A NEW PROOF OF VÁZSONYI'S CONJECTURE

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ABSTRACT. We present a self-contained proof that the number of diameter pairs among n points in Euclidean 3-space is at most $2n-2$. The proof avoids the ball polytopes used in the original proofs by Grünbaum, Heppes and Straszewicz. As a corollary we obtain that any three-dimensional diameter graph can be embedded in the projective plane.

Let S be a set of n points of diameter D in $\mathbb{R}^d$. Define the *diameter graph* on S by joining all *diameters*, i.e., point pairs at distance D. The following theorem was conjectured by Vázsonyi, as reported in [2]. It was subsequently independently proved by Grünbaum [3], Heppes [4] and Straszewicz [7].

Theorem 1. *The number of edges in a diameter graph on $n \geq 4$ points in $\mathbb{R}^3$ is at most $2n-2$.*

All three proofs (see [6, Theorem 13.14]) use the ball polytope obtained by taking the intersection of the balls of radius D centred at the points. However, these ball polytopes do not behave the same as ordinary polytopes. In particular, their graphs need not be 3-connected, as shown by Kupitz, Martini and Perles in [5], where a detailed study of the ball polytopes associated to the above theorem is made. The proof presented here avoids the use of ball polytopes.

Theorem 2. *Any diameter graph in $\mathbb{R}^3$ has a bipartite double covering that has a centrally symmetric drawing on the 2-sphere.*

In fact, each point $x \in S$ will correspond to an antipodal pair of points x_r and x_b on the sphere, with x_r coloured red and x_b blue. Each edge xy of the diameter graph will correspond to two antipodal edges $x_r y_b$ and $x_b y_r$ on the sphere, giving a properly 2-coloured graph on $2n$ vertices. The drawing will be made such that no edges cross. By Euler's formula there will be at most $4n-4$ edges, hence at most $2n-2$ edges in the diameter graph. By identifying opposite points of the sphere we further obtain:

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Corollary 3. *Any diameter graph in $\mathbb{R}^3$ can be embedded in the projective plane such that all odd cycles are noncontractible.*

Therefore, any two odd cycles intersect, and we regain the following theorem of Dol'nikov [1]:

Corollary 4. *Any two odd cycles in a diameter graph on a finite set in $\mathbb{R}^3$ intersect.*

Proof of Theorem 2. Without loss we assume from now on that $D = 1$. Let $\mathbb{S}^2$ denote the sphere in $\mathbb{R}^3$ with centre the origin and radius 1. We may repeatedly remove all vertices of degree at most 1 in the diameter graph. Since such vertices can easily be added later, this is no loss of generality. For each $x \in S$, let $R(x)$ be the intersection of $\mathbb{S}^2$ with the cone generated by $\{y - x : xy \text{ is a diameter}\}$. Each $R(x)$ is a convex spherical polygon with great circular arcs as edges. (If x has degree 2 then $R(x)$ is an arc). Colour $R(x)$ red and $B(x) := -R(x)$ blue. Assume for the moment the following two properties of these polygons:

Lemma 1. *If $x \neq y$, then $R(x)$ and $R(y)$ are disjoint.*

Lemma 2. *If $R(x)$ and $B(y)$ intersect, then xy is a diameter and $R(x) \cap B(y) = \{y - x\}$.*

For each $x \in S$ we choose any x_r in the interior of $R(x)$ and let $x_b = -x_r$. (If $R(x)$ is an arc we let x_r be in its relative interior.) Draw arcs inside $R(x)$ from x_r to all the vertices of $R(x)$, as well as antipodal arcs from x_b to the vertices of $B(x)$. This gives a centrally symmetric drawing of a 2-coloured double covering of the diameter graph. By Lemmas 1 and 2 no edges cross, and the theorem follows. ☐

The following proofs of Lemmas 1 and 2 are dimension independent, which gives a double covering on $\mathbb{S}^{d-1}$ of any diameter graph in $\mathbb{R}^d$.

Lemma 3. *Let $x_1, \dots, x_k$ and $\sum_{i=1}^k \lambda_i x_i$ be unit vectors in $\mathbb{R}^d$, with all $\lambda_i \geq 0$. Suppose that for some $y \in \mathbb{R}^d$, $\|y - x_i\| \leq 1$ for all $i = 1, \dots, k$. Then $\|y - \sum_{i=1}^k \lambda_i x_i\| \leq 1$.*

Proof. By the triangle inequality,

$$1 \leq \|\sum_{i=1}^{k} \lambda_i x_i\| \leq \sum_{i=1}^{k} \lambda_i. \tag{1}$$

Expanding $\|y - x_i\|^2 \leq 1$ by inner products,

$$-2\langle x_i, y\rangle \leq -\|y\|^2. \tag{2}$$

Therefore,

$$
\begin{aligned}
\|y - \sum_{i=1}^{k} \lambda_i x_i\|^2 &= \|y\|^2 - 2\sum_{i=1}^{k} \langle x_i, y\rangle + 1 \\
&\le \Big(1 - \sum_{i=1}^{k} \lambda_i\Big)\|y\|^2 + 1 \quad \text{by (2)} \\
&\le 1 \qquad \text{by (1).}
\end{aligned}
$$

□

Proof of Lemma 1. Let the neighbours of x be $x + x_i$, and the neighbours of y be $y + y_j$, with the x_i and y_j unit vectors. Suppose that

$$\sum_i \lambda_i x_i = \sum_j \mu_j y_j \in R(x) \cap R(y) \text{ with } \lambda_i, \mu_j \ge 0.$$

Since $\|x + x_i - y\| \le 1$ for all i, Lemma 3 gives

$$\|x + \sum_i \lambda_i x_i - y\| \le 1.$$

Similarly, Lemma 3 applied to $\|x - y - y_j\| \le 1$ gives

$$\|x - y - \sum_j \mu_j y_j\| \le 1.$$

By the triangle inequality,

$$
\begin{aligned}
2 &= \|2\sum_i \lambda_i x_i\| \\
&= \|(x + \sum_i \lambda_i x_i - y) - (x - y - \sum_j \mu_j y_j)\| \\
&\le \|x + \sum_i \lambda_i x_i - y\| + \|x - y - \sum_j \mu_j y_j\| \\
&\le 2.
\end{aligned}
$$

Since we have equality throughout, $x + \sum_i \lambda_i x_i - y$ and $-x + y + \sum_j \mu_j y_j$ are unit vectors in the same direction, hence are equal, which gives $x = y$. □

Proof of Lemma 2. Since $\|x_i - x_j\| \le 1$ for all i, j, $R(x)$ is properly contained in an open hemisphere of $\mathbb{S}^2$, hence $R(x) \cap B(x) = \emptyset$. Thus without loss of generality, $x \ne y$. As before, let the neighbours of x be $x + x_i$, and the neighbours of y be $y + y_j$, with the x_i and y_j unit vectors. Suppose that $\sum_i \lambda_i x_i = -\sum_j \mu_j y_j \in R(x) \cap B(y)$ with

$\lambda_i, \mu_j \geq 0$. For a fixed j we have that $\|x + x_i - y - y_j\| \leq 1$ for all i. Lemma 3 then gives

$$\|x + \sum_i \lambda_i x_i - y - y_j\| \leq 1 \quad \text{for all } j.$$

Again by Lemma 3,

$$\|x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j\| \leq 1.$$

By the triangle inequality,

$$\begin{aligned} 2 &= \|2 \sum_i \lambda_i x_i\| \\ &= \|(x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j) + (y - x)\| \\ &\leq \|x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j\| + \|y - x\| \\ &\leq 2. \end{aligned}$$

Since we have equality throughout, $x + \sum_i \lambda_i x_i - y - \sum_j \mu_j y_j$ and $y - x$ are unit vectors in the same direction, hence are equal, which gives $x + \sum_i \lambda_i x_i = y$ and $R(x) \cap B(y) = \{y - x\}$. □

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