# The Eulerian Distribution on Involutions is Indeed Unimodal 

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#### Abstract

Let $I_{n, k}$ (resp. $J_{n, k}$ ) be the number of involutions (resp. fixed-point free involutions) of $\{1, \ldots, n\}$ with $k$ descents. Motivated by Brenti's conjecture which states that the sequence $I_{n, 0}, I_{n, 1}, \ldots, I_{n, n-1}$ is log-concave, we prove that the two sequences $I_{n, k}$ and $J_{2 n, k}$ are unimodal in $k$, for all $n$. Furthermore, we conjecture that there are nonnegative integers $a_{n, k}$ such that


$$
\sum_{k=0}^{n-1} I_{n, k} t^{k}=\sum_{k=0}^{\lfloor(n-1) / 2\rfloor} a_{n, k} t^{k}(1+t)^{n-2 k-1} .
$$

This statement is stronger than the unimodality of $I_{n, k}$ but is also interesting in its own right.
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## 1 Introduction

A sequence $a_{0}, a_{1}, \ldots, a_{n}$ of real numbers is said to be unimodal if for some $0 \leq j \leq n$ we have $a_{0} \leq a_{1} \leq \cdots \leq a_{j} \geq a_{j+1} \geq \cdots \geq a_{n}$, and is said to be log-concave if $a_{i}^{2} \geq a_{i-1} a_{i+1}$ for all $1 \leq i \leq n-1$. Clearly a log-concave sequence of positive terms is unimodal. The reader is referred to Stanley's survey [10] for the surprisingly rich variety of methods to show that a sequence is log-concave or unimodal. As noticed by Brenti [2], even though log-concave and unimodality have one-line definitions, to prove the unimodality or logconcavity of a sequence can sometimes be a very difficult task requiring the use of intricate combinatorial constructions or of refined mathematical tools.

Let $\mathfrak{S}_{n}$ be the set of all permutations of $[n]:=\{1, \ldots, n\}$. We say that a permutation $\pi=a_{1} a_{2} \cdots a_{n} \in \mathfrak{S}_{n}$ has a descent at $i(1 \leq i \leq n-1)$ if $a_{i}>a_{i+1}$. The number of descents of $\pi$ is called its descent number and is denoted by $\mathrm{d}(\pi)$. A statistic on $\mathfrak{S}_{n}$ is said to be Eulerian, if it is equidistributed with the descent number statistic. Recall that the polynomial

$$
A_{n}(t)=\sum_{\pi \in \mathfrak{S}_{n}} t^{1+\mathrm{d}(\pi)}=\sum_{k=1}^{n} A(n, k) t^{k}
$$

is called an Eulerian polynomial. It is well-known that the Eulerian numbers $A(n, k)$ $(1 \leq k \leq n)$ form a unimodal sequence, of which several proofs have been published: such
as the analytical one by showing that the polynomial $A_{n}(t)$ has only real zeros [3, p. 294], by induction based on the recurrence relation of $A(n, k)$ (see [9]), or by combinatorial techniques (see [7, [1]).

Let $\mathcal{I}_{n}$ be the set of all involutions in $\mathfrak{S}_{n}$ and $\mathcal{J}_{n}$ the set of all fixed-point free involutions in $\mathfrak{S}_{n}$. Define

$$
\begin{aligned}
& I_{n}(t)=\sum_{\pi \in \mathcal{I}_{n}} t^{\mathrm{d}(\pi)}=\sum_{k=0}^{n-1} I_{n, k} t^{k} \\
& J_{n}(t)=\sum_{\pi \in \mathcal{J}_{n}} t^{\mathrm{d}(\pi)}=\sum_{k=0}^{n-1} J_{n, k} t^{k} .
\end{aligned}
$$

The first values of these polynomials are given in Table 1.

Table 1: The polynomials $I_{n}(t)$ and $J_{n}(t)$ for $n \leq 6$.

| $n$ | $I_{n}(t)$ | $J_{n}(t)$ |
| :--- | :--- | :--- |
| 1 | 1 | 0 |
| 2 | $1+t$ | $t$ |
| 3 | $1+2 t+t^{2}$ | 0 |
| 4 | $1+4 t+4 t^{2}+t^{3}$ | $t+t^{2}+t^{3}$ |
| 5 | $1+6 t+12 t^{2}+6 t^{3}+t^{4}$ | 0 |
| 6 | $1+9 t+28 t^{2}+28 t^{3}+9 t^{4}+t^{5}$ | $t+3 t^{2}+7 t^{3}+3 t^{4}+t^{5}$ |

As one may notice from Table that the coefficients of $I_{n}(t)$ and $J_{n}(t)$ are symmetric and unimodal for $1 \leq n \leq 6$. Actually, the symmetries had been conjectured by Dumont and were first proved by Strehl [12]. Recently, Brenti (see [5]) conjectured that the coefficients of the polynomial $I_{n}(t)$ are log-concave and Dukes [5] has obtained some partial results on the unimodality of the coefficients of $I_{n}(t)$ and $J_{2 n}(t)$. Note that, in contrast to Eulerian polynomials $A_{n}(t)$, the polynomials $I_{n}(t)$ and $J_{2 n}(t)$ may have non-real zeros.

In this paper we will prove that for $n \geq 1$, the two sequences $I_{n, 0}, I_{n, 1}, \ldots, I_{n, n-1}$ and $J_{2 n, 1}, J_{2 n, 2}, \ldots, J_{2 n, 2 n-1}$ are unimodal. Our starting point is the known generating functions of polynomials $I_{n}(t)$ and $J_{n}(t)$ :

$$
\begin{align*}
& \sum_{n=0}^{\infty} I_{n}(t) \frac{u^{n}}{(1-t)^{n+1}}=\sum_{r=0}^{\infty} \frac{t^{r}}{(1-u)^{r+1}\left(1-u^{2}\right)^{r(r+1) / 2}}  \tag{1.1}\\
& \sum_{n=0}^{\infty} J_{n}(t) \frac{u^{n}}{(1-t)^{n+1}}=\sum_{r=0}^{\infty} \frac{t^{r}}{\left(1-u^{2}\right)^{r(r+1) / 2}} \tag{1.2}
\end{align*}
$$

which have been obtained by Désarménien and Foata [4] and Gessel and Reutenauer [8] using different methods. We first derive linear recurrence formulas for $I_{n, k}$ and $J_{2 n, k}$ in the next section and then prove the unimodality by induction in Section 3. We end this paper with further conjectures beyond the unimodality of the two sequences $I_{n, k}$ and $J_{2 n, k}$.

## 2 Linear recurrence formulas for $I_{n, k}$ and $J_{2 n, k}$

Since the recurrence formula for the numbers $I_{n, k}$ is a little more complicated than $J_{2 n, k}$, we shall first prove it for the latter.
Theorem 2.1. For $n \geq 2$ and $k \geq 0$, the numbers $J_{2 n, k}$ satisfy the following recurrence formula:

$$
\begin{align*}
2 n J_{2 n, k}= & {[k(k+1)+2 n-2] J_{2 n-2, k}+2[(k-1)(2 n-k-1)+1] J_{2 n-2, k-1} } \\
& +[(2 n-k)(2 n-k+1)+2 n-2] J_{2 n-2, k-2} . \tag{2.1}
\end{align*}
$$

Here and in what follows $J_{2 n, k}=0$ if $k<0$.
Proof. Equating the coefficients of $u^{2 n}$ in (1.2), we obtain

$$
\begin{equation*}
\frac{J_{2 n}(t)}{(1-t)^{2 n+1}}=\sum_{r=0}^{\infty}\binom{r(r+1) / 2+n-1}{n} t^{r} \tag{2.2}
\end{equation*}
$$

Since

$$
\binom{r(r+1) / 2+n-1}{n}=\frac{r(r-1) / 2+r+n-1}{n}\binom{r(r+1) / 2+n-2}{n-1}
$$

it follows from (2.2) that

$$
\frac{J_{2 n}(t)}{(1-t)^{2 n+1}}=\frac{t^{2}}{2 n}\left(\frac{J_{2 n-2}(t)}{(1-t)^{2 n-1}}\right)^{\prime \prime}+\frac{t}{n}\left(\frac{J_{2 n-2}(t)}{(1-t)^{2 n-1}}\right)^{\prime}+\frac{n-1}{n} \frac{J_{2 n-2}(t)}{(1-t)^{2 n-1}}
$$

or

$$
\begin{align*}
J_{2 n}(t)= & \frac{t^{2}(1-t)^{2}}{2 n} J_{2 n-2}^{\prime \prime}(t)+\left[\frac{(2 n-1) t^{2}(1-t)}{n}+\frac{t(1-t)^{2}}{n}\right] J_{2 n-2}^{\prime}(t) \\
& +\left[(2 n-1) t^{2}+\frac{(2 n-1)(1-t) t}{n}+\frac{(n-1)(1-t)^{2}}{n}\right] J_{2 n-2}(t) \\
= & \frac{t^{4}-2 t^{3}+t^{2}}{2 n} J_{2 n-2}^{\prime \prime}(t)+\left[\frac{(2-2 n) t^{3}}{n}+\frac{(2 n-3) t^{2}}{n}+\frac{t}{n}\right] J_{2 n-2}^{\prime}(t) \\
& +\left[(2 n-2) t^{2}+\frac{t}{n}+\frac{n-1}{n}\right] J_{2 n-2}(t) \tag{2.3}
\end{align*}
$$

Equating the coefficients of $t^{n}$ in (2.3) yields

$$
\begin{aligned}
J_{2 n, k}= & \frac{(k-2)(k-3)}{2 n} J_{2 n-2, k-2}-\frac{(k-1)(k-2)}{n} J_{2 n-2, k-1}+\frac{k(k-1)}{2 n} J_{2 n-2, k} \\
& +\frac{(2-2 n)(k-2)}{n} J_{2 n-2, k-2}+\frac{(2 n-3)(k-1)}{n} J_{2 n-2, k-1}+\frac{k}{n} J_{2 n-2, k} \\
& +(2 n-2) J_{2 n-2, k-2}+\frac{1}{n} J_{2 n-2, k-1}+\frac{n-1}{n} J_{2 n-2, k} .
\end{aligned}
$$

After simplification, we obtain (2.1).

Theorem 2.2. For $n \geq 3$ and $k \geq 0$, the numbers $I_{n, k}$ satisfy the following recurrence formula:

$$
\begin{align*}
n I_{n, k}= & (k+1) I_{n-1, k}+(n-k) I_{n-1, k-1}+\left[(k+1)^{2}+n-2\right] I_{n-2, k} \\
& +[2 k(n-k-1)-n+3] I_{n-2, k-1}+\left[(n-k)^{2}+n-2\right] I_{n-2, k-2} . \tag{2.4}
\end{align*}
$$

Here and in what follows $I_{n, k}=0$ if $k<0$.
Proof. Extracting the coefficients of $u^{2 n}$ in (1.1), we obtain

$$
\begin{equation*}
\frac{I_{n}(t)}{(1-t)^{n+1}}=\sum_{r=0}^{\infty} t^{r} \sum_{k=0}^{\lfloor n / 2\rfloor}\binom{r(r+1) / 2+k-1}{k}\binom{r+n-2 k}{n-2 k} . \tag{2.5}
\end{equation*}
$$

Let

$$
T(n, k):=\binom{x+k-1}{k}\binom{y-2 k}{n-2 k}
$$

and

$$
s(n):=\sum_{k=0}^{\lfloor n / 2\rfloor} T(n, k) .
$$

Applying Zeilberger's algorithm, the Maple package ZeilbergerRecurrence (T, n, k, s, 0. .n) gives

$$
\begin{equation*}
(2 x+y+n+1) s(n)+(y+1) s(n+1)-(n+2) s(n+2)=0, \tag{2.6}
\end{equation*}
$$

i.e.,

$$
s(n)=\frac{y+1}{n} s(n-1)+\frac{2 x+y+n-1}{n} s(n-2) .
$$

When $x=r(r+1) / 2$ and $y=r$, we get

$$
\begin{equation*}
s(n)=\frac{r+1}{n} s(n-1)+\frac{r(r-1)+3 r+n-1}{n} s(n-2) . \tag{2.7}
\end{equation*}
$$

Now, from (2.5) and (2.7) it follows that

$$
\begin{aligned}
\frac{n I_{n}(t)}{(1-t)^{n+1}}= & t\left(\frac{I_{n-1}(t)}{(1-t)^{n}}\right)^{\prime}+\frac{I_{n-1}(t)}{(1-t)^{n}}+t^{2}\left(\frac{I_{n-2}(t)}{(1-t)^{n-1}}\right)^{\prime \prime}+3 t\left(\frac{I_{n-2}(t)}{(1-t)^{n-1}}\right)^{\prime} \\
& +(n-1) \frac{I_{n-2}(t)}{(1-t)^{n-1}}
\end{aligned}
$$

or

$$
\begin{align*}
n I_{n}(t)= & \left(t-t^{2}\right) I_{n-1}^{\prime}(t)+[1+(n-1) t] I_{n-1}(t)+t^{2}(1-t)^{2} I_{n-2}^{\prime \prime}(t) \\
& +t(1-t)[3+(2 n-5) t] I_{n-2}^{\prime}(t)+(n-1)\left[1+t+(n-2) t^{2}\right] I_{n-2}(t) \tag{2.8}
\end{align*}
$$

Comparing the coefficients of $t^{k}$ in both sides of (2.8), we obtain

$$
\begin{aligned}
n I_{n, k}= & k I_{n-1, k}-(k-1) I_{n-1, k-1}+I_{n-1, k}+(n-1) I_{n-1, k-1} \\
& +k(k-1) I_{n-2, k}-2(k-1)(k-2) I_{n-2, k-1}+(k-2)(k-3) I_{n-2, k-2} \\
& +3 k I_{n-2, k}+(2 n-8)(k-1) I_{n-2, k-1}-(2 n-5)(k-2) I_{n-2, k-2} \\
& +(n-1) I_{n-2, k}+(n-1) I_{n-2, k-1}+(n-1)(n-2) I_{n-2, k-2}
\end{aligned}
$$

which, after simplification, equals the right-hand side of (2.4).
Remark. The recurrence formula (2.6) can also be proved by hand as follows. It is easy to see that the generating function of $s(n)$ is

$$
\begin{equation*}
\sum_{n=0}^{\infty} s(n) u^{n}=\left(1-u^{2}\right)^{-x}(1-u)^{-y-1} \tag{2.9}
\end{equation*}
$$

Differentiating (2.9) with respect to $u$ implies that

$$
\sum_{n=0}^{\infty} n s(n) u^{n-1}=\left(\frac{2 u x}{1-u^{2}}+\frac{y+1}{1-u}\right)\left(1-u^{2}\right)^{-x}(1-u)^{-y-1}
$$

consequently,

$$
\begin{align*}
\left(1-u^{2}\right) \sum_{n=0}^{\infty} n s(n) u^{n-1} & =[(2 x+y+1) u+y+1]\left(1-u^{2}\right)^{-x}(1-u)^{-y-1} \\
& =[(2 x+y+1) u+y+1] \sum_{n=0}^{\infty} s(n) u^{n} \tag{2.10}
\end{align*}
$$

Comparing the coefficients of $u^{n+1}$ in both sides of (2.10), we obtain

$$
(n+2) s(n+2)-n s(n)=(2 x+y+1) s(n)+(y+1) s(n+1)
$$

which is equivalent to (2.6).
Note that the right-hand side of (2.1) (resp. (2.4)) is invariant under the substitution $k \rightarrow 2 n-k$ (resp. $k \rightarrow n-1-k$ ), provided that the sequence $I_{n-1, k}$ (resp. $J_{2 n-2, k}$ ) is symmetric. Thus, by induction we derive immediately the symmetry properties of $J_{2 n, k}$ and $I_{n, k}$ (see [4, 8, (12).

Corollary 2.3. For $n, k \in \mathbb{N}$, we have

$$
I_{n, k}=I_{n, n-1-k}, \quad J_{2 n, k}=J_{2 n, 2 n-k} .
$$

It would be interesting to find a combinatorial proof of the recurrence formulas (2.1) and (2.4), since such a proof could hopefully lead to a combinatorial proof of the unimodality of these two sequences.

## 3 Unimodality of the sequences $I_{n, k}$ and $J_{2 n, k}$

The following observation is crucial in our inductive proof of the unimodality of the sequences $I_{n, k}(0 \leq k \leq n-1)$ and $J_{2 n, k}(1 \leq k \leq 2 n-1)$.

Lemma 3.1. Let $x_{0}, x_{1}, \ldots, x_{n}$ and $a_{0}, a_{1}, \ldots, a_{n}$ be real numbers such that $x_{0} \geq x_{1} \geq$ $\cdots \geq x_{n} \geq 0$ and $a_{0}+a_{1}+\cdots+a_{k} \geq 0$ for all $k=0,1, \ldots, n$. Then

$$
\sum_{i=0}^{n} a_{i} x_{i} \geq 0
$$

Indeed, the above inequality follows from the identity:

$$
\sum_{i=0}^{n} a_{i} x_{i}=\sum_{k=0}^{n}\left(x_{k}-x_{k+1}\right)\left(a_{0}+a_{1}+\cdots+a_{k}\right),
$$

where $x_{n+1}=0$.
Theorem 3.2. The sequence $J_{2 n, 1}, J_{2 n, 2}, \ldots, J_{2 n, 2 n-1}$ is unimodal.
Proof. By the symmetry of $J_{2 n, k}$, it is enough to show that $J_{2 n, k} \geq J_{2 n, k-1}$ for all $2 \leq k \leq n$. We proceed by induction on $n$. Clearly, the $n=2$ case is obvious. Suppose the sequence $J_{2 n-2, k}$ is unimodal in $k$. By Theorem [2.1], one has

$$
\begin{equation*}
2 n\left(J_{2 n, k}-J_{2 n, k-1}\right)=A_{0} J_{2 n-2, k}+A_{1} J_{2 n-2, k-1}+A_{2} J_{2 n-2, k-2}+A_{3} J_{2 n-2, k-3} \tag{3.1}
\end{equation*}
$$

where

$$
\begin{aligned}
& A_{0}=k^{2}+k+2 n-2, \quad A_{1}=4 n k-3 k^{2}-6 n+k+6 \\
& A_{2}=3 k^{2}+4 n^{2}-8 n k-5 k+12 n-4, \quad A_{3}=3 k-k^{2}+4 n k-4 n^{2}-8 n .
\end{aligned}
$$

We have the following two cases:

- If $2 \leq k \leq n-1$, then

$$
J_{2 n-2, k} \geq J_{2 n-2, k-1} \geq J_{2 n-2, k-2} \geq J_{2 n-2, k-3}
$$

by the induction hypothesis, and clearly

$$
\begin{aligned}
& A_{0} \geq 0, \quad A_{0}+A_{1}=2(k-1)(2 n-k)+4 \geq 0 \\
& A_{0}+A_{1}+A_{2}=(2 n-k)^{2}-3 k+8 n \geq 0, \quad A_{0}+A_{1}+A_{2}+A_{3}=0
\end{aligned}
$$

Therefore, by Lemma 3.1 we have

$$
J_{2 n, k}-J_{2 n, k-1} \geq 0
$$

- If $k=n$, then

$$
J_{2 n-2, n-1} \geq J_{2 n-2, n}=J_{2 n-2, n-2} \geq J_{2 n-2, n-3}
$$

by symmetry and the induction hypothesis. In this case, we have $A_{1}=(n-2)(n-$ $3) \geq 0$ and thus the corresponding condition of Lemma 3.1 is satisfied. Therefore, we have

$$
J_{2 n, n}-J_{2 n, n-1} \geq 0
$$

This completes the proof.
Theorem 3.3. The sequence $I_{n, 0}, I_{n, 1}, \ldots, I_{n, n-1}$ is unimodal.
Proof. By the symmetry of $I_{n, k}$, it suffices to show that $I_{n, k} \geq I_{n, k-1}$ for all $1 \leq k \leq$ $(n-1) / 2$. From Table 1 it is clear that the sequences $I_{n, k}$ are unimodal in $k$ for $1 \leq n \leq 6$.

Now suppose $n \geq 7$ and the sequences $I_{n-1, k}$ and $I_{n-2, k}$ are unimodal in $k$. Replacing $k$ by $k-1$ in (2.4), we obtain

$$
\begin{align*}
n I_{n, k-1}= & k I_{n-1, k-1}+(n-k+1) I_{n-1, k-2}+\left(k^{2}+n-2\right) I_{n-2, k-1} \\
& +[2(k-1)(n-k)-n+3] I_{n-2, k-2}+\left[(n-k+1)^{2}+n-2\right] I_{n-2, k-3} \tag{3.2}
\end{align*}
$$

Combining (2.4) and (3.2) yields

$$
\begin{align*}
n\left(I_{n, k}-I_{n, k-1}\right)= & B_{0} I_{n-1, k}+B_{1} I_{n-1, k-1}+B_{2} I_{n-1, k-2} \\
& +C_{0} I_{n-2, k}+C_{1} I_{n-2, k-1}+C_{2} I_{n-2, k-2}+C_{3} I_{n-2, k-3} \tag{3.3}
\end{align*}
$$

where

$$
\begin{aligned}
& B_{0}=k+1, \quad B_{1}=n-2 k, \quad B_{2}=-(n-k+1), \\
& C_{0}=(k+1)^{2}+n-2, \quad C_{1}=2 n k-3 k^{2}-2 k-2 n+5, \\
& C_{2}=n^{2}-4 n k+3 k^{2}+4 n-2 k-5, \quad C_{3}=-(n-k+1)^{2}-n+2 .
\end{aligned}
$$

Notice that $I_{n-1, k} \geq I_{n-1, k-1} \geq I_{n-1, k-2}$ for $1 \leq k \leq(n-1) / 2$. By Lemma 3.1, we have

$$
\begin{equation*}
B_{0} I_{n-1, k}+B_{1} I_{n-1, k-1}+B_{2} I_{n-1, k-2} \geq 0 \tag{3.4}
\end{equation*}
$$

It remains to show that

$$
\begin{equation*}
C_{0} I_{n-2, k}+C_{1} I_{n-2, k-1}+C_{2} I_{n-2, k-2}+C_{3} I_{n-2, k-3} \geq 0, \quad \forall 1 \leq k \leq(n-1) / 2 \tag{3.5}
\end{equation*}
$$

We need to consider the following two cases:

- If $1 \leq k \leq(n-2) / 2$, then

$$
I_{n-2, k} \geq I_{n-2, k-1} \geq I_{n-2, k-2} \geq I_{n-2, k-3}
$$

by the induction hypothesis, and

$$
\begin{aligned}
& C_{0}=(k+1)^{2}+n-2 \geq 0, \quad C_{0}+C_{1}=(2 k-1)(n-k-1)+k+3 \geq 0 \\
& C_{0}+C_{1}+C_{2}=(n-k+1)^{2}+n-2 \geq 0, \quad C_{0}+C_{1}+C_{2}+C_{3}=0
\end{aligned}
$$

- If $k=(n-1) / 2$, then by symmetry and the induction hypothesis,

$$
I_{n-2, k-1} \geq I_{n-2, k}=I_{n-2, k-2} \geq I_{n-2, k-3}
$$

In this case, we have $C_{1}=(n-3)(n-7) / 4 \geq 0$ for $n \geq 7$.
Therefore, by Lemma 3.1 the inequality (3.5) holds. It follows from (3.3)-(3.5) that

$$
I_{n, k}-I_{n, k-1} \geq 0, \quad \forall 1 \leq k \leq(n-1) / 2
$$

This completes the proof.

## 4 Further remarks and open problems

Since $I_{n, k}=I_{n, n-1-k}$, we can rewrite $I_{n}(t)$ as follows:

$$
I_{n}(t)=\sum_{k=0}^{n-1} I_{n, k} t^{k}= \begin{cases}\sum_{k=0}^{n / 2-1} I_{n, k} t^{k}\left(1+t^{n-2 k-1}\right), & \text { if } n \text { is even } \\ I_{n,(n-1) / 2} t^{(n-1) / 2}+\sum_{k=0}^{(n-3) / 2} I_{n, k} t^{k}\left(1+t^{n-2 k-1}\right), & \text { if } n \text { is odd }\end{cases}
$$

Applying the well-known formula

$$
x^{n}+y^{n}=\sum_{j=0}^{\lfloor n / 2\rfloor}(-1)^{j} \frac{n}{n-j}\binom{n-j}{j}(x y)^{j}(x+y)^{n-2 j},
$$

we obtain

$$
\begin{equation*}
I_{n}(t)=\sum_{k=0}^{\lfloor(n-1) / 2\rfloor} a_{n, k} t^{k}(1+t)^{n-2 k-1} \tag{4.1}
\end{equation*}
$$

where

$$
a_{n, k}= \begin{cases}\sum_{j=0}^{k}(-1)^{k-j} \frac{n-2 j-1}{n-k-j-1}\binom{n-k-j-1}{k-j} I_{n, j}, & \text { if } 2 k+1<n \\ I_{n, k}+\sum_{j=0}^{k-1}(-1)^{k-j} \frac{n-2 j-1}{n-k-j-1}\binom{n-k-j-1}{k-j} I_{n, j}, & \text { if } 2 k+1=n\end{cases}
$$

The first values of $a_{n, k}$ are given in Table 2, which seems to suggest the following conjecture.

Conjecture 4.1. For $n \geq 1$ and $k \geq 0$, the coefficients $a_{n, k}$ are nonnegative integers.

Table 2: Values of $a_{n, k}$ for $n \leq 16$ and $0 \leq k \leq\lfloor(n-1) / 2\rfloor$.

| $k \backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 |  |  | 0 | 1 | 2 | 4 | 6 | 9 | 12 | 16 | 20 | 25 | 30 | 36 | 42 | 49 |
| 2 |  |  |  |  | 2 | 6 | 18 | 39 | 79 | 141 | 239 | 379 | 579 | 849 | 1211 | 1680 |
| 3 |  |  |  |  |  |  | 0 | 18 | 78 | 272 | 722 | 1716 | 3626 | 7160 | 13206 | 23263 |
| 4 |  |  |  |  |  |  |  |  | 20 | 124 | 668 | 2560 | 8360 | 23536 | 59824 | 139457 |
| 5 |  |  |  |  |  |  |  |  |  |  | 32 | 700 | 4800 | 24160 | 95680 | 325572 |
| 6 |  |  |  |  |  |  |  |  |  |  |  |  | 440 | 5480 | 44632 | 257964 |
| 7 |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 2176 | 44376 |

Since the coefficients of $t^{k}(1+t)^{n-2 k-1}$ are symmetric and unimodal with center of symmetry at $(n-1) / 2$, Conjecture 4.1, is stronger than the fact that the coefficients of $I_{n}(t)$ are symmetric and unimodal. A more interesting question is to give a combinatorial interpretation of $a_{n, k}$. Note that the Eulerian polynomials can be written as

$$
A_{n}(t)=\sum_{k=1}^{\lfloor(n+1) / 2\rfloor} c_{n, k} t^{k}(1+t)^{n-2 k+1}
$$

where $c_{n, k}$ is the number of increasing binary trees on $[n]$ with $k$ leaves and no vertices having left children only (see [1, 6, 7]).

We now proceed to derive a recurrence relation for $a_{n, k}$. Set $x=x(t)=t /(1+t)^{2}$ and

$$
P_{n}(x)=\sum_{k=0}^{\lfloor(n-1) / 2\rfloor} a_{n, k} x^{k} .
$$

Then we can rewrite (4.1) as

$$
\begin{equation*}
I_{n}(t)=(1+t)^{n-1} P_{n}(x) . \tag{4.2}
\end{equation*}
$$

Differentiating (4.2) with respect to $t$ we get

$$
\begin{align*}
I_{n}^{\prime}(t)= & (n-1)(1+t)^{n-2} P_{n}(x)+(1+t)^{n-1} P_{n}^{\prime}(x) x^{\prime}(t),  \tag{4.3}\\
I_{n}^{\prime \prime}(t)= & (n-1)(n-2)(1+t)^{n-3} P_{n}(x)+2(n-1)(1+t)^{n-2} P_{n}^{\prime}(x) x^{\prime}(t) \\
& +(1+t)^{n-1} P_{n}^{\prime \prime}(x)\left(x^{\prime}(t)\right)^{2}+(1+t)^{n-1} P_{n}^{\prime}(x) x^{\prime \prime}(t),  \tag{4.4}\\
x^{\prime}(t)= & \frac{1-t}{(1+t)^{3}}, \quad x^{\prime \prime}(t)=\frac{2 t-4}{(1+t)^{4}} . \tag{4.5}
\end{align*}
$$

Substituting (4.2)-(4.5) into (2.8), we obtain

$$
\begin{align*}
n(1 & +t)^{n-1} P_{n}(x) \\
= & {\left[1+(2 n-2) t+t^{2}\right](1+t)^{n-3} P_{n-1}(x)+t(1-t)^{2}(1+t)^{n-5} P_{n-1}^{\prime}(x) } \\
& +\left[-\left(t^{2}+14 t+1\right)(1-t)^{2}+\left(1+6 t-18 t^{2}+6 t^{3}+t^{4}\right) n+4 t^{2} n^{2}\right](1+t)^{n-5} P_{n-2}(x) \\
& +\left[3 t\left(t^{2}-4 t+1\right)(1-t)^{2}+4 t^{2}(1-t)^{2} n\right](1+t)^{n-7} P_{n-2}^{\prime}(x) \\
& +t^{2}(1-t)^{4}(1+t)^{n-9} P_{n-2}^{\prime \prime}(x) \tag{4.6}
\end{align*}
$$

Dividing the two sides of (4.6) by $(1+t)^{n-1}$ and noticing that $t /(1+t)^{2}=x$, after a little manipulation we get

$$
\begin{aligned}
n P_{n}(x)= & {[1+(2 n-4) x] P_{n-1}(x)+\left(x-4 x^{2}\right) P_{n-1}^{\prime}(x) } \\
& +\left[(n-1)+(2 n-8) x+4(n-3)(n-4) x^{2}\right] P_{n-2}(x) \\
& +\left[3 x+(4 n-30) x^{2}+(72-16 n) x^{3}\right] P_{n-2}^{\prime}(x)+\left(x^{2}-8 x^{3}+16 x^{4}\right) P_{n-2}^{\prime \prime}(x) .
\end{aligned}
$$

Extracting the coefficients of $x^{k}$ yields

$$
\begin{aligned}
n a_{n, k}= & a_{n-1, k}+(2 n-4) a_{n-1, k-1}+k a_{n-1, k}-4(k-1) a_{n-1, k-1} \\
& +(n-1) a_{n-2, k}+(2 n-8) a_{n-2, k-1}+4(n-3)(n-4) a_{n-2, k-2} \\
& +3 k a_{n-2, k}+(4 n-30)(k-1) a_{n-2, k-1}+(72-16 n)(k-2) a_{n-2, k-2} \\
& +k(k-1) a_{n-2, k}-8(k-1)(k-2) a_{n-2, k-1}+16(k-2)(k-3) a_{n-2, k-2} .
\end{aligned}
$$

After simplification, we obtain the following recurrence formula for $a_{n, k}$.
Theorem 4.2. For $n \geq 3$ and $k \geq 0$, there holds

$$
\begin{align*}
n a_{n, k}= & (k+1) a_{n-1, k}+(2 n-4 k) a_{n-1, k-1}+[k(k+2)+n-1] a_{n-2, k} \\
& +[(k-1)(4 n-8 k-14)+2 n-8] a_{n-2, k-1}+4(n-2 k)(n-2 k+1) a_{n-2, k-2}, \tag{4.7}
\end{align*}
$$

where $a_{n, k}=0$ if $k<0$ or $k>(n-1) / 2$.
Note that, if $n \geq 2 k+3$, then

$$
(k-1)(4 n-8 k-14)+2 n-8>0 \quad \text { for any } k \geq 1,
$$

and so are the other coefficients in (4.7). Therefore, Conjecture 4.1 would be proved if one can show that $a_{2 n+1, n} \geq 0$ and $a_{2 n+2, n} \geq 0$.

Finally, from (4.1) it is easy to see that

$$
\begin{aligned}
& a_{2 n+1, n}=(-1)^{n} I_{2 n+1}(-1)=\sum_{k=0}^{2 n}(-1)^{n-k} I_{2 n+1, k}, \\
& a_{2 n+2, n}=(-1)^{n} I_{2 n+2}^{\prime}(-1)=\sum_{k=1}^{2 n+1}(-1)^{n+1-k} k I_{2 n+2, k} .
\end{aligned}
$$

Thus, Conjecture 4.1 is equivalent to the nonnegativity of the above two alternating sums.
Since $J_{2 n, k}=J_{2 n, 2 n-k}$, in the same manner as $I_{n}(t)$ we obtain

$$
J_{2 n}(t)=\sum_{k=1}^{n} b_{2 n, k} t^{k}(1+t)^{2 n-2 k}
$$

where

$$
b_{2 n, k}= \begin{cases}\sum_{j=1}^{k}(-1)^{k-j} \frac{2 n-2 j}{2 n-k-j}\binom{2 n-k-j}{k-j} J_{2 n, j}, & \text { if } k<n, \\ J_{2 n, k}+\sum_{j=1}^{k-1}(-1)^{k-j} \frac{2 n-2 j}{2 n-k-j}\binom{2 n-k-j}{k-j} J_{2 n, j}, & \text { if } k=n\end{cases}
$$

Now, it follows from (2.2) that

$$
J_{2 n, k}=\sum_{i=0}^{k}(-1)^{k-i}\binom{2 n+1}{k-i}\binom{i(i+1) / 2+n-1}{i(i+1) / 2-1}
$$

is a polynomial in $n$ of degree $d:=k(k+1) / 2-1$ with leading coefficient $1 / d$ !, and so is $b_{2 n, k}$. Thus, we have $\lim _{n \rightarrow+\infty} b_{2 n, k}=+\infty$ for any fixed $k>1$.

The first values of $b_{2 n, k}$ are given in Table 3, which seems to suggest
Conjecture 4.3. For $n \geq 9$ and $k \geq 1$, the coefficients $b_{2 n, k}$ are nonnegative integers.
Similarly to the proof of Theorem 4.2, we can prove the following result.
Theorem 4.4. For $n \geq 2$ and $k \geq 1$, there holds

$$
\begin{aligned}
2 n b_{2 n, k}= & {[k(k+1)+2 n-2] b_{2 n-2, k}+[2+2(k-1)(4 n-4 k-3)] b_{2 n-2, k-1} } \\
& +8(n-k+1)(2 n-2 k+1) b_{2 n-2, k-2} .
\end{aligned}
$$

where $b_{2 n, k}=0$ if $k<1$ or $k>n$.
Theorem 4.4 allows us to reduce the verification of Conjecture 4.3 to the boundary case $b_{2 n, n} \geq 0$ for $n \geq 9$.

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Table 3: Values of $b_{2 n, k}$ for $2 n \leq 24$ and $1 \leq k \leq n$.

| $k \backslash 2 n$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 |  | -1 | -1 | 0 | 2 | 5 | 9 | 14 | 20 | 27 | 35 | 44 |
| 3 |  |  | 3 | 12 | 36 | 91 | 201 | 399 | 728 | 1242 | 2007 | 3102 |
| 4 |  |  |  | -7 | -10 | 91 | 652 | 2593 | 7902 | 20401 | 46852 | 98494 |
| 5 |  |  |  |  | 25 | 219 | 1710 | 10532 | 50165 | 194139 | 639968 | 1861215 |
| 6 |  |  |  |  |  | -65 | 249 | 11319 | 122571 | 841038 | 4377636 | 18747924 |
| 7 |  |  |  |  |  |  | 283 | 6586 | 135545 | 1737505 | 15219292 | 101116704 |
| 8 |  |  |  |  |  |  |  | -583 | 33188 | 1372734 | 24412940 | 277963127 |
| 9 |  |  |  |  |  |  |  |  | 4417 | 379029 | 16488999 | 367507439 |
| 10 |  |  |  |  |  |  |  |  |  | 1791 | 3350211 | 203698690 |
| 11 |  |  |  |  |  |  |  |  |  |  | 133107 | 36903128 |
| 12 |  |  |  |  |  |  |  |  |  |  |  | 761785 |

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