

FINITE FORM OF THE QUINTUPLE PRODUCT IDENTITY

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ABSTRACT. The celebrated quintuple product identity follows surprisingly from an almost-trivial algebraic identity, which is the limiting case of the terminating q -Dixon formula.

The celebrated quintuple product identity discovered by Watson [3] (cf. [2, P 147] also) states that

$$\sum_{k=-\infty}^{+\infty} (1-xq^k)q^{3\binom{k}{2}}(qx^3)^k = [q, x, q/x; q]_{\infty} [qx^2, q/x^2; q^2]_{\infty} \quad \text{for } |q| < 1 \quad (1)$$

where the q -shifted factorial is defined by

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for } n = 1, 2, \dots$$

with the following abbreviated multiple parameter notation

$$[\alpha, \beta, \dots, \gamma; q]_{\infty} = (\alpha; q)_{\infty} (\beta; q)_{\infty} \cdots (\gamma; q)_{\infty}.$$

This identity has several important applications in combinatorial analysis, number theory and special functions. For the historical note, we refer the reader to the paper [1]. In this short note, we shall show that identity (1) follows surprisingly from the following algebraic identity.

Theorem (Finite form of the quintuple product identity). *For a natural number m and a variable x , there holds an algebraic identity:*

$$1 \equiv \sum_{k=0}^m (1+xq^k) \begin{bmatrix} m \\ k \end{bmatrix} \frac{(x; q)_{m+1}}{(q^k x^2; q)_{m+1}} x^k q^{k^2}. \quad (2)$$

In fact, performing parameter replacements $m \rightarrow m+n$, $x \rightarrow -q^{-m}x$ and $k \rightarrow k+m$ and then simplifying the result through factorial-fraction relation

$$\begin{aligned} \frac{(-q^{-m}x; q)_{m+n+1}}{(q^{k-m}x^2; q)_{m+n+1}} &= \frac{(-q^{-m}x; q)_m (-x; q)_{1+n}}{(q^{k-m}x^2; q)_{m-k} (x^2; q)_{1+n+k}} \\ &= (-1)^{m-k} q^{\binom{k}{2}-mk} x^{2k-m} \times \frac{(-q/x; q)_m (-x; q)_{1+n}}{(q/x^2; q)_{m-k} (x^2; q)_{1+n+k}} \end{aligned}$$

we may restate the algebraic identity displayed in the theorem as the finite bilateral series identity

$$1 \equiv \sum_{k=-m}^n (1-xq^k) \begin{bmatrix} m+n \\ m+k \end{bmatrix} \frac{(-x; q)_{1+n} (-q/x; q)_m}{(x^2; q)_{1+n+k} (q/x^2; q)_{m-k}} x^{3k} q^{k^2+\binom{k}{2}}. \quad (3)$$

Letting $m, n \rightarrow \infty$ in this equation and applying the relation

$$(q; q)_{\infty} \frac{(x^2; q)_{\infty} (q/x^2; q)_{\infty}}{(-x; q)_{\infty} (-q/x; q)_{\infty}} = [q, x, q/x; q]_{\infty} [qx^2, q/x^2; q^2]_{\infty}$$

we derive immediately the quintuple product identity displayed in (1).

In terms of basic hypergeometric series, we remark that the finite sum identity (2) is just the limiting case $M \rightarrow \infty$ of the terminating q -Dixon formula (cf. [2, II-14]):

$${}_4\phi_3 \left[\begin{matrix} x^2, -qx, q^{-m}, M \\ -x, q^{1+m}x^2, qx^2/M \end{matrix} \middle| q; \frac{q^{1+m}x}{M} \right] = \frac{(qx^2; q)_m (qx/M; q)_m}{(qx; q)_m (qx^2/M; q)_m}.$$

REFERENCES

- [1] L. Carlitz - M. V. Subbarao, *A simple proof of the quintuple product identity*, Proc. Amer. Math. Society 32:1 (1972), 42-44.
- [2] G. Gasper - M. Rahman, *Basic Hypergeometric Series* (2nd edition), Cambridge University Press, 2004.
- [3] G. N. Watson, *Theorems stated by Ramanujan VII: Theorems on continued fractions*, J. London Math. Soc. 4 (1929), 39-48.

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