_j \in L(\mathbf{A})$ is witnessed by d_j such that the intervals (d_j, b_j) are pairwise disjoint. We define a family of $\mathrm{C}_{\mathcal{L}}$-order embeddings $\{h_k \colon A \to A\}_{k \in \mathbb{Z}^+}$ such that

- for each even $k \in \mathbb{Z}^+$, h_k is the identity function on $T_\Box$ and for each $i \in I$,

$$h_k[[a_i, c_i)] \subseteq [a_i, a_i + \tfrac{1}{k}),$$

- for each odd $k \in \mathbb{Z}^+$, h_k is the identity function on $T_\Diamond$ and for each $j \in J$,

$$h_k[(d_j, b_j]] \subseteq (b_j - \tfrac{1}{k}, b_j].$$

Note that Lemma 23 ensures for all $x \in W$ that $\{V(\Box\varphi, x), V(\Diamond\varphi, x) \colon \varphi \in \mathrm{Fm}\} \subseteq T_\Box \cap T_\Diamond$ and hence that for all $k \in \mathbb{Z}^+$ (even and odd), h_k is the identity function on $\{V(\Box\varphi, x), V(\Diamond\varphi, x) \colon \varphi \in \mathrm{Fm}\}$. Let h_0 be the identity on A, let $\widehat{W}_0 = W$, and for each $k \in \mathbb{Z}^+$, let $\widehat{W}_k$ be a copy of W with a distinct copy $\widehat{x}_k$ of each $x \in W$; also let $\widehat{x}_0 = x$ for each $x \in W$. We define the universal $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{V} \rangle$ where

$$\widehat{W} = \bigcup_{k \in \mathbb{N}} \widehat{W}_k \quad \text{and} \quad \widehat{V}(p, \widehat{x}_k) = h_k(V(p, x)) \text{ for } p \in \mathrm{Var}, x \in W, \text{ and } k \in \mathbb{N}.$$

It suffices now to prove that for all $\varphi \in \mathrm{Fm}$, $x \in W$, and $k \in \mathbb{N}$,

$$\widehat{V}(\varphi, \widehat{x}_k) = h_k(V(\varphi, x)),$$

proceeding by induction on $\ell(\varphi)$. The base case follows by definition, while for the non-modal connectives, the argument is the same as in the proof of Lemma 1. Consider $\varphi = \Diamond\psi$. Fix $x \in W$ and $k \in \mathbb{N}$. There are two cases.

(a) Suppose that $V(\Diamond\psi, x) = b_j$ for some $j \in J$. Note first that by Lemma 23, $V(\Diamond\psi, x) = b_j \in T_\Diamond \cup T_\Box$ and hence $h_k(b_j) = b_j$. Clearly $V(\psi, z) \leq b_j$ for all $z \in W$. Hence, by the induction hypothesis and the construction of $\{h_n \colon A \to A\}_{n \in \mathbb{N}}$, for all $n \in \mathbb{N}$ and $\widehat{z}_n \in \widehat{W}$,

$$\widehat{V}(\psi, \widehat{z}_n) = h_n(V(\psi, z)) \leq b_j.$$

Also, for some $y \in W$,

$$V(\psi, y) \in (d_j, b_j].$$

Hence for any odd $n \in \mathbb{N}$,

$$h_n(V(\psi, y)) \in (b_j - \tfrac{1}{n}, b_j].$$

Using the induction hypothesis,

$$\begin{aligned}
\widehat{V}(\Diamond\psi, \widehat{x}_k) &= \bigvee\{\widehat{V}(\psi, \widehat{y}_n) : y \in W, n \in \mathbb{N}\} \\
&= \bigvee\{h_n(V(\psi, y)) : y \in W, n \in \mathbb{N}\} \\
&= \bigvee\{b_j - \tfrac{1}{n} : n \in \mathbb{Z}^+\} \\
&= b_j \\
&= h_k(V(\Diamond\psi, x)).
\end{aligned}$$

(b) Suppose that $V(\Diamond\psi, x) = b \neq b_j$ for all $j \in J$. Note again that by Lemma 23, $V(\Diamond\psi, x) = b \in T_\Diamond \cup T_\Box$ and hence $h_k(b) = b$. Clearly, $V(\psi, z) \leq b$ for all $z \in W$. It follows again by the induction hypothesis and the construction of $\{h_n \colon A \to A\}_{n\in\mathbb{N}}$ that for all $n \in \mathbb{N}$ and $\widehat{z}_n \in \widehat{W}$,

$$\widehat{V}(\psi, \widehat{z}_n) = h_n(V(\psi, z)) \leq b.$$

Moreover, because W is finite, there is a $y \in W$ such that $V(\psi, y) = b = V(\Diamond\psi, x)$. Using the induction hypothesis and the fact that h_n is the identity function on $\{V(\Box\varphi, z), V(\Diamond\varphi, z) \colon \varphi \in \mathrm{Fm}\}$ for all $n \in \mathbb{N}$ and $z \in W$, it follows that

$$\begin{aligned}
\widehat{V}(\Diamond\psi, \widehat{x}) &= \bigvee\{\widehat{V}(\psi, \widehat{z}_n) : z \in W, n \in \mathbb{N}\} \\
&= \bigvee\{h_n(V(\psi, z)) : z \in W, n \in \mathbb{N}\} \\
&= \bigvee\{h_n(b) : n \in \mathbb{N}\} \\
&= b \\
&= h_k(V(\Diamond\psi, x)).
\end{aligned}$$

The case $\varphi = \Box\psi$ is very similar. $\Box$

Combining Lemmas 22, 24, and 25, we obtain the following equivalence.

Theorem 26. *Let $\mathbf{A}$ be a locally homogeneous order-based algebra. Then $\models_{\mathsf{S5}(\mathbf{A})^{\mathsf{C}}} \varphi$ if and only if φ is valid in all finite universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-models.*

The desired decidability and complexity results are now obtained by considering the number of truth values needed to check validity of formulas in finite universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-models. Recall (see Section 5) that if $A(m) = \{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\}$, then a finite system $\langle A(m), \Phi, \Psi, \{c'\}_{c \in \mathrm{C}_{\mathcal{L}}}\rangle$, where $\Phi, \Psi \subseteq A(m)^2$, is consistent with $\mathbf{A}$ if there exists an order-preserving embedding $h\colon A(m) \to A$ such that $h(c') = c$ for all $c \in \mathrm{C}_{\mathcal{L}}$, $h(c)$ witnesses right homogeneity at $h(a) \in R(\mathbf{A})$, for all $\langle a, c\rangle \in \Phi$, and $h(d)$ witnesses left homogeneity at $h(b) \in L(\mathbf{A})$, for all $\langle d, b\rangle \in \Phi$.

Theorem 27. *Let $\mathbf{A}$ be a locally homogeneous order-based algebra. Then the validity problem of $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ is co-NP reducible to the problem of consistency of finite systems with $\mathbf{A}$.*

Proof. Consider $\varphi \in \mathrm{Fm}$ and let $n = |\Sigma(\varphi)| = \ell(\varphi) + |\mathrm{C}_{\mathcal{L}}|$. To check if φ is not $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$-valid, it suffices, by Lemmas 24 and 25, to check that φ is not valid in a finite universal $\mathsf{FS5}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, V, T_{\Box}, T_{\Diamond}\rangle$ with $|W| \leq |\Sigma(\varphi)| \leq n$. To compute $V(\varphi, x)$ in such a model, we need to know only the values $V[\Sigma(\varphi), W]$ (that is, fewer than n^2 values) and the endpoints of the intervals defining $T_{\Box}$ and $T_{\Diamond}$ (that is, fewer than $2n$ values). So, we need at most $3n^2$ distinct values. Therefore, we may assume that these values are in a fixed finite set $A_n = A(p(n)) = \{0, \frac{1}{p(n)}, \ldots, \frac{p(n)-1}{p(n)}, 1\}$, containing properly spaced copies of constants, where $p(n) = 3|\mathrm{C}_{\mathcal{L}}|n^2$. We may assume also that $W = W_n \subseteq \{0, 1, \ldots, n-1\}$. Then checking non-deterministically that φ is not valid amounts to performing the following steps:

1. Guessing the values $V(p, i)$ in A_n for each $p \in \mathrm{Var}(\varphi)$ and $i \in W_n$ (at most $np(n)$ steps).
2. Guessing the sets $\Phi, \Psi \subseteq A_n^2$ such that Φ and Ψ define families of disjoint open intervals and using them to define, respectively, $T^*_{\Box}, T^*_{\Diamond} \subseteq A(p(n))$ (at most $2p(n)^2$ steps).
3. Checking that the system $\langle A_n, \Phi, \Psi, \{c'\}_{c \in \mathrm{C}_{\mathcal{L}}}\rangle$ is consistent with $\mathbf{A}$.
4. Computing $V(\varphi, 0)$ in the model $\langle W_n, V, T^*_{\Box}, T^*_{\Diamond}\rangle$ and checking $V(\varphi, 0) < 1$ (essentially n^3 steps).

Hence a counter-model for φ may be guessed in polynomial time if we have an oracle for the consistency problem. $\Box$

Corollary 28. . *The validity problems of $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, $\mathsf{S5}(\mathbf{G}_{\downarrow})^{\mathsf{C}}$, and $\mathsf{S5}(\mathbf{G}_{\uparrow})^{\mathsf{C}}$ are co-NP-complete. The same is true for $\mathsf{S5}(\mathbf{A})^{\mathsf{C}}$ if $\mathbf{A}$ is a finite combination of $\mathbf{G}$, $\mathbf{G}_{\downarrow}$,*

$\mathbf{G}_{\uparrow}$*, and finite algebras via ordered sums, lexicographical products, and fusion of consecutive points.*

Proof. The validity problem is co-NP hard already for the pure propositional logic over any $\mathbf{A}$, because classical propositional logic is interpretable in these logics. Moreover, for $\mathbf{G}$, $\mathbf{G}_{\downarrow}$, and $\mathbf{G}_{\uparrow}$, the consistency problem is checked in null or linear time. In the other cases, the consistency problem is solvable in polynomial time. $\square$

Moreover, recalling the relationship between crisp order-based S5 logics and one-variable fragments of corresponding first-order logics, we obtain:

Theorem 29. *The validity problems of the one-variable fragments of first-order Gödel logics based on* $\mathbf{G}$*,* $\mathbf{G}_{\downarrow}$*, and* $\mathbf{G}_{\uparrow}$ *are co-NP complete. The same is true for the one-variable fragments of first-order Gödel logics based on a finite combination of* $\mathbf{G}$*,* $\mathbf{G}_{\downarrow}$*, and* $\mathbf{G}_{\uparrow}$*, and finite algebras via ordered sums, lexicographical products, and fusion of consecutive points.*

7. Concluding Remarks

In this paper, we have established the decidability and PSPACE-completeness of the validity problem for certain "order-based" modal logics, including the Gödel modal logics investigated in [9, 10, 27]. We have also established decidability and co-NP-completeness for the validity problem of "crisp S5" versions of these logics corresponding to one-variable fragments of first-order logics. In particular, we have answered positively the open problem of the decidability (indeed, co-NP-completeness) of the validity problem for the one-variable fragment of first-order Gödel logic. There remain, however, a number of significant questions, notably:

- *Are order-based multi-modal logics also decidable?* This question is of particular interest as many-valued description logics (see, e.g., [5, 21, 35]) may be viewed as many-valued multi-modal logics. The challenge in this case is to extend the new semantics to a multi-modal setting.

- *Is the new semantics suitable for other classes of order-based modal logics?* We have focussed in this paper on "K" and "S5" order-based modal logics, but it would be useful to develop a more general approach that encompasses also decidability for logics based on frames satisfying combinations of conditions such as reflexivity, symmetry, and transitivity.

- *Is validity in the two-variable fragment of first-order Gödel logic decidable?* Notably, validity in the two-variable fragment of first-order classical logic (indeed, any first-order tabular intermediate logic) is decidable [29], while the same fragment of first-order intuitionistic logic is undecidable [24].

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