A Complete Classification of Ternary Self-Dual Codes of Length 24

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Abstract

Ternary self-dual codes have been classified for lengths up to 20. At length 24, a classification of only extremal self-dual codes is known. In this paper, we give a complete classification of ternary self-dual codes of length 24 using the classification of 24-dimensional odd unimodular lattices.

1 Introduction

As described in [16], self-dual codes are an important class of linear codes for both theoretical and practical reasons. It is a fundamental problem to classify self-dual codes of modest length and determine the largest minimum weight among self-dual codes of that length. By the Gleason–Pierce theorem, there are nontrivial divisible self-dual codes over \mathbb{F}_q for q = 2, 3 and 4 only, where \mathbb{F}_q denotes the finite field of order q, and this is one of the reasons why much work has been done concerning self-dual codes over these fields.

A code C over \mathbb{F}_3 is called ternary. All codes in this paper are ternary. A code C of length n is said to be *self-dual* if $C = C^{\perp}$, where the dual code C^{\perp} of C is defined as $C^{\perp} = \{x \in \mathbb{F}_3^n | x \cdot y = 0 \text{ for all } y \in C\}$ under the standard inner product $x \cdot y$. A self-dual code of length n exists if and only if $n \equiv 0 \pmod{4}$. It was shown in [13] that the minimum weight d of a self-dual code

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of length n is bounded by $d \leq 3[n/12] + 3$. If d = 3[n/12] + 3, then the code is called *extremal*. Two codes C and C' are *equivalent* if there exists a monomial matrix P with $C' = C \cdot P = \{xP \mid x \in C\}$. The *automorphism* group Aut(C) of C is the group of all monomial matrices P with $C = C \cdot P$.

All self-dual codes of length ≤ 20 have been classified [4, 12, 15]. At length 24, the complete classification has not been done yet, although it was shown by Leon, Pless and Sloane [11] that there are exactly two inequivalent extremal self-dual codes, namely, the extended quadratic residue code QR_{24} and the Pless symmetry code P_{24} . Moreover, it was shown in [11] that there are at least 13 and 96 inequivalent self-dual codes with minimum weights 3 and 6, respectively.

Applying the classification method in [8] to length 24, we give a classification of self-dual codes of length 24 with minimum weights 3 and 6, which completes the classification of self-dual codes of length 24.

Theorem 1. There are exactly 166 inequivalent ternary self-dual [24, 12, 6] codes. There are exactly 170 inequivalent ternary self-dual [24, 12, 3] codes.

Generator matrices of all self-dual codes of length 24 can be obtained electronically from [7]. All computer calculations in this paper were done by MAGMA [3].

2 Preliminaries

An *n*-dimensional (Euclidean) lattice L is *integral* if $L \subseteq L^*$, where the dual lattice L^* is defined as $L^* = \{x \in \mathbb{R}^n | (x, y) \in \mathbb{Z} \text{ for all } y \in L\}$ under the standard inner product (x, y). A lattice L with $L = L^*$ is called *unimodular*. The norm of a vector x is (x, x). The *minimum norm* of L is the smallest norm among all nonzero vectors of L. A unimodular lattice L is *even* if all vectors of L have even norms, and *odd* if some vector has an odd norm. The kissing number of L is the number of vectors of L with minimum norm. Two lattices L and L' are *isomorphic*, denoted $L \cong L'$, if there exists an orthogonal matrix A with $L' = L \cdot A$. The *automorphism group* $\operatorname{Aut}(L)$ of Lis the group of all orthogonal matrices A with $L = L \cdot A$.

If C is a self-dual code of length n and minimum weight d, then

$$A_3(C) = \frac{1}{\sqrt{3}} \{ (x_1, \dots, x_n) \in \mathbb{Z}^n \mid (x_1 \text{ mod } 3, \dots, x_n \text{ mod } 3) \in C \}$$

is a unimodular lattice with minimum norm $\min\{3, d/3\}$. This construction of lattices from codes is called Construction A.

Lemma 2. Let C be a ternary self-dual code of length n. Let α_i (resp. β_{3i}) be the number of vectors of norm i in $A_3(C)$ (resp. codewords of weight 3i in C) (i = 1, 2). Then

$$\alpha_2 = \beta_6 + 3\beta_3 \text{ and } \alpha_1 = \beta_3.$$

Proof. The straightforward proof is omitted.

The weight distribution of a self-dual code C of length 24 is determined by the numbers β_3 , β_6 (see [11, Table III]). Hence the weight distribution of C can be determined by the numbers α_1 , α_2 in $A_3(C)$.

There are 155 non-isomorphic 24-dimensional odd unimodular lattices with minimum norm 2, and there are 117 non-isomorphic 24-dimensional odd unimodular lattices with minimum norm 1 [2] (see also [6, Table 2.2]). We denote the *i*-th 24-dimensional odd unimodular lattice with minimum norm ≥ 2 in [6, Table 17.1] by $L_{24,i}$ (i = 1, 2, ..., 156). The lattices $L_{24,i}$ (i = 2, 3, ..., 156) are the 155 non-isomorphic 24-dimensional odd unimodular lattices with minimum norm 2. A 24-dimensional unimodular lattice with minimum norm 1 except \mathbb{Z}^{24} can be constructed as $M_i \oplus \mathbb{Z}^{24-i}$ where M_i is an *i*-dimensional unimodular lattice with minimum norm ≥ 2 . Here we denote a 24-dimensional unimodular lattice $M_{i,j} \oplus \mathbb{Z}^{24-i}$ with minimum norm 1 by $L_{i,j}$ where $M_{i,j}$ is the *j*-th *i*-dimensional unimodular lattice with minimum norm ≥ 2 in [6, Table 16.7]. All non-isomorphic unimodular lattices with minimum norm ≥ 2 can be constructed as neighbors of the standard lattices for dimensions up to 24 (see [1, Tables I, II and III]).

A set $\{f_1, \ldots, f_n\}$ of *n* vectors f_1, \ldots, f_n in an *n*-dimensional lattice *L* with $(f_i, f_j) = 3\delta_{ij}$ is called a 3-frame of *L*, where δ_{ij} is the Kronecker delta. Clearly, $A_3(C)$ has a 3-frame. Conversely, every self-dual code can be obtained from a 3-frame of some unimodular lattice. Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a 3-frame of *L*. Consider the mapping

$$\pi_{\mathcal{F}} : \frac{1}{3} \oplus_{i=1}^{n} \mathbb{Z}f_{i} \to \mathbb{F}_{3}^{n}$$
$$\pi_{\mathcal{F}}(x) = ((x, f_{i}) \bmod 3)_{1 \le i \le n}$$

Then Ker $\pi_{\mathcal{F}} = \bigoplus_{i=1}^{n} \mathbb{Z} f_i \subset L$, so the code $C = \pi_{\mathcal{F}}(L)$ satisfies $\pi_{\mathcal{F}}^{-1}(C) = L$. This implies $A_3(C) \cong L$ and every code C with $A_3(C) \cong L$ is obtained as $\pi_{\mathcal{F}}(L)$ for some 3-frame \mathcal{F} of L.

Lemma 3 ([8]). Let L be an n-dimensional integral lattice, and let $\mathcal{F} = \{f_1, \ldots, f_n\}$, $\mathcal{F}' = \{f'_1, \ldots, f'_n\}$ be 3-frames of L. Then the codes $\pi_{\mathcal{F}}(L)$ and $\pi_{\mathcal{F}'}(L)$ are equivalent if and only if there exists an orthogonal matrix $P \in \operatorname{Aut}(L)$ such that $\{\pm f_1, \ldots, \pm f_n\} \cdot P = \{\pm f'_1, \ldots, \pm f'_n\}$.

In order to establish the nonexistence of a 3-frame for some lattices, the shadows of lattices are considered. Let $L = L_0 \cup L_2$ be an odd unimodular lattice with even sublattice L_0 . Then L_0^* can be written as a union of cosets of L_0 : $L_0^* = L_0 \cup L_2 \cup L_1 \cup L_3$. The shadow S of L is defined to be $S = L_1 \cup L_3$ [5].

Lemma 4. Let $L = L_0 \cup L_2$ be a 24-dimensional odd unimodular lattice with shadow $S = L_1 \cup L_3$. Let v be a vector of L_2 with (v, v) = 3. If there exist vectors $a \in L_1$ and $b \in L_3$ such that (a, a) = (b, b) = 2, (a, b) = 1/2 and v = a - b, then v does not belong to any 3-frame of L.

Proof. Suppose that $\{f_1, \ldots, f_{24}\}$ is a 3-frame of L and $v = f_1$. By Lemma 2 in [9], a vector $a \in L_1$ can be written as

$$a = \frac{1}{6} \sum_{i=1}^{24} a_i f_i, \quad (a_i \in 1 + 2\mathbb{Z}).$$

Since

$$2 = (a, a) = \frac{1}{12} \sum_{i=1}^{24} a_i^2 \ge \frac{1}{12} \sum_{i=1}^{24} 1 = 2,$$

we have $a_i = \pm 1$ for all $1 \le i \le 24$. Then

$$\frac{3}{2} = (a, a) - (a, b) = (a, v) = \frac{a_1}{2} = \pm \frac{1}{2}.$$

This is a contradiction.

Let L be a 24-dimensional unimodular lattice and let V be the set of pairs $\{v, -v\}$ with $(v, v) = 3, v \in L$ satisfying the condition that there do not exist $a \in L_1$ and $b \in L_3$ such that (a, a) = (b, b) = 2, (a, b) = 1/2 and v = a - b. We define the simple undirected graph Γ , whose set of vertices is the set V and two vertices $\{v, -v\}, \{w, -w\} \in V$ are adjacent if (v, w) = 0. It follows that the 3-frames are precisely the 24-cliques in the graph Γ . It is clear that $\operatorname{Aut}(L)$ acts on the graph Γ as automorphisms, and Lemma 3 implies that the $\operatorname{Aut}(L)$ -orbits on the set of 24-cliques of Γ are in one-to-one correspondence with the equivalence classes of codes C satisfying $A_3(C) \cong L$. Therefore,

the classification of such codes reduces to finding a set of representatives of 24-cliques of Γ up to the action of Aut(L). This computation was performed by MAGMA [3], the results were then converted to 3-frames, and then to self-dual codes of length 24. In this way, by considering 3-frames of all 24-dimensional odd unimodular lattices with minimum norms 2 and 1, we have all inequivalent self-dual codes of length 24 with minimum weights 6 and 3, respectively.

Note that the graph Γ is an empty graph for the lattice $L_{i,j}$, where

 $\begin{aligned} (i,j) = & (20,1), (22,1), (23,2), (23,3), (24,94), (24,125), (24,126), \\ & (24,135), (24,136), (24,137), (24,143), (24,147), (24,148), \\ & (24,149), (24,151), (24,152), (24,153), (24,155), (24,156). \end{aligned}$

In particular, none of these lattices have a 3-frame.

3 Decomposable self-dual codes of length 24

As described in [11, Table II], there are 27 inequivalent decomposable selfdual codes D_i (i = 1, 2, ..., 27) of length 24. A decomposable self-dual code can be written as $C_1 \oplus C_2$ where C_1 and C_2 are self-dual codes of lengths 20 and 4, or both C_1 and C_2 are indecomposable self-dual codes of length 12. We denote the unique self-dual [4,2,3] code in [12, Table 1] by E_4 . We also denote the extended ternary Golay [12,6,6] code by G_{12} and the unique indecomposable [12,6,3] code by $4C_3(12)$ [12, Table 1]. We denote the self-dual codes of length 20 by $C_{20,1}, \ldots, C_{20,24}$ according to the order given in [15, Tables II and III]. In Table 1, we list the number β_3 of the codewords of weight 3 and the order $\# \operatorname{Aut}(D_i)$ of the automorphism group for D_i $(i = 1, 2, \ldots, 27)$.

There is a unique decomposable self-dual [24, 12, 6] code. This can also be established from the classification of odd unimodular lattices, by the following lemma.

Lemma 5. Let C be a ternary self-dual code of length n and minimum weight at least 6. Then C is decomposable if and only if $A_3(C)$ is decomposable.

Proof. If C is decomposable, then obviously $A_3(C)$ is decomposable. Conversely, suppose that $A_3(C) = L \oplus L'$ for some sublattices L, L'. Since $A_3(C)$ has minimum norm ≥ 2 , both L and L' have minimum norms ≥ 2 . Let x

i	(C_1, C_2)	β_3	$\#\operatorname{Aut}(D_i)$	i	(C_1, C_2)	β_3	$#\operatorname{Aut}(D_i)$
1	$(C_{20,1}, E_4)$	48	8806025134080	15	$(C_{20,15}, E_4)$	12	1327104
2	$(C_{20,2}, E_4)$	32	41278242816	16	$(C_{20,16}, E_4)$	12	1161216
3	$(C_{20,3}, E_4)$	24	126127964160	17	$(C_{20,17}, E_4)$	10	110592
4	$(C_{20,4}, E_4)$	24	1146617856	18	$(C_{20,18}, E_4)$	10	331776
5	$(C_{20,5}, E_4)$	20	477757440	19	$(C_{20,19}, E_4)$	8	184320
6	$(C_{20,6}, E_4)$	18	310542336	20	$(C_{20,20}, E_4)$	8	24576
7	$(C_{20,7}, E_4)$	16	198180864	21	$(C_{20,21}, E_4)$	8	491520
8	$(C_{20,8}, E_4)$	20	644972544	22	$(C_{20,22}, E_4)$	8	92160
9	$(C_{20,9}, E_4)$	18	89579520	23	$(C_{20,23}, E_4)$	8	497664
10	$(C_{20,10}, E_4)$	16	15925248	24	$(C_{20,24}, E_4)$	8	99532800
11	$(C_{20,11}, E_4)$	14	985374720	25	(G_{12}, G_{12})	0	72260812800
12	$(C_{20,12}, E_4)$	14	2985984	26	$(4C_3(12), 4C_3(12))$	16	7739670528
13	$(C_{20,13}, E_4)$	12	19906560	27	$(4C_3(12), G_{12})$	8	11824496640
14	$(C_{20,14}, E_4)$	12	2985984				

Table 1: Decomposable self-dual codes of length 24

be a vector of norm 3 in $A_3(C)$. Then x can be written as either $(x_1, 0)$ or $(0, x_2)$ where $x_1 \in L$ and $x_2 \in L'$. Hence, every 3-frame of $A_3(C)$ is a union of those of L and of L'. Therefore, C is decomposable.

If C is a decomposable self-dual [24, 12, 6] code, then Lemma 5 implies that $A_3(C)$ is a decomposable odd unimodular lattice with minimum norm 2. The only such lattices are $L_{24,153} = E_8 \oplus (D_8 \oplus D_8)^+$ and $L_{24,154} = D_{12}^+ \oplus D_{12}^+$. Clearly, $L_{24,153}$ has no 3-frame, since the lattice E_8 does not have one. Since the extended ternary Golay [12, 6, 6] code G_{12} is the unique code C with $A_3(C) \simeq D_{12}^+$, the decomposable code D_{25} is the unique code C with $A_3(C) \simeq L_{24,154}$, up to equivalence.

4 Self-dual [24, 12, 6] codes

In this section, we give a classification of self-dual [24, 12, 6] codes by considering 3-frames of 24-dimensional odd unimodular lattices with minimum norm 2.

By the approach described in Section 2, we completed the classification of self-dual [24, 12, 6] codes. In Table 2, we list the number N_i of inequivalent self-dual [24, 12, 6] codes C with $A_3(C) \cong L_{24,i}$. The columns # Aut in the table list the orders of automorphism groups. From Table 1, the only decomposable self-dual [24, 12, 6] code is $D_{25} = G_{12} \oplus G_{12}$ where the automorphism group order is marked in Table 2. We remark that there is no self-dual [24, 12, 6] code C with $A_3(C) \cong L_{24,i}$ unless i is listed in Table 2.

By Lemma 2, the weight enumerator of a self-dual [24, 12, 6] code C is determined by the kissing number of the lattice $A_3(C)$. Since $A_3(C)$ is isomorphic to one of the 155 lattices whose kissing numbers are given in [1, Table III], we do not give the weight enumerator of C or the number of codewords of weight 6 in C for codes C given in Table 2.

Let C_i denote the set of all inequivalent indecomposable self-dual codes of length 24 containing exactly 2i codewords of weight 3. The following values

(1)
$$T_i = \sum_{C \in \mathcal{C}_i} \frac{1}{\# \operatorname{Aut}(C)} \quad (i = 0, 1, \dots, 8)$$

were determined theoretically in [11, Table I], without finding the set C_i . We now have the set C_0 as

$$\mathcal{C}_0 = \{QR_{24}, P_{24}\} \cup \mathcal{C} \setminus \{D_{25}\},\$$

where C is the set of the 166 codes given in Table 2, and we verified that the value T_0 obtained from our classification coincided with the value determined in [11, Table I]. This shows that there is no other self-dual code with minimum weight $d \ge 6$.

We investigate the previously known self-dual [24, 12, 6] codes. In [10] and [11], self-dual codes generated by the rows of Hadamard matrices of order 24 were studied. The 60 inequivalent Hadamard matrices of order 24 give exactly two inequivalent extremal self-dual codes [11] and exactly seven inequivalent self-dual [24, 12, 6] codes $C(H1), \ldots, C(H7)$ which are generated by the matrices $H1, \ldots, H7$, respectively [10]. We verified that all of these seven codes appear in the present classification, and in Table 2 we mark the orders of automorphism groups for these codes.

Some properties of automorphism groups of self-dual [24, 12, 6] codes were given in [11]. For example, only the primes 2, 3, 5, 7, 11 and 13 can divide the orders of the automorphism groups. A self-dual [24, 12, 6] code with a trivial automorphism group was given in [11, Fig. 2] and the authors conjectured that the code is unique. Our classification shows that the conjecture is true. We mark the code by Tri in Table 2. The code $g_{10} + \eta_{14}$ given in [11, Fig. 4] is the unique self-dual [24, 12, 6] code with an automorphism of order 7 and the code $g_{11} + p_{13}$ given in [11, Fig. 5] is the unique self-dual [24, 12, 6] code with an automorphism of order 13 [11]. In Table 2 we mark the orders of automorphism groups for these two codes. From our classification, only

i	N_i	$\#\operatorname{Aut}$	i	N_i	$\#\operatorname{Aut}$
2	2	64, 64	32	2	$3456^{C(H7)}, 31104^{C(H2)}$
3	5	8, 12, 24, 24, 3072	33	2	48,576
4	4	16, 16, 32, 512	34	1	192
5	10	4, 4, 8, 8, 12, 16, 24, 32, 32, 72	35	1	64
6	2	$960^{C(H4)}, 3072^{C(H6)}$	36	1	512
7	11	$2^{Tri}, 4, 4, 4, 4, 4, 8, 8, 16, 16, 32$	37	1	256
8	8	4, 8, 12, 16, 16, 24, 32, 64	40	1	256
9	8	16, 24, 48, 64, 64, 96, 192, 384	41	1	256
10	12	4, 4, 4, 8, 12, 16, 16, 16, 24, 24, 48, 384	43	1	5184
11	5	12, 16, 48, 128, 1728	44	2	384, 3456
12	4	16, 48, 48, 1728	46	1	512
13	11	4, 4, 4, 8, 8, 8, 16, 16, 16, 16, 16	47	1	1728
14	1	64	50	1	8192
15	4	16, 32, 32, 2048	53	1	1024
16	4	8, 8, 16, 32	59	1	2304
17	7	8, 32, 32, 32, 32, 96, 1152	61	1	17280
18	4	16, 32, 64, 384	62	1	5184
19	6	32, 48, 48, 192, 648, 864	63	1	1728
20	1	384	64	1	34560
21	2	16, 64	65	1	3456
22	3	8, 16, 128	73	1	20736
23	2	64, 128	74	1	$49152^{C(H3)}$
24	4	32,64,64,256	78	2	$20736, 241920^{g_{10}+\eta_{14}}$
25	2	2304, 3072	87	1	276480
26	1	288	102	1	746496
27	2	288, 576	114	1	622080
28	1	96	115	1	746496
29	2	128, 768	130	1	$8294400^{C(H5)}$
30	2	256, 768	141	1	$88957440^{g_{11}+p_{13}}$
31	2	96, 1296	154	1	$72260812800^{D_{25},C(H1)}$

Table 2: Ternary self-dual [24, 12, 6] codes

 $g_{11} + p_{13}$ and the decomposable code D_{25} are self-dual [24, 12, 6] codes with an automorphism of order 11.

5 Self-dual [24, 12, 3] codes

In this section, we give a classification of self-dual [24, 12, 3] codes by considering 3-frames in 24-dimensional odd unimodular lattices with minimum norm 1. These are the lattices \mathbb{Z}^{24} and $L_{i,j}$ ($i \leq 23$).

In Table 3, we list the number N of inequivalent self-dual [24, 12, 3] codes C with $A_3(C) \cong L$ for $L = \mathbb{Z}^{24}, L_{i,j}$. The column # Aut in Table 3 lists the orders of automorphism groups. In Table 3, we mark the orders of automorphism groups for decomposable codes. We remark that there is no self-dual [24, 12, 3] code C with $A_3(C) \cong L_{i,j}$ unless the lattice $L_{i,j}$ is listed in Table 3.

By Lemma 2, the weight enumerator of a self-dual [24, 12, 3] code C is determined by the numbers of vectors of norms 1 and 2 in the lattice $A_3(C)$. Since $A_3(C)$ is isomorphic to $L_{i,j} = M_{i,j} \oplus \mathbb{Z}^{24-i}$ for some i, j, and the kissing numbers of $M_{i,j}$ are given in [6, Table 16.7], we do not give the weight enumerators of codes in Table 3.

Similar to Section 4, for each i = 1, 2, ..., 8 we computed T_i given in (1) and verified that it coincided with the value determined in [11, Table I]. This shows that there is no other self-dual code with minimum weight 3. In addition, the number of distinct self-dual codes of length n is known [12] as

$$N(n) = 2 \prod_{i=1}^{(n-2)/2} (3^i + 1).$$

As a check, we verified the mass formula

$$\sum_{D \in \mathcal{D}_{24}} \frac{2^{24} \cdot 24!}{\# \operatorname{Aut}(D)} = 96722522147893108730806108160000 = N(24),$$

where \mathcal{D}_{24} denotes the set of all inequivalent self-dual codes of length 24. The mass formula shows that there is no other self-dual code of length 24 and the classification is complete. As a corollary, we have the following:

Corollary 6. A 24-dimensional odd unimodular lattice L can be constructed from some ternary self-dual code of length 24 by Construction A if and only if L is isomorphic to one of the lattices given in Tables 2 and 3, and the odd Leech lattice.

L	N	# Aut
\mathbb{Z}^{24}	1	8806025134080^{D_1}
L_8	1	41278242816^{D_2}
L_{12}	2	$1146617856^{D_4}, 126127964160^{D_3}$
L_{14}	2	$477757440^{D_5}, 644972544^{D_8}$
L_{15}	2	$89579520^{D_9}, 310542336^{D_6}$
$L_{16,1}$	1	$7739670528^{D_{26}}$
$L_{16,3}$	2	$15925248^{D_{10}}, 198180864^{D_7}$
L_{17}	3	$2985984^{D_{12}}, 80621568, 985374720^{D_{11}}$
$L_{18,1}$	1	67184640
$L_{18,2}$	1	26873856
$L_{18,3}$	3	$1327104^{D_{15}}, 17915904, 19906560^{D_{13}}$
$L_{18,4}$	3	$1161216^{D_{16}}, 2985984^{D_{14}}, 6718464$
$L_{19,1}$	1	13436928
$L_{19,2}$	1	746496
$L_{19,3}$	3	$110592^{D_{17}}, 331776^{D_{18}}, 497664$
$L_{20,2}$	1	$11824496640^{D_{27}}$
$L_{20,4}$	1	22394880
$L_{20,6}$	1	1327104
$L_{20,7}$	2	559872,6718464
$L_{20,8}$	1	331776
$L_{20,9}$	1	165888
$L_{20,10}$	1	41472
$L_{20,11}$	3	$184320^{D_{19}}, 491520^{D_{21}}, 497664$
$L_{20,12}$	7	$24576^{D_{20}}, 92160^{D_{22}}, 124416, 373248, 497664^{D_{23}}, 14929920, 99532800^{D_{24}}$
$L_{21,2}$	1	29113344
$L_{21,3}$	1	82114560
$L_{21,4}$	1	1119744
$L_{21,5}$	2	248832, 3483648
$L_{21,6}$	3	41472, 124416, 1866240
$L_{21,7}$	2	20736, 1119744
$L_{21,8}$	2	13824, 27648
$L_{21,9}$	3	7776, 20736, 31104
$L_{21,10}$	4	6912, 15552, 41472, 248832
$L_{21,11}$	1	6912
$L_{21,12}$	2	62208, 186624

Table 3: Ternary self-dual $\left[24,12,3\right]$ codes

Table 3: Ternary self-dual $\left[24,12,3\right]$ codes (continued)

L	N	$\#\operatorname{Aut}$			
$L_{22,8}$	1	1658880			
$L_{22,10}$	1	373248			
$L_{22,12}$	1	248832			
$L_{22,14}$	2	36864, 110592			
$L_{22,15}$	1	96768			
$L_{22,16}$	1	27648			
$L_{22,17}$	1	20736			
$L_{22,18}$	2	6912, 20736			
$L_{22,19}$	2	9216, 829440			
$L_{22,20}$	3	2304, 6912, 10368			
$L_{22,21}$	3	1152, 3456, 3456			
$L_{22,22}$	2	13824, 18432			
$L_{22,23}$	2	576, 9216			
$L_{22,24}$	6	1152, 1152, 1728, 4608, 6912, 10368			
$L_{22,25}$	5	576, 576, 864, 1152, 10368			
$L_{22,26}$	5	576, 1728, 2304, 4608, 4608			
$L_{22,27}$	2	4608, 746496			
$L_{23,4}$	1	68428800			
$L_{23,16}$	1	622080			
$L_{23,18}$	1	186624			
$L_{23,24}$	1	13824			
$L_{23,27}$	2	31104, 311040			
$L_{23,30}$	1	3072			
$L_{23,31}$	1	3072			
$L_{23,32}$	2	2880, 3456			
$L_{23,33}$	2	2304, 27648			
$L_{23,35}$	2	1152, 3456			
$L_{23,37}$	3	192, 384, 3456			
$L_{23,38}$	2	864, 10368			
$L_{23,39}$	4	192, 288, 2592, 15552			
$L_{23,40}$	6	288, 384, 576, 768, 768, 3456			
$L_{23,41}$	2	384, 432			
$L_{23,42}$	3	96, 144, 192			
$L_{23,43}$	2	96, 384			
$L_{23,44}$	9	24, 48, 96, 96, 96, 144, 384, 384, 1728			
$L_{23,45}$	4	96, 192, 384, 384			
$L_{23,46}$	10	24, 48, 48, 48, 48, 96, 192, 384, 768, 13824			
$L_{23,47}$	4	60, 72, 96, 288			

6 Even unimodular neighbors of $A_3(C)$

Let C be a self-dual code of length $n \equiv 0 \pmod{12}$ containing the all-one's vector **1**. To construct the Niemeier lattices from self-dual codes of length 24, Montague [14] considered the following constructions of unimodular lattices

$$L_S(C) = \left\langle \frac{1}{2\sqrt{3}} \mathbf{1}, B_3(C) \right\rangle$$
 and $L_T(C) = \left\langle \frac{1}{2\sqrt{3}} \mathbf{1} - e_1, B_3(C) \right\rangle$

which are unimodular neighbors of $A_3(C)$, where $B_3(C) = \{v \in A_3(C) | (v, v) \in 2\mathbb{Z}\}$ and $e_1 = (\sqrt{3}, 0, \ldots, 0)$. In particular, if $n \equiv 0 \pmod{24}$, then $L_S(C)$ and $L_T(C)$ are the even unimodular neighbors of $A_3(C)$. These constructions are called the straight and twisted constructions, respectively [14]. Montague [14] demonstrated that the 23 Niemeier lattices other than the Leech lattice can be constructed by the straight construction, and the 22 Niemeier lattices other than the two lattices with root systems A_{24} and D_{24} can be constructed by the twisted constructed by the twisted construction. He also conjectured that the Niemeier lattice with root system D_{24} cannot be constructed by the twisted construction.

We verified that the code $g_{11} + p_{13}$ given in [11, Fig. 5] which is equivalent to the unique code C with $A_3(C) \cong L_{24,141}$ in Table 2, gives the Niemeier lattice with root system A_{24} by the twisted construction. More specifically, let C be the code with generator matrix

(2)
$$\begin{pmatrix} G_{11} & O_{5\times13} \\ O_{6\times11} & P_{13} \\ 00000011111 & 1101000001000 \end{pmatrix},$$

where $O_{m \times n}$ denotes the $m \times n$ zero matrix,

$$G_{11} = \begin{pmatrix} 10000201221\\ 01000210122\\ 00100221012\\ 00010222101\\ 00001212210 \end{pmatrix} \text{ and } P_{13} = \begin{pmatrix} 1000002212001\\ 0100001012202\\ 0010002010221\\ 0001001022021\\ 0000101220201\\ 0000011210022 \end{pmatrix}$$

Then the lattice $L_T(C)$ is the Niemeier lattice with root system A_{24} . Hence the 23 Niemeier lattices other than the lattice with root system D_{24} can be constructed by the twisted construction. In addition, we verified that the lattice with root system D_{24} cannot be constructed by the twisted construction. We note, however, that there is a delicate point regarding the distinction between the two constructions, as described by the following proposition.

Proposition 7. Let C be a ternary self-dual code of length $n \equiv 0 \pmod{12}$. Suppose that $\mathbf{1} \in C$ and $v = (v_1, \ldots, v_n)$ is a codeword of weight n in C. Let P be the diagonal matrix whose diagonal entries are the entries of the codeword v regarded as elements of $\{\pm 1\} \subset \mathbb{Z}$. Then

$$L_S(C) \cdot P = \begin{cases} L_S(C \cdot P) & \text{if } \prod_{i=1}^n v_i = 1, \\ L_T(C \cdot P) & \text{otherwise.} \end{cases}$$

Proof. Observe $\mathbf{1} \in C \cdot P$. It is easy to see that $B_3(C) \cdot P = B_3(C \cdot P)$. Thus

$$L_S(C) \cdot P = \left\langle \frac{1}{2\sqrt{3}} v, B_3(C \cdot P) \right\rangle,$$

where v is regarded as a vector of $\{\pm 1\}^n \subset \mathbb{Z}^n$. Therefore $L_S(C) \cdot P = L_S(C \cdot P)$ if and only if

$$\frac{1}{2\sqrt{3}}(v-\mathbf{1}) \in B_3(C \cdot P).$$

Since $(v - 1, v - 1) \equiv 0 \pmod{12}$, this is equivalent to $(v - 1, v - 1) \equiv 0 \pmod{8}$, or $\prod_{i=1}^{n} v_i = 1$.

Therefore, if a self-dual code C contains both **1** and a codeword v of weight n with $\prod_{i=1}^{n} v_i = -1$, then $L_S(C)$ (resp. $L_T(C)$) is isomorphic to $L_T(C')$ (resp. $L_S(C')$) for some code C' which is equivalent to C. This means that the definitions of the straight and the twisted constructions are independent of the choice of a representative in the equivalence class of codes only if $\prod v_i = 1$ holds for all codewords v of weight n in C. Self-dual codes satisfying this condition are called admissible [9]. For example, the code Cwith generator matrix given in (2) is not admissible. In fact, there is a code C' equivalent to C such that $L_T(C')$ is the Niemeier lattice with root system A_{12}^2 .

The constructions of Montague [14] have been generalized in [9] for any self-dual codes, not necessarily containing **1**. Let E_4 be the self-dual [4, 2, 3]

code with generator matrix $\begin{pmatrix} 1021\\0122 \end{pmatrix}$. The decomposable code D_1 is equivalent to the direct sum E_4^6 of six copies of E_4 , and the Niemeier lattice with root system D_{24} can be obtained as both $L_S(E_4^6)$ and $L_T(E_4^6)$, in the notation of [9]. In fact, for any self-dual code C of length 24, $\beta_{24} = 48 - 21\beta_3 + \beta_6$ holds, where β_i denotes the number of codewords of weight i in C (see [11, Table III]). Thus, if C has maximum weight less than 24, then C has minimum weight 3. This implies that $A_3(C)$ has minimum norm 1, and hence the two even neighbors of $A_3(C)$ are isomorphic.

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