# PERMUTATION TABLEAUX AND PERMUTATION PATTERNS 

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#### Abstract

In this paper we introduce and study a class of tableaux which we call permutation tableaux; these tableaux are naturally in bijection with permutations, and they are a distinguished subset of the J-diagrams of Alex Postnikov [15, 20. The structure of these tableaux is in some ways more transparent than the structure of permutations; therefore we believe that permutation tableaux will be useful in furthering the understanding of permutations. We give two bijections from permutation tableaux to permutations. The first bijection carries tableaux statistics to permutation statistics based on relative sizes of pairs of letters in a permutation and their places. We call these statistics weak excedance statistics because of their close relation to weak excedances. The second bijection carries tableaux statistics (via the weak excedance statistics to statistics based on generalized permutation patterns. We then give enumerative applications of these bijections. One nice consequence of these results is that the polynomial enumerating permutation tableaux according to their content generalizes both Carlitz' $q$-analog of the Eulerian numbers 6] and the more recent $q$-analog of the Eulerian numbers found in 20]. We conclude our paper with a list of open problems, as well as remarks on progress on these problems which has been made by $A$. Burstein, S. Corteel, N. Eriksen, A. Reifegerste, and X. Viennot.


## 1. Introduction

The aim of this article is to advertise a new class of tableaux together with two curious bijections for the study of permutations. We call these tableaux permutation tableaux; they are naturally in bijection with permutations, and are a distinguished subset of Alex Postnikov's J-diagrams [15], which were enumerated by the second author 20 because of their connection with the totally nonnegative part of the Grassmannian.

Recall that a partition $\lambda=\left(\lambda_{1}, \ldots, \lambda_{k}\right)$ is a weakly decreasing sequence of nonnegative integers. For a partition $\lambda$, where $\sum \lambda_{i}=m$, the Young diagram $Y_{\lambda}$ of shape $\lambda$ is a left-justified diagram of $m$ boxes, with $\lambda_{i}$ boxes in the $i$-th row.

We define a permutation tableau $\mathcal{T}_{n}^{k}$ to be a partition $\lambda$ such that $Y_{\lambda}$ is contained in a $k \times(n-k)$ rectangle, together with a filling of the boxes of $Y_{\lambda}$ with 0 's and 1's such that the following properties hold:
(1) Each column of the rectangle contains at least one 1.
(2) There is no 0 which has a 1 above it in the same column and a 1 to its left in the same row.

[^0]We call such a filling a valid filling of $Y_{\lambda}$. Observe that the requirement in (1) implies that the Young diagram must have $n-k$ columns, whereas the number of rows may be smaller than $k$.

If we forget the requirement (1) above we recover the definition of a J-diagram [15].
Figure 1 gives an example of a permutation tableau.


Figure 1. A permutation tableau

Note that there is a unique permutation tableau which has $n$ rows and no columns!

We will also think of a permutation tableau $\mathcal{T}_{n}^{k}$ as a $k \times(n-k)$ array of 0 's, 1's, and 2's, by simply taking the previous description of a permutation tableaux and putting a 2 in every box of the rectangle which is not in $Y_{\lambda}$, as in Figure 2 We position the partition shape so that its top row lies at the top of the rectangle; therefore the 2's cut out a (rotated) Young diagram in the southeast corner of the rectangle.


Figure 2. Another representation of a permutation tableau

Postnikov [16] has described a map that takes permutation tableaux contained in a $k \times(n-k)$ rectangle to permutations in $\mathcal{S}_{n}$ with $k$ weak excedances. In this paper we give a much simpler description of this map, prove that it is a bijection, and show that this map in fact preserves many more statistics. Namely, the numbers of 0's, 1's and 2's, respectively, in a permutation tableau equal certain linear combinations of certain statistics defined on the corresponding permutation. Each of these statistics counts pairs of indices $(i, j)$ in a permutation, according to the relative sizes of the
letters in those places and the place numbers themselves. These statistics were defined by Corteel in 88.

We then define another bijection, taking permutations to permutations and translating the statistics mentioned above into certain linear combinations of generalized permutation patterns. These combinations between them contain precisely half the generalized patterns of length 3 with one dash (see Section 4).

We conclude our paper by giving various enumerative applications of our bijections. The structure of permutation tableaux is in many ways more transparent than the structure of permutations, and hence lends itself more easily to enumeration. For example, by using our bijections together with results of the second author [20], we are able to give the entire distribution of permutations according to the number of occurrences of the generalized pattern $(2-31)$. This is the first such result for any pattern of length 3 (or more). However, this particular result, although first conjectured by the present authors, was first proved by Corteel [8, whose work provided us with a crucial piece of the puzzle solved here.

Another interesting consequence of the results presented here is that the statistic counting permutation tableaux according to the number of rows and number of 0 's is an Euler-Mahonian statistic, that is, has the same distribution as the bistatistic on permutations consisting of the number of descents and the major index. It follows that if we define $D_{k, n}(p, q, r)$ to be the polynomial enumerating the permutation tableaux contained in a $k \times(n-k)$ rectangle according to the number of 0 's, 1 's, and 2's, then $D_{k, n}(p, 1,1)$ is equal to Carlitz' classical $q$-analog $B_{n, k}(p)$ of the Eulerian numbers [6]. Additionally, $D_{k, n}(1, q, 1)$ is equal to the more recent $q$-analog $E_{k, n}(q)$ of the Eulerian numbers that was studied in [20].

## 2. Bijection from Permutation Tableaux to Permutations

In this section we describe a bijection $\Phi$ from permutation tableaux to permutations. More precisely, $\Phi$ is a bijection from the set of permutation tableaux contained in a $k \times(n-k)$ rectangle to permutations in the symmetric group $\mathcal{S}_{n}$ with $k$ weak excedances. Here, a weak excedance of a permutation $\pi$ is a value $\pi(i)$ such that $\pi(i) \geq i$. In this situation we say that $i$ is a weak excedance bottom of $\pi$ and that $\pi(i)$ is a weak excedance top of $\pi$. To make the notation less cumbersome, we abbreviate these as wexbottoms and wextops, respectively. Non-weak excedance bottoms and non-weak excedance tops are defined in the obvious way, in terms of $i$ and $\pi(i)$ such that $\pi(i)<i$, and are abbreviated non-wexbots and non-wextops, respectively. The number of weak excedances in $\pi$ will be denoted wex $\pi$. Also, we let wexbotsum be the sum of all the wexbottoms in $\pi$.

We remark that Postnikov 16 defined a map that is equal to $\Phi$ but his description was more complicated and went through the intermediate step of web diagrams. Additionally, his proof that it was a bijection used the geometry of the totally nonnegative part of the Grassmannian. Another indirect way to define the map $\Phi$ can be obtained from the Appendix of [21]: use the restriction from decorated permutations to permutations of the composition of the maps from Lemmas A. 4 and A. 5.

Our primary contributions in this section and the next are to: describe $\Phi$ in a simple enough way that we can actually give a combinatorial proofs of its properties; prove that $\Phi$ translates a remarkable number of statistics on tableaux to statistics on permutations.


Figure 3. The diagram of a tableau. The topmost 1 in each column becomes a white vertex, and the other 1's become black vertices.

Before giving the bijection $\Phi$, we must define the $\operatorname{diagram} D\left(\mathcal{T}_{n}^{k}\right)$ associated with $\mathcal{T}_{n}^{k}$ as follows. Regard the south-east border of the partition $Y_{\lambda}$ contained in the $k \times(n-k)$ rectangle as giving a path (the partition path) $P=\left\{P_{i}\right\}_{i=1}^{n}$ of length $n$ from the northeast corner of the rectangle to the southwest corner of the rectangle: label each of the (unit) steps in this path with a number from 1 to $n$ according to the order in which the step was taken. Then, remove the 0's from $\mathcal{T}_{n}^{k}$ and replace each 1 in $\mathcal{T}_{n}^{k}$ with a vertex. We will call the top vertex in each column a white vertex and all other vertices black vertices. Finally, from each vertex $v$, draw an edge to the east and an edge to the south; each such edge should connect $v$ to either a closest vertex in the same row or column, or to one of the labels from 1 to $n$. The resulting picture is the diagram $D\left(\mathcal{T}_{n}^{k}\right)$. See Figure 3,

We now define the permutation $\pi=\Phi\left(\mathcal{T}_{n}^{k}\right)$ via the following procedure. For each $i \in\{1, \ldots, n\}$, find the corresponding position on $D\left(\mathcal{T}_{n}^{k}\right)$ which is labeled by $i$. If the label $i$ is on a vertical step of $P$, start from this position and travel straight west as far as possible on edges of $D\left(\mathcal{T}_{n}^{k}\right)$. Then, take a "zig-zag" path southeast, by traveling on edges of $D\left(\mathcal{T}_{n}^{k}\right)$ south and east and turning at each opportunity (i.e. at each new vertex). This path will terminate at some label $j \geq i$, and we let $\pi(i)=j$. If $i$ is not connected to any edge (equivalently, if there are no vertices in the row of $i$ ) then we set $\pi(i)=i$. Similarly, if the label $i$ is on a horizontal step of $P$, start from this position and travel north as far as possible on edges of $D\left(\mathcal{T}_{n}^{k}\right)$. Then, as before, take a zig-zag path south-east, by traveling on edges of $D\left(\mathcal{T}_{n}^{k}\right)$ east and south, and turning at each opportunity. This path will terminate at some label $j<i$, and we let $\pi(i)=j$.

See Figure 4 for a picture of the path taken by $i$.
Example 1. If $\mathcal{T}_{n}^{k}$ is the permutation tableau whose diagram is given in Figures 3 and 4 then $\Phi\left(\mathcal{T}_{n}^{k}\right)=74836215$.

The rest of this section will be devoted to proving various properties of the map $\Phi$, and in particular, that $\Phi$ is actually a bijection.

The following three lemmas are clear from the construction above.
Lemma 2. In $\Phi\left(\mathcal{T}_{n}^{k}\right)$, the letter $i$ is a fixed point if and only if there is an entire row in $\mathcal{T}_{n}^{k}$ that has no 1's and whose right hand edge is labeled by i. In particular, $n, n-1, \ldots, n-m+1$ are fixed points in $\pi$ if and only if the bottom $m$ rows of $\mathcal{T}_{n}^{k}$ (in the $k \times(n-k)$ rectangle) consist entirely of 2's.


Figure 4. The paths taken by 1 and $6: \pi(1)=7, \pi(6)=2$.
Lemma 3. Any directed step in a path on $D\left(\mathcal{T}_{n}^{k}\right)$ determines the path completely.
Lemma 3 implies the following.
Corollary 4. $\Phi\left(\mathcal{T}_{n}^{k}\right)$ is a permutation.
Lemma 5. The weak excedance bottoms of $\pi=\Phi\left(\mathcal{T}_{n}^{k}\right)$ are precisely the labels on the vertical edges of $P$. The non-weak excedance bottoms of $\pi$ are precisely the labels on the horizontal edges of $P$. In particular, $\Phi\left(\mathcal{T}_{n}^{k}\right)$ is a permutation in $\mathcal{S}_{n}$ with precisely $k$ weak excedances.

We will now give some more definitions and prove a refinement of Lemma 2
First we define a relative permutation: this is a biword $\pi=\binom{a_{1} \ldots a_{n}}{b_{1} \ldots}$, where the $a_{i}$ 's are distinct integers and the $b_{j}$ 's are also distinct. This denotes the map sending $b_{i} \mapsto a_{i}$. For example, the permutation 31524 corresponds to the biword $\left(\begin{array}{llll}3 & 1 & 5 & 2\end{array}\right)$. Note that it is more common in the literature to write biwords in the other way, that is, with the rows interchanged compared to our notation. Because we want to emphasize weak excedance tops and bottoms, we feel it is more intuitive to have the tops in the top row and the bottoms in the bottom row.

We define the notion of congruence for relative permutations in the obvious way, as follows. Suppose that $a_{i_{1}}<a_{i_{2}}<\cdots<a_{i_{n}}$ and $b_{j_{1}}<b_{j_{2}}<\cdots<b_{j_{n}}$; then define the reduction of $\pi$ to be the permutation that one obtains by replacing $a_{i_{k}}$ by $k$ and $b_{j_{m}}$ by $m$. We now say that two relative permutations are congruent if their reductions are equal.

In keeping with the above definition, we define a relative fixed point to be a pair $b_{p} \mapsto a_{p}$ such that if $a_{p}$ is the $j$-th smallest letter among the $a_{i}$ then $b_{p}$ is also the $j$-th smallest letter among the $b_{i}$.

For example, in $\left(\begin{array}{lll}5 & 3 & 1\end{array}\right)$, the pair $2 \mapsto 3$ is a relative fixed point, since each is the second smallest letter in its row.

Note that we will use the biword notation $\binom{a_{1} \ldots a_{n}}{1 \ldots}$ as an alternative representation of the permutation $a_{1} \ldots a_{n}$.
Definition 6. Fix a permutation tableau $\mathcal{T}_{n}^{k}$ and let $m:=n-k$ be the number of columns in $\mathcal{T}_{n}^{k}$. We will construct a sequence of $m+1$ relative permutations $\pi_{0}, \pi_{1}, \ldots, \pi_{m}$ associated to $\mathcal{T}_{n}^{k}$ and will prove a generalization of Lemma 2 for
these relative permutations. Let $\pi_{0}:=\Phi\left(\mathcal{T}_{n}^{k}\right)$. Let $r_{1}<\cdots<r_{m}$ be the nonexcedence bottoms of $\pi_{1}$, in increasing order. We construct $\pi_{1}$ by deleting the pair $r_{1} \mapsto \pi_{0}\left(r_{1}\right)$ from the biword for $\pi_{0}$, and then deleting any resulting relative fixed points. In general, we construct $\pi_{i+1}$ from $\pi_{i}$ by deleting the pair $r_{i} \mapsto \pi_{0}\left(r_{i}\right)$ from the biword for $\pi_{i}$, and then deleting any resulting relative fixed points.

Example 7. Let $\mathcal{T}_{n}^{k}$ be the permutation tableau in Figure 3 Then we have that
 the empty biword.

Lemma 8. Use the notation of Definition [6. Fix $\mathcal{T}_{n}^{k}$ and choose some number $i$ which is less than the number $m$ of columns of $\mathcal{T}_{n}^{k}$. Construct a new permutation tableau $\widetilde{\mathcal{T}_{n}^{k}}$ by deleting the rightmost $i$ columns of $\mathcal{T}_{n}^{k}$ and then deleting all of the resulting rows which contain only 0 's. Then $\Phi\left(\widetilde{\mathcal{T}_{n}^{k}}\right)$ is congruent to $\pi_{i}$, as relative permutations.

Example 9. Let $\mathcal{T}_{n}^{k}$ be the permutation tableau in Figure 3. Then if we consider $\mathcal{T}_{n}^{k}$ and the resulting tableaux we get by cutting off the rightmost $i$ columns for $1 \leq i \leq 4$, the corresponding permutations that we get are: $\binom{74836215}{12345678}$, $\left(\begin{array}{llll}6 & 3 & 7 & 5\end{array} 2 \begin{array}{ll}1 & 4 \\ 1 & 2\end{array} 345457\right),\left(\begin{array}{lll}3 & 4 & 1\end{array} 2\right.$ congruent to the relative permutations which we computed in Example 7

Proof. By induction, it is enough to prove Lemma 8 for the case $i=1$. Suppose that $r$ is the smallest non-excedence bottom of $\pi_{0}$, that is, $r$ is the number indexing the rightmost column of $\mathcal{T}_{n}^{k}$, and suppose that $r \mapsto h$ in $\pi_{0}$ (for some $h<r$ ). Then by definition, $\pi_{1}$ is equal to the result of deleting $r \mapsto h$ and any resulting relative fixed points from the relative permutation $\pi_{0}$.

Let $\widetilde{\mathcal{T}_{n}^{k}}$ be the permutation tableau formed by deleting the rightmost column of $\mathcal{T}_{n}^{k}$ and then deleting any resulting all-zero rows. Since it will not affect the congruence class of $\Phi\left(\widetilde{\mathcal{T}_{n}^{k}}\right)$, we can label the partition path of $\widetilde{\mathcal{T}_{n}^{k}}$ with the labels inherited from $\mathcal{T}_{n}^{k}$; in other words, we can label the new partition path with the numbers $\{1, \ldots, n\} \backslash\left\{r, j_{1}, j_{2}, \ldots\right\}$, where the $j_{i}$ 's index the all-zero rows that we deleted. Now let us analyze the difference between $\Phi\left(\widetilde{\mathcal{T}_{n}^{k}}\right)$ and $\pi_{0}$. By consideration of the map $\Phi, \Phi\left(\widetilde{\mathcal{T}_{n}^{k}}\right)$ is identical to $\pi_{0}$ except that:

- we have removed $r \mapsto h$ from the biword for $\pi_{0}$
- we have removed resulting relative fixed points
- if in $\pi_{0}$ we had $i \mapsto j$ where either $j=r$ or $j$ indexed a row that is not present in $\widetilde{\mathcal{T}_{n}^{k}}$, then in $\Phi\left(\widetilde{\mathcal{T}_{n}^{k}}\right)$ we have that $i \mapsto g$, where $g$ is the maximal label in $\{1, \ldots, n\} \backslash\left\{r, j_{1}, j_{2}, \ldots\right\}$ which is less than $j$.
It is clear now that $\Phi\left(\widetilde{\mathcal{T}_{n}^{k}}\right)$ is congruent to $\pi_{1}$.
Lemma 10. Use the notation of Definition[6] After we delete $r_{i} \mapsto \pi_{0}\left(r_{i}\right)$ from the biword for $\pi_{i-1}$, we will have a relative fixed point $j \mapsto j^{\prime}$ in the resulting relative permutation if and only if every entry in row $j$ of $\mathcal{T}_{n}^{k}$ to the left of the column indexed by $r_{i}$ is a zero.

Proof. We will prove this lemma by induction. Suppose that it is true for $i \leq I$; we will now prove it for $I+1$. Let $\widetilde{\mathcal{T}_{n}^{k}}$ be the tableau obtained by deleting the rightmost $I$ columns of $\mathcal{T}_{n}^{k}$ and then deleting any resulting all-zero rows. By Lemma we have that $\Phi\left(\widetilde{\mathcal{T}_{n}^{k}}\right)$ is congruent to $\pi_{I}$, and by the induction hypothesis, these relative permutations have no relative fixed points.

Suppose that with the exception of the rightmost column of $\widetilde{\mathcal{T}_{n}^{k}}$ (which corresponds to the column in $\mathcal{T}_{n}^{k}$ indexed by $r_{I+1}$ ), all entries in row $j$ of $\widetilde{\mathcal{T}_{n}^{k}}$ are 0 . Label the partition path of $\widetilde{\mathcal{T}_{n}^{k}}$ with the consecutive numbers $1, \ldots, n^{\prime}$ for some $n^{\prime}$ and suppose that $r$ is the label indexing the rightmost column. Clearly $r>j$. Since $\pi_{I}$ has no relative fixed points, the entry of $\widetilde{\mathcal{T}_{n}^{k}}$ in column $r$ and row $j$ must be a 1 , and additionally every entry in the same column below this 1 must also be a 1. It follows from the definition of $\Phi$ that in the permutation $\Phi\left(\widetilde{\mathcal{T}_{n}^{k}}\right)$, we have $j \mapsto j+1$. Again by the definition of $\Phi$, we have $r \mapsto h$ in $\Phi\left(\widetilde{\mathcal{T}_{n}^{k}}\right)$ for some $h<j$. Recalling that $\Phi\left(\widetilde{\mathcal{T}_{n}^{k}}\right)$ is congruent to $\pi_{I}$ and $r>j$, it follows that when we delete $r \mapsto h$ in $\pi_{I}, j \mapsto j+1$ will become a relative fixed point in the resulting relative permutation.

All of these steps can be reversed, so we are done.

We are finally ready to prove the following theorem.
Theorem 11. The map $\Phi$ is a bijection from permutation tableaux to permutations.
Proof. To prove that $\Phi$ is a bijection, we will give an explicit description of its inverse, by reverse-engineering $\Phi$ so as to be consistent with Lemma 10

Let $\pi$ be the permutation $\binom{a_{1} \ldots a_{n}}{1 \ldots}$. We now give the procedure for computing $\Phi^{-1}(\pi)$, column by column, from right to left.

0 . Compute the weak excedance bottoms of $\pi$ to get the shape of the partition in $\mathcal{T}_{n}^{k}$ (see Lemma 5). Let $\tilde{\pi}=\pi$.

1. Check for relative fixed points in $\tilde{\pi}$. If $j \mapsto j^{\prime}$ is a relative fixed point then by Lemma [10 $\mathcal{T}_{n}^{k}$ must have 0's in the as-yet-undetermined part of the row corresponding to the weak excedance bottom $j$; fill in these entries. Recompute $\tilde{\pi}$ by removing the relative fixed points.
2. Suppose we have determined the content of the $i$ rightmost columns (but have not determined the content of the other columns). Then look at the next column to the left, which is indexed by a non-excedance bottom $r$ (that is, by the label on the horizontal step at the bottom of that column (Lemma (5)). Knowing that $r \rightarrow a_{r}$ in $\pi$ uniquely determines the position $p$ of the highest 1 in the column corresponding to $r$, since there is a unique zig-zag path going backwards (north-west) from $a_{r}$ to a box in the column above $r$. Insert a 1 at that position and 0's in all boxes above it which are in the same column. Also, insert 1's into all undetermined boxes below $p$. (Note that we know that all nonzero boxes below position $p$ must also be 1's; otherwise, if there were some 0 below the 1 then everything to its left would have to be a 0 , and then by Lemma 10, we would have a relative fixed point in $\tilde{\pi}$, contradiction.) Reduce $\tilde{\pi}$ by removing the column $r \mapsto a_{r}$ from the biword for $\tilde{\pi}$. Go to step 1 .


Figure 5. The permutation tableau for $\pi=514263$

It is now clear from this construction that our resulting tableau $\mathcal{T}_{n}^{k}$ will be a permutation tableau, and moreover, that it will be the inverse image $\Phi^{-1}(\pi)$.

Example 12. Let $\pi=514263$. Since $\pi$ is in $S_{6}$ and has three weak excedances and $a_{6} \neq 6$, our permutation tableau $\mathcal{T}_{n}^{k}$ will be contained in a $3 \times 3$ rectangle (the resulting tableau is shown in Figure (5). As in step 0, we want to first compute the shape of the associated partition. Since $1,3,5$ are the wexbottoms, and $2,4,6$ are the non-wexbottoms, this uniquely determines a path (the partition path) from the northeast corner of the rectangle to the southwest corner of the rectangle with vertical steps in positions $1,3,5$ and horizontal steps in positions $2,4,6$. That is, our partition has the shape $(3,2,1)$. We now draw this partition, labeling the edges of its southeast border accordingly with the numbers $1, \ldots, 6$, and set $\tilde{\pi}=(5,1,4,2,6,3)$.

Going to step 1, we see that $\tilde{\pi}$ has no relative fixed points.
Going to step 2, the fact that $2 \rightarrow 1$ in $\pi$ implies that the rightmost column (which consists of a single box) contains a 1 in the top row. We now reduce the permutation $\tilde{\pi}=\binom{514263}{123456}$ by removing $(2,1)$, obtaining $\tilde{\pi}=\binom{54263}{13}$.

Going back to step 1 , we see that there are no relative fixed points in $\tilde{\pi}$.
Going to step 2 , since $4 \rightarrow 2$ in $\tilde{\pi}$ it is clear that the highest box in the column indexed by 4 must contain a 1 . All undetermined boxes below this 1 must contain 1's also. We now reduce the permutation $\tilde{\pi}=\binom{54263}{13456}$ by removing $4 \rightarrow 2$, obtaining $\tilde{\pi}=\left(\begin{array}{lll}5 & 4 & 6 \\ 1 & 3 & 5\end{array}\right)$.

Going back to step 1 , we now see that $\tilde{\pi}$ has the relative fixed point $(3,4)$. Therefore the undetermined part of the row corresponding to 3 consists of zeros. Now we reduce the permutation $\tilde{\pi}=\left(\begin{array}{lll}5 & 4 & 6\end{array} 3\right)$, obtaining $\tilde{\pi}=\left(\begin{array}{lll}5 & 6 & 3 \\ 1 & 5 & 5\end{array}\right)$.

Going to step 2 , since $6 \rightarrow 3$ in $\tilde{\pi}$ the top box in the column corresponding to 6 has a 1. All undetermined boxes below that contain 1's. We have now filled in all columns of the tableau-obtaining the permutation tableau in Figure 5-so we are done.

## 3. How $\Phi$ translates statistics

The six permutation statistics in the following definition will be related to the statistics recording the numbers of 0's, 1's and 2's in permutation tableaux. The first four of these refine Postnikov's definition of alignment [15] (see [20]); all of these statistics were defined by Corteel [8].
Definition 13. Given a permutation $\pi=a_{1} a_{2} \ldots a_{n}$, let

$$
\begin{aligned}
\mathrm{A}_{\mathrm{EE}}(i) & =\left\{j \mid j<i \leq a_{i}<a_{j}\right\}, \\
\mathrm{A}_{\mathrm{NN}}(i) & =\left\{j \mid a_{j}<a_{i}<i<j\right\}, \\
\mathrm{A}_{\mathrm{EN}}(i) & =\left\{j \mid j \leq a_{j}<a_{i}<i\right\}, \\
\mathrm{A}_{\mathrm{NE}}(i) & =\left\{j \mid a_{i}<i<j \leq a_{j}\right\}, \\
\mathrm{C}_{\mathrm{EE}}(i) & =\left\{j \mid j<i \leq a_{j}<a_{i}\right\}, \\
\mathrm{C}_{\mathrm{NN}}(i) & =\left\{j \mid a_{i}<a_{j}<i<j\right\} .
\end{aligned}
$$

We then set

$$
\mathrm{A}_{\mathrm{EE}}(\pi)=\sum_{i}\left|\mathrm{~A}_{\mathrm{EE}}(i)\right|
$$

and likewise for the other five statistics.
Observe that if we draw the permutation as a chord diagram on a circle, as in Figure [6] then $j \in A_{* *}(i)$ means that the chords starting at $i$ and $j$ do not intersect and roughly "point in the same direction" (see 20 for more details); we will say that this is an alignment of type $\mathrm{A}_{* *}$. And if $j \in \mathrm{C}_{* *}(i)$ then the chords starting at $i$ and $j$ cross each other; we will say that this is a crossing of type $\mathrm{C}_{* *}$. Note that the subscripts in our notation refer to whether the positions $i$ and $j$ are wexbottoms or non-wexbottoms of the permutation. For example, in Figure 6 the chords beginning at 3 and 5 form an alignment of type $A_{\mathrm{NE}}$, and the chords beginning at 2 and 4 form a crossing of type $\mathrm{C}_{\mathrm{NN}}$.


Figure 6. A chord diagram for the permutation 65187243

Theorem 14. Let $T(k, a, b, c)$ be the set of permutation tableaux with $k$ rows, $(n-k)$ columns, a 0's, b 1's and c 2's. Let $M(k, a, b, c)$ be the set of all permutations $\pi \in \mathcal{S}_{n}$ with

- $k=\operatorname{wex}(\pi)$,
- $a=\mathrm{A}_{\mathrm{EE}}(\pi)+\mathrm{A}_{\mathrm{NN}}(\pi)+\mathrm{A}_{\mathrm{EN}}(\pi)$,
- $b=\mathrm{C}_{\mathrm{EE}}(\pi)+\mathrm{C}_{\mathrm{NN}}(\pi)+(n-k)$,
- $c=\mathrm{A}_{\mathrm{NE}}(\pi)$.

Then $|T(k, a, b, c)|=|M(k, a, b, c)|$. Moreover, the map $\Phi$ is a bijection from $T(k, a, b, c)$ to $M(k, a, b, c)$ such that the weak excedence bottoms of $\pi=\Phi\left(\mathcal{T}_{n}^{k}\right)$ are precisely the labels on the vertical edges of the partition path $P$ associated with $\mathcal{T}_{n}^{k}$.

Because of Lemma [5] is enough to prove that $\Phi$ is a bijection. This will be done using a lemma of Corteel [8, and Propositions 16 and 17 below.

Lemma 15 (Corteel [8]). Let $k, n, \mathrm{~A}_{\mathrm{EE}}(\pi), \mathrm{A}_{\mathrm{NN}}(\pi), \mathrm{A}_{\mathrm{EN}}(\pi), \mathrm{A}_{\mathrm{NE}}(\pi), \mathrm{C}_{\mathrm{EE}}(\pi)$, $\mathrm{C}_{\mathrm{NN}}(\pi)$ be as above. Then

$$
\mathrm{A}_{\mathrm{EE}}(\pi)+\mathrm{A}_{\mathrm{NN}}(\pi)+\mathrm{A}_{\mathrm{EN}}(\pi)+\mathrm{A}_{\mathrm{NE}}(\pi)+\mathrm{C}_{\mathrm{EE}}(\pi)+\mathrm{C}_{\mathrm{NN}}(\pi)=(k-1)(n-k)
$$

Proposition 16. If $\Phi\left(\mathcal{T}_{n}^{k}\right)=\pi$ then the number of 2 's in $\mathcal{T}_{n}^{k}$ is equal to the number of alignments of type $\mathrm{A}_{\mathrm{NE}}$ in $\pi$.

Proof. Recall that if $\pi=\Phi\left(\mathcal{T}_{n}^{k}\right)$ then the wexbottoms and the non-wexbottoms of $\pi$ correspond to the labels of the vertical and horizontal steps, respectively, in the south east border of the partition underlying $\mathcal{T}_{n}^{k}$. Note that the position of every 2 in $\mathcal{T}_{n}^{k}$ can be given by specifying the label of the edge above it and the edge to its left. The label $i$ of the edge above it will be a non-wexbottom, and the label $j>i$ of the edge to its left will be a weak excedance bottom. Since $j>i$, and $j$ is a wexbottom, and $i$ is a non-wexbottom, the pair $(i, j)$ is precisely an alignment of type $\mathrm{A}_{\mathrm{NE}}$. Conversely, any alignment of type $\mathrm{A}_{\mathrm{NE}}$ is a pair $(i, j)$ where $i<j$, and $i$ is a non-wexbottom, and $j$ is a wexbottom. This implies that $i$ is the label of a horizontal step, and $j$ is the label of a vertical step. The fact that $i<j$ implies that the box of the tableau indexed as above by $i$ and $j$ contains a 2 .

Proposition 17. Under the bijection $\Phi$, there is a one-to-one correspondence between black vertices in the diagram of the permutation tableau, and crossings of types $\mathrm{C}_{\mathrm{EE}}$ and $\mathrm{C}_{\mathrm{NN}}$ in the permutation.

Before proving this proposition, we will illustrate the main idea with an example.
Example 18. Again consider the tableau $\mathcal{T}_{n}^{k}$ in Figure 3 This tableau corresponds to the permutation $\pi=74836215$, which has a total of four crossings. The pair of chords $2 \mapsto 4,3 \mapsto 8$, and the pair of chords $1 \mapsto 7,3 \mapsto 8$ are crossings of type $\mathrm{C}_{\mathrm{EE}}$, while the pair of chords $6 \mapsto 2,8 \mapsto 5$ and the pair of chords $7 \mapsto 1,8 \mapsto 5$ are crossings of type $\mathrm{C}_{\mathrm{NN}}$.

Observe that under the bijection $\Phi$, the paths $(2 \rightarrow 4)$ and $(3 \rightarrow 8)$ intersect in a unique horizontal edge: the edge between the horizontally adjacent black and white vertices in row 3 of the tableau. We will associate this crossing to the black vertex which is the leftmost of the two vertices.

Similarly, the paths $(1 \rightarrow 7)$ and $(3 \rightarrow 8)$ intersect in a unique horizontal edge: the edge between the two leftmost black vertices in row 3 . We will associate this crossing to the black vertex which is the leftmost of the two vertices.

On the other hand, the paths $(6 \rightarrow 2)$ and $(8 \rightarrow 5)$ intersect in the vertical edge between the two black vertices in the column indexed by 6 . We will associate this crossing to the black vertex which is the bottom of these two vertices.

Simiarly, the paths $(7 \rightarrow 1)$ and $(8 \rightarrow 5)$ intersect in the vertical edge between the two vertices in the column indexed by 7 . We will associate this crossing to the black vertex which is the bottom of these two vertices.

In this way, the crossings of $\pi$ get associated bijectively with the black vertices of $\mathcal{T}_{n}^{k}$.

We will now make rigorous the idea that the above example suggests.
Proof. Recall that black vertices correspond to those 1's in a tableau that are not topmost in their columns. Let $D$ be the diagram of $\mathcal{T}_{n}^{k}$, and let $\pi=\left(a_{1}, \ldots, a_{n}\right)$ be $\Phi\left(\mathcal{T}_{n}^{k}\right)$. We will construct a map $\phi$ (induced by $\Phi$ ) which takes each crossing $(i, j)$ (where $i<j$ ) of type $\mathrm{C}_{\mathrm{EE}}$ or $\mathrm{C}_{\mathrm{NN}}$ in $\pi$ to a black vertex $d$ in $D$, and show that this is a bijection. The map $\phi$ is defined as follows. Let $(i, j)$ be a crossing of type $\mathrm{C}_{\mathrm{EE}}$ or $\mathrm{C}_{\mathrm{NN}}$. We claim that the paths $\left(i \rightarrow a_{i}\right)$ and $\left(j \rightarrow a_{j}\right)$ intersect in a unique edge. If that edge is horizontal, then let $d$ be the left vertex of the edge. If that edge is vertical, then let $d$ be the bottom vertex of the edge.

First we need to show that the paths $\left(i \rightarrow a_{i}\right)$ and $\left(j \rightarrow a_{j}\right)$ intersect in an edge. We will prove this when $(i, j)$ is a crossing of type $\mathrm{C}_{\mathrm{EE}}$; the proof for $\mathrm{C}_{\mathrm{NN}}$ is similar. Since $i<j$ and $a_{i}<a_{j}$, it is clear that the paths must cross each other at least once.

Consider the first point $x$ at which the path $\left(j \rightarrow a_{j}\right)$ intersects the path $\left(i \rightarrow a_{i}\right)$. We will show that the intersection here will contain an edge. Clearly this intersection must be in the zig-zag portion of the path $\left(i \rightarrow a_{i}\right)$. If we let $d_{1}, d_{2}, \ldots, d_{t}$ be the sequence of vertices encountered by the path $\left(i \rightarrow a_{i}\right)$ in its zig-zag portion, then, by construction of that path, there are no vertices in the diagram $D$ between any $d_{r}$ and $d_{r+1}$. Note that if the path $\left(j \rightarrow a_{j}\right)$ intersects the path $\left(i \rightarrow a_{i}\right)$ in only the point $x$ (rather than an edge containing $x$ ), then it is easy to see-using condition (2) in the definition of permutation tableaux - that $x$ must actually be a vertex in $D$, located between some $d_{r}$ and $d_{r+1}$. This is a contradiction.

Next, we show that the paths $\left(i \rightarrow a_{i}\right)$ and $\left(j \rightarrow a_{j}\right)$ intersect in a unique edge. If the two paths were to intersect a second time (and they may indeed intersect again in a vertex), then this intersection must take place in the zig-zag portion of both paths. Such a point $e$ of intersection must be approached via a south step by $\left(i \rightarrow a_{i}\right)$ and must be approached via an east step by $\left(j \rightarrow a_{j}\right)$. But then, according to the procedure defining $\Phi$, the path $\left(i \rightarrow a_{i}\right)$ will immediately turn east, and the path $\left(j \rightarrow a_{j}\right)$ will immediately turn south. Therefore this intersection is not an edge intersection.

We have thus shown that $\Phi$ induces a well-defined map from crossings to black vertices. We will now show that this map is a bijection by constructing its inverse. Namely, to each black vertex in $D$ we need to produce a crossing of type $\mathrm{C}_{\text {EE }}$ or $\mathrm{C}_{\text {NN }}$. We do this as follows. Given a black vertex $d$, there is a path $\left(i \rightarrow a_{i}\right)$ on $D$ which enters $d$ by going south, and then leaves $d$ going east. (It is easy to see that such a path exists by tracing backwards through the algorithm that defined the $\operatorname{map} \Phi$.)

If the path $\left(i \rightarrow a_{i}\right)$ is an excedance, then consider the unique path $\left(j \rightarrow a_{j}\right)$ which enters $d$ traveling west. This path must be a weak excedance, as it is only the paths of weak excedances which contain steps to the west. Moreover, $(i, j)$ must form a crossing of type $\mathrm{C}_{E E}$, since the two paths intersect in an edge (and we have seen that two paths which are both weak excedances may not intersect in an edge more than once).

On the other hand, if the path $\left(i \rightarrow a_{i}\right)$ is a non-excedance, then consider the unique path $\left(j \rightarrow a_{j}\right)$ which enters $d$ traveling north. Clearly this path must be a non-excedance, as it is only the paths of non-excedances which contain steps north. Moreover, $(i, j)$ must form a crossing of type $\mathrm{C}_{\mathrm{NN}}$, since the two paths must intersect in a unique edge.

Therefore $\phi$ is a bijection between the set of $\mathrm{C}_{\mathrm{EE}^{-}}$and $\mathrm{C}_{\mathrm{NN}}$-crossings in $\pi$, and the set of black vertices in $D$.

We now finish the proof of Theorem [14 with the following argument.
Proof. The first, third, and fourth parts of the theorem follow from Lemma 5 Proposition 16, and Proposition 17, respectively. It remains to prove the second part of the theorem. Let $m=\mathrm{A}_{\mathrm{EE}}(\pi)+\mathrm{A}_{\mathrm{NN}}(\pi)+\mathrm{A}_{\mathrm{EN}}(\pi)$. We know that $a+b+c=$ $k(n-k)$. By Lemma 15 we have that $\mathrm{A}_{\mathrm{EE}}(\pi)+\mathrm{A}_{\mathrm{NN}}(\pi)+\mathrm{A}_{\mathrm{EN}}(\pi)+\mathrm{A}_{\mathrm{NE}}(\pi)+$ $\mathrm{C}_{\mathrm{EE}}(\pi)+\mathrm{C}_{\mathrm{NN}}(\pi)=(k-1)(n-k)$. Therefore $m+c+b-(n-k)=(k-1)(n-k)$, which implies that $m+c+b=k(n-k)=a+b+c$, and hence $m=a$.

## 4. Permutation patterns

In this section we introduce necessary terminology and definitions that will be used in the next section, where we construct a bijection $\Psi: \mathcal{S}_{n} \longrightarrow \mathcal{S}_{n}$. This bijection proves the equidistribution of certain linear combinations of the statistics in Definition 13 (alignments and crossings) with certain linear combinations of generalized permutation patterns, which we define below. The composition of $\Psi$ and the bijection $\Phi$ from Section 2 then proves the equidistribution of our tableaux statistics (numbers of 0 's, 1 's and 2 's) with the pattern statistics to be defined here.

A classical permutation pattern $p=p_{1} p_{2} \ldots p_{k}$ is simply a permutation, and an occurrence of $p$ in a permutation $\pi=a_{1} a_{2} \ldots a_{n}$ is a subsequence $a_{i_{1}}, a_{i_{2}}, \ldots, a_{i_{k}}$ of $\pi$ (where $i_{1}<i_{2}<\cdots<i_{k}$ ) whose letters are in the same relative order as in $p$. For example, the permutation 416235 has two occurrences of the pattern (2-3-1), namely the subsequences 462 and 463 .

In the literature, the pattern $(2-3-1)$ is usually denoted simply by 231 . We write it here with dashes between consecutive letters in order to emphasize that there are no restrictions on the distance between the letters in a permutation that form an occurrence of the pattern. A generalized pattern is a pattern where some pairs of adjacent letters may lack a dash between them. Such an absence indicates that the corresponding letters must be adjacent in an occurrence of the pattern in a permutation. For example, the pattern $(2-31)$ occurs only once in 416235 , namely as 462 . In the subsequence 463, whose letters are in the same relative order as those of $(2-31)$, the last two letters are not adjacent in 416235 as required for an occurrence of $(2-31)$.

Generalized patterns were first introduced systematically by Babson and Steingrimsson in [1, but some instances had been treated previously in various contexts. For example, the pattern $(31-2)$ is implicit in [11] and in 18 (where the similar patterns $(2-31),(13-2)$ and $(2-13)$ are also treated), and dashless patterns, such as (123), appeared already in 13.

The reason for writing patterns in parentheses is that we will consider them as functions from the set of permutations to the natural numbers, where the value of a pattern $p$ on a permutation $\pi$ is the number of different occurrences of $p$ in $\pi$. For example, if $\pi=416235$, as above, then $(2-3-1) \pi=2$ and $(2-31) \pi=1$.

It is easy to see that there are exactly twelve different patterns of length 3 with one dash. Six of these will be considered here, namely $(1-32),(2-31),(3-21)$, $(21-3),(31-2)$ and $(32-1)$. These are all the patterns of length 3 with one dash whose two letters not separated by a dash are in decreasing order.

A descent in a permutation $\pi=a_{1} a_{2} \ldots a_{n}$ is an $i$ such that $a_{i}>a_{i+1}$. We say that $a_{i}$ is a descent top and $a_{i+1}$ a descent bottom. The set of descent tops is denoted DESTOPSET, and the set of descent bottoms DESBOTSET. Moreover, we let DESTOPSUM be the sum of the elements of DESTOPSET, and likewise for DESBOTSUM.

We now define the linear combinations of patterns whose joint distribution on permutations matches the distribution of 0's, 1's and 2's on permutation tableaux.

Definition 19. Given a permutation $\pi$, let

$$
\begin{aligned}
& a(\pi)=(21-3) \pi+(3-21) \pi+(31-2) \pi-\binom{\operatorname{des} \pi}{2} \\
& b(\pi)=(2-31) \pi+n-1-\operatorname{des} \pi \\
& c(\pi)=(1-32) \pi+(32-1) \pi-\binom{\operatorname{des} \pi}{2} .
\end{aligned}
$$

It is important to note that since we will be considering the quadruple statistic consisting of $a, b, c$ and the number of descents, the terms $\binom{\operatorname{des} \pi}{2}$ and $n-1-\operatorname{des} \pi$ in the above definition only effect a shift of the statistics involved, but not an essential modification. For example, the bistatistic (des, $b$ ) can be seen as a 2dimensional array of numbers, with the $k$-th entry in the $i$-th row consisting of the number of permutations $\pi$ with $i$ descents and with $b(\pi)=k$. Thus, replacing $b=(2-31)+n-1-\operatorname{des} \pi$ by $b^{\prime}=(2-31)$, we would only shift the nonzero entries in each row $n-1-i$ steps to the left. A similar, but more complicated, statement is true for the quadruple statistic (des, $a, b, c$ ). The upshot of this is that if we delete the terms $\binom{\operatorname{des} \pi}{2}$ and ( $n-1-\operatorname{des} \pi$ ) from the definitions of $a, b$, and $c$, the resulting quadruple statistic changes only in that we are disregarding some "initial" zeros.

It is also important to note that the sum $(1-32) \pi+(32-1) \pi$ in $c$ is equal to Desbotsum $\pi-\operatorname{des} \pi$. Namely, for each descent ...yx... in $\pi$, the pattern $(1-32)$ counts the letters to the left of the descent that are smaller than its descent bottom $(x)$. The letters to the right of the descent, and smaller than $x$, are counted by $(32-1)$, so clearly we are counting all letters in $\pi$ that are smaller than $x$. Analogously, the sum $(21-3) \pi+(3-21) \pi$ in $a$ equals the sum of $n-t$ over all descent tops $t$ in $\pi$.

This leaves $(31-2)$ in $a$, which sums the left embracing numbers in $\pi$, so called because (31-2) counts, for each letter $x$ in $\pi$, the descents to the left of $x$ that embrace $x$, that is, where the letters of the descent are larger and smaller, respectively, than $x$. Analogously, the pattern $(2-31)$ in $b$ sums the right embracing numbers in $\pi$.

To be more precise, we define the right embracing number of each letter $\ell$ in $\pi$, denoted $\operatorname{REmbr}(\ell)$, as the number of descents $\ldots y x \ldots$ to the right of $\ell$ in $\pi$ such that $x<\ell<y$.

Lemma 20. Let des be the number of descents in a permutation $\pi$, and let $a(\pi)$, $b(\pi)$ and $c(\pi)$ be as above. Then

$$
a(\pi)+b(\pi)+c(\pi)=(\operatorname{des}+1)(n-\operatorname{des}-1)
$$

Proof. Each of the patterns involved in $a+b+c$ counts certain letters to the left or to the right of each descent in $\pi$. Together they count, for each descent in $\pi$, all the letters in $\pi$ not belonging to the descent itself. There are, of course, $n-2$ such letters for each descent. Thus, the sum of all the patterns in $a(\pi)+b(\pi)+c(\pi)$ is des $\cdot(n-2)$. Completing the proof now only requires a routine calculation.

## 5. Another bijection

We now describe the construction of a bijection $\Psi: \mathcal{S}_{n} \longrightarrow \mathcal{S}_{n}$ that takes a permutation $\pi$ to a permutation $\tau$ such that the set of descent tops in $\pi$ determines the set of weak excedance tops in $\tau$ and the set of descent bottoms in $\pi$ determines the set of weak excedance bottoms in $\tau$. Moreover, the right embracing number of $i$ in $\pi$ becomes $\mathrm{C}_{\mathrm{EE}}(i)$ in $\tau$ if $i$ is a wexbottom in $\tau$ and becomes $\mathrm{C}_{\mathrm{NN}}(i)$ in $\tau$ otherwise.

This bijection is based on the same idea as the central bijection in 4, which in turn was shown, in [4], to be essentially equivalent to several seemingly different bijections in the literature, due to Foata-Zeilberger [11, Françon-Viennot [12, de Médicis-Viennot 14] and Simion-Stanton [18], respectively. More precisely, our bijection $\Psi$ here uses the same data as the bijection $\Phi$ in [4], and these data (descent tops and bottoms and right embracing numbers) completely determine a permutation. However, our bijection $\Psi$ uses this data in a different way than $\Phi$ in [4], and produces a different permutation, turning descent tops into weak excedance tops etc., whereas $\Phi$ in [4] turns descent tops into excedance tops, etc.

Recall the biword notation for permutations. For example, we write the permutation 31524 as

$$
\left(\begin{array}{lllll}
3 & 1 & 5 & 2 & 4 \\
1 & 2 & 3 & 4 & 5
\end{array}\right) .
$$

In order to construct $\Psi(\pi)$ (where $\pi \in \mathcal{S}_{n}$ ), we first construct two biwords, $\binom{f^{\prime}}{f}$ and $\binom{g^{\prime}}{g}$, and then form the biword $\tau^{\prime}=\left(\begin{array}{cc}f^{\prime} & g^{\prime} \\ f & g\end{array}\right)$ by concatenating $f$ and $g$, and $f^{\prime}$ and $g^{\prime}$, respectively. The words $f, f^{\prime}, g, g^{\prime}$ are defined as follows (we will prove later, in Theorem 21] that this is possible):

- The letters of $f$ consist of the set obtained by adding 1 to each of the descent bottoms in $\pi$ and then adjoining the letter 1 . The letters of $f$ are ordered increasingly. These letters will be the wexbottoms of $\tau$.
- The letters of $g$ consist of the set obtained from the non-descent bottoms in $\pi$ by removing the letter $n$ and adding 1 to the remaining letters. The letters of $g$ are ordered increasingly. These letters will be the non-wexbots of $\tau$.
- The letters of $f^{\prime}$ consist of the set obtained by subtracting 1 from each of the descent tops of $\pi$ and then adjoining the letter $n$. The letters of $f^{\prime}$ are ordered so that, for $a$ in $f^{\prime}, \mathrm{C}_{\mathrm{EE}}(a)$ in $\tau$ is the right embracing number of $a$ in $\pi$. (Observe that $\mathrm{C}_{\mathrm{EE}}(a)$ only depends on the relative order of the wextops in $\tau$, together with their corresponding wexbottoms.) These letters will be the wextops of $\tau$.
- The letters of $g^{\prime}$ consist of the set obtained by removing 1 from the set of non-descent tops in $\pi$ and then subtracting 1 from the remaining letters. The letters of $g^{\prime}$ are ordered so that, for $a$ in $g^{\prime}, \mathrm{C}_{\mathrm{NN}}(a)$ is the right embracing number of $a$ in $\pi$. These letters will be the non-wextops of $\tau$.
Rearranging the columns of $\tau^{\prime}$, so that the bottom row is in increasing order, we obtain the permutation $\tau=\Psi(\pi)$ as the top row of the rearranged biword. Before we prove that this can always be done in the way described, we give an example.

Let $\pi=215896374$. Then $\pi$ has

```
Descent bottoms: 1 3 4 6 Non-descent bottoms: 2 5 7 8 9
Descent tops: 2679 Non-Descent tops: 1 3 4 5 8
```

The right embracing numbers are 2 for 5,1 for 6 and 8 , and 0 for all others:

$$
\begin{gathered}
21-5-8-963-74 \\
21
\end{gathered}
$$

We construct a permutation with the corresponding wexbottoms and wextops, and the corresponding nonzero values for $\mathrm{C}_{\mathrm{EE}}$ and $\mathrm{C}_{\mathrm{NN}}$, that is, with $\mathrm{C}_{\mathrm{EE}}(5)=2$ and $\mathrm{C}_{\mathrm{NN}}(6)=\mathrm{C}_{\mathrm{NN}}(8)=1$. First, the wexbottoms are obtained by adding 1 to each descent bottom, and adjoining 1, which is always a wexbottom. The wextops are obtained by subtracting 1 from the descent tops, and adjoining $n$, which is always a wextop. Thus, we get


That is, $f$ is the word $12457, g$ is the word 3689 , $f^{\prime}$ will be some permutation (to be determined) of the letters 15689 , and $g^{\prime}$ will be some permutation (to be determined) of the letters 2347.

We construct the permutation in two parts, one for the weak excedances, the other for the non-weak excedances.

Now, the definitions of $\mathrm{C}_{\mathrm{EE}}$ and $\mathrm{C}_{\mathrm{NN}}$ are such that $\mathrm{C}_{\mathrm{EE}}$ only applies to pairs of weak excedances, and $\mathrm{C}_{\mathrm{NN}}$ only to pairs of non-weak excedances. We first construct the weak excedance part of the permutation, by deciding where to place each of the wextops, in the places given by the wexbots:

$$
\overline{1} \overline{2} \overline{4} \overline{5} \overline{7}
$$

We start from the right, in place 7 , which has a 0 associated to it (since REMBR $(7)=$ 0 in $\pi$ ). We need to put there the smallest number among the wextops that is at least as large as 7 (otherwise, $\mathrm{C}_{\mathrm{EE}}(7)$ would exceed 0 in the resulting permutation). This is the number 8:

$$
\begin{array}{rrrrr} 
& & & 8 \\
- & 2 & 4 & 5 & 7
\end{array}
$$

This leaves the wextops $1,5,6,9$. The next place, 5 , has a 2 associated to it (since $\operatorname{REMBR}(5)=2$ in $\pi$ ), so we have to put there a wextop that it is bigger than exactly two of the remaining wextops that are at least as big as 5 . This forces us
to make this 9 (and the two remaining wextops between 5 and 9 in size are 5 and $6)$.

We continue in this way until we have placed all the wextops, in such a way that the values of $\mathrm{C}_{\mathrm{EE}}$ for the remaining places are 0 , since 5 is the only letter among the wexbottoms here with a nonzero right embracing number in $\pi$ :

$$
\frac{1}{1} \frac{6}{1} \frac{5}{2} \frac{5}{4} \frac{9}{5} \frac{8}{7}
$$

The non-wex part is done in a similar way, but starting from the left, and we get:

$$
\begin{array}{llll}
2 & \frac{3}{2} & \frac{4}{7} \\
3 & \frac{7}{6} & 9
\end{array}
$$

Observe that $\mathrm{C}_{\mathrm{NN}}(6)=\mathrm{C}_{\mathrm{NN}}(8)=1$ and $\mathrm{C}_{\mathrm{NN}}(3)=\mathrm{C}_{\mathrm{NN}}(9)=0$, as required. Concatenating these two biwords, and sorting the columns to get the bottom row in increasing order, the permutation we obtain is $\Psi(215896374)=162593847$.

We now prove that the above procedure can always be carried out in the way described.

Theorem 21. Let $\mathrm{DB}^{\prime}(\pi)$ be the set obtained from $\operatorname{DESBOTSET}(\pi)$ by adding 1 to each of its elements, and adjoining the letter 1.

Let $\mathrm{DT}^{\prime}(\pi)$ be the set obtained from $\operatorname{DESTOPSET}(\pi)$ by subtracting 1 from each of its elements, and adjoining the letter $n$.

For a permutation $\tau$ let $\mathrm{WB}(\tau)$ be the set of weak excedance bottoms of $\tau$ and let $\mathrm{WT}(\tau)$ be the set of weak excedance tops of $\tau$.

The map $\Psi$ described above is well defined, and has the following properties, where $\tau=\Psi(\pi)$ :
(i) $\mathrm{WB}(\tau)=\mathrm{DB}^{\prime}(\pi)$,
(ii) $\mathrm{WT}(\tau)=\mathrm{DT}^{\prime}(\pi)$,
(iii) $\mathrm{C}_{\mathrm{EE}}(\tau)+\mathrm{C}_{\mathrm{NN}}(\tau)=\operatorname{REMBR}(\pi)$.

Moreover, $\Psi$ is a bijection.
Proof. Recall that $\mathrm{C}_{\mathrm{EE}}(i)=0$ unless $i$ is an excedance bottom, and that $\mathrm{C}_{\mathrm{NN}}(i)=0$ unless $i$ is a non-excedance bottom.

Let the letters of $\operatorname{WB}(\tau)$ be $b_{1}, b_{2}, \ldots, b_{\ell}$, ordered so that $b_{\ell}<\cdots<b_{2}<b_{1}$. Look at the largest letter in $\operatorname{WB}(\tau)$, that is, $b_{1}$. Suppose the embracing number of $b_{1}$ in $\pi$ is $e_{1}$. Then there are at least $e_{1}$ descent tops in $\pi$ that are larger than $b_{1}$. Thus, by the construction of $\mathrm{WT}(\tau)$ from the descent top set of $\pi$, there are at least $e_{1}+1$ elements $x$ in $\mathrm{WT}(\tau)$ such that $b_{1} \leq x$. So, we can find an element $t_{1}$ in $\mathrm{WT}(\tau)$ such that $\mathrm{WT}(\tau)$ contains precisely $e_{1}$ elements $x$ satisfying $b_{1} \leq x \leq t_{1}$. Setting $\tau\left(b_{1}\right)=t_{1}$ guarantees that $\mathrm{C}_{\mathrm{EE}}\left(b_{1}\right)=e_{1}$ in $\tau$.

Look next at $b_{2}$, the second largest element in $\mathrm{WB}(\tau)$. Suppose its embracing number in $\pi$ is $e_{2}$. There are then at least $e_{2}+1$ elements $x$ in WT $(\tau)$ such that $b_{2} \leq x$. However, one of these elements is $t_{1}$, which has already been placed to the right of place $b_{2}$ in $\tau$, and so $t_{1}$ cannot contribute to $\mathrm{C}_{\mathrm{EE}}(b)$ in $\tau$. But, $b_{1}+1$ is a descent bottom in $\pi$ and so its corresponding descent top, $d$, must be larger than $b_{1}+1$ and hence larger than $b_{2}$. Thus, $b_{2}$ cannot be embraced by the descent $\ldots d\left(b_{1}+1\right) \ldots$ in $\pi$. Hence, the embracing number of $b_{2}$ in $\pi$ can be at most one
less than the number of elements $x$ in $\mathrm{WT}(\tau)$ satisfying $b_{2} \leq x$. We can therefore find an element $t_{2} \neq t_{1}$ in $\mathrm{WT}(\tau)$ such that precisely $e_{2}$ of the elements $x$ in $\mathrm{WT}(\tau)$ apart from $t_{1}$ satisfy $b_{2} \leq x \leq t_{2}$.

An analogous argument shows that the embracing number of $b_{i}$ in $\pi$ can be at most $N+1-i$, where $N$ is the number of elements $x$ in wT $(\tau)$ with $b_{i} \leq x$. We can thus place each of the elements $t_{i}$ of $\mathrm{WT}(\tau)$ in $\tau$ so that $\mathrm{C}_{\mathrm{EE}}\left(b_{i}\right)$ in $\tau$ equals $\operatorname{REMBR}\left(b_{i}\right)$ in $\pi$.

In particular, each placement according to the above algorithm will result in the creation of a weak excedance. Namely, clearly the $k$-th largest wexbottom is smaller than or equal to the $k$-th largest wextop. Thus, by induction, since we consider the wexbots in decreasing order, the largest wextop unused at each stage of the algorithm is greater than or equal to the wexbottom being considered.

To construct the subword of $\tau$ consisting of non-wextops, we proceed in a similar way, except that we start from the smallest non-wexbottom. At each stage, for the non-wexbottom $b_{i}$ we find a non-wextop $d$ that satisfies $d<x<b_{i}$ for precisely $e$ elements $x$ among the remaining non-wextops, where $e=\operatorname{REmbR}\left(b_{i}\right)$ in $\pi$. The argument showing that this is always possible, and that each placement results in a non-weak excedance, is analogous to the case of the weak excedance subword, and is omitted.

To prove that $\Psi$ is a bijection, it suffices to show that it is injective, since it is a map from $\mathcal{S}_{n}$ to itself. Let $\sigma_{1}$ and $\sigma_{2}$ be two permutations with $\Psi\left(\sigma_{1}\right)=\Psi\left(\sigma_{2}\right)$. From the definition of $\Psi$ it is clear that $\sigma_{1}$ and $\sigma_{2}$ must have the same descent tops and descent bottoms and also the same right embracing numbers for each letter. It follows from the proof of Theorem 4 in [4] p. 249] that a permutation is uniquely determined by its sets of descent bottoms and tops, respectively, together with the right embracing numbers of its letters. Thus, we must have $\sigma_{1}=\sigma_{2}$.

In fact, the proof of Theorem 4 in [4] can be applied directly to our situation with trivial modifications, and yields a description of the inverse of $\Psi$. In short, given the sets of weak excedance tops and weak excedance bottoms, respectively, of a permutation $\pi$, and the numbers $\left(\mathrm{C}_{\mathrm{EE}}(i)+\mathrm{C}_{\mathrm{NN}}(i)\right)$, there is a unique permutation $\Psi^{-1}(\pi)$ with the corresponding descent tops and bottoms, respectively, and whose vector of right embracing numbers $\operatorname{REMBR}(i)$ equals the vector of numbers $\mathrm{C}_{\mathrm{EE}}(i)+$ $\mathrm{C}_{\mathrm{NN}}(i)$. The permutation $\Psi^{-1}(\pi)$ can be constructed in the exact same way as is done in the proof of Theorem 4 in [4], except that the data we start with come from weak excedance tops and bottoms (and $\mathrm{C}_{\mathrm{EE}}$ and $\mathrm{C}_{\mathrm{NN}}$ ), instead of excedance tops and bottoms and the inversion numbers defined in 4.

Recall that $\operatorname{wexbotsum}(\pi)$ is the sum of all the wexbottoms in $\pi$. The following corollary of Theorem 21] requires only straightforward calculations.

## Corollary 22.

$$
\begin{aligned}
\text { WEXtopsum } \Psi(\pi) & =\text { DESTOPSUM } \pi+n-\operatorname{des} \pi \\
\text { WEXbotsum } \Psi(\pi) & =\text { DESBOTSUM } \pi+\operatorname{des} \pi+1
\end{aligned}
$$

We will use the following lemma in our proofs of the equidistribution results between our tableaux statistics and permutation statistics.

## Lemma 23.

$$
\begin{align*}
& \mathrm{A}_{\mathrm{EN}}(\pi)=\binom{n}{2}-\binom{n-\mathrm{wex}}{2}+\text { wex }- \text { WEXTOPSUM }  \tag{1}\\
& \mathrm{A}_{\mathrm{NE}}(\pi)=\text { WEXBOTSUM }-\binom{\text { wex }}{2} . \tag{2}
\end{align*}
$$

Proof. Equation (11) in the statement of the lemma is equivalent to

$$
\mathrm{A}_{\mathrm{EN}}(\pi)+\text { WEXTOPSUM }- \text { wex }=\binom{n}{2}-\binom{n-\mathrm{wex}}{2}
$$

We will show that the sum in the left-hand-side above counts all pairs $(i, j)$, with $1 \leq i<j \leq n$, except those for which neither of $i$ and $j$ is a weak excedance.

Recall that $A_{\text {EN }}$ counts the pairs $(i, j)$ such that $j \leq a_{j}<a_{i}<i$. Each such pair can be described as consisting of a wextop $w$ in the permutation, and a non-wextop that is larger than $w$ and to the right of $w$.

We can interpret (WExTOPSUM - wex) as the sum, over all wextops, of the size of the wextop, minus 1 . Counting this for each wextop $w$ can be done by counting all the letters in the permutation that are strictly smaller than $w$. Doing this for all wextops is equivalent to counting all pairs of letters in the permutation that either consist of two wextops, or a wextop and a non-wextop, where the wextop is the larger of the two.

Therefore $A_{\text {En }}$ and (wextopsum - wex) together count all pairs of letters in the permutation, except those consisting of two non-wextops. (Observe that it is impossible to have a non-wextop $z$ and a wextop $w$ such that $z$ is left of $w$ and $z>w$.) The total number of pairs of letters in a permutation in $\mathcal{S}_{n}$ is of course $\binom{n}{2}$, and the number of pairs of non-wexbots is $\binom{n-$ wex }{2} , which completes the proof. Equation (2) in the statement of the lemma can be proved in a similar manner.

We can now prove the main results about the equidistribution implied by the bijection $\Psi$.

Theorem 24. Let $\sigma=\Psi(\pi)$. We have

$$
\begin{align*}
\operatorname{des} \pi & =\mathrm{wex} \sigma-1,  \tag{3}\\
(31-2) \pi & =\mathrm{A}_{\mathrm{EE}}(\sigma)+\mathrm{A}_{\mathrm{NN}}(\sigma),  \tag{4}\\
(21-3) \pi+(3-21) \pi-\binom{\operatorname{des} \pi}{2} & =\mathrm{A}_{\mathrm{EN}}(\sigma)  \tag{5}\\
(2-31) \pi & =\mathrm{C}_{\mathrm{EE}}(\sigma)+\mathrm{C}_{\mathrm{NN}}(\sigma),  \tag{6}\\
(1-32) \pi+(32-1) \pi-\binom{\operatorname{des} \pi}{2} & =\mathrm{A}_{\mathrm{NE}}(\sigma) . \tag{7}
\end{align*}
$$

Proof. Equations (3) and (6) in the statement of the theorem follow directly from Theorem [21] since $(2-31) \pi$ is the sum of the right embracing numbers for all the letters in $\pi$. We will prove (5) here; the proof of (7) is analogous and is omitted. Having done this, Equation (4) follows from the other four identites in the present theorem, together with Lemmas 15 and 20 and routine calculations.

To prove Equation (5), observe that

$$
(21-3) \pi+(3-21) \pi=n \cdot \operatorname{des} \pi-\text { DESTOPSUM } \pi
$$

This is because $(21-3) \pi+(3-21) \pi$ counts the letters in $\pi$ larger than the descent top $b$ for each descent $\ldots b a \ldots$ in $\pi$. According to Corollary 22] the right-hand-side in the equation above can be rewritten as follows:

$$
n \cdot \operatorname{des} \pi-\operatorname{DESTOPSUM}=n \cdot \operatorname{des} \pi-\text { WEXTOPSUM } \sigma+n-\operatorname{des} \pi
$$

By Lemma 23 this is equal to

$$
n \cdot \operatorname{des} \pi+\left(\operatorname{A} \operatorname{EN}(\sigma)-\operatorname{wex} \sigma-\binom{n}{2}+\binom{n-\operatorname{wex} \sigma}{2}\right)+n-\operatorname{des} \pi
$$

which, in turn, is equal to

$$
\mathrm{A}_{\mathrm{EN}}(\sigma)+n \cdot \operatorname{des} \pi-(\operatorname{des} \pi+1)-\binom{n}{2}+\binom{n-(\operatorname{des} \pi+1)}{2}+n-\operatorname{des} \pi
$$

To show that this last expression is equal to $\mathrm{A}_{\mathrm{EN}}(\sigma)+\binom{\operatorname{des} \pi}{2}$ is straightforward.
Note that Equations (4) and (5) in Theorem 24 together imply that

$$
(31-2) \pi+(21-3) \pi+(3-21) \pi-\binom{\operatorname{des}}{2}=\mathrm{A}_{\mathrm{EE}}(\sigma)+\mathrm{A}_{\mathrm{NN}}(\sigma)+\mathrm{A}_{\mathrm{EN}}(\sigma)
$$

This last equation, together with Theorem 14 leads to the following corollary
Corollary 25. Let $T(k, a, b, c)$ be the set of permutation tableaux with $k$ rows and $(n-k)$ columns, which are filled with precisely a 0's, b 1's and c 2's. Let P $(k, a, b, c)$ be the set of all permutations $\pi \in \mathcal{S}_{n}$, such that

- $k-1=\operatorname{des}(\pi)$,
- $a=[(31-2)+(21-3)+(3-21)] \pi-\binom{\operatorname{des} \pi}{2}$,
- $b=(2-31) \pi+n-1-\operatorname{des} \pi$,
- $c=[(1-32)+(32-1)] \pi-\binom{\operatorname{des} \pi}{2}$.

Then $|T(d, a, b, c)|=|P(d, a, b, c)|$.

## 6. EnUMERATION RESULTS

One nice application of permutation tableaux is that they facilitate enumeration of permutations according to various statistics. This is because permutation tableaux satisfy a rather simple recurrence, which we now explain.

Fix a partition $\lambda=\left(\lambda_{1}, \ldots, \lambda_{k}\right)$. Let $F_{\lambda}(p, q)$ be the polynomial in $p$ and $q$ such that the coefficient of $p^{s} q^{t}$ is the number of valid fillings of the Young diagram $Y_{\lambda}$ which contain $s$ 0's and $t$ 1's. As Figure 7 illustrates, there is a simple recurrence for $F_{\lambda}(p, q)$. (Note, however, that this is not the same as the recurrence given for J-diagrams in 20.)

Explicitly, any valid filling of $\lambda$ is obtained in one of the following ways:

- inserting a column whose bottom entry is 1 and whose other entries are 0 after the $\left(\lambda_{k}-1\right)$ st column of a valid filling of $\left(\lambda_{1}-1, \lambda_{2}-1, \ldots, \lambda_{k}-1\right)$;
- adding a 1 to the end of the bottom row of a valid filling of the shape $\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k-1}, \lambda_{k}-1\right)$;
- adding an all-zero row of length $\lambda_{k-1}$ to a valid filling of $\left(\lambda_{1}, \ldots, \lambda_{k-1}\right)$.

Thus, we have the following recurrence.

## Proposition 26.

$$
\begin{aligned}
& F_{\lambda}(p, q)=p^{k-1} q F_{\left(\lambda_{1}-1, \lambda_{2}-1, \ldots, \lambda_{k}-1\right)}(p, q)+q F_{\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k-1}, \lambda_{k}-1\right)}(p, q) \\
&+p^{\lambda_{k}} F_{\left(\lambda_{1}, \ldots, \lambda_{k-1}\right)}(p, q)
\end{aligned}
$$



Figure 7. Recurrence for $F_{\lambda}(p, q)$
It is straightforward to compute $F_{\lambda}(p, q)$ when $k$ (the number of rows of $\lambda$ ) is small. Here are the first two formulas.

Proposition 27.

$$
\begin{aligned}
F_{\left(\lambda_{1}\right)}(p, q) & =q^{\lambda_{1}} \\
F_{\left(\lambda_{1}, \lambda_{2}\right)}(p, q) & =-q^{\lambda_{1}-1} p^{\lambda_{2}+1}+q^{\lambda_{1}-1}[2]_{p, q}^{\lambda_{2}+1}
\end{aligned}
$$

In the above expression, $[2]_{p, q}$ is the $p, q$-analog of 2 . Recall that the $p, q$-analog of the number $n$ is $p^{n-1}+p^{n-2} q+p^{n-3} q^{2}+\cdots+q^{n-1}$, denoted $[n]_{p, q}$.

Let $D_{k, n}(p, q, r):=\sum_{\lambda} F_{\lambda}(p, q) r^{k(n-k)-|\lambda|}$, where $\lambda$ ranges over all partitions contained in a $k \times(n-k)$ rectangle. By Theorem 14] $D_{k, n}(p, q, r)$ enumerates permutations according to the number of weak excedances, several kinds of alignments, and crossings. And by Corollary 25] $D_{k, n}(p, q, r)$ enumerates permutations according to the number of descents and occurrences of various generalized patterns. Therefore it would be nice to get an explicit expression for $D_{k, n}(p, q, r)$, for example by solving explicitly for $F_{\lambda}(p, q)$ and then by summing over partitions $\lambda$ contained in a $k \times(n-k)$ rectangle.

For fixed small $k$, it is not too difficult to compute the generating function $D_{k}(p, q, r, x):=\sum_{n} D_{k, n}(p, q, r) x^{n}$. Here are the first few formulas. Note that it is easy to determine what the denominator should be for $D_{k}(p, q, r, x)$, but the numerator is significantly more complicated.

## Proposition 28.

$$
\begin{aligned}
& D_{1}(p, q, r, x)=\frac{x}{1-q x} \\
& D_{2}(p, q, r, x)=\frac{x^{2}}{(1-p q x)(1-q r x)\left(1-q[2]_{p, q} x\right)} \\
& D_{3}(p, q, r, x)= \\
& \frac{x^{3}\left(1+p q^{2} x-p^{3} q^{2} r x^{2}-2 p^{2} q^{3} r x^{2}-p q^{4} r x^{2}\right)}{\left(1-p^{2} q x\right)(1-p q r x)\left(1-q r^{2} x\right)\left(1-p q[2]_{p, q} x\right)\left(1-q r[2]_{p, q} x\right)\left(1-q[3]_{p, q} x\right)}
\end{aligned}
$$

One can derive these formulas by either using the methodology outlined above (i.e. by summing $F_{\lambda}(p, q)$ ), or else by translating the problem of enumerating permutation tableaux into a problem about enumerating certain weighted lattice paths, and then by enumerating these lattice paths. In order to sketch the latter method,
let us define a $b a d$ zero in a permutation tableau to be a 0 which lies directly underneath some 1. Note that if some column $C$ in a permutation tableau $\mathcal{T}_{n}^{k}$ contains a bad zero in the $r$ th row, then every column to the left of $C$ must also contain a zero in the $r$ th row.

In the lattice path method for enumeration of permutation tableaux, we associate to each permutation tableau $\mathcal{T}_{n}^{k}$ a weighted lattice path $L=\left\{L_{i}\right\}_{i=1}^{n}$ consisting of $n$ steps in the plane, which must be of the following types: $(1,1)$ (a northeast step), $(1,0)$ (an east step), and $(1,-j)$, where $1 \leq j \leq k-1$ (a southeast step). Each step $L_{i}$ in the lattice path represents the step $P_{i}$ in the partition path $\left\{P_{i}\right\}_{i=1}^{n}$. (Recall that the partition path follows the shape of the partition $Y_{\lambda}$ and travels from the northeast corner to the southwest corner of the $k \times(n-k)$ rectangle containing $\left.\mathcal{T}_{n}^{k}.\right)$ The steps $(1,1)$ in $L$ correspond to vertical steps in the partition path, and have weight $x$. A step $(1,0)$ in $L$ corresponds to a horizontal step in the partition path such that the corresponding column $C$ of $\mathcal{T}_{n}^{k}$ does not introduce any bad zeros (except those that were forced by bad zeros to the right of $C$ ). Such a step has weight $p^{a} q^{b} r^{c} x$, where $a, b$, and $c$ are the numbers of 0 's, 1 's, and 2 's, respectively, in column $C$. (Note that $a+b+c=k$.) Finally, a step $(1,-j)$ in $L$ corresponds to a horizontal step in the partition path such that the corresponding column $C$ of $\mathcal{T}_{n}^{k}$ introduces exactly $j$ new bad zeros (that were not forced by bad zeros in columns to the right of $C$ ). As before, such a step has weight $p^{a} q^{b} r^{c} x$, where $a, b$, and $c$ are the numbers of 0 's, 1's, and 2's, respectively, in column $C$. Observe that the height of any point in the lattice path $L$ is equal to the number of boxes of the corresponding column of $\mathcal{T}_{n}^{k}$ which can be filled with either a 0 or a 1 . By associating weighted lattice paths to permutation tableaux in this way, we can facilitate computation of the generating functions $D_{k}(p, q, r, x)$ for small $k$.

Now we will give complete results about a certain specialization of $D_{k, n}(p, q, r)$. Let $E_{k, n}(q):=D_{k, n}(1, q, 1)$. An explicit formula for $E_{k, n}(q)$ was found in [20]; the proof utilized a recurrence similar to that in Proposition 26

Theorem 29 (Williams [20]).

$$
E_{k, n}(q)=q^{n-k^{2}} \sum_{i=0}^{k-1}(-1)^{i}[k-i]^{n} q^{k i-k}\left(\binom{n}{i} q^{k-i}+\binom{n}{i-1}\right)
$$

In the above formula, the notation $[k-i]$ refers to the $q$-analog of the number $k-i$, that is, $1+q+q^{2}+\cdots+q^{k-i-1}$.

The polynomials above have many nice properties. It was observed in 20 that if one renormalizes $E_{k, n}(q)$ by defining $\hat{E}_{k, n}(q):=q^{k-n} E_{k, n}(q)$, then $\hat{E}_{k, n}(q)$ is a new $q$-analog of the Eulerian numbers (distinct from Carlitz' classical $q$-analog of the Eulerian numbers [6]). Furthermore, $\hat{E}_{k, n}(q)$ specializes at $q=-1,0,1$ to the binomial coefficients, the Narayana numbers, and the Eulerian numbers. Additionally, $\hat{E}_{k, n}(q)=\hat{E}_{n+1-k, n}(q)$. It was shown more recently by Corteel [8] that the polynomials $\hat{E}_{k, n}(q)$ naturally relate to the ASEP model in statistical physics.

Theorem 29 together with Corollary 25 implies the following result.

Corollary 30. The number of permutations in $\mathcal{S}_{n}$ with $k-1$ descents and $m$ occurrences of the pattern $(2-31)$ is equal to the coefficient of $q^{m}$ in

$$
\hat{E}_{k, n}(q)=q^{-k^{2}} \sum_{i=0}^{k-1}(-1)^{i}[k-i]^{n} q^{k i}\left(\binom{n}{i} q^{k-i}+\binom{n}{i-1}\right) .
$$

This result was first conjectured by the authors of this paper, and first proved by Corteel [8]. The formula $\hat{E}_{k, n}(q)$ is the first known polynomial expression which gives the complete distribution of a permutation pattern of length greater than 2 (the two cases of length 2 correspond to the Eulerian numbers and the coefficients of $[n]$ !, respectively).

The generating function for the polynomials $\hat{E}_{k, n}(q)$ has been expressed in two ways: as a formal power series and as a continued fraction. That is, it can be shown [20] that $\hat{E}(q, x, y):=\sum_{n, k} \hat{E}_{k, n}(q) y^{k} x^{n}$ is equal to

$$
\sum_{i=0}^{\infty} \frac{y^{i}\left(q^{2 i+1}-y\right)}{q^{i^{2}+i+1}\left(q^{i}-q^{i+1}[i] x+[i] x y\right)}
$$

Additionally, Corteel [8 used results of Clark, Steingrímsson, and Zeng [4] to show the following:

Theorem 31 (Corteel [8]).

$$
\hat{E}(q, x, y)=\frac{1}{1-b_{0} x-\frac{\lambda_{1} x^{2}}{1-b_{1} x-\frac{\lambda_{2} x^{2}}{1-b_{2} x-\frac{\lambda_{3} x^{2}}{\ddots}}},}
$$

where $b_{n}=y[n+1]_{q}+[n]_{q}, \lambda_{n}=y[n]_{q}^{2}$, and $[n]_{q}=1+q+\cdots+q^{n-1}$.

### 6.1. The Euler-Mahonian distribution and Carlitz' $q$-analog of the Eulerian numbers

Recall that the Euler-Mahonian distribution is the joint distribution of the number of descents and the major index for permutations in $S_{n}$. The major index is the sum of the places of the descents in a permutation.

The generating function for this joint distribution is given by Carlitz' $q$-Eulerian polynomials $B_{n, k}(q)$ 6], which one can define by

$$
\begin{equation*}
B_{n, k}(q)=\sum_{\pi} q^{\mathrm{MAJ}(\pi)-\binom{k}{2}} \tag{8}
\end{equation*}
$$

where the sum is over permutations in $S_{n}$ which have $k-1$ descents. Note that one subtracts $\binom{k}{2}$ from the exponent because when the number of descents of $\pi$ is $k-1$, the quantity maj $(\pi)$ is at least $\binom{k}{2}$.

Analogous to the Eulerian numbers, the coefficients of the $q$-Eulerian polynomial satisfy the recurrence 6]

$$
\begin{equation*}
B_{n, k}(q)=[k+1] B_{n-1, k}(q)+q^{k}[n-k] B_{n-1, k-1}(q), \tag{9}
\end{equation*}
$$

subject to the initial conditions $B_{0, k}(q)=1$ for $k=0$, and $B_{0, k}(q)=0$ otherwise.

We will now show that Carlitz' $q$-analog is simply a specialization of the polynomial $D_{k, n}(p, q, r)$.

Proposition 32. With the notation above, we have that $D_{k, n}(p, 1,1)=B_{n, k}(p)$.
Proof. We will prove this by showing that if (rows, zeros) is the bistatistic counting rows and 0's in tableaux, then (rows, zeros) has the same distribution as the pair $\left(\right.$ des +1, MAJ $-\binom{$ des +1}{2}$)$.

Note first that the statistic $a$ in Corollary 25, when stripped of $\binom{$ des }{2} , has the same distribution as the statistic

$$
\begin{equation*}
(1-32)+(32-1)+(2-31) \tag{10}
\end{equation*}
$$

This is because the statistic in (10) is obtained by taking the reverse complement of the statistic $(31-2)+(21-3)+(3-21)$, that is, by reversing each of the patterns and then replacing each letter $i$ by $4-i$. Doing the same with each permutation in $\mathcal{S}_{n}$ (with 4 replaced by $n+1$ ) is a bijection from $\mathcal{S}_{n}$ to itself, and this bijection clearly proves the equidistribution of $(31-2)+(21-3)+(3-21)$ with $(1-32)+(32-1)+(2-31)$, even when each statistic is taken jointly with the number of descents (which is invariant under reverse complement). The statistic $(1-32)+(32-1)+(2-31)+$ des is equal to the statistic MAK, as pointed out in [1, and it was shown by Foata and Zeilberger 11 that (des, MAK) has the EulerMahonian distribution, that is, the same distribution as (des, MAJ). ${ }^{1}$ It follows that $\left(\right.$ des, maJ $-\binom{$ des +1}{2}$)=\left(\right.$ des, MAJ $-\operatorname{des}-\binom{$ des }{2}$)$ has the same distribution as (des, MAK - des $-\binom{$ des }{2} ), which, in turn, has the same distribution as (des, $(1-32)+$ $(32-1)+(2-31)-\binom{$ des }{2}$)$. Hence, by Corollary 25 (des +1 , MAJ $-\binom{$ des +1}{2} ) has the same distribution as (rows, zeros).

Therefore $D_{k, n}(p, 1,1)=B_{n, k}(p)$.
Thus the polynomials $D_{k, n}(p, q, r)$ generalize both the classical $q$-analog of the Eulerian numbers and the new $q$-analog of the Eulerian numbers found in 20 .

## 7. Open problems and other remarks

When this section of the paper was first written, it contained a collection of open problems that we thought were worth studying. We are happy to report that in the nine months following our posting of this paper on the electronic arXiv, there was a great deal of progress on our open problems by Alexander Burstein [3], Sylvie Corteel [7], Niklas Eriksen [10], Astrid Reifegerste 17, and Xavier Viennot [19]. We will list the full set of open problems below, with remarks at the end about the progress that has been made on them.
(1) Find an explicit expression for $D_{k, n}(p, q, r)$.
(2) Can one prove Proposition 32 by checking the analogous recurrence (9) for permutation tableaux?
(3) We say that a 1 in a permutation tableau is essential if it is the topmost one in its column or the leftmost 1 in its row. A tableau is determined by its essential 1's: all the other 1's are determined by these, because of condition

[^1](2) in the definition of a permutation tableau. What do the essential 1's correspond to in the corresponding permutation?

We conjecture-based on experimental evidence for $n$ up to 9 -that the distribution of permutation tableaux according to the number of essential 1 's is equal to that of permutations according to $(n-c)$, where $n$ is the length of the permutations and $c$ the number of cycles when each permutation is written in standard cycle form. This distribution is the same as that for Left-to-Right minima. Moreover, we conjecture that the joint distribution of tableaux according to the number of rows and the number of essential 1's equals that of permutations according to ( $n-1-\operatorname{des}$ ) and ( $n-\mathrm{LR}$ ), where des is the number of descents, and LR the number of Left-to-Rightminima. The bistatistic (des $+1, L R$ ), in turn, has the same distribution as the number of weak excedances and the number of cycles of a permutation, when written in standard cycle form.
(4) The number of 0 's in a tableau corresponds to the total number of occurrences of the patterns $(3-21),(21-3)$ and $(31-2)$. It is easy to see that these patterns have the same distributions as $(1-32),(32-1)$ and $(2-31)$, respectively. To prove this, simply reverse each permutation in $\mathcal{S}_{n}$. Can we partition the 0 's in a tableau into two sets, one corresponding to occurrences of $(3-21)+(21-3)$ and the other to occurrences of $(31-2)$ ? Observe that the first one of these sets would correspond to descent tops and the second one to left embracings. Thus, these sets would be symmetric counterparts of 2's and 1's respectively, although this symmetry is not transparent in the tableaux.
(5) The reflection of a permutation tableau $\mathcal{T}_{n}^{k}$ in its north-west/south-east diagonal yields a permutation tableau if and only if $\mathcal{T}_{n}^{k}$ has a 1 in each row. That is equivalent to the associated permutation being fixed point free. Which permutation is associated to the reflected tableau (that tableau is also fixed point free because it has a 1 in each row)?
(6) A permutation tableau $T$ must have at least one 1 in each column. If it has only this minimum number of 1 's, then the corresponding permutation, that is, $\Psi^{-1}(\Phi(T))$, has no occurrences of the pattern $(2-31)$. It has been shown (see (5), that permutations avoiding this pattern are enumerated by the Catalan numbers. Is there a bijection from these tableaux to any of the well known objects enumerated by Catalan numbers, such as Dyck paths?
(7) Find a better description of the bijection $\Psi$, and of the composition of maps $\Psi^{-1} \circ \Phi$ 。

As mentioned earlier, much progress on these problems has been made by Alexander Burstein [3, Sylvie Corteel [7, Niklas Eriksen [10], Astrid Reifegerste [17, and Xavier Viennot [19].

Independently, Alexander Burstein [3] and Niklas Eriksen 10 have solved open problems 3] 5] Additionally, Sylvie Corteel [7] and Astrid Reifegerste [17] have (independently) solved problem 6. Xavier Viennot 19] has found a new bijection from permutation tableaux to permutations which answers the open problems 2 and also 6. Additionally, it seems to be related to the recent work of the second author [9] on the asymmetric exclusion process.

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[^1]:    ${ }^{1}$ Actually, the statistic $(1-32)+(32-1)+(2-31)$ is a slight variation on MAK, as defined by Foata and Zeilberger, but is easily seen to have the same distribution when taken together with the number of descents. Foata and Zeilberger's MAK is actually equal to the statistic called makl in 1 Table 1]

