# A Complete Classification of Ternary Self-Dual Codes of Length 24 

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#### Abstract

Ternary self-dual codes have been classified for lengths up to 20. At length 24 , a classification of only extremal self-dual codes is known. In this paper, we give a complete classification of ternary self-dual codes of length 24 using the classification of 24 -dimensional odd unimodular lattices.


## 1 Introduction

As described in [16], self-dual codes are an important class of linear codes for both theoretical and practical reasons. It is a fundamental problem to classify self-dual codes of modest length and determine the largest minimum weight among self-dual codes of that length. By the Gleason-Pierce theorem, there are nontrivial divisible self-dual codes over $\mathbb{F}_{q}$ for $q=2,3$ and 4 only, where $\mathbb{F}_{q}$ denotes the finite field of order $q$, and this is one of the reasons why much work has been done concerning self-dual codes over these fields.

A code $C$ over $\mathbb{F}_{3}$ is called ternary. All codes in this paper are ternary. A code $C$ of length $n$ is said to be self-dual if $C=C^{\perp}$, where the dual code $C^{\perp}$ of $C$ is defined as $C^{\perp}=\left\{x \in \mathbb{F}_{3}^{n} \mid x \cdot y=0\right.$ for all $\left.y \in C\right\}$ under the standard inner product $x \cdot y$. A self-dual code of length $n$ exists if and only if $n \equiv 0$ $(\bmod 4)$. It was shown in [13] that the minimum weight $d$ of a self-dual code

[^0]of length $n$ is bounded by $d \leq 3[n / 12]+3$. If $d=3[n / 12]+3$, then the code is called extremal. Two codes $C$ and $C^{\prime}$ are equivalent if there exists a monomial matrix $P$ with $C^{\prime}=C \cdot P=\{x P \mid x \in C\}$. The automorphism group $\operatorname{Aut}(C)$ of $C$ is the group of all monomial matrices $P$ with $C=C \cdot P$.

All self-dual codes of length $\leq 20$ have been classified [4, 12, 15]. At length 24, the complete classification has not been done yet, although it was shown by Leon, Pless and Sloane [11] that there are exactly two inequivalent extremal self-dual codes, namely, the extended quadratic residue code $Q R_{24}$ and the Pless symmetry code $P_{24}$. Moreover, it was shown in [11] that there are at least 13 and 96 inequivalent self-dual codes with minimum weights 3 and 6 , respectively.

Applying the classification method in [8] to length 24, we give a classification of self-dual codes of length 24 with minimum weights 3 and 6 , which completes the classification of self-dual codes of length 24 .

Theorem 1. There are exactly 166 inequivalent ternary self-dual $[24,12,6]$ codes. There are exactly 170 inequivalent ternary self-dual $[24,12,3]$ codes.

Generator matrices of all self-dual codes of length 24 can be obtained electronically from [7]. All computer calculations in this paper were done by Magma [3].

## 2 Preliminaries

An $n$-dimensional (Euclidean) lattice $L$ is integral if $L \subseteq L^{*}$, where the dual lattice $L^{*}$ is defined as $L^{*}=\left\{x \in \mathbb{R}^{n} \mid(x, y) \in \mathbb{Z}\right.$ for all $\left.y \in L\right\}$ under the standard inner product $(x, y)$. A lattice $L$ with $L=L^{*}$ is called unimodular. The norm of a vector $x$ is $(x, x)$. The minimum norm of $L$ is the smallest norm among all nonzero vectors of $L$. A unimodular lattice $L$ is even if all vectors of $L$ have even norms, and odd if some vector has an odd norm. The kissing number of $L$ is the number of vectors of $L$ with minimum norm. Two lattices $L$ and $L^{\prime}$ are isomorphic, denoted $L \cong L^{\prime}$, if there exists an orthogonal matrix $A$ with $L^{\prime}=L \cdot A$. The automorphism group $\operatorname{Aut}(L)$ of $L$ is the group of all orthogonal matrices $A$ with $L=L \cdot A$.

If $C$ is a self-dual code of length $n$ and minimum weight $d$, then

$$
A_{3}(C)=\frac{1}{\sqrt{3}}\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{Z}^{n} \mid\left(x_{1} \bmod 3, \ldots, x_{n} \bmod 3\right) \in C\right\}
$$

is a unimodular lattice with minimum norm $\min \{3, d / 3\}$. This construction of lattices from codes is called Construction A.

Lemma 2. Let $C$ be a ternary self-dual code of length n. Let $\alpha_{i}$ (resp. $\beta_{3 i}$ ) be the number of vectors of norm $i$ in $A_{3}(C)$ (resp. codewords of weight $3 i$ in C) $(i=1,2)$. Then

$$
\alpha_{2}=\beta_{6}+3 \beta_{3} \text { and } \alpha_{1}=\beta_{3}
$$

Proof. The straightforward proof is omitted.
The weight distribution of a self-dual code $C$ of length 24 is determined by the numbers $\beta_{3}, \beta_{6}$ (see [11, Table III]). Hence the weight distribution of $C$ can be determined by the numbers $\alpha_{1}, \alpha_{2}$ in $A_{3}(C)$.

There are 155 non-isomorphic 24 -dimensional odd unimodular lattices with minimum norm 2, and there are 117 non-isomorphic 24-dimensional odd unimodular lattices with minimum norm 1 [2] (see also [6, Table 2.2]). We denote the $i$-th 24 -dimensional odd unimodular lattice with minimum norm $\geq 2$ in [6, Table 17.1] by $L_{24, i}(i=1,2, \ldots, 156)$. The lattices $L_{24, i}$ $(i=2,3, \ldots, 156)$ are the 155 non-isomorphic 24 -dimensional odd unimodular lattices with minimum norm 2. A 24-dimensional unimodular lattice with minimum norm 1 except $\mathbb{Z}^{24}$ can be constructed as $M_{i} \oplus \mathbb{Z}^{24-i}$ where $M_{i}$ is an $i$-dimensional unimodular lattice with minimum norm $\geq 2$. Here we denote a 24-dimensional unimodular lattice $M_{i, j} \oplus \mathbb{Z}^{24-i}$ with minimum norm 1 by $L_{i, j}$ where $M_{i, j}$ is the $j$-th $i$-dimensional unimodular lattice with minimum norm $\geq 2$ in [6, Table 16.7]. All non-isomorphic unimodular lattices with minimum norm $\geq 2$ can be constructed as neighbors of the standard lattices for dimensions up to 24 (see [1, Tables I, II and III]).

A set $\left\{f_{1}, \ldots, f_{n}\right\}$ of $n$ vectors $f_{1}, \ldots, f_{n}$ in an $n$-dimensional lattice $L$ with $\left(f_{i}, f_{j}\right)=3 \delta_{i j}$ is called a 3 -frame of $L$, where $\delta_{i j}$ is the Kronecker delta. Clearly, $A_{3}(C)$ has a 3 -frame. Conversely, every self-dual code can be obtained from a 3 -frame of some unimodular lattice. Let $\mathcal{F}=\left\{f_{1}, \ldots, f_{n}\right\}$ be a 3 -frame of $L$. Consider the mapping

$$
\begin{aligned}
& \pi_{\mathcal{F}}: \frac{1}{3} \oplus_{i=1}^{n} \mathbb{Z} f_{i} \rightarrow \mathbb{F}_{3}^{n} \\
& \pi_{\mathcal{F}}(x)=\left(\left(x, f_{i}\right) \bmod 3\right)_{1 \leq i \leq n}
\end{aligned}
$$

Then $\operatorname{Ker} \pi_{\mathcal{F}}=\oplus_{i=1}^{n} \mathbb{Z} f_{i} \subset L$, so the code $C=\pi_{\mathcal{F}}(L)$ satisfies $\pi_{\mathcal{F}}^{-1}(C)=L$. This implies $A_{3}(C) \cong L$ and every code $C$ with $A_{3}(C) \cong L$ is obtained as $\pi_{\mathcal{F}}(L)$ for some 3 -frame $\mathcal{F}$ of $L$.

Lemma 3 ([8]). Let $L$ be an $n$-dimensional integral lattice, and let $\mathcal{F}=$ $\left\{f_{1}, \ldots, f_{n}\right\}, \mathcal{F}^{\prime}=\left\{f_{1}^{\prime}, \ldots, f_{n}^{\prime}\right\}$ be 3 -frames of $L$. Then the codes $\pi_{\mathcal{F}}(L)$ and $\pi_{\mathcal{F}^{\prime}}(L)$ are equivalent if and only if there exists an orthogonal matrix $P \in \operatorname{Aut}(L)$ such that $\left\{ \pm f_{1}, \ldots, \pm f_{n}\right\} \cdot P=\left\{ \pm f_{1}^{\prime}, \ldots, \pm f_{n}^{\prime}\right\}$.

In order to establish the nonexistence of a 3-frame for some lattices, the shadows of lattices are considered. Let $L=L_{0} \cup L_{2}$ be an odd unimodular lattice with even sublattice $L_{0}$. Then $L_{0}^{*}$ can be written as a union of cosets of $L_{0}: L_{0}^{*}=L_{0} \cup L_{2} \cup L_{1} \cup L_{3}$. The shadow $S$ of $L$ is defined to be $S=L_{1} \cup L_{3}$ [5].

Lemma 4. Let $L=L_{0} \cup L_{2}$ be a 24-dimensional odd unimodular lattice with shadow $S=L_{1} \cup L_{3}$. Let $v$ be a vector of $L_{2}$ with $(v, v)=3$. If there exist vectors $a \in L_{1}$ and $b \in L_{3}$ such that $(a, a)=(b, b)=2,(a, b)=1 / 2$ and $v=a-b$, then $v$ does not belong to any 3-frame of $L$.

Proof. Suppose that $\left\{f_{1}, \ldots, f_{24}\right\}$ is a 3 -frame of $L$ and $v=f_{1}$. By Lemma 2 in [9], a vector $a \in L_{1}$ can be written as

$$
a=\frac{1}{6} \sum_{i=1}^{24} a_{i} f_{i}, \quad\left(a_{i} \in 1+2 \mathbb{Z}\right)
$$

Since

$$
2=(a, a)=\frac{1}{12} \sum_{i=1}^{24} a_{i}^{2} \geq \frac{1}{12} \sum_{i=1}^{24} 1=2
$$

we have $a_{i}= \pm 1$ for all $1 \leq i \leq 24$. Then

$$
\frac{3}{2}=(a, a)-(a, b)=(a, v)=\frac{a_{1}}{2}= \pm \frac{1}{2} .
$$

This is a contradiction.
Let $L$ be a 24-dimensional unimodular lattice and let $V$ be the set of pairs $\{v,-v\}$ with $(v, v)=3, v \in L$ satisfying the condition that there do not exist $a \in L_{1}$ and $b \in L_{3}$ such that $(a, a)=(b, b)=2,(a, b)=1 / 2$ and $v=a-b$. We define the simple undirected graph $\Gamma$, whose set of vertices is the set $V$ and two vertices $\{v,-v\},\{w,-w\} \in V$ are adjacent if $(v, w)=0$. It follows that the 3 -frames are precisely the 24 -cliques in the graph $\Gamma$. It is clear that Aut $(L)$ acts on the graph $\Gamma$ as automorphisms, and Lemma 3 implies that the $\operatorname{Aut}(L)$-orbits on the set of 24 -cliques of $\Gamma$ are in one-to-one correspondence with the equivalence classes of codes $C$ satisfying $A_{3}(C) \cong L$. Therefore,
the classification of such codes reduces to finding a set of representatives of 24 -cliques of $\Gamma$ up to the action of $\operatorname{Aut}(L)$. This computation was performed by Magma [3, the results were then converted to 3 -frames, and then to self-dual codes of length 24 . In this way, by considering 3 -frames of all 24 dimensional odd unimodular lattices with minimum norms 2 and 1 , we have all inequivalent self-dual codes of length 24 with minimum weights 6 and 3, respectively.

Note that the graph $\Gamma$ is an empty graph for the lattice $L_{i, j}$, where

$$
\begin{aligned}
(i, j)= & (20,1),(22,1),(23,2),(23,3),(24,94),(24,125),(24,126), \\
& (24,135),(24,136),(24,137),(24,143),(24,147),(24,148), \\
& (24,149),(24,151),(24,152),(24,153),(24,155),(24,156) .
\end{aligned}
$$

In particular, none of these lattices have a 3-frame.

## 3 Decomposable self-dual codes of length 24

As described in [11, Table II], there are 27 inequivalent decomposable selfdual codes $D_{i}(i=1,2, \ldots, 27)$ of length 24 . A decomposable self-dual code can be written as $C_{1} \oplus C_{2}$ where $C_{1}$ and $C_{2}$ are self-dual codes of lengths 20 and 4 , or both $C_{1}$ and $C_{2}$ are indecomposable self-dual codes of length 12. We denote the unique self-dual $[4,2,3]$ code in [12, Table 1] by $E_{4}$. We also denote the extended ternary Golay $[12,6,6]$ code by $G_{12}$ and the unique indecomposable $[12,6,3]$ code by $4 C_{3}(12)$ [12, Table 1]. We denote the self-dual codes of length 20 by $C_{20,1}, \ldots, C_{20,24}$ according to the order given in [15, Tables II and III]. In Table 1, we list the number $\beta_{3}$ of the codewords of weight 3 and the order $\# \operatorname{Aut}\left(D_{i}\right)$ of the automorphism group for $D_{i}(i=1,2, \ldots, 27)$.

There is a unique decomposable self-dual $[24,12,6]$ code. This can also be established from the classification of odd unimodular lattices, by the following lemma.

Lemma 5. Let $C$ be a ternary self-dual code of length $n$ and minimum weight at least 6. Then $C$ is decomposable if and only if $A_{3}(C)$ is decomposable.

Proof. If $C$ is decomposable, then obviously $A_{3}(C)$ is decomposable. Conversely, suppose that $A_{3}(C)=L \oplus L^{\prime}$ for some sublattices $L, L^{\prime}$. Since $A_{3}(C)$ has minimum norm $\geq 2$, both $L$ and $L^{\prime}$ have minimum norms $\geq 2$. Let $x$

Table 1: Decomposable self-dual codes of length 24

| $i$ | $\left(C_{1}, C_{2}\right)$ | $\beta_{3}$ | $\# \operatorname{Aut}\left(D_{i}\right)$ | $i$ | $\left(C_{1}, C_{2}\right)$ | $\beta_{3}$ | \# Aut $\left(D_{i}\right)$ |
| :---: | :---: | :---: | ---: | :---: | :---: | :---: | ---: |
| 1 | $\left(C_{20,1}, E_{4}\right)$ | 48 | 8806025134080 | 15 | $\left(C_{20,15}, E_{4}\right)$ | 12 | 1327104 |
| 2 | $\left(C_{20,2}, E_{4}\right)$ | 32 | 41278242816 | 16 | $\left(C_{20,16}, E_{4}\right)$ | 12 | 1161216 |
| 3 | $\left(C_{20,3}, E_{4}\right)$ | 24 | 126127964160 | 17 | $\left(C_{20,17}, E_{4}\right)$ | 10 | 110592 |
| 4 | $\left(C_{20,4}, E_{4}\right)$ | 24 | 1146617856 | 18 | $\left(C_{20,18}, E_{4}\right)$ | 10 | 331776 |
| 5 | $\left(C_{20,5}, E_{4}\right)$ | 20 | 477757440 | 19 | $\left(C_{20,19}, E_{4}\right)$ | 8 | 184320 |
| 6 | $\left(C_{20,6}, E_{4}\right)$ | 18 | 310542336 | 20 | $\left(C_{20,20}, E_{4}\right)$ | 8 | 24576 |
| 7 | $\left(C_{20,7}, E_{4}\right)$ | 16 | 198180864 | 21 | $\left(C_{20,21}, E_{4}\right)$ | 8 | 491520 |
| 8 | $\left(C_{20,8}, E_{4}\right)$ | 20 | 644972544 | 22 | $\left(C_{20,22}, E_{4}\right)$ | 8 | 92160 |
| 9 | $\left(C_{20,9}, E_{4}\right)$ | 18 | 89579520 | 23 | $\left(C_{20,23}, E_{4}\right)$ | 8 | 497664 |
| 10 | $\left(C_{20,10}, E_{4}\right)$ | 16 | 15925248 | 24 | $\left(C_{20,24}, E_{4}\right)$ | 8 | 99532800 |
| 11 | $\left(C_{20,11}, E_{4}\right)$ | 14 | 985374720 | 25 | $\left(G_{12}, G_{12}\right)$ | 0 | 72260812800 |
| 12 | $\left(C_{20,12}, E_{4}\right)$ | 14 | 2985984 | 26 | $\left(4 C_{3}(12), 4 C_{3}(12)\right)$ | 16 | 7739670528 |
| 13 | $\left(C_{20,13}, E_{4}\right)$ | 12 | 19906560 | 27 | $\left(4 C_{3}(12), G_{12}\right)$ | 8 | 11824496640 |
| 14 | $\left(C_{20,14}, E_{4}\right)$ | 12 | 2985984 |  |  |  |  |

be a vector of norm 3 in $A_{3}(C)$. Then $x$ can be written as either $\left(x_{1}, 0\right)$ or $\left(0, x_{2}\right)$ where $x_{1} \in L$ and $x_{2} \in L^{\prime}$. Hence, every 3 -frame of $A_{3}(C)$ is a union of those of $L$ and of $L^{\prime}$. Therefore, $C$ is decomposable.

If $C$ is a decomposable self-dual $[24,12,6]$ code, then Lemma 5 implies that $A_{3}(C)$ is a decomposable odd unimodular lattice with minimum norm 2. The only such lattices are $L_{24,153}=E_{8} \oplus\left(D_{8} \oplus D_{8}\right)^{+}$and $L_{24,154}=$ $D_{12}^{+} \oplus D_{12}^{+}$. Clearly, $L_{24,153}$ has no 3-frame, since the lattice $E_{8}$ does not have one. Since the extended ternary Golay $[12,6,6]$ code $G_{12}$ is the unique code $C$ with $A_{3}(C) \simeq D_{12}^{+}$, the decomposable code $D_{25}$ is the unique code $C$ with $A_{3}(C) \simeq L_{24,154}$, up to equivalence.

## 4 Self-dual [24, 12, 6] codes

In this section, we give a classification of self-dual $[24,12,6]$ codes by considering 3 -frames of 24 -dimensional odd unimodular lattices with minimum norm 2.

By the approach described in Section 2, we completed the classification of self-dual $[24,12,6]$ codes. In Table 2, we list the number $N_{i}$ of inequivalent self-dual $[24,12,6]$ codes $C$ with $A_{3}(C) \cong L_{24, i}$. The columns \# Aut in the table list the orders of automorphism groups. From Table 11, the only decomposable self-dual $[24,12,6]$ code is $D_{25}=G_{12} \oplus G_{12}$ where the automorphism group order is marked in Table 2. We remark that there is no
self-dual $[24,12,6]$ code $C$ with $A_{3}(C) \cong L_{24, i}$ unless $i$ is listed in Table 2.
By Lemma 2, the weight enumerator of a self-dual $[24,12,6]$ code $C$ is determined by the kissing number of the lattice $A_{3}(C)$. Since $A_{3}(C)$ is isomorphic to one of the 155 lattices whose kissing numbers are given in [1, Table III], we do not give the weight enumerator of $C$ or the number of codewords of weight 6 in $C$ for codes $C$ given in Table 2.

Let $\mathcal{C}_{i}$ denote the set of all inequivalent indecomposable self-dual codes of length 24 containing exactly $2 i$ codewords of weight 3 . The following values

$$
\begin{equation*}
T_{i}=\sum_{C \in \mathcal{C}_{i}} \frac{1}{\# \operatorname{Aut}(C)} \quad(i=0,1, \ldots, 8) \tag{1}
\end{equation*}
$$

were determined theoretically in [11, Table I], without finding the set $\mathcal{C}_{i}$. We now have the set $\mathcal{C}_{0}$ as

$$
\mathcal{C}_{0}=\left\{Q R_{24}, P_{24}\right\} \cup \mathcal{C} \backslash\left\{D_{25}\right\},
$$

where $\mathcal{C}$ is the set of the 166 codes given in Table 2, and we verified that the value $T_{0}$ obtained from our classification coincided with the value determined in [11, Table I]. This shows that there is no other self-dual code with minimum weight $d \geq 6$.

We investigate the previously known self-dual $[24,12,6]$ codes. In 10 and [11], self-dual codes generated by the rows of Hadamard matrices of order 24 were studied. The 60 inequivalent Hadamard matrices of order 24 give exactly two inequivalent extremal self-dual codes [11] and exactly seven inequivalent self-dual $[24,12,6]$ codes $C(H 1), \ldots, C(H 7)$ which are generated by the matrices $H 1, \ldots, H 7$, respectively [10]. We verified that all of these seven codes appear in the present classification, and in Table 2 we mark the orders of automorphism groups for these codes.

Some properties of automorphism groups of self-dual [24,12, 6] codes were given in [11]. For example, only the primes $2,3,5,7,11$ and 13 can divide the orders of the automorphism groups. A self-dual $[24,12,6]$ code with a trivial automorphism group was given in [11, Fig. 2] and the authors conjectured that the code is unique. Our classification shows that the conjecture is true. We mark the code by Tri in Table 2. The code $g_{10}+\eta_{14}$ given in [11, Fig. 4] is the unique self-dual $[24,12,6]$ code with an automorphism of order 7 and the code $g_{11}+p_{13}$ given in [11, Fig. 5] is the unique self-dual [24, 12, 6] code with an automorphism of order 13 [11]. In Table 2] we mark the orders of automorphism groups for these two codes. From our classification, only

Table 2: Ternary self-dual $[24,12,6]$ codes

| $i$ | $N_{i}$ | \# Aut | $i$ | $N_{i}$ | \# Aut |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 2 | 64, 64 | 32 | 2 | $3456{ }^{\text {C(H7) }}, 31104^{\text {C(H2) }}$ |
| 3 | 5 | 8, 12, 24, 24, 3072 | 33 | 2 | 48, 576 |
| 4 | 4 | 16, 16, 32, 512 | 34 | 1 | 192 |
| 5 | 10 | $4,4,8,8,12,16,24,32,32,72$ | 35 | 1 | 64 |
| 6 | 2 | $960^{C(H 4)}, 3072^{C(H 6)}$ | 36 | 1 | 512 |
| 7 | 11 | $2^{\text {Tri }}, 4,4,4,4,4,8,8,16,16,32$ | 37 | 1 | 256 |
| 8 | 8 | $4,8,12,16,16,24,32,64$ | 40 | 1 | 256 |
| 9 | 8 | 16, 24, 48, 64, 64, 96, 192, 384 | 41 | 1 | 256 |
| 10 | 12 | $4,4,4,8,12,16,16,16,24,24,48,384$ | 43 | 1 | 5184 |
| 11 | 5 | $12,16,48,128,1728$ | 44 | 2 | 384, 3456 |
| 12 | 4 | 16, 48, 48, 1728 | 46 | 1 | 512 |
| 13 | 11 | $4,4,4,8,8,8,16,16,16,16,16$ | 47 | 1 | 1728 |
| 14 | 1 | 64 | 50 | 1 | 8192 |
| 15 | 4 | 16, 32, 32, 2048 | 53 | 1 | 1024 |
| 16 | 4 | 8, 8, 16, 32 | 59 | 1 | 2304 |
| 17 | 7 | 8, 32, 32, 32, 32, 96, 1152 | 61 | 1 | 17280 |
| 18 | 4 | 16, 32, 64, 384 | 62 | 1 | 5184 |
| 19 | 6 | 32, 48, 48, 192, 648, 864 | 63 | 1 | 1728 |
| 20 | 1 | 384 | 64 | 1 | 34560 |
| 21 | 2 | 16, 64 | 65 | 1 | 3456 |
| 22 | 3 | 8, 16, 128 | 73 | 1 | 20736 |
| 23 | 2 | 64, 128 | 74 | 1 | $49152^{C(H 3)}$ |
| 24 | 4 | 32, 64, 64, 256 | 78 | 2 | 20736, $241920^{g_{10}+\eta_{14}}$ |
| 25 | 2 | 2304, 3072 | 87 | 1 | 276480 |
| 26 | 1 | 288 | 102 | 1 | 746496 |
| 27 | 2 | 288, 576 | 114 | 1 | 622080 |
| 28 | 1 | 96 | 115 | 1 | 746496 |
| 29 | 2 | 128, 768 | 130 | 1 | $8294400^{C(H 5)}$ |
| 30 | 2 | 256, 768 | 141 | 1 | $88957440^{g_{11}+p_{13}}$ |
| 31 | 2 | 96, 1296 | 154 | 1 | $72260812800^{D_{25}, C(H 1)}$ |

$g_{11}+p_{13}$ and the decomposable code $D_{25}$ are self-dual $[24,12,6]$ codes with an automorphism of order 11 .

## 5 Self-dual [24, 12, 3] codes

In this section, we give a classification of self-dual $[24,12,3]$ codes by considering 3 -frames in 24 -dimensional odd unimodular lattices with minimum norm 1. These are the lattices $\mathbb{Z}^{24}$ and $L_{i, j}(i \leq 23)$.

In Table 3, we list the number $N$ of inequivalent self-dual [24, 12, 3] codes $C$ with $A_{3}(C) \cong L$ for $L=\mathbb{Z}^{24}, L_{i, j}$. The column \# Aut in Table 3 lists the orders of automorphism groups. In Table 3, we mark the orders of automorphism groups for decomposable codes. We remark that there is no self-dual $[24,12,3]$ code $C$ with $A_{3}(C) \cong L_{i, j}$ unless the lattice $L_{i, j}$ is listed in Table 3.

By Lemma 2, the weight enumerator of a self-dual $[24,12,3]$ code $C$ is determined by the numbers of vectors of norms 1 and 2 in the lattice $A_{3}(C)$. Since $A_{3}(C)$ is isomorphic to $L_{i, j}=M_{i, j} \oplus \mathbb{Z}^{24-i}$ for some $i, j$, and the kissing numbers of $M_{i, j}$ are given in [6, Table 16.7], we do not give the weight enumerators of codes in Table 3,

Similar to Section 4, for each $i=1,2, \ldots, 8$ we computed $T_{i}$ given in (1) and verified that it coincided with the value determined in [11, Table I]. This shows that there is no other self-dual code with minimum weight 3 . In addition, the number of distinct self-dual codes of length $n$ is known [12] as

$$
N(n)=2 \prod_{i=1}^{(n-2) / 2}\left(3^{i}+1\right)
$$

As a check, we verified the mass formula

$$
\sum_{D \in \mathcal{D}_{24}} \frac{2^{24} \cdot 24!}{\# \operatorname{Aut}(D)}=96722522147893108730806108160000=N(24)
$$

where $\mathcal{D}_{24}$ denotes the set of all inequivalent self-dual codes of length 24 . The mass formula shows that there is no other self-dual code of length 24 and the classification is complete. As a corollary, we have the following:

Corollary 6. A 24-dimensional odd unimodular lattice $L$ can be constructed from some ternary self-dual code of length 24 by Construction $A$ if and only if $L$ is isomorphic to one of the lattices given in Tables圆 and 图, and the odd Leech lattice.

Table 3: Ternary self-dual $[24,12,3]$ codes

| $L$ | $N$ |  |
| :---: | ---: | :--- |
| $\mathbb{Z}^{24}$ | 1 | $8806025134080^{D_{1}}$ |
| $L_{8}$ | 1 | $41278242816^{D_{2}}$ |
| $L_{12}$ | 2 | $1146617856^{D_{4}}, 126127964160^{D_{3}}$ |
| $L_{14}$ | 2 | $477757440^{D_{5}}, 644972544^{D_{8}}$ |
| $L_{15}$ | 2 | $89579520^{D_{9}}, 310542336^{D_{6}}$ |
| $L_{16,1}$ | 1 | $7739670528^{D_{26}}$ |
| $L_{16,3}$ | 2 | $15925248^{D_{10}}, 198180864^{D_{7}}$ |
| $L_{17}$ | 3 | $2985984^{D_{12}, 80621568,985374720^{D_{11}}}$ |
| $L_{18,1}$ | 1 | 67184640 |
| $L_{18,2}$ | 1 | 26873856 |
| $L_{18,3}$ | 3 | $1327104^{D_{15}}, 17915904,19906560^{D_{13}}$ |
| $L_{18,4}$ | 3 | $1161216^{D_{16},}, 2985984^{D_{14}}, 6718464$ |
| $L_{19,1}$ | 1 | 13436928 |
| $L_{19,2}$ | 1 | 746496 |
| $L_{19,3}$ | 3 | $110592^{D_{17}}, 331776^{D_{18}}, 497664$ |
| $L_{20,2}$ | 1 | $11824496640^{D_{27}}$ |
| $L_{20,4}$ | 1 | 22394880 |
| $L_{20,6}$ | 1 | 1327104 |
| $L_{20,7}$ | 2 | 559872,6718464 |
| $L_{20,8}$ | 1 | 331776 |
| $L_{20,9}$ | 1 | 165888 |
| $L_{20,10}$ | 1 | 41472 |
| $L_{20,11}$ | 3 | $184320^{D_{19}}, 491520^{D_{21}}, 497664$ |
| $L_{20,12}$ | 7 | $24576^{D_{20}, 92160^{D_{22}}, 124416,373248,497664^{D_{23}}, 14929920,99532800^{D_{24}}}$ |
| $L_{21,2}$ | 1 | 29113344 |
| $L_{21,3}$ | 1 | 82114560 |
| $L_{21,4}$ | 1 | 1119744 |
| $L_{21,5}$ | 2 | 248832,3483648 |
| $L_{21,6}$ | 3 | $41472,124416,1866240$ |
| $L_{21,7}$ | 2 | 20736,1119744 |
| $L_{21,8}$ | 2 | 13824,27648 |
| $L_{21,9}$ | 3 | $7776,20736,31104$ |
| $L_{21,10}$ | 4 | $6912,15552,41472,248832$ |
| $L_{21,11}$ | 1 | 6912 |
| $L_{21,12}$ | 2 | 62208,186624 |
|  |  |  |

Table 3: Ternary self-dual $[24,12,3]$ codes (continued)

| $L$ | $N$ |  |
| :---: | ---: | :--- |
| $L_{22,8}$ | 1 | 1658880 |
| $L_{22,10}$ | 1 | 373248 |
| $L_{22,12}$ | 1 | 248832 |
| $L_{22,14}$ | 2 | 36864,110592 |
| $L_{22,15}$ | 1 | 96768 |
| $L_{22,16}$ | 1 | 27648 |
| $L_{22,17}$ | 1 | 20736 |
| $L_{22,18}$ | 2 | 6912,20736 |
| $L_{22,19}$ | 2 | 9216,829440 |
| $L_{22,20}$ | 3 | $2304,6912,10368$ |
| $L_{22,21}$ | 3 | $1152,3456,3456$ |
| $L_{22,22}$ | 2 | 13824,18432 |
| $L_{22,23}$ | 2 | 576,9216 |
| $L_{22,24}$ | 6 | $1152,1152,1728,4608,6912,10368$ |
| $L_{22,25}$ | 5 | $576,576,864,1152,10368$ |
| $L_{22,26}$ | 5 | $576,1728,2304,4608,4608$ |
| $L_{22,27}$ | 2 | 4608,746496 |
| $L_{23,4}$ | 1 | 68428800 |
| $L_{23,16}$ | 1 | 622080 |
| $L_{23,18}$ | 1 | 186624 |
| $L_{23,24}$ | 1 | 13824 |
| $L_{23,27}$ | 2 | 31104,311040 |
| $L_{23,30}$ | 1 | 3072 |
| $L_{23,31}$ | 1 | 3072 |
| $L_{23,32}$ | 2 | 2880,3456 |
| $L_{23,33}$ | 2 | 2304,27648 |
| $L_{23,35}$ | 2 | 1152,3456 |
| $L_{23,37}$ | 3 | $192,384,3456$ |
| $L_{23,38}$ | 2 | 864,10368 |
| $L_{23,39}$ | 4 | $192,288,2592,15552$ |
| $L_{23,40}$ | 6 | $288,384,576,768,768,3456$ |
| $L_{23,41}$ | 2 | 384,432 |
| $L_{23,42}$ | 3 | $96,144,192$ |
| $L_{23,43}$ | 2 | 96,384 |
| $L_{23,44}$ | 9 | $24,48,96,96,96,144,384,384,1728$ |
| $L_{23,45}$ | 4 | $96,192,384,384$ |
| $L_{23,46}$ | 10 | $24,48,48,48,48,96,192,384,768,13824$ |
| $L_{23,47}$ | 4 | $60,72,96,288$ |
|  |  |  |

## 6 Even unimodular neighbors of $A_{3}(C)$

Let $C$ be a self-dual code of length $n \equiv 0(\bmod 12)$ containing the all-one's vector 1. To construct the Niemeier lattices from self-dual codes of length 24, Montague [14] considered the following constructions of unimodular lattices

$$
L_{S}(C)=\left\langle\frac{1}{2 \sqrt{3}} \mathbf{1}, B_{3}(C)\right\rangle \text { and } L_{T}(C)=\left\langle\frac{1}{2 \sqrt{3}} \mathbf{1}-e_{1}, B_{3}(C)\right\rangle
$$

which are unimodular neighbors of $A_{3}(C)$, where $B_{3}(C)=\left\{v \in A_{3}(C) \mid(v, v) \in\right.$ $2 \mathbb{Z}\}$ and $e_{1}=(\sqrt{3}, 0, \ldots, 0)$. In particular, if $n \equiv 0(\bmod 24)$, then $L_{S}(C)$ and $L_{T}(C)$ are the even unimodular neighbors of $A_{3}(C)$. These constructions are called the straight and twisted constructions, respectively [14]. Montague [14] demonstrated that the 23 Niemeier lattices other than the Leech lattice can be constructed by the straight construction, and the 22 Niemeier lattices other than the two lattices with root systems $A_{24}$ and $D_{24}$ can be constructed by the twisted construction. He also conjectured that the Niemeier lattice with root system $D_{24}$ cannot be constructed by the twisted construction.

We verified that the code $g_{11}+p_{13}$ given in [11, Fig. 5] which is equivalent to the unique code $C$ with $A_{3}(C) \cong L_{24,141}$ in Table 2, gives the Niemeier lattice with root system $A_{24}$ by the twisted construction. More specifically, let $C$ be the code with generator matrix

$$
\left(\begin{array}{cc}
G_{11} & O_{5 \times 13}  \tag{2}\\
O_{6 \times 11} & P_{13} \\
00000011111 & 1101000001000
\end{array}\right)
$$

where $O_{m \times n}$ denotes the $m \times n$ zero matrix,

$$
G_{11}=\left(\begin{array}{l}
10000201221 \\
01000210122 \\
00100221012 \\
00010222101 \\
00001212210
\end{array}\right) \text { and } P_{13}=\left(\begin{array}{l}
1000002212001 \\
0100001012202 \\
0010002010221 \\
0001001022021 \\
0000101220201 \\
0000011210022
\end{array}\right)
$$

Then the lattice $L_{T}(C)$ is the Niemeier lattice with root system $A_{24}$. Hence the 23 Niemeier lattices other than the lattice with root system $D_{24}$ can be constructed by the twisted construction.

In addition, we verified that the lattice with root system $D_{24}$ cannot be constructed by the twisted construction. We note, however, that there is a delicate point regarding the distinction between the two constructions, as described by the following proposition.

Proposition 7. Let $C$ be a ternary self-dual code of length $n \equiv 0(\bmod 12)$. Suppose that $\mathbf{1} \in C$ and $v=\left(v_{1}, \ldots, v_{n}\right)$ is a codeword of weight $n$ in $C$. Let $P$ be the diagonal matrix whose diagonal entries are the entries of the codeword $v$ regarded as elements of $\{ \pm 1\} \subset \mathbb{Z}$. Then

$$
L_{S}(C) \cdot P= \begin{cases}L_{S}(C \cdot P) & \text { if } \prod_{i=1}^{n} v_{i}=1 \\ L_{T}(C \cdot P) & \text { otherwise }\end{cases}
$$

Proof. Observe $1 \in C \cdot P$. It is easy to see that $B_{3}(C) \cdot P=B_{3}(C \cdot P)$. Thus

$$
L_{S}(C) \cdot P=\left\langle\frac{1}{2 \sqrt{3}} v, B_{3}(C \cdot P)\right\rangle
$$

where $v$ is regarded as a vector of $\{ \pm 1\}^{n} \subset \mathbb{Z}^{n}$. Therefore $L_{S}(C) \cdot P=$ $L_{S}(C \cdot P)$ if and only if

$$
\frac{1}{2 \sqrt{3}}(v-\mathbf{1}) \in B_{3}(C \cdot P)
$$

Since $(v-1, v-1) \equiv 0(\bmod 12)$, this is equivalent to $(v-\mathbf{1}, v-\mathbf{1}) \equiv 0$ $(\bmod 8)$, or $\prod_{i=1}^{n} v_{i}=1$.

Therefore, if a self-dual code $C$ contains both 1 and a codeword $v$ of weight $n$ with $\prod_{i=1}^{n} v_{i}=-1$, then $L_{S}(C)$ (resp. $L_{T}(C)$ ) is isomorphic to $L_{T}\left(C^{\prime}\right)$ (resp. $L_{S}\left(C^{\prime}\right)$ ) for some code $C^{\prime}$ which is equivalent to $C$. This means that the definitions of the straight and the twisted constructions are independent of the choice of a representative in the equivalence class of codes only if $\prod v_{i}=1$ holds for all codewords $v$ of weight $n$ in $C$. Self-dual codes satisfying this condition are called admissible [9]. For example, the code $C$ with generator matrix given in (2) is not admissible. In fact, there is a code $C^{\prime}$ equivalent to $C$ such that $L_{T}\left(C^{\prime}\right)$ is the Niemeier lattice with root system $A_{12}^{2}$.

The constructions of Montague [14] have been generalized in 9] for any self-dual codes, not necessarily containing 1 . Let $E_{4}$ be the self-dual $[4,2,3]$
code with generator matrix $\binom{1021}{0122}$. The decomposable code $D_{1}$ is equivalent to the direct sum $E_{4}^{6}$ of six copies of $E_{4}$, and the Niemeier lattice with root system $D_{24}$ can be obtained as both $L_{S}\left(E_{4}^{6}\right)$ and $L_{T}\left(E_{4}^{6}\right)$, in the notation of [9]. In fact, for any self-dual code $C$ of length $24, \beta_{24}=48-21 \beta_{3}+\beta_{6}$ holds, where $\beta_{i}$ denotes the number of codewords of weight $i$ in $C$ (see [11, Table III]). Thus, if $C$ has maximum weight less than 24 , then $C$ has minimum weight 3 . This implies that $A_{3}(C)$ has minimum norm 1 , and hence the two even neighbors of $A_{3}(C)$ are isomorphic.

## References

[1] R. Bacher, Tables de réseaux entiers unimodulaires construits comme $k$-voisins de $Z^{n}$, J. Thé. Nombres Bordeaux 9 (1997), 479-497.
[2] R.E. Borcherds, The Leech lattice and other lattices, Ph.D. Dissertation, Univ. of Cambridge, 1984.
[3] W. Bosma and J. Cannon, Handbook of Magma Functions, Department of Mathematics, University of Sydney, Available online at http://magma.maths.usyd.edu.au/magma/.
[4] J.H. Conway, V. Pless and N.J.A. Sloane, Self-dual codes over GF(3) and GF(4) of length not exceeding 16, IEEE Trans. Inform. Theory 25 (1979), 312-322.
[5] J.H. Conway and N.J.A. Sloane, A new upper bound for the minimum of an integral lattice of determinant 1, Bull. Amer. Math. Soc. (N.S.) 23 (1990), 383-387.
[6] J.H. Conway and N.J.A. Sloane, Sphere Packing, Lattices and Groups (3rd ed.), Springer-Verlag, New York, 1999.
[7] M. Harada and A. Munemasa, Database of Self-Dual Codes, Available online at http://www.math.is.tohoku.ac.jp/~munemasa/ selfdualcodes.htm.
[8] M. Harada, A. Munemasa and B. Venkov, Classification of ternary extremal self-dual codes of length 28, (submitted).
[9] M. Harada, M. Kitazume and M. Ozeki, Ternary code construction of unimodular lattices and self-dual codes over $\mathbb{Z}_{6}$, J. Algebraic Combin. 16 (2002), 209-223.
[10] C.W.H. Lam, L. Thiel and A. Pautasso, On ternary codes generated by Hadamard matrices of order 24, Congr. Numer. 89 (1992), 7-14.
[11] J.S. Leon, V. Pless and N.J.A. Sloane, On ternary self-dual codes of length 24, IEEE Trans. Inform. Theory 27 (1981), 176-180.
[12] C.L. Mallows, V. Pless and N.J.A. Sloane, Self-dual codes over GF(3), SIAM J. Appl. Math. 31 (1976), 649-666.
[13] C.L. Mallows and N.J.A. Sloane, An upper bound for self-dual codes, Inform. Control 22 (1973), 188-200.
[14] P.S. Montague, A new construction of lattices from codes over GF(3), Discrete Math. 135 (1994), 193-223.
[15] V. Pless, N.J.A. Sloane and H.N. Ward, Ternary codes of minimum weight 6 and the classification of length 20, IEEE Trans. Inform. Theory 26 (1980), 305-316.
[16] E. Rains and N.J.A. Sloane, "Self-dual codes," Handbook of Coding Theory, V.S. Pless and W.C. Huffman (Editors), Elsevier, Amsterdam 1998, pp. 177-294.


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