CYCLIC SIEVING, PROMOTION, AND REPRESENTATION THEORY

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ABSTRACT. We prove a collection of conjectures of D. White [37], as well as some related conjectures of Abuzzahab-Korson-Li-Meyer [1] and of Reiner and White [21], [37], regarding the cyclic sieving phenomenon of Reiner, Stanton, and White [22] as it applies to jeu-de-taquin promotion on rectangular tableaux. To do this, we use Kazhdan-Lusztig theory and a characterization of the dual canonical basis of $\mathbb{C}[x_{11}, \ldots, x_{nn}]$ due to Skandera [27]. Afterwards, we extend our results to analyzing the fixed points of a dihedral action on rectangular tableaux generated by promotion and evacuation, suggesting a possible sieving phenomenon for dihedral groups. Finally, we give applications of this theory to cyclic sieving phenomena involving reduced words for the long elements of hyperoctahedral groups and noncrossing partitions.

1. INTRODUCTION

Suppose that we are given a finite set X equipped with the action of a finite cyclic group C generated by c. In studying the combinatorial structure of the action of C on X, it is natural to ask for the sizes of the fixed point sets $X^1 = X, X^c, X^{c^2}, \ldots, X^{c^{(|C|-1)}}$. Indeed, the cardinalities of the above sets determine the cycle structure of the image of c under the canonical homomorphism $C \to S_X$, so from a purely enumerative standpoint these fixed point set sizes determine the action of C on X.

Reiner, Stanton, and White studied such actions and associated polynomials to them which encode the sizes of all of the above fixed point sets at the same time [22]. Following their 2004 paper, we make the following definition.

Definition 1.1. Let C be a finite cyclic group acting on a finite set X and let c be a generator of C. Let $\zeta \in \mathbb{C}$ be a root of unity having the same multiplicative order as c and let $X(q) \in \mathbb{Q}[q]$ be a polynomial. We say that the triple $(X, C, X(q))$ *exhibits the cyclic sieving phenomenon (CSP)* if for any integer $d \geq 0$ we have that the fixed point set cardinality $|X^{c^d}|$ is equal to the polynomial evaluation $X(\zeta^d)$.

This definition generalizes Stembridge's notion of the $q = -1$ phenomenon [34], [35] which restricts the above definition to the case where the cyclic group C has order 2 or, equivalently, when we are given a set X equipped with an involution $X \to X$.

A few remarks are in order. First, notice that since the identity element of C fixes every element of X, we have that $X(1) = |X|$ whenever $(X, C, X(q))$ exhibits the CSP. Also, it is easy to show that given a finite cyclic group C acting on a finite set X, a polynomial $X(q)$ such that $(X, C, X(q))$ exhibits the CSP is unique modulo the ideal in $\mathbb{Q}[q]$ generated by the cyclotomic polynomial $\Phi_{|C|}(q)$. Finally, given any finite cyclic group C acting on a finite set X, it is possible to show that the triple $(X, C, X(q))$ exhibits the CSP, where $X(q) = \sum_{i=0}^{|C|-1} a_i q^i$ and a_i is the number of C-orbits in X with stabilizer order dividing i. Typically the interest in a CSP is that $X(q)$ may be taken to be a natural polynomial deformation of a formula enumerating $|X|$.

These polynomial deformations often come from the theory of q-numbers. For any $n \in \mathbb{N}$, define the q-analogue $[n]_q := \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}$. Following the case of natural numbers, we define further $[n]!_q := [n]_q[n-1]_q \ldots [1]_q$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q}$. It is well known that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the generating function for partitions which fit inside an $(n-k)$ by k rectangle and is, therefore, a polynomial [29]. We first give the 'canonical' example of the CSP.

Theorem 1.1. *(Reiner-Stanton-White 2004 [22]) Fix two positive integers $k \leq n$. Let X be the set of all subsets of $[n]$ having size k and let $C = \mathbb{Z}/n\mathbb{Z}$ act on X via the long cycle $(1, 2, \ldots, n) \in S_n$.*

Then, the triple $(X, C, X(q))$ exhibits the CSP, where $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$.

Reiner et. al. also proved a version of the above result for the case of multisets.

Theorem 1.2. *(Reiner-Stanton-White 2004 [22]) Fix two positive integers k and n. Let X be the set of all k-element multisets of $[n]$. Let $C = \mathbb{Z}/n\mathbb{Z}$ act on X by the long cycle $(1, 2, \ldots, n) \in S_n$.*

Then, the triple $(X, C, X(q))$ exhibits the CSP, where $X(q) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$.

In each of these results and in all of the rest of the CSPs appearing in this paper, the set X is a set of combinatorial objects and C is generated by a natural combinatorial operator on X. We shall see that Theorems 1.1 and 1.2 are both implied by CSPs regarding certain sets of tableaux and an action given by the sliding algorithm of jeu-de-taquin promotion. Postponing definitions until Section 2, we state our three main results. The following was conjectured by D. White [37].

Theorem 1.3. *Let $\lambda \vdash n$ be a rectangular partition and let $X = SYT(\lambda)$. Let $C = \mathbb{Z}/n\mathbb{Z}$ act on X by jeu-de-taquin promotion.*

Then, the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where $X(q)$ is the $q-$analogue of the hook length formula

$$X(q) = f^\lambda(q) := \frac{[n]!_q}{\Pi_{(i,j)\in\lambda}[h_{ij}]_q}.$$

There are several interpretations of the polynomial $X(q)$ in the above result. In addition to being the natural q-analogue of the hook length formula, up to a power of q we may also interpret $X(q)$ to be the fake degree polynomial corresponding to the λ-isotypic component of the action of S_n on the coinvariant algebra $\mathbb{C}[x_1, \ldots, x_n]/\mathbb{C}[x_1, \ldots, x_n]_+^{S_n}$ (see Proposition 4.1 Stanley [32] together with Corollary 7.21.5 of [30]). Moreover, we have that $X(q)$ is equal to the Kostka-Foulkes polynomial $K_{\lambda,1^n}(q)$ corresponding to λ and the composition (1^n) of length n consisting entirely of 1's. Finally, up to a power of q, $X(q)$ is equal to the $q-$analogue of the Weyl dimension formula for the $(1, 1, \ldots, 1)$-weight space of the irreducible representation of $GL_n(\mathbb{C})$ having highest weight λ. There is a column strict version of the above result, also conjectured by White [37].

Theorem 1.4. *Let $k \geq 0$ and let $\lambda \vdash n$ be a rectangular partition. Let $X = CST(\lambda, k)$ and let $C = \mathbb{Z}/k\mathbb{Z}$ act on X via jeu-de-taquin promotion.*

Then, the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where $X(q)$ is a $q-$shift of the principal specialization of the Schur function

$$X(q) := q^{-\kappa(\lambda)} s_\lambda(1, q, q^2, \ldots, q^{k-1}),$$

where κ is the statistic on partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ given by

$$\kappa(\lambda) = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + \ldots.$$

There are several interpretations of the polynomial $X(q)$ in the above theorem, as well. Just as standard tableaux of a fixed shape are enumerated by the hook length formula, the hook content formula enumerates column strict tableaux of a fixed shape (with uniformly bounded entries). Up to a power of q, the polynomial $X(q)$ is the $q-$analogue of the hook content formula. Also, $X(q)$ is equal to MacMahon's generating function for plane partitions which fit inside a box having dimensions λ_1 by λ_1' by $k - \lambda_1$ weighted by number of boxes [18]. This latter interpretation can be easily seen via the obvious bijection between column strict tableaux and plane partitions. Finally, up to a power of q, $X(q)$ is equal to the $q-$analogue of the Weyl dimension formula corresponding to the irreducible representation of $GL_k(\mathbb{C})$ having highest weight λ.

The previous two results have concerned standard tableaux of fixed rectangular shape and column strict tableaux of fixed rectangular shape and arbitrary content, respectively. We can also formulate a result suggested by Reiner and White [21] [37] concerning column strict tableaux of fixed shape and fixed content. Specifically,

suppose that $\lambda \vdash n$ is rectangular and $\alpha \models n$ is a composition with length $\ell(\alpha) = k$. Also suppose that α has some cyclic symmetry, i.e., there exists some $d|k$ such that the dth cyclic shift operator preserves α. It can be shown that the action of promotion on column strict tableaux acts cyclically on content compositions. Therefore, the dth power of promotion acts on the set of column strict tableaux of shape λ and content α.

Theorem 1.5. *Let $\lambda \vdash n$ and $\alpha \models n$ be as above. Let $C = \mathbb{Z}/(\frac{k}{d}\mathbb{Z}) = \langle c \rangle$ act on the set of column strict tableaux of shape λ and content α by the dth power of promotion. Let ζ be a primitive $\frac{k}{d}^{th}$ root of unity.*

Then, for any $m \geq 0$, the number of fixed points under the action of the mth power of promotion is equal to the modulus $|K_{\lambda,\alpha}(\zeta^m)|$, where $K_{\lambda,\alpha}(q)$ is the Kostka-Foulkes polynomial.

This result is almost, but not quite, a CSP. Specifically, since the evaluation of a Kostka-Foulkes polynomial at a root of unity may be strictly negative, we do not have that the action of the dth power of promotion on the relevant set of tableaux together with $K_{\lambda,\alpha}(q)$ exhibits the CSP. However, we do have cyclic sieving 'up to modulus'. Moreover, since the q-hook length formula is (up to a power of q) a special case of a Kostka-Foulkes polynomial, it follows that (again up to modulus) our result on standard tableaux is a special case of this latter result. It will turn out that Theorem 1.5 is a weight space refinement of Theorem 1.4.

The bulk of the remainder of this paper is devoted to the proofs of the above Theorems 1.3, 1.4, and 1.5. We now give our overarching philosophy which was also used by Stembridge [35] in relation to the action of evacuation on tableaux and the $q = -1$ phenomenon. Suppose that we are given some finite set X equipped with the action a finite cyclic group $C = \langle c \rangle$ and a polynomial $X(q)$ and we wish to show that $(X, C, X(q))$ exhibits the CSP. On its face, this is a purely enumerative problem - if we could find some formula for the evaluation of $X(q)$ at appropriate roots of unity and equate this with some formula enumerating the fixed point set of X under the action of appropriate elements of C, we would be done. This direct approach, when possible, has its merits. CSPs are certainly of enumerative interest and in many cases show that previously studied natural q analogues of counting formulas for fundamental combinatorial sets encode information about fixed point sets under fundamental combinatorial operators.

However, this naïve approach can often be very difficult. For example, while direct combinatorial proofs of the CSPs for subsets and multisets in [22] exist, there is no known enumerative proof of the CSPs for rectangular tableaux given above. We instead take the viewpoint that, instead of being purely enumerative, CSPs often conceal deeper algebraic structure which can be exploited in their proofs.

Specifically, suppose that we have some complex vector space V with basis $\{f_x \mid x \in X\}$ indexed by the elements of X. Suppose further that V is equipped with the action of a group G and denote the corresponding representation by $\rho : G \to GL(V)$. Finally suppose that we can find some element $g \in G$ whose action on V is given by the formula

$$(1.1) \qquad \rho(g)(f_x) = f_{c \cdot x},$$

for all $x \in X$. That is, the matrix for the action of g with respect to the given basis of V is the permutation matrix corresponding to the action of c on X. It is immediate that for any $d \geq 0$, the number of fixed points $|X^{c^d}|$ of the action of c^d on X is equal to the trace of the linear operator $\rho(g^d)$. In symbols, if $\chi : G \to \mathbb{C}$ is the character of our representation,

$$(1.2) \qquad |X^{c^d}| = \chi(g^d),$$

for all $d \geq 0$. So, we have reduced our problem of enumerating fixed point sets to the evaluation of a certain character at a certain group element. If we can interpret this character evaluation as an appropriate root of unity evaluation of $X(q)$, then our CSP will be proved.

At first glance, this approach may seem to complicate matters. We must first model our action of C on X in a representation theoretic context. Moreover, the evaluation of the character $\chi(g^d)$ may be no easier than the enumeration of the fixed point set X^{c^d}, particularly if the action of C on X is easy to understand. However, character theory is a well studied subject and provides us with much artillery with which to attack the former problem. An elementary and fundamentally important example of this is that if $h \in G$ is a group element which is conjugate to g, we have that $\chi(g^d) = \chi(h^d)$, where the latter character may be easier to compute. So, under nice conditions, the problem of proving a CSP may be reduced to a problem of studying conjugacy in a group. And, as a bonus, this method of proving a CSP reveals representation theoretic structure that may have previously gone unnoticed. In our situation, we will gain some understanding of why the hypothesis that our partitions be rectangular is necessary in our results. It is this approach that we will use in proving our CSPs.

These representation theoretic methods, however, come with some fine print. It is often too much to ask that an equation of the form (1.1) hold. Sometimes we must content ourselves with finding a group element $g \in G$ such that for all $x \in X$ we have

$$(1.3) \qquad \rho(g)(f_x) = \gamma(x) f_{c \cdot x},$$

where $\gamma : X \to \mathbb{C}$ is some function, hopefully as simple as possible. Moreover, the identification of a polynomial evaluation at a root of unity with a character evaluation may require more involved techniques than elementary group conjugacy. In particular,

this part of the proof of our result on standard tableaux will use results from Springer's theory of regular elements.

The remainder of this paper is organized as follows. In Section 2 we review some tableaux theoretic definitions, the algorithm of jeu-de-taquin promotion, and some representation theoretic tools we will be using in the proofs of our CSPs (specifically Kazhdan-Lusztig theory). In Section 3 we prove our CSP for standard tableaux. In Section 4 we derive a slightly new perspective on the irreducible representations of the general linear group which will lead to a proof of our CSP for column strict tableaux in Section 5. In Section 6 we extend the general philosophy of the proofs in Section 5 to get our result concerning column strict tableaux of fixed content. In Section 7 we extend our results on cyclic actions and prove some results which enumerate fixed points under combinatorial *dihedral* actions. In Section 8 we derive corollaries of our fixed point results on tableaux for other combinatorial actions involving handshake patterns and the reflection group B_n. We close in Section 9 with some open questions.

2. Tableaux and Representation Theory Background

We begin this section by gonig over the definitions of standard and column strict tableaux, as well as the definition of the action of jeu-de-taquin promotion. For a more leisurely introduction to this material, see [25] or [30].

Given a positive integer n, a *partition* λ of n is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The number k is the *length* $\ell(\lambda)$ of λ. The number n is the *size* $|\lambda|$ of λ. We write $\lambda \vdash n$ to indicate that λ is a partition of n. We will sometimes use exponential notation to write repeated parts of partitions, so that $4^2 3^3 1$ is the partition $(4, 4, 3, 3, 3, 1) \vdash 18$.

We identify partitions $\lambda \vdash n$ with their *Ferrers diagrams*, i.e., λ is identified with the subset of the lower right quadrant of $\mathbb{Z} \times \mathbb{Z}$ given by $\{(i, -j) \,|\, i \in [\lambda_j]\}$. For example, the Ferrers diagram of the partition $(4, 4, 3, 1) \vdash 12$ is given by

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{array} \quad .$$

Given a coordinate $(i, -j)$ in the Ferrers diagram of λ, the *hook length* h_{ij} at $(i, -j)$ is the number of dots directly south or directly east of $(i, -j)$, the dot (i, j) included. In the above partition $h_{12} = 6$. A partition is said to be *rectangular* if its Ferrers diagram is a rectangle. The *conjugate* λ' of a partition λ is the partition whose Ferrers diagram is obtained by reflecting the Ferrers diagram for λ across the line $y = -x$. So, $(4, 4, 3, 1)' = (4, 3, 3, 2)$.

Given partitions λ and μ such that we have a set theoretic containment of Ferrers diagrams $\mu \subseteq \lambda$, we define the *skew partition* λ/μ to be the set theoretic difference $\lambda \smallsetminus \mu$ of Ferrers diagrams. The *size* of λ/μ is the difference $|\lambda| - |\mu|$ and we write $\lambda/\mu \vdash n$ to denote that the skew partition λ/μ has size n.

Given a positive integer n, a *composition* α of n is a finite sequence of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ which satisfies $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$. In particular, some of the α_i may be zero. We write $\alpha \models n$ to denote that α is a composition of n. The number n is called the *size* $|\alpha|$ of α and the number k is called the *length* $\ell(\alpha)$ of α. If $\alpha \models n$ and $\ell(\alpha) = k$, the composition α defines a function $[n] \to [k]$ given by sending every number in the interval $(\alpha_1 + \cdots + \alpha_{i-1}, \alpha_1 + \cdots + \alpha_i]$ to the number i for $i = 1, 2, \ldots, k$. We denote this function by α, as well. For example, if $\alpha = (0, 2, 1, 0, 1)$, then $\alpha : [4] \to [5]$ is given by $\alpha(1) = 2, \alpha(2) = 2, \alpha(3) = 3, \alpha(4) = 5$. Given a partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of length k, define $c_k \cdot \alpha$ to be the cyclically rotated composition $(\alpha_2, \alpha_3, \ldots, \alpha_k, \alpha_1)$. This defines an action of the order k cyclic subgroup of S_k generated by $(1, 2, \ldots, k)$ on the set of all compositions of n having length k.

Let $\lambda \vdash n$ and let $\alpha \models n$ be a composition with $\ell(\alpha) = k$. A *column strict tableau* T of shape λ with content α is a filling of the Ferrers diagram of λ with α_1 $1's$, α_2 $2's$, $\ldots, \alpha_k$ $k's$ such that the numbers increase strictly down every column and weakly across every row. An example of a column strict tableau of shape $(4, 4, 3, 1)$ is

$$
\begin{array}{cccc}
1 & 1 & 3 & 4 \\
3 & 3 & 4 & 6 \\
4 & 5 & 5 & \\
6 & & &
\end{array}.
$$

Given a partition $\lambda \vdash n$ and a composition $\alpha \models n$, the set of all column strict tableaux of shape λ and content α is denoted $CST(\lambda, k, \alpha)$, where $k := \ell(\alpha)$ reminds us of the maximum possible entry in our tableau. The set of *all* column strict tableaux of shape λ with entries at most k is denoted $CST(\lambda, k)$, so that $CST(\lambda, k) = \bigcup_{\alpha \models n, \ell(\alpha) = k} CST(\lambda, k, \alpha)$. The tableaux obtained by requiring that entries increase weakly down columns and strictly across rows are called *row strict* and we have the analogous definitions of $RST(\lambda, k, \alpha)$ and $RST(\lambda, k)$. Finally, a tableau is called *standard* if its content composition consists entirely of $1's$. We denote by $SYT(\lambda)$ the set of all standard tableaux of shape λ and make the canonical identification of $SYT(\lambda)$ with $CST(\lambda, n, 1^n)$ for $\lambda \vdash n$. Given any partition $\lambda \vdash n$, we have the *column superstandard* tableaux $CSS(\lambda)$ obtained by filling in the boxes of λ within each column, going from left to right. For example,

$$
CSS((3, 3, 2)) = \begin{array}{ccc}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6 &
\end{array}.
$$

For a partition $\lambda \vdash n$, $f^\lambda := |SYT(\lambda)|$ is famously enumerated by the Frame-Robinson-Thrall hook length formula [8]:

$$(2.1) \qquad\qquad f^\lambda = \frac{n!}{\Pi_{(i,-j)\in\lambda} h_{ij}}.$$

Also given a positive integer k, the *Schur function* associated to the partition λ in k variables is the element of the polynomial ring $\mathbb{C}[x_1,\ldots,x_k]$ given by

$$(2.2) \qquad\qquad s_\lambda(x_1,\ldots,x_k) = \sum_{\alpha \models n, \ell(\alpha)=k} \sum_{T \in CST(\lambda,k,\alpha)} x_1^{\alpha_1}\ldots x_k^{\alpha_k}.$$

That is, the Schur function associated to λ is the generating function for column strict tableaux of shape λ weighted by their content compositions.

The sets X involved in the most important CSPs proved in this paper will be standard tableaux with fixed rectangular shape, column strict tableaux with fixed rectangular shape and uniformly bounded entries, and column strict tableaux with fixed rectangular shape and specified content. The cyclic actions on each of these sets will be based on jeu-de-taquin promotion, a combinatorial algorithm which we presently outline. The action of promotion has received recent attention from Stanley [31], who considers a related action on linear extensions of finite posets, and from Bandlow, Schilling, and Thiery [2], who derive crystal theoretic uniqueness results related to the action of promotion on column strict tableaux.

Roughly speaking, promotion acts on tableaux by deleting all of the highest possible entries, sliding the remaining entries out while preserving the column strict condition, and then increasing all entries by one and filling holes with one so that the resulting object is a column strict tableaux. More formally, suppose that we are given a partition $\lambda \vdash n$ and a positive integer k. We define the jeu-de-taquin promotion operator $j : CST(\lambda,k) \to CST(\lambda,k)$ as follows. Given a tableau T in $CST(\lambda,k)$, first replace every k appearing in T with a dot. Suppose that there is some dot in the figure so obtained that is not contained in a continuous strip of dots in the northwest corner. Then, choose the westernmost dot contained in a connected component of dots not in the northwest corner of the figure. Say this dot has coordinates (a,b), where $a > 0$ and $b < 0$. By our choice of dot, at least one of the positions $(a-1,b)$ or $(a,b+1)$ must be filled with a number - i.e., the positions immediately west or north of our dot. If only one of these positions has a number, interchange that number and the dot. If both of these positions have a number, interchange the greater of these two numbers and the dot (if the two numbers are equal, interchange the number at the northern position $(a,b+1)$ and the dot). This interchange moves the dot one unit north or one unit west. Continue interchanging the dot with numbers in this fashion until the dot lies in a connected component of dots in the northwest corner of the resulting figure. If all of the dots in the resulting figure are not in a connected component in the northwest corner, choose the westernmost dot contained

in a connected component of dots not in the northwest corner of the figure and slide this new dot to the northwest corner in the same way. Iterate this process until all dots are contained in a connected component in the northwest corner of the figure. Now increase all the numbers in the figure by 1 and replace the dots with $1's$. The resulting figure is $j(T)$.

Example 2.1. Let T be the following element of $CST((4, 4, 3, 1), 6)$:

$$T = \begin{array}{cccc} 1 & 1 & 3 & 4 \\ 3 & 3 & 4 & 6 \\ 4 & 5 & 5 & \\ 6 & & & \end{array}.$$

We compute the image $j(T)$ of T under jeu-de-taquin promotion.

$$T = \begin{array}{cccc} 1 & 1 & 3 & 4 \\ 3 & 3 & 4 & 6 \\ 4 & 5 & 5 & \\ 6 & & & \end{array} \mapsto \begin{array}{cccc} 1 & 1 & 3 & 4 \\ 3 & 3 & 4 & \bullet \\ 4 & 5 & 5 & \\ \bullet & & & \end{array} \mapsto \begin{array}{cccc} \bullet & 1 & 3 & 4 \\ 1 & 3 & 4 & \bullet \\ 3 & 5 & 5 & \\ 4 & & & \end{array}$$

$$\mapsto \begin{array}{cccc} \bullet & \bullet & 1 & 3 \\ 1 & 3 & 4 & 4 \\ 3 & 5 & 5 & \\ 4 & & & \end{array} \mapsto \begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 4 & 5 & 5 \\ 4 & 6 & 6 & \\ 5 & & & \end{array} = j(T)$$

Notice that the content of T is $(2, 0, 3, 3, 2, 2)$ and the content of $j(T)$ is $(2, 2, 0, 3, 3, 2)$ - that is, j acts by cyclic rotation on content compositions. Also notice that the result of applying j to T depended on our considering T as an element of $CST((4, 4, 3, 1), 6)$. If we had considered T as an element of $CST((4, 4, 3, 1), k)$ for any $k > 6$, then we would have

$$j(T) = \begin{array}{cccc} 2 & 2 & 4 & 5 \\ 4 & 4 & 5 & 7 \\ 5 & 6 & 6 & \\ 7 & & & \end{array}.$$

For the proof of the following lemma about promotion, see [25] or [30].

Lemma 2.1. Let T be in $CST(\lambda, k)$.
1. The tableau $j(T)$ is well-defined, i.e., independent of the order in which we chose dots to slide northwest in the algorithm.
2. The tableau $j(T)$ is column strict with entries $\leq k$.
3. If T has content α, then $j(T)$ has content $c_k \cdot \alpha$.

So, j is indeed a well defined function $CST(\lambda, k) \to CST(\lambda, k)$. By running the defining algorithm for j in reverse, we see that j is a bijection and we call its inverse j^{-1} jeu-de-taquin *demotion*. By Part 3 of the above lemma, j restricts to an operator

$SYT(\lambda) \to SYT(\lambda)$ where we consider the 'upper bound' k to be equal to $|\lambda|$. We call this restriction promotion, as well, and retain the notation j for it.

We now review the construction of the Kazhdan-Lusztig basis for the Hecke algebra $H_n(q)$ and the symmetric group algebra $\mathbb{C}[S_n]$, the construction of the Kazhdan-Lusztig cellular representations of S_n, and the interaction of these representations with the combinatorial insertion algorithm of RSK. See [4], [9], [25], and [30] for a more leisurely introduction to this material.

A fundamental result in the representation theory of the symmetric group S_n is that we have an isomorphism of $\mathbb{C}[S_n]$-modules

$$\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} f^\lambda W_\lambda,$$

where W_λ is the irreducible representation of S_n indexed by the partition λ. Viewing the left regular representation as a well understood object, it is natural to study bases of the left hand side of the above isomorphism which allow the visualization of the decomposition on the right hand side and, in particular, facilitate the study of the irreducibles W_λ. The 'natural' basis $\{w \,|\, w \in S_n\}$ fails miserably in this regard - every element of the symmetric group acts as an $n!$ by $n!$ permutation matrix with respect to this basis, rendering the above isomorphism invisible. It turns out a basis for $\mathbb{C}[S_n]$ in which the above isomorphism is evident arises in a natural way when one studies algebras which generalize the symmetric group algebra.

In a fundamental 1978 paper [14], Kazhdan and Lusztig studied the representation theory of the Hecke algebra $H_n(q)$, which is a quantum deformation of the symmetric group algebra $\mathbb{C}[S_n]$ and reduces to $\mathbb{C}[S_n]$ in the specialization $q = 1$. $H_n(q)$ admits an involution which restricts to the identity on $\mathbb{C}[S_n]$ and it is natural to ask whether there are bases of $H_n(q)$ which are invariant under this involution. The answer is 'yes', and up to certain normalization conditions these bases are unique. Specialization of this basis of $H_n(q)$ at $q = 1$ yields a basis of $\mathbb{C}[S_n]$ which, as we will see, can be viewed as 'upper triangular' with respect to the above isomorphism. Also amazingly, the avatars of the irreducibles W_λ so obtained will interact very nicely with combinatorial notions such as descent sets of tableaux and RSK insertion.

The symmetric group S_n has a Coxeter presentation with generators $s_1, s_2, \ldots, s_{n-1}$ and relations

$$
\begin{aligned}
(2.3) \qquad\qquad s_i s_j &= s_j s_i & \text{for } |i - j| > 1, \\
(2.4) \qquad\qquad s_i s_j s_i &= s_j s_i s_j & \text{for } |i - j| = 1, \\
(2.5) \qquad\qquad s_i^2 &= 1 & \text{for all } i.
\end{aligned}
$$

We interpret s_i to be the adjacent transposition $(i, i+1)$. The *length* $\ell(w)$ of a permutation $w \in S_n$ is the minimum value of r so that $w = s_{i_1} \ldots s_{i_r}$ for some adjacent transpositions s_{i_j}. Call such a minimal length word *reduced*. We define the *left descent set of* w to be the subset $D_L(w)$ of $[n-1]$ given by $D_L(w) := \{i \mid \ell(s_i w) < \ell(w)\}$. The *right descent set of* w, $D_R(w)$, is defined analogously. In this paper, we will denote permutations in S_n by their cycle decomposition, their expressions as words in the Coxeter generators s_i, and by their one line notation. In this latter system, writing $w = w_1 w_2 \ldots w_n$ means that w sends 1 to w_1, 2 to w_2, and so on.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$, we define the *Young subgroup of* S_n *indexed by* λ to be the subgroup S_λ of S_n which stabilizes the sets

$$\{1, 2, \ldots, \lambda_1\}, \{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}, \ldots, \{\lambda_1 + \cdots + \lambda_{k-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_k = n\}.$$

For example, if $\lambda = (3, 2, 2) \vdash 7$, then the associated Young subgroup of S_7 is given by $S_\lambda = S_{\{1,2,3\}} \times S_{\{4,5\}} \times S_{\{6,7\}}$. In general, we have a natural direct product decomposition $S_\lambda \cong S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ which implies the corresponding order formula $|S_\lambda| = \lambda_1! \cdots \lambda_k!$.

S_n comes equipped with a (strong) Bruhat order, the partial order given by the transitive closure of $w \prec v$ if and only if there exists some (not necessarily simple) reflection $t \in T := \bigcup_{w \in S_n, i \in [n-1]} w s_i w^{-1}$ such that $v = tw$ and $\ell(w) \leq \ell(v)$. Unless otherwise indicated, writing $w \leq v$ for permutations w and v will always mean comparability in Bruhat order. The identity permutation 1 is the unique minimal element of S_n under Bruhat order. The *long element* w_o whose one-line notation is $n(n-1) \ldots 1$ is the unique maximal element in S_n under Bruhat order.

Let q be a formal indeterminate. The *Hecke algebra* $H_n(q)$ is the $\mathbb{C}(q^{1/2})$-algebra with generators $T_{s_1}, T_{s_2}, \ldots, T_{s_{n-1}}$ subject to the relations:

$$\text{(2.6)} \qquad T_{s_i} T_{s_j} = T_{s_j} T_{s_i} \qquad \qquad \text{for } |i - j| > 1,$$

$$\text{(2.7)} \qquad T_{s_i} T_{s_j} T_{s_i} = T_{s_j} T_{s_i} T_{s_j} \qquad \qquad \text{for } |i - j| = 1,$$

$$\text{(2.8)} \qquad T_{s_i}^2 = (1 - q) T_{s_i} + q \qquad \qquad \text{for all } i.$$

It turns out that if w is a permutation in S_n and $s_{i_1} \ldots s_{i_r}$ is a reduced expression for w, then the Hecke algebra element $T_w := T_{s_{i_1}} \ldots T_{s_{i_r}}$ is independent of the choice of reduced word for w. Moreover, the set $\{T_w \mid w \in S_n\}$ forms a basis for $H_n(q)$ over the field $\mathbb{C}(q^{1/2})$. Finally, it is obvious that the specialization of the defining relations of $H_n(q)$ to $q = 1$ yields the classical group algebra $\mathbb{C}[S_n]$.

It follows from the definition of $H_n(q)$ that the generator T_{s_i} is invertible for all i, with $T_{s_i}^{-1} = \frac{1}{q}(T_{s_i} - (1 - q))$. Therefore, for any permutation w in S_n we have that the algebra element T_w is invertible, being a product of invertible elements. With this in mind, define an involution D of $H_n(q)$ by $D(q^{1/2}) = q^{-1/2}$ and $D(T_w) = (T_{w^{-1}})^{-1}$ and

extending linearly over $\mathbb{C}$. In the latter formula, the inverse in the subscript is taken in the symmetric group S_n and the inverse in the exponent is taken in the Hecke algebra $H_n(q)$. Notice that in the specialization at $q = 1$, the involution D is just the identity map on $\mathbb{C}[S_n]$.

Theorem 2.2. *(Kazhdan-Lusztig* [14]*) There is a unique basis*
$\{C'_w(q) = (q^{-1/2})^{\ell(w)} \sum_{v \in S_n} P_{v,w}(q) T_v \,|\, w \in S_n\}$ *of $H_n(q)$ such that*
1. (Invariance) $D(C'_w(q)) = C'_w(q)$ for all permutations $w \in S_n$,
2. (Polynomality) $P_{w,v}(q) \in \mathbb{Z}[q]$ always,
3. (Normalization) $P_{w,w}(q) = 1$ for any $w \in S_n$,
4. (Bruhat compatibility) $P_{v,w} = 0$ unless $v \leq w$ in Bruhat order, and
5. (Degree bound) The degree of $P_{v,w}(q)$ is at most $(1/2)(\ell(w) - \ell(v) - 1)$.

The basis in the above theorem is called the *Kazhdan-Lusztig (KL) basis* of $H_n(q)$. Its specialization to $q = 1$ is a basis of $\mathbb{C}[S_n]$, but to make some results in this paper look cleaner we will throw in some signs and call $\{C'_w(1) \,|\, w \in S_n\}$ the *KL basis* of $\mathbb{C}[S_n]$, where $C'_w(1) = \sum_{v \in S_n} (-1)^{\ell(w)-\ell(v)} P_{v,w}(1) v$. These signs will not seriously affect the representation theory. The polynomials $P_{v,w}(q)$ are the *KL polynomials* and are notoriously difficult to compute for general v and w. The KL basis of either $H_n(q)$ or $\mathbb{C}[S_n]$ leads to the definition of the *KL representation*, which is just the left regular representation of either algebra viewed with respect to this basis.

In light of the degree bound in the above theorem, we recall a statistic $\mu(v, w)$ on ordered pairs of permutations $v, w \in S_n$ by letting $\mu(v, w) = [q^{(\ell(w)-\ell(v)-1)/2}] P_{v,w}(q)$. So, $\mu(v, w)$ is the coefficient of the maximum possible power of q in $P_{v,w}(q)$. By Bruhat compatibility and polynomality, we have that $\mu(v, w) = 0$ unless $v \leq w$ and also $\ell(v, w) := \ell(w) - \ell(v)$ is odd. Moreover, we introduce a symmetrized version of μ given by $\mu[u, v] := \max\{\mu(u, v), \mu(v, u)\}$.

The KL μ function can be used to get a recursive formula for the $P_{u,v}(q)$ which will be of technical importance to us in what follows.

Lemma 2.3. *(*[14]*, Equation 2.2c) If $u \leq w$ and $i \in D_L(w)$, then*

$$(2.9) \qquad P_{u,w}(q) = q^{1-c} P_{s_i u, s_i w}(q) + q^c P_{u, s_i w}(q) - \sum_{s_i v < v} q^{\frac{\ell(v,w)}{2}} \mu(v, s_i w) P_{u,v}(q),$$

where $c = 1$ if $i \in D_L(u)$ and $c = 0$ otherwise.

The KL μ function is well behaved with respect to left and right multiplication by the long element $w_o \in S_n$, as well as taking the inverses of the permutations involved. For a proof of the following lemma, see for example [4].

Lemma 2.4. *Let $u, v \in S_n$. We have that $\mu(u, v) = \mu(w_o v, w_o u) = \mu(v w_o, u w_o) = \mu(w_o u w_o, w_o v w_o)$. Also, we have that $\mu(u, v) = \mu(u^{-1}, v^{-1})$.*

Starting from the KL representation, we get a natural preorder on permutations in S_n. Specifically, for u and v in S_n, say that $u \leq_L^* v$ if and only if there exists some i in $[n-1]$ such that $C'_v(1)$ appears with nonzero coefficient in the expansion of the product $s_i C'_u(1)$ in the KL basis of $\mathbb{C}[S_n]$. The transitive closure of $\leq_L^*$ defines a preorder $\leq_L$ on S_n (i.e., a reflexive, transitive relation which need not be antisymmetric). The preorder $\leq_L$ is called the *left KL preorder*. The *right KL preorder* $\leq_R$ is defined in exactly the same way, but by instead considering the *right* regular representation of $H_n(q)$ or $\mathbb{C}[S_n]$. The *two sided KL preorder* $\leq_{LR}$ is the transitive closure of the union of $\leq_L$ and $\leq_R$.

Given any set X equipped with a preorder $\leq$, we can define an equivalence relation on X by $x \sim y$ if and only if $x \leq y$ and $y \leq x$. The preorder $\leq$ induces a partial order on the equivalence classes $X/\sim$ also denoted $\leq$ defined by $[x] \leq [y]$ if and only if for some elements $x \in [x]$ and $y \in [y]$, $x \leq y$. The equivalence classes of permutations in S_n so defined via the preorders $\leq_L, \leq_R,$ and $\leq_{LR}$ are called the *left, right, and two sided KL cells*, respectively. Remarkably, these cells can be identified via an explicit combinatorial algorithm.

It follows from counting the dimensions on both sides of the isomorphism $\mathbb{C}[S_n] \cong \bigoplus_{\lambda \vdash n} f^\lambda W^\lambda$ that the sets of permutations w in S_n and ordered pairs (P, Q) of standard tableaux of the same shape with n boxes are in bijection. The *Robinson-Schensted-Knuth (RSK)* algorithm gives an explicit bijection between these sets. For a detailed definition of its algorithm see [25] or [30]. In this paper, $w \mapsto (P(w), Q(w))$ will always mean that w *row* inserts to $(P(w), Q(w))$. For example, in S_6 we have that

$$623415 \mapsto \begin{pmatrix} \begin{array}{cccc} 1 & 3 & 4 & 5 \\ 2 & & & \\ 6 & & & \end{array} & , & \begin{array}{cccc} 1 & 3 & 4 & 6 \\ 2 & & & \\ 5 & & & \end{array} \end{pmatrix}.$$

Define the *shape* $\mathrm{sh}(w)$ of a permutation w to be the shape of either $P(w)$ or $Q(w)$. So, $\mathrm{sh}(623415) = (4,1,1)$.

To further explore the interaction between the RSK algorithm and the algebraic properties of the Coxeter group S_n, we introduce the notion of a descent set of a tableau. Given $\lambda \vdash n$ and $T \in SYT(\lambda)$, the *descent set* $D(T)$ of T is the subset of $[n-1]$ defined by $i \in D(T)$ if and only if $i+1$ occurs strictly south and weakRoughly speaking, promotion acts on tableaux by deleting all of the highest possible entries, sliding the remaining entries out while preserving the column strict condition, and then altering entries and filling holes so that the resulting object is a column strict tableaux.ly west of i in T. For example, if

$$T = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 6 & & \end{array} \quad ,$$

then $D(T) = \{1, 3, 5\}$. In the case of rectangular tableaux, we will later generalize the descent set to another combinatorial object called an *extended descent set* which will help us greatly in proving our representation theoretic results.

The RSK algorithm behaves in a predictable way with respect to taking inverses, finding descent sets, and left and right multiplication by the long permutation w_o. To show this, we recall Schützenberger's combinatorial algorithm of *evacuation*. Given an arbitrary partition $\lambda \vdash n$ and a tableau T in $CST(\lambda, k)$ for some $k \geq 0$, define the image $e(T)$ of T under evacuation as follows. First, embed T in the northwest corner of a very large rectangle. Then rotate the rectangle by 180 degrees, moving T to the southeast corner. Then, for $i = 1, 2, \ldots, k$, replace each i occurring in T with $k - i + 1$. Finally, use the jeu-de-taquin sliding algorithm to move the boxes of T from the southeast corner to the northwest corner of the rectangle. The resulting tableau is $e(T)$.

Lemma 2.5. *Let $\lambda \vdash n$ and let $T \in CST(\lambda, k, \alpha)$ for some $k \geq 0$ and some composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$.*
1. $e(T)$ is a well-defined element of $CST(\lambda, k)$, that is, independent of the choices involved in embedding or sliding.
2. The content of $e(T)$ is $(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$.
3. The operator e is an involution, that is, $e(e(T)) = T$ always.

With Part 2 of the above lemma as motivation, given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, we define $w_{o_k} \cdot \alpha$ to be the reverse composition $(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$. It follows from the above lemma that e restricts to an involution on $SYT(\lambda)$ for arbitrary partitions λ. Evacuation interacts nicely with RSK. Proofs of these results can be found, for example, in [25] and [4].

Lemma 2.6. *Let $w \in S_n$ and suppose $w \mapsto (P, Q)$.*
1. $w^{-1} \mapsto (Q, P)$.
2. $w_o w \mapsto (P', e(Q)')$.
3. $w w_o \mapsto (e(P)', Q')$.
4. $w_o w w_o \mapsto (e(P), e(Q))$.
5. $i \in D_L(w)$ if and only if $i \in D(P)$.
6. $i \in D_R(w)$ if and only if $i \in D(Q)$,
where T' denotes the conjugate of a standard tableau T.

RSK leads to a natural triple of equivalence relations on S_n. We say that two permutations $v, w \in S_n$ are *left Knuth equivalent* if we have $P(w) = P(v)$. Analogously, v and w are *right Knuth equivalent* if we have the equality $Q(w) = Q(v)$. Finally, we have an equivalence relation given by $w \sim v$ if and only if w and v have the same shape. Amazingly, these algorithmic, combinatorial equivalence classes agree with the KL cells.

Theorem 2.7. *Let w and v be permutations in S_n.*
1. w and v are left Knuth equivalent if and only if w and v lie in the same left KL cell.
2. w and v are right Knuth equivalent if and only if w and v lie in the same right KL cell.
3. w and v have the same shape if and only if w and v lie in the same two-sided KL cell.

So, we may interpret the partial orders induced by the left and right KL preorders as partial orders on the set of standard tableaux with n boxes. It can be shown that these two partial orders are identical. Similarly, the induced partial order on two-sided KL cells can be identified with a partial order on all partitions of n. It can be shown that this latter partial order is just the dominance order $\leq_{dom}$ defined by $\mu \leq_{dom} \lambda$ if and only if for all $i \geq 0$ we have the comparability of partial sums $\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i$. We also have the following change-of-label result, which shows that the symmetrized KL μ-function behaves well with respect to a translation of Knuth class.

Theorem 2.8. *(Change of label) Identify permutations with their images under RSK. Let U_1, U_2, T_1, and T_2 be standard tableaux with n boxes, all having the same shape. We have that*
1. $\mu[(U_1, T_1), (U_1, T_2)] = \mu[(U_2, T_1), (U_2, T_2)]$ and
2. $\mu[(U_1, T_1), (U_2, T_1)] = \mu[(U_1, T_2), (U_2, T_2)]$.

Thanks to change of label, we can define $\mu[P, Q]$ for two standard tableaux P and Q of the same shape to be the common value of $\mu[(P, T), (Q, T)] = \mu[(T, P), (T, Q)]$, where T is any standard tableau having the same shape as P and Q.

With these results in hand, we are ready to define a powerful avatar of the Specht modules. Let $\lambda \vdash n$ be a partition and choose an arbitrary tableau $T \in SYT(\lambda)$. By Theorem 2.7 and its following remarks we have a left action of S_n on $S_0^{\lambda,T} := \bigoplus_w \mathbb{C}\{C'_w(1)\}$, where the w in the direct sum ranges over all permutations such that either $\text{sh}(T) <_{dom} \text{sh}(w)$ or $T = P(w)$. Also by the paragraph following Theorem 2.7 the module $S_0^{\lambda,T}$ contains an S_n-invariant submodule $S_1^{\lambda,T} := \bigoplus_w \mathbb{C}\{C'_w(1)\}$, where now w ranges over all permutations such that $\text{sh}(T) <_{dom} \text{sh}(w)$. It therefore makes sense to define $S^{\lambda,T}$ to be the quotient module

$$S^{\lambda,T} := S_0^{\lambda,T}/S_1^{\lambda,T}.$$

The vector space $S^{\lambda,T}$ carries an action of the group algebra $\mathbb{C}[S_n]$. We can identify the basis vectors of $S^{\lambda,T}$ with elements of $SYT(\lambda)$ via associating to a tableau U of shape λ the image of $C'_{(T,U)}(1)$ in the above quotient.

It is natural to ask to what degree the module $S^{\lambda,T}$ depends on the choice of standard tableau T. In this direction, it turns out that the action of the Coxeter

generators s_i on the space $S^{\lambda,T}$ is completely determined by the symmetrized KL μ-function and the purely combinatorial descent set of a tableau. Identifying the basis elements of $S^{\lambda,T}$ with tableaux in $SYT(\lambda)$, we have that the action of s_i on $S^{\lambda,T}$ is given by the explicit formula

$$(2.10) \qquad s_i P = \begin{cases} -P & \text{if } i \in D(P) \\ P + \sum_{i \in D(Q)} \mu[P,Q]Q & \text{if } i \notin D(P). \end{cases}$$

This formula can be proven via some reasonably explicit manipulations of the KL basis and KL polynomials [9]. In particular, the action of s_i does not depend on the tableau T that we chose in the construction of $S^{\lambda,T}$. It follows that the matrices giving the left action of s_i with respect to the given bases of $S^{\lambda,T}$ and $S^{\lambda,U}$ are literally equal (up to reordering basis elements) for any two standard tableaux $T, U \in SYT(\lambda)$.

With this strong isomorphism in hand, we define the $\mathbb{C}[S_n]$-module S^λ to be the $\mathbb{C}[S_n]$-module $S^{\lambda,T}$ for any choice of $T \in SYT(\lambda)$. Again remarkably, the modules S^λ are precisely the irreducible representations of S_n. For an exposition and extended version of the following result, see [9].

Theorem 2.9. *The module S^λ is isomorphic as a $\mathbb{C}[S_n]$-module to the irreducible representation of S_n indexed by the partition λ.*

S^λ is called the *(left) KL cellular representation* indexed by λ. The basis elements of S^λ are homomorphic images of a subset of the KL basis of $\mathbb{C}[S_n]$ and are in natural bijection with the elements of $SYT(\lambda)$. This avatar of the Specht modules will turn out to be very useful in the representation theoretic modeling of combinatorial operators.

An earlier use of the modules S^λ in this way is due to Berenstein, Zelevinsky [3] and Stembridge [35] and concerns the action of the evacuation operator e. Since e has order 2, one would hope to find an order 2 element in the symmetric group S_n which maps (up to sign) to the permutation matrix corresponding the evacuation under the KL cellular representation. It turns out that the long element does the trick.

Theorem 2.10. (*Berenstein-Zelevinsky* [3], *Stembridge* [35]) *Identify the basis vectors of the Kazhdan-Lusztig (left) cellular representation corresponding to λ with elements $P \in SYT(\lambda)$. Denote this representation by $\rho : S_n \to GL(S^\lambda)$. Let $w_o \in S_n$ be the long element.*

Then, up to a plus or minus sign, we have that $\rho(w_o)$ is the linear operator which sends P to $e(P)$, where e is evacuation.

Informally, this result states that given any partition $\lambda \vdash n$, the image of the long element $w_o \in S_n$ models the action of evacuation on $SYT(\lambda)$. It will turn out that for *rectangular* partitions λ, the image of the long *cycle* $(1, 2, \ldots, n)$ in S_n will model the

action of promotion on $SYT(\lambda)$. A straightforward application of the Murnaghan-Nakayama rule determines whether the sign appearing in the above theorem is a plus or a minus. The issue of resolving this sign will be slightly more involved for us and will be taken care of by direct manipulation of the KL basis. In the next section we show precisely how this is done.

3. Promotion on Standard Tableaux

Given a rectangular shape $\lambda = b^a$ with $ab = n$, we want to determine how the operation of jeu-de-taquin promotion on $SYT(\lambda)$ interacts with the left KL cellular representation S^λ of shape λ. Our first goal is to show that the promotion operator j interacts nicely with the μ function.

Define a deletion operator $d : SYT(b^a) \to SYT(b^{a-1}(b-1))$ by letting $d(U)$ be the (standard) tableau obtained by deleting the n in the lower right hand corner of U. It is easy to see that d is a bijection. Our first lemma shows that d is well behaved with respect to the μ function.

Lemma 3.1. *For any $U, T \in SYT(b^a)$, we have that $\mu[U, T] = \mu[d(U), d(T)]$.*

Proof. The line of reasoning here is similar to one used by Taskin in Lemma 3.12 of [36].

Recall that for any partition $\lambda \vdash n$ we have the associated column superstandard tableau $CSS(\lambda)$ defined by inserting the numbers $1, 2, \ldots, n$ into the diagram of λ first from top to bottom within each column, then from left to right across columns. Define permutations $u = u_1 \ldots u_n$, $v = v_1 \ldots v_n \in S_n$ and $t = t_1 \ldots t_{n-1}$, $w = w_1 \ldots w_{n-1} \in S_{n-1}$ by their images under RSK:

$$u \mapsto (U, CSS(b^a)),$$
$$v \mapsto (T, CSS(b^a)),$$
$$w \mapsto (d(U), CSS(b^{a-1}(b-1))),$$
$$t \mapsto (d(T), CSS(b^{a-1}(b-1))).$$

Using the definition of the RSK algorithm it's easy to check that, using one-line notation for permutations in S_n,

$$u_1 \ldots u_n = w_1 w_2 \ldots w_{n-a}(n)w_{n-a+1} \ldots w_{n-1},$$
$$v_1 \ldots v_n = t_1 t_2 \ldots t_{n-a}(n)t_{n-a+1} \ldots t_{n-1}.$$

By a result of Brenti ([5], Theorem 4.4), we have an equality of Kazhdan-Lusztig polynomials $P_{w,t}(q) = P_{w(n),t(n)}(q)$. For $1 \le k \le n$, define permutations $w^{(k)}$ and $t^{(k)}$ in S_n by $w^{(k)} := s_k s_{k+1} \cdots s_{n-1}(w(n))$, $t^{(k)} := s_k s_{k+1} \cdots s_{n-1}(t(n))$. Here $w(n)$ (respectively $t(n)$) is the permutation in S_n whose one-line notation is $w_1 \ldots w_{n-1}n$

(respectively $t_1 \ldots t_{n-1} n$). By the above remarks it follows that $w^{(n-a+1)} = u$ and $t^{(n-a+1)} = v$.

We claim that for all k, $P_{w^{(k)}, t^{(k)}}(q) = P_{w^{(k+1)}, t^{(k+1)}}(q)$. It will then follow by induction and Brenti's result that $P_{u,v}(q) = P_{w,t}(q)$. To see this, notice that $w^{(k)} = s_k w^{(k+1)}$, $t^{(k)} = s_k t^{(k+1)}$, and the transposition s_k satisfies $s_k \in D_L(w^{(k+1)}) \cap D_L(t^{(k+1)})$. These conditions together with Lemma 2.3 in the case $c = 1$ imply that we have the following polynomial relation:

$$(3.1) \quad P_{w^{(k)}, t^{(k)}}(q) = P_{s_k w^{(k+1)}, s_k t^{(k+1)}}(q)$$

$$= P_{w^{(k+1)}, t^{(k+1)}}(q) + q P_{s_k w^{(k+1)}, t^{(k+1)}}(q) - \sum_{s_k r < r} q^{\frac{\ell(r, t^{(k)})}{2}} \mu(r, t^{(k+1)}) P_{s_k w^{(k+1)}, r}(q).$$

However, we also have that both $w^{(i-1)}$ and $t^{(i-1)}$ map i to n for all i. Therefore, $w^{(k)} \not\leq t^{(k+1)}$ and the Bruhat interval $[w^{(k)}, t^{(k+1)}]$ is empty. This implies that the only surviving term in the above sum is $P_{w^{(k+1)}, t^{(k+1)}}(q)$ and we get that $P_{u,v}(q) = P_{w,t}(q)$, as desired. To complete the proof, we make the easy observations that $\ell(u,v) = \ell(w,t)$ and that the pairs (u,v) and (w,t) both lie in the same right cell. Therefore, by taking the left tableaux of the pairs (u,v) and (w,t), we get the desired equality of μ coefficients. $\square$

Observe that the proof of Lemma 3.1 shows a stronger statement regarding an equality of KL polynomials rather than just an equality of their top coefficients.

It is easy to show using Lemma 2.4 and Part 4 of Lemma 2.6 that $\mu[e(P), e(Q)] = \mu[P, Q]$ for any standard tableaux P and Q having the same *arbitrary* shape. We use this fact and the above result to get the desired fact about the action of j.

Proposition 3.2. *Let P, Q be standard tableaux which are either both in $SYT(b^a)$ or $SYT(b^{a-1}(b-1))$. We have that $\mu[P, Q] = \mu[j(P), j(Q)]$.*

This proposition says that, in the special case that our shape is a rectangle or a rectangle missing its outer corner, the action of promotion preserves the μ function. This does not hold in general for other shapes. Indeed, a counterexample may be found in the smallest shapes which do not meet the criteria of Proposition 3.2: $\lambda = (3,1)$ and $\lambda = (2,1,1)$. In the case of $\lambda = (3,1)$, promotion acts on the set $SYT((3,1))$ via a single 3-cycle:

$$\begin{pmatrix} \begin{array}{ccc} 1 & 2 & 3 \\ 4 \end{array} & , & \begin{array}{ccc} 1 & 3 & 4 \\ 2 \end{array} & , & \begin{array}{ccc} 1 & 2 & 4 \\ 3 \end{array} \end{pmatrix}.$$

However, it is easy to check that in S_4 we have $\mu[4123, 2134] = 0$, $\mu[2134, 3124] = 1$, and $\mu[3124, 4123] = 1$. Since for any standard tableau T the column reading word of T (i.e., the element of $S_{|T|}$ obtained by reading off the letters in T from bottom to top, then from left to right) has insertion tableau T, we obtain the claimed counterexample.

This lemma and the above example give some representation theoretic justification for why we need our hypotheses that λ be rectangular in Theorems 1.5 through 1.7, but perhaps leave mysterious why these results fail in general for rectangles minus outer corners. We will soon give combinatorial intuition for why the strictly rectangular hypotheses are needed.

Proof. (of Proposition 3.2) We prove this first for the case of rectangular shapes. The case of shapes which are rectangular minus an outer corner will follow easily. We introduce a family of operators on tableaux. Define a creation operator $c : SYT(b^{a-1}(b-1)) \to SYT(b^a)$ by letting $c(T)$ be the tableau obtained by adding a box labeled n to the lower right hand corner of T. Given a tableau T of shape λ/μ, let $r(T)$ be the tableau obtained by rotating T by 180 degrees and let $\delta(T)$ be the tableau obtained by replacing every entry i in T with $|T| + 1 - i$. Given a standard tableau T which is not skew, let $J(T)$ be the tableau obtained by embedding T in the northwest corner of a very large rectangle, and playing jeu-de-taquin to move the boxes of T to the southeast corner. Observe that we have the following relations of operators:

$$\text{I. } \delta r = r \delta$$
$$\text{II. } e = \delta r J$$
$$\text{III. } j = r \delta c \delta r J d$$

Let P and Q be standard tableaux of shape b^a. With the above definitions, we have the following chain of equalities:

$$
\begin{aligned}
\mu[P, Q] &= \mu[d(P), d(Q)] & \text{(Lemma 3.1)} \\
&= \mu[ed(P), ed(Q)] \\
&= \mu[\delta r J d(P), \delta r J d(Q)] & \text{(II)} \\
&= \mu[c \delta r J d(P), c \delta r J d(Q)] & \text{(Lemma 3.1)} \\
&= \mu[ec \delta r J d(P), ec \delta r J d(Q)] \\
&= \mu[\delta r J c \delta r J d(P), \delta r J c \delta r J d(Q)] & \text{(II)} \\
&= \mu[\delta r c \delta r J d(P), \delta r c \delta r J d(Q)] & (c \delta r J d(P),\ c \delta r J d(Q) \text{ are rectangles}) \\
&= \mu[r \delta c \delta r J d(P), r \delta c \delta r J d(Q)] & \text{(I)} \\
&= \mu[j(P), j(Q)]. & \text{(III)}
\end{aligned}
$$

This completes the proof for the case of rectangular shapes. For the case of shapes which are rectangular minus an outer corner, notice that we have the following equality of operators on $SYT(b^{a-1}(b-1))$:

$$\text{IV. } j = djc.$$

Let P and Q be in $SYT(b^{a-1}(b-1))$. We have the following chain of equalities:

$$\mu[P, Q] = \mu[c(P), c(Q)] \qquad\qquad\qquad \text{(Lemma 3.1)}$$
$$= \mu[jc(P), jc(Q)] \qquad\qquad (c(P), \, c(Q) \text{ are rectangles})$$
$$= \mu[djc(P), djc(Q)] \qquad\qquad\qquad \text{(Lemma 3.1)}$$
$$= \mu[j(P), j(Q)]. \qquad\qquad\qquad\qquad \text{(IV)}$$

This completes the proof of the lemma.

$\square$

To better understand the action of promotion, we introduce a new combinatorial set related to rectangular tableaux which will be the ordinary tableau descent set with the possible addition of n. Given a standard tableau P of rectangular shape with n boxes, define the *extended descent set of P* denoted $D_e(P) \subseteq [n]$, as follows. For $i = 1, 2, \ldots, n-1$, say that $i \in D_e(P)$ if and only if i is contained in the ordinary descent set $D(P)$. To determine if n is contained in $D_e(P)$, consider the tableau U with entries $\{2, 3, \ldots, n\}$ obtained by deleting the 1 in P and playing jeu-de-taquin to move the resulting hole to the southeastern corner. The entry n is either immediately north or immediately west of this hole in U. Say that n is contained in $D_e(P)$ if and only if n appears north of this hole in U. The fundamental combinatorial fact about the extended descent set is that promotion acts on it by cyclic rotation.

Lemma 3.3. *Let $P \in SYT(b^a)$. For any i, we have that $i \in D_e(P)$ (mod n) if and only if $i + 1 \in D_e(j(P))$ (mod n).*

Example 3.1. Let $b = 4$ and $a = 3$ and let P be the following element of $SYT(4, 4, 4)$:

$$P = \begin{array}{cccc} 1 & 2 & 4 & 9 \\ 3 & 5 & 8 & 11 \\ 6 & 7 & 10 & 12 \end{array} .$$

Applying the promotion operator to P yields

$$j(P) = \begin{array}{cccc} 1 & 2 & 3 & 5 \\ 4 & 6 & 9 & 10 \\ 7 & 8 & 11 & 12 \end{array} .$$

It can be shown that $D_e(P) = \{2, 4, 5, 9, 11\}$ and $D_e(j(P)) = \{3, 5, 6, 10, 12\}$.

For partitions $\lambda \vdash n$ other than rectangles, it is not in general possible to define an extended descent set which agrees with the ordinary descent set on the letters $[n-1]$ and on which j acts with order n. Indeed, it can be shown that promotion acts with order 6 on $SYT((3, 2))$ and $(3, 2)$ is a partition of 5. This gives some combinatorial intuition for why we needed the strictly rectangular hypotheses in our results.

Proof. We prove the equivalent statement involving jeu-de-taquin demotion. Let $Q := j^{-1}(P)$ denote the image of P under the demotion operator and suppose that $i \in D_e(P)$ for $i \in [n]$. We want to show that $i - 1 \in D_e(Q) \pmod{n}$. This is shown in several cases depending on the value of i. Let U be as in the above paragraph and let Γ be the jeu-de-taquin path in the rectangle b^a involved in obtaining U from P.

Case I. $i \in \{2, 3, \ldots, n-1\}$.

For this case, observe that a single jeu-de-taquin slide does not affect the (ordinary) descent set of a tableau. Here we extend the definition of the descent set of a tableau to a tableau with a hole in it in the obvious way.

Case II. $i = n$.

Since P is rectangular, Γ must contain n. Since n is in $D_e(P)$, the entry immediately above n is also contained in Γ. Thus, n moves up when U is constructed and we get that $n - 1$ is immediately above n in Q. Therefore, $n - 1$ is a descent of Q, as desired.

Case III. $i = 1$.

We must show that $n \in D_e(Q)$. To do this, we consider the action of demotion on Q. Let Δ be the jeu-de-taquin path in Q involved in sliding out a hole at the upper left, so that Δ ends in the lower right hand corner of Q at the entry n. We want to show that the final edge of Δ is a downward edge into n.

To do this, define a subset S of the path Γ by $(x, y) \in S$ if and only if $(x+1, y) \in \Gamma$. Here we consider diagrams of partitions to be subsets of the fourth quadrant of the plane intersected with the integer lattice $\mathbb{Z}^2$ which have their upper leftmost coordinate equal to $(1, -1)$. Since Γ starts at $(1, -1)$ and ends at $(b, -a)$, the projection of S onto the x-axis is the full interval $[1, b-1]$. If the final edge of Δ were not a downward edge into n, it would follow that Δ intersected the set S nontrivially. Choose (x, y) with x minimal so that $(x, y) \in \Delta \cap S$.

We claim that $(x, y+1)$ is not contained in Δ. Indeed, if $(x, y+1)$ were contained in Δ, then since Δ is a jeu-de-taquin path the (x, y)-entry of Q is less than the $(x+1, y+1)$-entry of Q, contradicting the fact that P is standard. So, either $x = 1$ or $(x-1, y) \in \Delta$. If $x = 1$, then since $(1, y) \in \Delta$ but $(1, y+1) \notin \Delta$, we must have that $y = -1$. But $(1, -1) \in S$ contradicts 1 being a descent of P. If $(x-1, y) \in \Delta$, then the fact that the projection of S onto the x-axis is $[1, b-1]$ contradicts the minimality of x. We conclude that the final edge of Δ is a downward edge into n, as desired. $\square$

We prove a technical lemma about the image of the long cycle under the KL representation. This lemma will be used to pin down a constant in a Schur's Lemma argument in the proof of Proposition 3.5.

Lemma 3.4. *Let $c_n = (1, 2, \ldots, n) \in S_n$ be the long cycle. Let $\lambda = b^a$ be a rectangle. Identify permutations with their images under RSK. The coefficient of the KL basis*

element $C'_{(j(CSS(\lambda)),CSS(\lambda))}(1)$ in the expansion of $C'_{(CSS(\lambda),CSS(\lambda))}(1)c_n$ in the KL basis of $\mathbb{C}[S_n]$ is $(-1)^{a-1}$.

Proof. Define permutations $u, v \in S_n$ by

$$u \mapsto (CSS(\lambda), CSS(\lambda)),$$
$$v \mapsto (j(CSS(\lambda)), CSS(\lambda)).$$

It follows that the one-line notation for u and v is

$$
\begin{aligned}
u =& a(a-1)\ldots 1(2a)(2a-1)\ldots(a+1)\ldots n\ldots(n-a+1),\\
v =& (a+1)a\ldots 31(2a+1)(2a)\ldots(a+3)2(3a+1)(3a)\ldots(2a+3)(a+2)\ldots\\
& n(n-1)\ldots(n-a+2)(n+2-2a).
\end{aligned}
$$

By inspection, both of these are 3412 and 4231 avoiding, so therefore by smoothness considerations (see Theorem A2 of [14]) all of the KL polynomials $P_{x,u}(q)$ for $x \in S_{a^b}$ are equal to 1 and we have the corresponding formulas:

$$C'_u(1) = \sum_{x \in S_{a^b}} (-1)^{\ell(x,u)} x,$$

$$C'_u(1)c_n = \sum_{x \in S_{a^b}} (-1)^{\ell(x,u)} x c_n.$$

The unique Bruhat maximal element $x_o c_n$ for which $x_o \in S_{a^b}$ has one-line notation
$x_o c_n = (a+1)a\ldots 32(2a+1)(2a)\ldots(a+2)(3a+1)\ldots(2a+2)\ldots n(n-1)\ldots(n-a+2)1$.
Right multiplying by c_n^{-1} yields that
$x_o = a(a-1)\ldots 21(2a)\ldots(n-a)(n-a-1)\ldots(n-2a+1)(n-1)(n-2)\ldots(n-a+1)(n)$.
Therefore, x_o and u differ by a cycle of length a and $(-1)^{\ell(x_o,u)} = (-1)^{a-1}$. It follows that there exist numbers $\gamma_y \in \mathbb{C}$,

$$C'_u(1)c_n = (-1)^{a-1}C'_{x_o c_n}(1) + \sum_{y < x_o c_n} \gamma_y C'_y(1).$$

The coefficient we are interested in is γ_v. First, notice that the Bruhat interval $[v, x_o c_n]$ consists of all permutations in S_n whose one line notation has the form

$$(a+1)a\ldots 3\beta_1(2a+1)(2a)\ldots(a+3)\beta_2\ldots n(n-1)\ldots(n-a+2)\beta_b,$$

where $\beta_1 \in \{1, 2\}$, $\beta_2 \in \{2, a+2\}$, $\ldots\beta_{b-1} \in \{n-3a+2, n-2a+2\}$, $\beta_b \in \{n-2a+2, 1\}$. It follows from the definition of the Bruhat order that the subposet of such permutations is isomorphic to the Boolean lattice B_{b-1} of rank $b-1$. Moreover, all permutations of the above form are 3412 and 4231 avoiding, meaning that for

every $w \in [v, x_o c_n]$ we have that $C'_w(1) = \sum_{z \leq w} (-1)^{\ell(z,w)} z$. Finally, we observe that $[v, x_o c_n] \cap S_{a^b} c_n = \{x_o c_n\}$.

All of the conditions in the above paragraph imply that $\gamma_v = (-1)^{a-1}$, as desired. □

We now have all of the ingredients necessary to analyze the relationship between j and the KL representation. Specifically, we show that up to a predictable scalar, j acts like the long cycle $(1, 2, \ldots, n) \in S_n$. This gives our desired analogue of Equation 1.5.

Proposition 3.5. *Let* $\lambda = b^a$ *be a rectangular shape with* $ab = n$ *and let* $\rho : S_n \to GL(S^\lambda)$ *be the associated KL cellular representation, with basis identified with* $SYT(\lambda)$. *Let* c_n *denote the long cycle* $(1, 2, \ldots n) \in S_n$. *Define a* $\mathbb{C}$-*linear map* $J : S^\lambda \to S^\lambda$ *by extending* $J(P) = j(P)$. *We have that*

$$(3.2) \qquad \rho(c_n) = (-1)^{a-1} J.$$

Equivalently, for any standard tableau $P \in SYT(\lambda)$, *we have that*

$$(3.3) \qquad \rho(c_n)(P) = (-1)^{a-1} j(P).$$

Proof. We want to show that the operator $J^{-1} \rho(c_n)$ commutes with the action of S_n. If we can do this, by Schur's Lemma it will follow that for some constant $\gamma \in \mathbb{C}$ we have the equality of operators $J = \gamma \rho(c_n)$. In [35], Stembridge used this sort of argument to prove Theorem 2.10.

Since S_n is generated by the simple reflections s_i for $i = 1, 2, \ldots, n-1$, it is enough to show that $J^{-1} \rho(c_n)$ commutes with $\rho(s_i)$ for each $i = 1, 2, \ldots, n-1$. From conjugation within the symmetric group we know that $\rho(c_n) \rho(s_i) = \rho(s_{i+1}) \rho(c_n)$ always, where we interpret s_n to be the 'affine' transposition $(1, n)$. So, it is enough to show that $J^{-1} \rho(s_{i+1}) = \rho(s_i) J^{-1}$ always. For $i = 1, 2, \ldots, n-2$ this will follow from the action of Coxeter generators on S^λ, but for $i = n-1$ this poses a problem since there is no known nice formula in general for the action of s_n on these modules. In fact, we show that $J^{-1} \rho(c_n)$ commutes with $\rho(s_i)$ for only $i = 1, 2, \ldots, n-2$ and then apply the branching rule for restriction of irreducible symmetric group modules to show that these commutation relations are in fact sufficient for our purposes.

We first show that for $i = 1, 2, \ldots, n-2$ we have

$$J^{-1} \rho(c_n) \rho(s_i) = \rho(s_i) J^{-1} \rho(c_n).$$

Given a standard tableau P, let $j^{-1}(P)$ be the unique tableau which maps to P under promotion. From the corresponding conjugation relation in the symmetric group it follows that

$$\rho(c_n) \rho(s_i) = \rho(s_{i+1}) \rho(c_n).$$

On the other hand, because $1 \leq i \leq n-2$ and J acts as j on basis elements indexed by standard tableaux, we have that for any $P \in SYT(\lambda)$,

$$
\begin{aligned}
J^{-1}\rho(s_{i+1})(P) &= \begin{cases} -j^{-1}(P) & \text{if } i+1 \in D_e(P) \\ j^{-1}(P) + \sum_{i+1 \in D_e(Q)} \mu[P,Q]j^{-1}(Q) & \text{if } i+1 \notin D_e(P) \end{cases} \\
&= \begin{cases} -j^{-1}(P) & \text{if } i \in D_e(j^{-1}(P)) \\ j^{-1}(P) + \sum_{i \in D_e(j^{-1}(Q))} \mu[P,Q]j^{-1}(Q) & \text{if } i \notin D_e(j^{-1}(P)) \end{cases} \\
&= \begin{cases} -j^{-1}(P) & \text{if } i \in D_e(j^{-1}(P)) \\ j^{-1}(P) + \sum_{i \in D_e(j^{-1}(Q))} \mu[j^{-1}(P),j^{-1}(Q)]j^{-1}(Q) & \text{if } i \notin D_e(j^{-1}(P)) \end{cases} \\
&= \rho(s_i)J^{-1}(P).
\end{aligned}
$$

The first equality is the action of the simple transpositions on the cellular KL basis, the second is Lemma 3.3, the third is Proposition 3.2, and the fourth is again the definition of the action of simple transpositions.

The above discussion implies that the operator $J^{-1}\rho(c_n)$ commutes with the action of the parabolic subgroup S_{n-1} of S_n on the irreducible S_n-module S^λ. Since λ is a rectangle, λ has a unique outer corner and by the branching rule for symmetric groups, the restriction $S^\lambda \downarrow^{S_n}_{S_{n-1}}$ remains an irreducible S_{n-1}-module. Therefore, by Schur's Lemma, we conclude that there is a number $\gamma \in \mathbb{C}$ so that

$$J = \gamma\rho(c_n).$$

We want to show that $\gamma = (-1)^{a-1}$. This follows from Lemma 3.4. $\square$

The above result states that for Specht modules of rectangular shape the image of the long cycle c_n under the KL representation is plus or minus the permutation matrix which encodes jeu-de-taquin promotion. It is not true that any conjugate of c_n in S_n enjoys this property - indeed, many are not even permutation matrices. Observe the analogy to Theorem 2.10 which states that the image of the long *element* w_o under the KL representation for an arbitrary shape is plus or minus a permutation matrix which encodes evacuation.

Also notice that if λ is a partition of any nonrectangular shape, then λ has more than one outer corner. This implies that the restricted module $S^\lambda \downarrow^{S_n}_{S_{n-1}}$ is not irreducible and the above proof breaks down.

As a corollary to this we get a classical result on the order of promotion [11].

Corollary 3.6. *For $\lambda \vdash n$ rectangular, every element of $SYT(\lambda)$ is fixed by j^n.*

Remark 3.2. For arbitrary partitions λ it is not true that the order of promotion on $SYT(\lambda)$ divides $|\lambda|$. In fact, this order for arbitrary λ is unknown. So, the hypothesis that λ be rectangular in the statement of the the corollary is necessary. Moreover, it

is not in general true that for a rectangular partition $\lambda \vdash n$ the order of promotion on $SYT(\lambda)$ is equal to n. For an extreme counterexample, if λ is either a single row or a single column, the set $SYT(\lambda)$ consists of a single element and j has order 1. Other counterexamples exist, minimal among them $\lambda = (2,2)$. The author does not know of a formula for the order of j on $SYT(\lambda)$ for arbitrary shapes λ. As a possible indication of the subtlety here, given *any* nonrectangular partition $\lambda \vdash n$, there exists a tableau $P \in SYT(\lambda)$ which is not fixed by the nth power of promotion [19].

For the special case of staircase shapes $\nu = (k, k-1, \ldots, 1) \vdash n$, the operator j^{2n} fixes every element of $SYT(\nu)$. This leads one to hope that the KL cellular basis may be exploited in studying the action of promotion on $SYT(\nu)$ for staircase shapes ν. Unfortunately, even for the staircase $(3, 2, 1)$ we do not have the result analogous to Proposition 3.2 that j preserves the μ function. It is, however, possible to show that for any staircase ν, the action of $j^{|\nu|}$ on $SYT(\nu)$ is given by transposition, which is known to preserve the μ function. Taken together, the operators of transposition and evacuation generate a group isomorphic to the Klein four group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, leaving some hope that the study of the action of this smaller group might be analyzed by our methods. Unfortunately, the case of the staircase $(3, 2, 1)$ also shows that no power j^d for $1 \leq d \leq 5$ preserves the μ function, so it would seem difficult to use our methods to analyze the action of any of the larger groups on the set $SYT(\nu)$ generated by lower positive powers of j together with evacuation.

Even in the case of rectangular shapes λ, the previous corollary does not give the order of j on a specific tableau $T \in SYT(\lambda)$. For the case where λ has ≤ 3 rows, this order can be interpreted as the order of the cyclic symmetry of a combinatorial object associated to T called an A_2 web [20].

Let s_n denote the "affine" transposition $(1, n)$ in S_n. As another corollary, we get a nice formula for the image of s_n under the KL representation.

Corollary 3.7. *Let $\lambda = b^a$ be a rectangular shape and identify the basis of the corresponding left KL cellular representation with $SYT(\lambda)$.*

We have that

$$s_n P = \begin{cases} -P & \text{if } n \in D_e(P) \\ P + \sum_{n \in D_e(Q)} \mu[P, Q]Q & \text{if } n \notin D_e(P). \end{cases}$$

Proof. Using the notation of the proof of the above proposition, we have that s_{n-1} commutes with the action of $J^{-1}\rho(c)$. It follows that $J^{-1}s_n(P) = s_{n-1}J^{-1}(P)$. The desired formula follows. $\square$

Finally, the above analysis leads to many equalities of μ coefficients.

Corollary 3.8. *Let $\lambda \vdash n$ be a rectangle and let Y and Y' be two orbits of $SYT(\lambda)$ under the action of j which have relatively prime orders. Given $P, Q \in Y$ and $P', Q' \in Y'$ we have that $\mu[P, P'] = \mu[Q, Q']$.*

Example 3.3. For $n = 6$ and $\lambda = (2, 2, 2)$, we have that $|SYT(\lambda)| = 5$ and the cyclic action of j on $SYT(\lambda)$ breaks $SYT(\lambda)$ into one cycle of size 2 and one cycle of size 3. Explicitly, the operator j acts on $SYT((2, 2, 2))$ via:

$$
\begin{pmatrix}
\begin{smallmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{smallmatrix} & , & \begin{smallmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 6 \end{smallmatrix} & , & \begin{smallmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{smallmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{smallmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 6 \end{smallmatrix} & , & \begin{smallmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{smallmatrix}
\end{pmatrix}
$$

Mapping the elements of these orbits to their column reading words in S_6, the above Corollary implies that when v is any element of the set $\{321654, 521643, 431652\} \subset S_6$ and w is any element of the set $\{421653, 531642\}$ we have that $\mu[v, w]$ is a constant. Since 421653 covers 321654 in Bruhat order, we see that this common value of μ is equal to 1.

An application of the above proposition yields Theorem 1.3, our desired result on cyclic sieving in the action of jeu-de-taquin on standard tableaux. Implicit in the statement of Theorem 1.3 is the fact that the nth power of promotion fixes every element of $SYT(\lambda)$. This fails if λ is nonrectangular. Even if we were to replace C with the cyclic group having order equal to the order of promotion on $SYT(\lambda)$ for nonrectangular λ, the action of promotion together with the appropriate q-hook length formula would still not in general yield a CSP. For example, consider the shape $\lambda = (3, 3, 1) \vdash 7$. There are 21 standard tableaux of shape λ and the action of promotion breaks $SYT(\lambda)$ up into three orbits having sizes 3, 5, and 13. Thus, the order of promotion on $SYT(\lambda)$ is $3 * 5 * 13 = 195$. It can be shown that the q-hook length formula for the shape lambda is $X(q) = (1+q+q^2+q^3+q^4+q^5+q^6)(1+q^2+q^4)$. The evaluation of $X(q)$ at a primitive 195th root of unity is not even real, let alone equal to zero, the number of fixed points of one iteration of promotion on $SYT(\lambda)$.

Example 3.4. In the above example of the action of promotion on $SYT((2, 2, 2))$, we have that

$$
f^{(2,2,2)}(q) = \frac{[6]!_q}{[4]_q[3]_q[2]_q[3]_q[2]_q[1]_q} = (1 - q + q^2)(1 + q + q^2 + q^3 + q^4).
$$

Letting $\zeta = e^{\frac{\pi i}{3}}$, we compute directly that

$$
\begin{array}{ccc}
f^{(2,2,2)}(1) = 5 & f^{(2,2,2)}(\zeta) = 0 & f^{(2,2,2)}(\zeta^2) = 2 \\
f^{(2,2,2)}(\zeta^3) = 3 & f^{(2,2,2)}(\zeta^4) = 2 & f^{(2,2,2)}(\zeta^5) = 0.
\end{array}
$$

This is in agreement with the fixed point set sizes:

$$
\begin{array}{ccc}
|SYT((2,2,2))^1| = 5 & |SYT((2,2,2))^j| = 0 & |SYT((2,2,2))^{j^2}| = 2 \\
|SYT((2,2,2))^{j^3}| = 3 & |SYT((2,2,2))^{j^4}| = 2 & |SYT((2,2,2))^{j^5}| = 0,
\end{array}
$$

as predicted by Theorem 1.3.

Proof. (of Theorem 1.3) By Corollary 3.6, C does indeed act on X by promotion. Let $\zeta = e^{\frac{2\pi i}{n}}$. Viewing $c_n = (1, 2, \ldots, n)$ as a permutation matrix in $S_n \subset GL_n(\mathbb{C})$, we get that c is conjugate to $\operatorname{diag}(1, \zeta, \zeta^2, \ldots, \zeta^{n-1})$. This means that for any $d \geq 0$ c_n^d is conjugate to $\operatorname{diag}(1, \zeta^d, \zeta^{2d}, \ldots, \zeta^{d(n-1)})$.

Let χ^λ denote the character of the irreducible representation of S_n corresponding to λ. It is well-known that c_n is a regular element of S_n, that is, there is an eigenvector of the image of c_n under the reflection representation which avoids all reflecting hyperplanes. A result of Springer (Proposition 4.5 of [28]) on regular elements implies that, for any $d \geq 0$, we have the character evaluation $\chi^\lambda(c_n^d) = \zeta^{d\kappa(\lambda)} f^\lambda(\zeta^d)$, where $\kappa(\lambda) = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + \cdots$. Since $\lambda = b^a$, we can compute that $\kappa(\lambda) = \frac{ba(a-1)}{2}$, which implies that $\zeta^{d\kappa(\lambda)} = (-1)^{d(a-1)}$.

On the other hand by Proposition 3.5, $\chi^\lambda(c_n^d)$ is equal to the trace of $(-1)^{d(a-1)} J^d$, where J is the permutation matrix which records promotion. This trace is $(-1)^{d(a-1)}$ times the number of fixed points $|X^{j^d}|$ of the action of j^d on X. The desired CSP follows. $\qquad\square$

4. A Construction of the Finite Dimensional Irreducible $GL_k(\mathbb{C})$ Representations

In the last section we proved a CSP regarding rectangular standard tableaux by modeling the action of promotion by the image of the long cycle in S_n under the Kazhdan-Lusztig cellular representation of appropriate shape. We want to prove an analogue of this CSP for column strict tableaux of rectangular shape having n boxes and entries uniformly bounded by k. In analogy with the last section, we will construct a representation having dimension $CST(\lambda, k)$ for rectangular $\lambda \vdash n$ under which some group element will act as the promotion operator with respect to some basis.

It is well known that the irreducible finite dimensional representations of the general linear group $GL_k(\mathbb{C})$ are parameterized by partitions λ having at most k rows and that the dimension of the irreducible representation labeled by λ is equal to $|CST(\lambda, k)|$. The symmetric group S_k embeds naturally into $GL_k(\mathbb{C})$ as the subgroup of permutation matrices and it will turn out that the long cycle $(1, 2, \ldots, k)$ can be taken to model the action of promotion on column strict tableaux provided our shape λ is rectangular. It remains, however, to find a basis for our representation under which $(1, 2, \ldots, k)$ acts as a monomial matrix corresponding to promotion. This will involve a construction of finite dimensional $GL_k(\mathbb{C})$ irreducible representations of *arbitrary* shape which we now present. This construction is essentially due to Du [7] together with results of Skandera [27] but we rederive it here for self-containment. Our basis

will essentially be a homomorphic image of a subset of the dual canonical basis for the polynomial ring $\mathbb{C}[x_{11}, \ldots, x_{kk}]$ and, as such, will bear spiritual similarities to Lusztig's canonical basis [17]. As a starting point of our construction, we introduce a family of polynomials called immanants. The homomorphic images of an appropriate subset of these polynomials will be the zero weight space of our representations.

For a positive integer $n \in \mathbb{N}$, let $x = (x_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix of commuting variables and let $\mathbb{C}[x_{11}, \ldots, x_{nn}]$ be the complex polynomial ring in these variables. We will sometimes abbreviate the latter ring as $\mathbb{C}[x_{ij}]_{1 \leq i,j \leq n}$. Call a polynomial in $\mathbb{C}[x_{ij}]_{1 \leq i,j \leq n}$ an *immanant* if it belongs to the $\mathbb{C}$-linear span of the permutation monomials $\{x_{1,w(1)} \cdots x_{n,w(n)} \mid w \in S_n\}$. Thus, immanants form an $n!$-dimensional complex vector space. Given any polynomial $f(x_{11}, \ldots, x_{nn}) \in \mathbb{C}[x_{ij}]_{1 \leq i,j \leq n}$ and an $n \times n$ matrix $A = (a_{ij})$ with entries in any commutative $\mathbb{C}-$algebra R, define $f(A)$ to be the element $f(a_{11}, \ldots, a_{nn})$ of R obtained by applying f to A.

Following [24], define for any $w \in S_n$ the *w-Kazhdan-Lusztig immanant* $\mathrm{Imm}_w(x)$ by the equation

$$(4.1) \qquad \mathrm{Imm}_w(x) = \sum_{v \geq w} (-1)^{\ell(w,v)} P_{w_o v, w_o w}(1) x_{1,v(1)} \cdots x_{n,v(n)}.$$

Specializing to the identity permutation, we have that $\mathrm{Imm}_1(x) = \det(x)$. So, at least superficially, KL immanants are deformations of the determinant. It can be shown that the KL immanants share the properties of Schur nonnegativity and total nonnegativity with the determinant [24], and that for the case of permutations w which are 321-avoiding, the application of the KL immanant $\mathrm{Imm}_w(x)$ to the path matrix of a planar network has a combinatorial interpretation which naturally generalizes Lindström's Lemma [23]. By the Bruhat triangularity of the KL polynomials, the set $\{\mathrm{Imm}_w(x) \mid w \in S_n\}$ forms a basis for the vector space of immanants.

We will find it necessary to work with KL immanants defined on variable sets with repeated entries. More precisely, given any pair of compositions $\alpha, \beta \models n$, we define the matrix $x_{\alpha,\beta}$ to be $(x_{\alpha(i),\beta(j)})_{1 \leq i,j \leq n}$. Note that either or both of $\ell(\alpha)$ or $\ell(\beta)$ may exceed n. We also construct the corresponding polynomial ring $\mathbb{C}[x_{\alpha(i),\beta(j)}]_{1 \leq i,j \leq n}$. For a permutation $w \in S_n$, denote by $\mathrm{Imm}_w(x_{\alpha,\beta})$ the element of $\mathbb{C}[x_{\alpha(i),\beta(j)}]_{1 \leq i,j \leq n}$ which results in applying the w-KL immanant to the matrix $x_{\alpha,\beta}$. So, for example, we have that $\mathrm{Imm}_w(x) = \mathrm{Imm}_w(x_{1^n,1^n})$. In this paper we will mostly be interested in the case where $\beta = 1^n$.

Example 4.1. Take $n = 3$ and let $\alpha = (1,2)$ and $\beta = (1,1,1)$. Our matrix $x_{\alpha,\beta}$ is given by

$$x_{(1,2),(1,1,1)} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23}. \end{pmatrix}$$

Let $w = 213 \in S_3$. Recalling that every KL polynomial for pairs of permutations in S_3 is either 0 or 1 depending on whether the pairs are Bruhat comparable, we see that

$$\mathrm{Imm}_{213}(x_{(1,2),(1,1,1)}) = x_{12}x_{11}x_{23} - x_{12}x_{13}x_{21} - x_{13}x_{11}x_{22} + x_{13}x_{12}x_{21}.$$

On the other hand, computing the KL immanant corresponding to the permutation $231 \in S_3$ yields the result

$$\mathrm{Imm}_{231}(x_{(1,2),(1,1,1)}) = x_{12}x_{13}x_{21} - x_{13}x_{12}x_{21} = 0.$$

As the above example shows, it can happen that a polynomial of the form $\mathrm{Imm}_w(x_{\alpha,\beta})$ is equal to zero. Indeed, since $\mathrm{Imm}_1(x) = \det(x)$, such a polynomial is nonzero for $w = 1$ if and only if all of the parts of α and β are at most 1. It is possible to derive a criterion based on the RSK correspondence for determining precisely when the polynomials $\mathrm{Imm}_w(x_{\alpha,\beta})$ vanish [27]. A remarkable fact, due to Skandera, is that the nonvanishing polynomials of the form $\mathrm{Imm}_w(x_{\alpha,\beta})$ form a *basis* for the polynomial ring $\mathbb{C}[x_{11}, \ldots, x_{kk}]$ [27]. Moreover, Skandera showed that the basis so constructed is essentially equivalent to Lusztig's dual canonical basis. We will not need the full force of this result, and now state the theorem which will be relevant for our purposes.

Theorem 4.1. (*Skandera [27]*) *Let $k \geq 0$. The nonzero elements of the set $\{\mathrm{Imm}_w(x_{\alpha,\beta})\}$, where w ranges over S_n and α and β range over all possible compositions of n having length k, are linearly independent and a subset of the dual canonical basis of the polynomial ring $\mathbb{C}[x_{11}, \ldots, x_{kk}]$ in k^2 variables.*

We now relate polynomial rings to representation theory. Let $Y = \mathbb{C}^k$ and $Z = \mathbb{C}^n$ be two complex vector spaces of dimensions k and n, respectively. Let Y^* and Z^* denote their dual spaces with standard bases $\{y_1, \ldots, y_k\}$ and $\{z_1, \ldots, z_n\}$, respectively. Now the tensor product $Y^* \otimes Z^*$ has basis $x_{ij} := y_i \otimes z_j$, for $1 \leq i \leq k$ and $1 \leq j \leq n$. In this way, we identify the symmetric algebra $\mathrm{Sym}(Y^* \otimes Z^*)$ with the polynomial ring $\mathbb{C}[x_{ij}]_{1 \leq i \leq k, 1 \leq j \leq n}$. This space carries an action of the general linear group $GL(Y) = GL_k(\mathbb{C})$, where matrices act on the first component of simple tensors by $g \cdot (f \otimes h) := (fg^{-1}) \otimes h$. Taking $n = k$ and $i \in [n-1]$, viewing the adjacent transposition s_i as an element of $S_n \subset GL_k(\mathbb{C})$, we quote a result from [24] about this action.

Lemma 4.2. *Let $w \in S_n$. We have that*

$$(4.2) \quad s_i\mathrm{Imm}_w(x) = \begin{cases} -\mathrm{Imm}_w(x) & s_iw > w, \\ \mathrm{Imm}_w(x) + \mathrm{Imm}_{s_iw}(x) + \sum_{s_iz > z} \mu(w, z)\mathrm{Imm}_z(x) & s_iw < w. \end{cases}$$

Using the formula in the above lemma, one can show that the KL immanants form a cellular basis for the vector space of immanants. More precisely, identifying permutations with their images under RSK, for any $\lambda \vdash n$ and $T \in SYT(\lambda)$, we have

that the space

$$V'_{T,n,1^n} := \mathbb{C}\{\mathrm{Imm}_{(T,P)}(x) \,|\, P \in SYT(\lambda)\} \oplus \bigoplus_{\nu >_{dom} \lambda} \mathbb{C}\{\mathrm{Imm}_{(U,S)}(x) \,|\, U, S \in SYT(\nu)\}$$

is closed under the left action of S_n. This closure is essentially a consequence of the fact that the formula in Lemma 4.2 is almost the same as the formula for the action of s_i on the KL basis of $\mathbb{C}[S_n]$.

Since $V'_{T,n,1^n}$ is closed under the action of $\mathbb{C}[S_n]$ for *any* T, it follows immediately that the quotient space

$$V_{T,n,1^n} := V'_{T,n,1^n} / (\bigoplus_{\nu >_{dom} \lambda} \mathbb{C}\{\mathrm{Imm}_{(U,S)}(x) \,|\, U, S \in SYT(\nu)\})$$

carries the irreducible S_n-representation corresponding to the shape λ. A basis for $V_{T,n,1^n}$ is given by the image of the set $\{\mathrm{Imm}_{(T,P)}(x) \,|\, P \in SYT(\lambda)\}$ under the canonical projection map. Letting $I_{1^n}(P)$ denote the image of $\mathrm{Imm}_{(T,P)}(x)$ under this projection, we can write this basis as $\{I_{1^n}(P) \,|\, P \in SYT(\lambda)\}$. The 1^n appearing in this notation will be fully justified when we broaden our scope to nonstandard tableaux. By a change of label argument, the representation of S_n on the quotient space $V_{T,n,1^n}$ does not depend on the choice of the tableau T, and is given by

$$(4.3) \qquad s_i I_{1^n}(P) = \begin{cases} -I_{1^n}(P) & \text{if } i \in D(P) \\ I_{1^n}(P) + \sum_{i \in D(Q)} \mu[P,Q] I_{1^n}(Q) & \text{if } i \notin D(P). \end{cases}$$

So far we have used KL immanants to construct representations of the group algebra $\mathbb{C}[S_n]$. The goal of this section is to construct modules over $GL_k(\mathbb{C})$. To do this, we first recall combinatorial notions of standardization and semistandardization on tableaux. These operations give us a means to transform row strict tableaux into standard tableaux and vice versa, when possible.

It is always possible to transform a row strict tableau into a standard tableau. Given a partition $\lambda \vdash n$ and a row strict tableau $P \in RST(\lambda, k, \alpha)$ for some $\alpha \models n$, define the *standardization* $std(P)$ of P to be the element of $SYT(\lambda)$ given by replacing the α_1 $1s$ in P with the numbers $[1, \alpha_1]$ increasing down columns, replacing the α_2 $2s$ in P with the numbers $[\alpha_1 + 1, \alpha_1 + \alpha_2]$ increasing down columns, and so on.

Example 4.2. Suppose that P is the tableau given by

$$P = \begin{matrix} 1 & 7 & 8 \\ 1 & 9 & \\ 2 & 9 & \end{matrix} \quad \in RST((3,2,2), 9, (2,1,0,0,0,0,1,1,2)),$$

then we have that

$$std(P) = \begin{matrix} 1 & 4 & 5 \\ 2 & 6 & \\ 3 & 7 & \end{matrix} \quad \in SYT((3,2,2)).$$

The reverse case is more complicated. Given $\lambda \vdash n$ and a standard tableau $T \in SYT(\lambda)$ along with a composition $\alpha \models n$, say that T is α-*semistandardizable* if the sequences of numbers $[1, \alpha_1]$, $[\alpha_1+1, \alpha_1+\alpha_2]$, ..., all occur in vertical strips in T which increase down columns. Equivalently, the standard tableau T is α-semistandardizable if and only if $D(T)$ contains the union of the intervals $[1, \alpha_1)$, $[\alpha_1 + 1, \alpha_2)$, ... Define the α-*semistandardization* $rst_\alpha(T)$ of T α-semistandardizable to be the element of $RST(\lambda, k, \alpha)$ formed by replacing the numbers in $[1, \alpha_1]$ in T by $1s$, the numbers in $[\alpha_1 + 1, \alpha_1 + \alpha_2]$ in T by $2s$, and so on.

Example 4.3. The tableau $T \in SYT((3, 2, 2))$ shown below is not $(1, 2, 3, 1)$-semistandardizable because the 3 occurs in a higher row than the 2 and therefore $rst_{(1,2,3,1)}(T)$ is undefined. Notice, however, that the tableau obtained by replacing the 3 in T with a 2, the $4, 5,$ and 6 with 3s, and the 7 with a 4 is in fact row strict.

$$T = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & 7 & \end{array}$$

On the other hand, the tableau $U \in SYT((3, 2, 2))$ shown is $(1, 2, 3, 1)$-semistandardizable and its $(1, 2, 3, 1)$-semistandardization is shown.

$$U = \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \\ 6 & 7 & \end{array} \qquad rst_{(1,2,3,1)}(U) = \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & \\ 3 & 4 & \end{array}$$

Our terminology here may be slightly misleading since semistandard tableaux are usually defined in the literature to be column strict rather than row strict. This difference is cosmetic, though, and this notion of semistandardization will allow for cleaner statements of our results.

The following is immediate.

Lemma 4.3. *Let $T \in SYT(\lambda)$ be α-semistandardizable. We have that $std(rst_\alpha(T)) = T$. Moreover, if $U \in RST(\lambda, k, \alpha)$, then $std(U)$ is α-standardizable.*

Therefore, for any composition $\alpha \models n$, standardization injects $RST(\lambda, k, \alpha)$ into $SYT(\lambda)$ and α-semistandardization gives a bijection between the α-semistandardizable elements of $SYT(\lambda)$ and $RST(\lambda, k, \alpha)$. The combinatorial notion of semistandardizable tableaux can be related to the vanishing of KL immanants as follows. This will be a key point in our construction of modules over the general linear group.

Lemma 4.4. *Let $U, T \in SYT(\lambda)$ and let $k \in \mathbb{N}$. We have that $\mathrm{Imm}_{(T,U)}(x_{\alpha,1^n}) = 0$ if and only if U is not α-semistandardizable. Moreover, the set $\{\mathrm{Imm}_{(T',U')}(x_{\alpha',1^n})\}$ ranging over all possible $\alpha' \models n$ with $\ell(\alpha') \leq k$, $U', T' \in SYT(\lambda)$, and U' that are α'-semistandardizable, is linearly independent.*

Proof. Let S_α denote the parabolic subgroup of S_n which setwise fixes the intervals $[1, \alpha_1]$, $[\alpha_1 + 1, \alpha_1 + \alpha_2]$, and so on. By Skandera's results on the dual canonical basis [27], we have that $\mathrm{Imm}_{(T,U)}(x_{\alpha,1^n}) \neq 0$ if and only if the inverse image w of (T, U) under RSK is a Bruhat maximal element of some left coset of S_α in S_n. It is easy to show using the properties of jeu-de-taquin and RSK that this happens if and only if U is α-semistandardizable. The claim about linear independence follows from Theorem 4.1. $\qquad\square$

Example 4.4. As we have seen in Example 4.1, we have that $\mathrm{Imm}_{213}(x_{(2,1),(1,1,1)})$ is nonzero and $\mathrm{Imm}_{231}(x_{(2,1),(1,1,1)}) = 0$. It is easy to see that the permutation 213 and 231 row insert as follows:

$$213 \mapsto \left(\begin{array}{cc} 1 & 3 \\ 2 & \end{array} , \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \right)$$

$$231 \mapsto \left(\begin{array}{cc} 1 & 3 \\ 2 & \end{array} , \begin{array}{cc} 1 & 2 \\ 3 & \end{array} \right).$$

Therefore, the recording tableau for 213 is $(2, 1)$-semistandardizable whereas the recording tableau for 231 is not. The nonvanishing and vanishing of the associated polynomials is therefore predicted by Lemma 4.4.

We are finally ready to define our general linear group modules. For any $T \in SYT(\lambda)$, define $V'_{T,k}$ to be the space

$$V'_{T,k} := \bigoplus_\alpha \bigoplus_U \mathbb{C}\{\mathrm{Imm}_{(T,U)}(x_{\alpha,1^n}) \oplus \bigoplus_{\nu <_{dom} \lambda} \bigoplus_{\alpha'} \bigoplus_{P,Q} \mathbb{C}\{\mathrm{Imm}_{(P,Q)}(x_{\alpha',1^n})\},$$

where the first set ranges over all compositions $\alpha \models n$ such that $\ell(\alpha) = k$ and all $U \in SYT(\lambda)$ which are α-semistandardizable and the second set ranges over all compositions $\alpha' \models n$ with $\ell(\alpha') = k$ and all pairs of tableaux $P, Q \in SYT(\nu)$. We first show that $V'_{T,k}$ is closed under the action of $GL_k(\mathbb{C})$.

Lemma 4.5. $V'_{T,k}$ *is a left* $GL_k(\mathbb{C})$-*module.*

Proof. For $1 \leq i < k$ and $z \in \mathbb{C}$, let $E_{i,z}$ (resp. $F_{i,z}$) denote the elementary matrices in $GL_k(\mathbb{C})$ which are obtained by replacing the 0 in position $(i, i+1)$ (resp. $(i+1, i)$) with a z. We show that $V'_{T,k}$ is closed under the action of the permutation matrices $S_k \subset GL_k(\mathbb{C})$, the Cartan subgroup H of diagonal matrices in $GL_k(\mathbb{C})$, and $E_{i,z}$ and $F_{i,z}$ for $1 \leq i < k$ and $z \in \mathbb{C}$. Since $GL_k(\mathbb{C})$ is generated by these matrices, the result will follow.

To show that $V'_{T,k}$ is closed under the action of S_k, let $1 \leq i < k$. We show that $V'_{T,k}$ is closed under the action of $s_i \in S_k$. Let $\mathrm{Imm}_{(P,Q)}(x_{\beta,1^n})$ be a basis element of $V'_{T,k}$. If we let $s_i \cdot \beta$ be the composition given by $s_i \cdot \beta := (\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \beta_i, \beta_{i+2}, \ldots, \beta_k) \models n$, it follows that the polynomial $s_i \cdot \mathrm{Imm}_{(P,Q)}(x_{\beta,1^n})$ is equal to the image of $w \cdot$

$\text{Imm}_{(P,Q)}(x_{1^n,1^n})$ under the $\mathbb{C}$-algebra homomorphism $x_{i,j} \mapsto x_{\beta'(i),j}$. Here w is the permutation in S_n which is obtained by replacing the square diagonal submatrix of size $|\beta_i + \beta_{i+1}|$ in the $n \times n$ identity matrix corresponding to rows and columns in $(\beta_1 + \cdots + \beta_{i-1}, \beta_1 + \cdots + \beta_{i+1}]$ with an antidiagonal matrix of 1's. It follows from Theorem 2.7 and the analogous actions of Equations 2.10 and 4.4 that this latter polynomial remains in $V'_{T,k}$, as desired.

Given any diagonal matrix $A = \text{diag}(a_1, \ldots, a_k)$ in H, it is easily seen that the immanant $\text{Imm}_{(P,Q)}(x_{\beta,1^n})$ is an eigenvector for the operator A with eigenvalue $a^\beta :=$ $a_1^{\beta(1)} \cdots a_k^{\beta(n)}$. Thus, $V'_{T,k}$ is closed under the action H.

Finally, we show that $V'_{T,k}$ is closed under the action of the $E_{i,z}$ and $F_{i,z}$. To simplify notation, we show this only for the case of nonzero z and $F_{i,z^{-1}}$. The other cases are similar. Let $\beta = (\beta_1, \ldots, \beta_k)$ be a composition of n of length k and let w be a permutation in S_n with $w \mapsto (P, Q)$. Let ν be the shape of P and Q. Let $I := \{\ell \in [n] \mid \beta(\ell) = i\}$. The image of $\text{Imm}_{(P,Q)}(x_{\beta,1^n})$ under the action of $F_{i,z^{-1}}$ is the polynomial

$$\sum_{v \in S_n} (-1)^{\ell(v,w)} P_{w_o v, w_o w}(1) \prod_{\ell \in I} (x_{iv(\ell)} + z x_{(i+1)v(\ell)}) \prod_{m \in [n] - I} x_{\beta(m)v(m)}.$$

Expanding out the terms in parenthesis and regrouping, we see that the above expression is equal to

$$\sum_{J \subseteq I} z^{|J|} \sum_{v \in S_n} (-1)^{\ell(v,w)} P_{w_o v, w_o w}(1) \prod_{j \in J} x_{(i+1)v(j)} \prod_{\ell \in I - J} x_{iv(\ell)} \prod_{m \in [n] - I} x_{\beta(m),v(m)}.$$

Fix a subset $J \subseteq I$. Let y be any permutation in S_n which fixes every letter not in I and rearranges the letters of I so that the letters in J are mapped into a contiguous suffix. Let $\gamma \models n$ be the composition of n defined by

$$\gamma := (\beta_1, \ldots, \beta_{i-1}, \beta_i - |J|, \beta_{i+1} + |J|, \beta_{i+1}, \ldots, \beta_k).$$

By the discussion following Lemma 4.2, the image of

$$\sum_{v \in S_n} (-1)^{\ell(v,w)} P_{w_o v, w_o w}(1) \prod_{j \in J} x_{(i+1)v(j)} \prod_{\ell \in I - J} x_{iv(\ell)} \prod_{m \in [n] - I} x_{\beta(m),v(m)}$$

under the action of y^{-1} is a complex linear combination of terms of the form $\text{Imm}_{(P',Q')}(x_{\gamma,1^n})$, where either P' and Q' are both standard tableaux of shape ν and $P = P'$ or P' and Q' are both standard tableaux of shape strictly dominating ν. Since the subset J was arbitrary and since we already know that $V'_{T,k}$ is closed under the action of S_k, it follows that $V'_{T,k}$ is stable under the action of $F_{i,z^{-1}}$, as desired. $\square$

The above result can be found in [7] together with [27]. At any rate, we define $V_{T,k}$ to be the quotient $GL_k(\mathbb{C})$-module given by

$$V_{T,k} := V'_{T,k} / \left(\bigoplus_{\nu >_{dom} \lambda} \bigoplus_{\alpha'} \bigoplus_{P,Q} \mathbb{C}\{\mathrm{Imm}_{(P,Q)}(x_{\alpha',1^n})\} \right),$$

where the second direct sum ranges over compositions $\alpha' \models n$ with $\ell(\alpha') = k$ and the third direct sum ranges over all standard tableaux P and Q of shape ν. Basis elements of $V_{T,k}$ are given by the images of the polynomials $\mathrm{Imm}_{(T,U)}(x_{\alpha,1^n})$ for compositions $\alpha = (\alpha_1, \dots, \alpha_k) \models n$ and α-semistandardizable U. The image of the above polynomial in $V_{T,k}$ shall be abbreviated $I_\alpha(U')$, where U' is the unique element of $RST(\lambda, k, \alpha)$ such that $std(U') = U$.

Theorem 4.6. *The $GL_k(\mathbb{C})$-module $V_{T,k}$ is isomorphic to the dual of the irreducible finite dimensional $GL_k(\mathbb{C})$-module with highest weight λ', where λ is the shape of T. Moreover, for any $\alpha \models n$ with $\ell(\alpha) = k$, the weight space of $V_{T,k}$ corresponding to $-\alpha$ is equal to the $\mathbb{C}$-linear span of $\{I_\alpha(U)\}$, where U ranges over $RST(\lambda, k, \alpha)$.*

Proof. We compute the Weyl character of $V_{T,k}$. Let $h := \mathrm{diag}(a_1, \dots, a_k)$ be an element of the Cartan subgroup of $GL_k(\mathbb{C})$ for some nonzero complex numbers $a_1, \dots, a_k$. It is easy to see that the action of h on some basis element $I_\alpha(U)$ for U row strict with content α is given by

$$h \cdot I_\alpha(U) = a_1^{-\alpha_1} \cdots a_k^{-\alpha_k} I_\alpha(U).$$

(Here we recall that our action of $GL_k(\mathbb{C})$ came from the *contragredient* action.) Since the set of all $I_\alpha(U)$ where U ranges over all row strict tableaux of shape λ and entries $\leq k$ forms a basis for $V_{T,k}$, it follows that h acts on $V_{T,k}$ with trace

$$\sum_{\alpha \models n, \ell(\alpha) = k} \sum_{U \in RST(\lambda, k, \alpha)} a_1^{-\alpha_1} \cdots a_k^{-\alpha_k}.$$

This latter sum is the combinatorial definition of the Schur function $s_{\lambda'}(a_1^{-1}, \dots, a_k^{-1})$, which is the Weyl character of the dual of the irreducible finite dimensional $GL_k(\mathbb{C})$-module with highest weight λ'.

The claim about the weight space decomposition of $V_{T,k}$ is obvious. $\square$

Example 4.5. We illustrate our construction for the case $k = 3$, $n = 4$, and $\lambda = (2,2)$. Let T be the standard tableau of shape $(2,2)$ given by

$$T = \begin{matrix} 1 & 2 \\ 3 & 4. \end{matrix}$$

The space $V'_{T,3}$ is equal to the complex span of

$$|RST((2,2),3)| + |RST((2,1,1),3)| * |SYT(2,1,1)| +$$
$$|RST((1,1,1,1),3)| * |SYT(1,1,1,1)| = 6 + 15 * 3 + 35 * 1 = 86$$

linearly independent polynomials which are obtained as above by applying KL immanants corresponding to permutations in S_4 which row insert to any of the shapes $(2, 2)$, $(2, 1, 1)$, or $(1, 1, 1, 1)$ to 4×4 matrices in the 9 variables $x_{11}, \ldots, x_{33}$ with possibly repeated rows. Also as explained above, we only consider KL immanants corresponding to the permutations with shape $(2, 2)$ whose insertion tableaux are equal to T. By Lemma 4.5, the 86 dimensional space $V'_{T,3}$ is closed under the action of $GL_3(\mathbb{C})$.

The space $V_{T,3}$ is obtained from $V'_{T,3}$ by modding out by the complex span of all polynomials obtained by the application of KL immanants whose associated permutations do not have shape $(2, 2)$ under RSK. The space $V_{T,3}$ has dimension $|RST((2, 2), 3)| = 6$. A complex basis of $V_{T,3}$ is the set $\{I_\alpha(U)\}$ as defined above, where U is a row strict tableau of shape $(2, 2)$ and content composition α satisfying $\ell(\alpha) = 3$. We write down an element of this basis explicitly.

First note that the permutations in S_4 with insertion tableau T are precisely 3412 and 3142. It can be shown that the KL immanants corresponding to these permutations are

$$\mathrm{Imm}_{3412}(x) = x_{13}x_{24}x_{31}x_{42} - x_{14}x_{23}x_{31}x_{42} - x_{13}x_{24}x_{32}x_{41} + x_{14}x_{23}x_{32}x_{41}$$

and

$$\mathrm{Imm}_{3142}(x) = x_{13}x_{21}x_{34}x_{42} - x_{13}x_{22}x_{34}x_{41} - x_{13}x_{24}x_{31}x_{42} - x_{14}x_{21}x_{33}x_{42}$$
$$+ x_{14}x_{22}x_{33}x_{41} + x_{13}x_{24}x_{32}x_{41} + x_{14}x_{23}x_{31}x_{42} - x_{14}x_{23}x_{32}x_{41}.$$

Let T_o be the tableau given by

$$T_o = \begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array}$$

The set $RST((2, 2), 3)$ is equal to

$$\{rst_{(1,2,1)}(T), rst_{(2,2,0)}(T_o), rst_{(2,0,2)}(T_o), rst_{(0,2,2)}(T_o), rst_{(1,1,2)}(T_o), rst_{(2,1,1)}(T_o)\}.$$

To find the basis element $I_{(1,2,1)}(rst_{(1,2,1)}(T))$, we apply the 3412-KL immanant to the matrix $x_{(1,2,1),(1,1,1,1)}$. This results in

$$\mathrm{Imm}_{3412}(x_{(1,2,1),(1,1,1,1)}) = x_{13}x_{24}x_{21}x_{32} - x_{14}x_{23}x_{21}x_{32} - x_{13}x_{24}x_{22}x_{31} + x_{14}x_{23}x_{22}x_{31}$$

and $I_{(1,2,1)}(rst_{(1,2,1)}(T))$ is the homomorphic image of this in the quotient $V'_{T,3}$.

5. PROMOTION ON COLUMN STRICT TABLEAUX

The goal in this section is to prove a CSP for column strict tableau using the description of irreducible $GL_k(\mathbb{C})$-modules given in the last section. This will essentially involve showing that the long cycle $(1, 2, \ldots, k) \in S_k \subset GL_k(\mathbb{C})$ acts as a monomial

matrix corresponding to promotion in the *rectangular* irreducible representations constructed in Section 4.

We start by defining a collection of epimorphisms which will allow us to work with our representations more easily. Given $\alpha \models n$, we define an epimorphism $\mathbb{C}$-algebras $\pi_\alpha : \mathbb{C}[x_{ij}]_{1 \leq i,j \leq n} \to \mathbb{C}[x_{\alpha(i)j}]_{1 \leq i,j \leq n}$ by the formula $\pi_\mu(x_{ij}) = x_{\alpha(i)j}$. The relevant vanishing properties of KL immanants may be stated cleanly in terms of the maps π_α.

Lemma 5.1. *Let $U, T \in SYT(\lambda)$. We have that $\pi_\alpha(\mathrm{Imm}_{(U,T)}(x))$ is nonzero if and only if U is α-semistandardizable, in which case $\pi_\alpha(\mathrm{Imm}_{(U,T)}(x)) = \mathrm{Imm}_{(U,T)}(x_{\alpha,1^n})$.*

Proof. This is straightforward from the definition of the KL immanants and π_α together with Lemma 4.4. $\square$

We record the action of the long cycle in S_n on immanants (which correspond to standard tableaux). This will be related to the action of the long cycle in S_k on semistandard tableaux via the maps π_α.

Lemma 5.2. *Let $\lambda = b^a$ be a rectangle and let $P \in SYT(\lambda)$. Let $c_n = (1, 2, \ldots n) \in S_n$ be the long cycle. We have that $c_n \cdot I_{1^n}(P) = (-1)^{b-1} I_{1^n}(j(P))$.*

Proof. This is straightforward using the corresponding result for the KL basis of $\mathbb{C}[S_n]$ (Proposition 3.5) and the action of S_n on immanants given in Lemma 4.2. $\square$

We also record how the combinatorial operations of promotion and standardization commute.

Lemma 5.3. *Let $\lambda \vdash n$ be a rectangle and $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$. We have the following equality of operators on $SYT(\lambda)$:*

(5.1) $$j \circ rst_\alpha = rst_{c_k \cdot \alpha} \circ j^{\alpha_k},$$

where $c_k \cdot \alpha$ is the composition of n of length k given by $c_k \cdot \alpha := (\alpha_k, \alpha_1, \cdots, \alpha_{k-1})$ and the right hand side is defined if and only if the left hand side is defined.

Proof. Observe that for any tableau $P \in SYT(\lambda)$, $i + 1$ occurs strictly south and weakly west of i in P if and only if $i \in D(P)$. For $i < n$ this is equivalent to the condition $i \in D_e(P)$. Therefore, this follows from the definition of jeu-de-taquin as well as the cyclic action of jeu-de-taquin on the extended descent set of rectangular tableaux from Lemma 3.3. $\square$

The following lemma shows how the π_α allow the transfer of information from the standard to the semistandard case.

Lemma 5.4. *Let $c_n = (1, 2, \ldots, n)$ be the long cycle in S_n and let $c_k = (1, 2, \ldots, k)$ be the long cycle in S_k. For $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$ we have a left action of c_n on $\mathbb{C}[x_{ij}]_{1 \leq i,j \leq n}$ and d maps $\mathbb{C}[x_{\alpha(i)j}]_{1 \leq i,j \leq n}$ into $\mathbb{C}[x_{c_k \cdot \alpha(i)j}]_{1 \leq i,j \leq n}$. We have the following commutative square.*

(5.2)
$$
\begin{array}{ccc}
& c_n^{\alpha_k} & \\
\mathbb{C}[x_{ij}]_{1 \leq i,j \leq n} & \longrightarrow & \mathbb{C}[x_{ij}]_{1 \leq i,j \leq n} \\
\pi_\alpha \downarrow & & \downarrow \pi_{c_k \cdot \alpha} \\
\mathbb{C}[x_{\alpha(i)j}]_{1 \leq i,j \leq n} & \longrightarrow & \mathbb{C}[x_{c_k \cdot \alpha(i)j}]_{1 \leq i,j \leq n} \\
& c_k &
\end{array}
$$

Proof. The commutativity of this diagram is easily checked on the generators x_{ij} of the algebra in the upper left. $\square$

Recall that in the last section we constructed representations $V_{T,k}$ of $GL_k(\mathbb{C})$ for a fixed standard tableau T with n boxes. Let $V_{T,k,\alpha}$ be the subspace of $V_{T,k}$ generated by the elements $\{I_\alpha(U)\}$, where U ranges over $RST(\lambda, k, \alpha)$. By Theorem 4.6, the spaces $V_{T,k,\alpha}$ give the weight space decomposition of the irreducible representation $V_{T,k}$. In terms of this weight space decomposition, the commutative square (5.2) implies the following commutative square:

(5.3)
$$
\begin{array}{ccc}
& c_n^{\alpha_k} & \\
V_{T,n,1^n} & \longrightarrow & V_{T,n,1^n} \\
\pi_\alpha \downarrow & & \downarrow \pi_{c_k \cdot \alpha} \\
V_{T,k,\alpha} & \longrightarrow & V_{T,k,c_k \cdot \alpha} \\
& c_k &
\end{array}
\quad .
$$

Using this square, we can relate the action of c_k on $V_{T,k}$ to promotion.

Proposition 5.5. *Given $U \in RST(\lambda, k, \alpha)$ for $\lambda = b^a$ rectangular and $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$, we have that*

(5.4)
$$
c_k \cdot I_\alpha(U) = (-1)^{\alpha_k(b-1)} I_{c_k \cdot \alpha}(j(U)).
$$

Proof. We have the following chain of equalities:
$$
\begin{aligned}
c_k I_\alpha(U) &= \pi_{c_k \cdot \alpha} c_n^{\alpha_k} I_{1^n}(std(U)) \\
&= \pi_{c_k \cdot \alpha}((-1)^{\alpha_k(b-1)} I_{c_k \cdot \alpha}(rst_{c_k \cdot \alpha} \circ j^{\alpha_k} \circ std(U)) \\
&= (-1)^{\alpha_k(b-1)} I_{c_k \cdots \alpha}(j(U)).
\end{aligned}
$$
The first equality comes from the commutative square (5.3) and Lemma 5.2. The second comes from Lemmas 5.2 and 5.3. The third is again Lemma 5.3 and the definition of $\pi_{c_k \cdot \alpha}$. $\square$

Corollary 5.6. *For λ rectangular, the order of j on $RST(\lambda, k)$ (or $CST(\lambda, k)$) is equal to k unless λ consists of a single column (respectively, single row) and $k = |\lambda|$.*

Proof. By Proposition 5.5, every element of $RST(\lambda, k)$ is fixed by the operator j^k. If λ is any rectangular shape other than a column (provided $RST(\lambda, k)$ is nonempty), it's easy to produce a row strict tableau of content composition having no nontrivial cyclic symmetry. $\square$

Example 5.1. If $\lambda = (2, 2)$ and $k = 3$, the set $CST((2, 2), 3)$ contains 6 elements and promotion acts as the following permutation:

$$\begin{pmatrix} \begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix} & , & \begin{matrix} 2 & 2 \\ 3 & 3 \end{matrix} & , & \begin{matrix} 1 & 1 \\ 3 & 3 \end{matrix} \end{pmatrix} \begin{pmatrix} \begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix} & , & \begin{matrix} 1 & 2 \\ 3 & 3 \end{matrix} & , & \begin{matrix} 1 & 1 \\ 2 & 3 \end{matrix} \end{pmatrix},$$

which does indeed have order $k = 3$.

Finally we are able to prove Theorem 1.4. As in the standard tableau case of Theorem 1.3, the hypothesis that λ is rectangular is necessary. For arbitrary shapes λ, the order of the operator j on $CST(\lambda, k)$ is unknown. Also as with the standard case, even if we removed the rectangular condition and replaced C by the cyclic group of size the same as the order of j on $CST(\lambda, k)$ (still taking $X(q)$ to be the principal specialization of s_λ), this result would not be true.

Example 5.2. Keeping with the earlier example of $\lambda = (2, 2)$ and $k = 3$, we compute that $\kappa((2, 2)) = 2$ and $s_{(2,2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_2^2 x_3^2 + x_1^2 x_3^2 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 + x_1^2 x_2 x_3$. Maintaining the notation of Theorem 5.7, we therefore have that $X(q) = 1 + q + 2q^2 + q^3 + q^4$. Letting $\zeta = e^{\frac{2\pi i}{3}}$, we see that

$$X(1) = 6 \quad X(\zeta) = 0 \quad X(\zeta^2) = 0.$$

These numbers agree with the fixed point set sizes:

$$|X^1| = 6 \quad |X^j| = 0 \quad |X^{j^2}| = 0,$$

as predicted by Theorem 1.4.

Proof. (of Theorem 1.4) We prove the equivalent assertion which is obtained by replacing X by $RST(\lambda, k)$ and $X(q)$ by $q^{-\kappa(\lambda')} s_{\lambda'}(1, q, q^2, \ldots, q^{k-1})$. We fix a standard tableau T of shape λ' and consider the action of $(1, 2, \ldots, k)$ on $V_{T,k}$.

Let $\zeta = e^{\frac{2\pi i}{k}} \in \mathbb{C}$. Suppose $U \in RST(\lambda, k, \alpha), \alpha = (\alpha_1, \ldots, \alpha_k)$ satisfies $c_k^m \cdot I_\alpha(U) = I_\alpha(U)$. Since c_k maps elements of the form $I_{(\alpha_1, \ldots, \alpha_k)}(P)$ for $P \in RST(\lambda, k, (\alpha_1, \ldots, \alpha_k))$ to elements of the form $I_{(\alpha_k, \alpha_1, \ldots, \alpha_{k-1})}(Q)$ for $Q \in RST(\lambda, k, (\alpha_k, \alpha_1, \ldots, \alpha_{k-1}))$, it follows that $i \equiv j \pmod{m}$ implies that $\alpha_i = \alpha_j$. Using the facts that $\kappa(\lambda') = \frac{ab(b-1)}{2}$

and $\alpha_1 + \cdots + \alpha_k = n = ab$ we get that

$$(\zeta^m)^{\kappa(\lambda')} = (e^{\frac{2\pi i}{k}})^{\frac{mab(b-1)}{2}}$$
$$= (e^{\frac{\pi i}{k}})^{mn(b-1)}$$
$$= (e^{\pi i})^{(\alpha_1 + \cdots + \alpha_m)(b-1)}$$
$$= (-1)^{(\alpha_1 + \cdots + \alpha_m)(b-1)}.$$

However, we already know that

$$c_k^m I_\alpha(U) = (-1)^{(\alpha_1 + \cdots + \alpha_m)(b-1)} I_\alpha(j^m(U)).$$

It follows that for any $r \geq 0$, the coefficient of $I_\alpha(U)$ in $c_k^r I_\alpha(U)$ is $\zeta^{r\kappa(\lambda')}$ if $j^r(U) = U$ and 0 otherwise. Therefore, the trace of the operator c_k^r on the space $V_{T,k}$ is equal to $\zeta^{r\kappa(\lambda')} |X^{j^r}|$.

On the other hand, the operator c_k^r is conjugate to $\mathrm{diag}(1, \zeta^r, \zeta^{(2r)}, \ldots, \zeta^{(k-1)r})$ in $GL_k(\mathbb{C})$. It is easy to see that if P is row strict with content α, then $I_\mu(P)$ is an eigenvector for the latter operator with eigenvalue $(1^r)^{-\alpha_1}(\zeta^r)^{-\alpha_2}(\zeta^{2r})^{-\alpha_3} \cdots$. It follows that the trace of c_k^r on V_λ is the specialization of the Schur function $s_{\lambda'}(1, q, q^2, \ldots, q^{k-1})$ at $q = \zeta^{-r}$. The desired CSP follows. $\qquad\square$

As a pair of corollaries to Theorem 1.4, we can prove Reiner-Stanton-White's results from the introduction quite easily.

Proof. (of Theorem 1.1) Interchange n and k in Theorem 1.4 and take λ to be a single column. $\qquad\square$

Proof. (of Theorem 1.2) Interchange n and k in Theorem 1.4 and take λ to be a single row. $\qquad\square$

6. Promotion on Column Strict Tableaux with Fixed Content

In this section we fix a rectangular partition $\lambda = b^a$ with $ab = n$, a positive integer k and a composition $\alpha \models n$ with $\ell(\alpha) = k$ such that α has some cyclic symmetry. We prove a near-CSP involving the action of certain powers of promotion on the set $CST(\lambda, k, \alpha)$ and the Kostka-Foulkes polynomials. In representation theoretic terms, this corresponds to a weight space refinement of our results in the last section.

Assume that for some integer $d|k$, we have that the composition α has cyclic symmetry of order d. That is, $\alpha_i = \alpha_j$ whenever $i \equiv j \pmod{d}$. Since j maps the set $CST(\lambda, k, (\alpha_1, \ldots, \alpha_k))$ into the set $CST(\lambda, k, (\alpha_k, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}))$, we have that the dth power j^d of j maps the set $CST(\lambda, k, \alpha)$ into itself. Note that for the special

case $d = 1$, $k = n$, and $\alpha = 1^n$ this is the statement that j acts on the set $SYT(\lambda)$ of standard tableaux of shape λ. Since j acts with order k on the set $CST(\lambda, k)$, we have that j^d generates an action of the cyclic group $\mathbb{Z}/(\frac{k}{d}\mathbb{Z})$ on $CST(\lambda, k, \alpha)$.

For a partition $\lambda \vdash n$ and a composition $\alpha \models n$, let $K_{\lambda,\alpha}(q) \in \mathbb{N}[q]$ be the associated Kostka-Foulkes polynomial. The Kostka-Foulkes polynomials are q-analogues of the Kostka numbers $K_{\lambda,\alpha}$ which enumerate the number of column strict tableaux with shape λ and content α. In particular $K_{\lambda,\alpha}(1) = K_{\lambda,\alpha}$ always. Moreover, we have the polynomial equality $K_{\lambda,\alpha}(q) = K_{\lambda,\alpha'}(q)$ for any rearrangement α' of the composition α. The Kostka-Foulkes polynomials are the generating function for the charge statistic on tableaux and are also the coefficients of the change of basis matrix from Schur functions to Hall-Littlewood symmetric functions. For more details on these polynomials, see [16]. Up to a power of q, the Kostka-Foulkes polynomials will play the role of $X(q)$ in our CSP.

It should be noted that the Kostka-Foulkes polynomials have an interesting representation theoretic interpretation. Let $\mathfrak{g}$ denote the simple Lie algebra $\mathfrak{sl}_k(\mathbb{C})$ and let $\mathfrak{g} \cong \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ denote the Cartan decomposition of $\mathfrak{g}$. For any partition $\lambda \vdash n$, let V^λ denote the irreducible representation of $\mathfrak{g}$ indexed by λ. The module V^λ has a weight space decomposition $V^\lambda \cong \oplus_\mu V^\lambda_\mu$, where the μ are elements of the dual algebra $\mathfrak{h}^*$. Let X be a *generic* element of the nilpotent subalgebra $\mathfrak{n}^+$ of $\mathfrak{g}$ generated by the positive roots. For $j = 0, 1, 2, \ldots$, let V_j denote the subspace of the weight space V^λ_μ which is killed by X^j, so that $V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots$. Since X is an element of the positive subalgebra, some sufficiently high power of X must carry every element of V^λ_μ outside of the weight support for the representation V^λ, so this filtration terminates in V^λ_μ for large j. This filtration of the weight space V^λ_μ is called the *Brylinski-Kostant (BK) filtration*. It turns out that the Kostka-Foulkes polynomial $K_{\lambda,\mu}(q)$ is equal to the associated jump polynomial ([6], [13]), that is,

$$(6.1) \qquad K_{\lambda,\mu}(q) = \sum_{j \geq 0} \dim(V_{j+1}/V_j) q^j.$$

To get our fixed point result, we will need some information about the evaluation of Kostka-Foulkes polynomials at roots of unity. Lascoux, Leclerc, and Thibon [16] have interpreted these evaluations in terms of ribbon tableaux. We define the relevant combinatorial objects.

For a positive integer m, an *m-ribbon* is a connected skew shape with m boxes which contains no 2 by 2 squares. The southwesternmost box of an m-ribbon is called the *head* of the m-ribbon and the northeasternmost box of an m-ribbon is called the *tail* of the m-ribbon.

For any skew partition μ/ν, an *m-ribbon tableau* of shape μ/ν is a tiling of the diagram of μ/ν by m-ribbons with a number attached to every m-ribbon in the tiling. Observe that the set of m-ribbon tableaux of shape μ is empty unless m divides $|\mu/\nu|$.

The content of an m-ribbon tableau T of shape $|\mu/\nu|$ is the composition of $|\mu/\nu|/m$ given by (number of 1s in T, number of 2s in T, ...). If an m-ribbon tableau T of shape μ/ν exists, the sign $\epsilon_m(\mu/\nu)$ of μ/ν is the number $(-1)^h(T)$, where $h(T)$ is the sum of the heights of the ribbons in T. It can be shown that the sign of a skew shape μ/ν is independent of the tableau T chosen.

An m-ribbon tableau T of shape μ/ν is said to be *column strict* if for any ribbon R in T with label, j the head of R does not lie to the right of a ribbon with label $i > j$ and the tail of R does not lie below a ribbon with label $i \geq j$. Observe that for $m = 1$ this definition reduces to the ordinary definition of column strict tableaux. For any composition $\beta \models \frac{|\mu/\nu|}{m}$, let $K^m_{\mu/\nu,\beta}$ denote the number of column strict ribbon tableaux of shape μ/ν and content β.

Say that a skew shape μ/ν is a *horizontal m-ribbon strip* if there exists a column strict m-ribbon tableau of shape μ/ν in which every ribbon has the same label. If μ/ν is a horizontal m-ribbon strip, it is easy to see that there exists a unique such column strict ribbon tableau (for fixed choice of label). The following result is mentioned on page 12 of [16].

Theorem 6.1. *(Lascoux-Leclerc-Thibon [16]) Let $\lambda \vdash n$ be a partition and $\alpha \models n$ be a composition. For $d|n$, let $\zeta \in \mathbb{C}$ be a root of unity of order d. If the multiplicity of any part of α is not divisible by d, then $K_{\lambda,\alpha}(\zeta) = 0$. If the multiplicity of every part of α is divisible by d, then the modulus $|K_{\lambda,\alpha}(\zeta)|$ is equal to the number of column strict d-ribbon tableaux with content $\tilde{\alpha}$. Here $\tilde{\alpha}$ is any composition of $\frac{n}{d}$ whose part multiplicities are all $\frac{1}{d}$ times the part multiplicities of α.*

We will also need a minor lemma which relates Schur function specialization to ribbon tableau enumeration. For this lemma there is no need for our rectangular shape hypothesis.

Lemma 6.2. *Let $\lambda \vdash n$ be a partition of* arbitrary *shape. Let $k > 0$ be a positive integer and let $d|k$. Let $\zeta = e^{2\pi i/k}$ be a primitive k^{th} root of unity and let $a_1, \ldots, a_d$ be arbitrary complex numbers.*

We have that

$$(6.2) \quad (\zeta^d)^{\kappa(\lambda)} s_\lambda(a_1, \zeta^d a_1, \ldots, \zeta^{k-d} a_1, \ldots, a_d, \ldots, \zeta^{k-d} a_d) = \sum_{\tilde{\beta}} K^{\frac{k}{d}}_{\lambda,\tilde{\beta}} a_1^{\frac{k}{d}\tilde{\beta}_1} \cdots a_d^{\frac{k}{d}\tilde{\beta}_d},$$

where the sum on the right hand side is over all compositions $\tilde{\beta} \models \frac{nd}{k}$.

The 'domino' case $d = \frac{k}{2}$ of the above lemma was known, for example, to Stembridge [35]. We also remark that the right hand side of the above identity can be interpreted as a product of specializations of Schur functions. In particular let $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\frac{k}{d})})$ be the $\frac{k}{d}$-quotient of the partition λ and let ν be the $\frac{k}{d}$-core of λ.

By the Stanton-White correspondence [33] there exists a content preserving bijection from column strict $\frac{k}{d}$-ribbon tableaux of skew shape λ/ν and $\frac{k}{d}$-tuples of ordinary column strict tableaux of shapes $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\frac{k}{d})})$. From this bijection it is fairly easy to derive an alternative formulation of Lemma 6.2. Namely, the expression $s_\lambda(a_1, \zeta^d a_1, \ldots, \zeta^{k-d} a_1, \ldots, a_d, \ldots, \zeta^{k-d} a_d)$ is equal to 0 unless λ has empty $\frac{k}{d}$-core and, if λ does have empty $\frac{k}{d}$-core, we have the identity

$$(6.3) \qquad |s_\lambda(a_1, \zeta^d a_1, \ldots, \zeta^{k-d} a_1, \ldots, a_d, \ldots, |\zeta^{k-d} a_d)| = |\Pi_{i=1}^{\frac{k}{d}} s_{\lambda^{(i)}}(a_1, a_2, | \ldots, a_d)|.$$

Proof. We show that both sides of this equation satisfy the same recursion.

For the left hand side of the above equation, observe that in any column strict tableaux in $CST(\lambda, k)$, the last $\frac{k}{d}$ letters $\{k - d + 1, k - d + 2, \cdots k\}$ of $[k]$ must appear in a skew shape flush with the exterior of the shape λ. This gives a partition of $CST(\lambda, k)$ according to this skew shape λ/ν.

We consider the skew Schur function specialization $s_{\lambda/\nu}(1, \zeta^d, \ldots, \zeta^{k-d})$ for the various skew shapes λ/ν. By a result of Lascoux, Leclerc, and Thibon [16], we have that the above is equal to 0 if λ/ν is not a horizontal $\frac{k}{d}$-ribbon strip and is equal to $\epsilon_{\frac{k}{d}}(\lambda/\nu)$ if λ/ν is a horizontal $\frac{k}{d}$-ribbon strip.

For the right hand side, given any column strict $\frac{k}{d}$-ribbon tableau T of content $\tilde\beta = (\tilde\beta_1, \ldots, \tilde\beta_d)$, we have that the restriction of T to the number $\tilde\beta_d$ is a horizontal $\frac{k}{d}$-ribbon strip of skew shape λ/ν for some normal shape $\nu \subseteq \lambda$. On the other hand, given any normal shape $\nu \subseteq \lambda$ such that λ/ν is a horizontal $\frac{k}{d}$-ribbon strip of with $\tilde\beta_k$ ribbons and any column strict $\frac{k}{d}$-ribbon tableau T_o of shape ν and content $(\tilde\beta_1, \tilde\beta_2, \ldots, \widetilde{\beta_{k-1}})$, we have that T_o extends uniquely to a column strict $\frac{k}{d}$-ribbon tableau T of shape λ and content $\tilde\beta$. It follows that both sides of the above equation satisfy the same recursion relation and, thus, the above equation holds for arbitrary partitions λ. Also, we have that both sides of the above equation are equal to zero when λ is not a $\frac{k}{d}$-ribbon tableau. $\qquad \square$

Proof. (of Theorem 1.5)

For any composition $\beta \models n$ of length k, let $N_{\lambda,\beta}^d$ denote the number of column strict tableaux of shape λ and content β which are fixed by the operator j^d. Observe that $N_{\lambda,\beta}^d$ is nonzero only if $\beta_i = \beta_j$ whenever $i \equiv j \pmod{d}$.

Let $c_k = (1, 2, \ldots, k)$ denote the long cycle in $S_k \subset GL_k(\mathbb{C})$. Let T be an arbitrary standard tableau of shape λ' and let $V_{T,k}$ be the irreducible $GL_k(\mathbb{C})$-representation constructed in Section 4. Identify the basis elements of $V_{T,K}$ with symbols $I_\beta(P)$ for compositions β of n of length k and *column* strict tableaux P of shape λ and content

β in the obvious way. By Theorem 4.6 have a weight space decomposition

$$V_{T,k} \cong \bigoplus_{\beta} V_{T,k,\beta},$$

where $V_{T,k,\beta}$ is the $\mathbb{C}$-linear subspace of $V_{T,k}$ spanned by the vectors $I_\beta(P)$ for $P \in CST(\lambda, k, \beta)$. Recall that each of the subspaces $V_{T,k,\beta}$ is stabilized by the Cartan subgroup H of diagonal matrices in $GL_k(\mathbb{C})$ and the action of c_k maps the space $V_{T,k,\beta}$ into the space $V_{T,k,c_k \cdot \beta}$.

Let $\zeta = e^{\frac{2\pi i}{k}}$ and let $a_1, \ldots, a_d \in \mathbb{C}^\times$ be any nonzero complex numbers. From the discussion in the above paragraph and Theorem 4.6, the action of the composition of c_k^d with an element of the Cartan subgroup yields the following character evaluation.

(6.4) $$\epsilon_{\frac{k}{d}}(\lambda)\kappa(\lambda)s_\lambda(a_1, \zeta^d a_1, \ldots, \zeta^{k-d}a_1, \ldots, a_d, \ldots, \zeta^{k-d}a_d) =$$

(6.5) $$\epsilon_{\frac{k}{d}}(\lambda)\kappa(\lambda)[s_\lambda(x_1, \ldots, x_k)]_{x_i = \zeta^{di}a_{\lfloor i/d \rfloor}} =$$

(6.6) $$\sum_{\beta} N_{\lambda,\beta}^d a_1^{\frac{k}{d}\beta_1} \cdots a_d^{\frac{k}{d}\beta_d},$$

where the sum on the lattermost expression ranges over all compositions β of n of length k such that $\beta_i = \beta_j$ whenever $i \equiv j \pmod{d}$. By Lemma 6.2 we can interpret the first expression in the above sequence of equalities as follows.

(6.7) $$\sum_{\beta} N_{\lambda,\beta}^d a_1^{\frac{k}{d}\beta_1} \cdots a_d^{\frac{k}{d}\beta_d} = \sum_{\tilde{\beta}} K_{\lambda,\tilde{\beta}}^{\frac{k}{d}} a_1^{\frac{k}{d}\tilde{\beta}_1} \cdots a_d^{\frac{k}{d}\tilde{\beta}_d},$$

where the sum on the left hand side ranges over all compositions β of n having length k such that $\beta_i = \beta_j$ whenever $i \equiv j \pmod{d}$ and the sum on the right hand side ranges over all compositions $\tilde{\beta}$ of $\frac{nd}{k}$ having length d. Choosing the complex numbers $a_1, \ldots, a_d$ to be algebraically independent over $\mathbb{Q}$, we can view this latter equation as a polynomial identity. It follows that if β is any composition of n of length k satisfying $\beta_i = \beta_j$ whenever $i \equiv j \pmod{d}$ and if we set $\tilde{\beta} := (\beta_1, \beta_2, \ldots, \beta_d)$, we have the equality

$$N_{\lambda,\beta}^d = K_{\lambda,\tilde{\beta}}^{\frac{k}{d}}.$$

It remains to relate the right hand side of the above equation to the evaluation of a Kostka-Foulkes polynomial at an appropriate root of unity. This is Theorem 6.1. $\square$

Remark 6.1. It would be desirable to strengthen Theorem 1.5 to a genuine CSP. One obvious way of doing this would be to find an expression $f(\lambda, \alpha)$ potentially

depending on both the rectangular partition λ and the composition α such that the triple $(CST(\lambda, k, \alpha), \langle j^d \rangle, q^{f(\lambda, \alpha)} K_{\lambda, \alpha}(q))$ exhibits the CSP. That is, our polynomial $X(q)$ would just be a $q-$shift of a Kostka-Foulkes polynomial. By Lascoux-Leclerc-Thibon we know that the evaluation of the Kostka-Foulkes polynomial $K_{\lambda, \alpha}(q)$ itself at the relevant roots of unity is plus or minus the correct positive number, but we do not know of an expression $f(\lambda, \alpha)$ which takes this sign into account.

7. Dihedral Actions

The cyclic sieving phenomenon is concerned with the enumeration of the fixed point set sizes when we are given an action on a finite set X by a finite cyclic group C. However, there are many interesting combinatorial actions on finite sets where the group in question is not cyclic. In general, given a finite group G acting on a finite set X, we can ask for the fixed point set sizes $|X^g|$ for all $g \in G$. For the case where G is a product of two cyclic groups, Barcelo, Reiner, and Stanton have proven an instance of a *bicyclic* sieving phenomenon involving complex reflection groups where the fixed point set sizes involved are obtained by evaluating a polynomial $X(q, t) \in \mathbb{Q}[q, t]$ in two variables at two appropriate roots of unity. In our case, we are interested in actions of *dihedral* groups D generated by promotion and evacuation on either of the sets $SYT(\lambda)$ or $CST(\lambda, k)$, where λ is a fixed rectangular shape and $k \geq 0$. While it is not yet known how to interpret the associated fixed point set sizes as evaluations of a naturally associated polynomial, we hope that this work will give motivation to the study of a 'dihedral sieving phenomenon'. Some work regarding dihedral actions is in the REU report [1].

Let $\lambda = b^a$ be a rectangular partition with $ab = n$ and let $k \geq 0$. It is possible to show that we have $eje = j^{-1}$ as operators on $CST(\lambda, k)$. Moreover, we know that the order of the operator j is equal to k while the order of the operator e is equal to 2. This implies that the group $\langle e, j \rangle$ generated by e and j, considered as a subgroup of the symmetric group $S_{CST(\lambda, k)}$, is dihedral of order $2k$. Similar remarks hold for the action of e and j on $RST(\lambda, k)$ and $SYT(\lambda)$.

We have already determined that the action of j on the above sets of tableaux is modeled by the action of the long cycle of an appropriate symmetric group on an appropriate module. By Stembridge's result, the action of e can be modeled by the action of the long element w_o. So, corresponding to the dihedral group acting on the sets $CST(\lambda, k)$, $RST(\lambda, k)$, and $SYT(\lambda)$, we get an isomorphic dihedral group generated by the long cycle and w_o sitting inside S_k, S_k, and S_n, respectively. Our first observation gives the cycle types of the operators w_o and $w_o c_n$ in the symmetric group S_n.

Lemma 7.1. *Let $w_o \in S_n$ be the long element and let $c_n \in S_n$ be the long cycle $(1, 2, \ldots, n)$. The permutation w_o has cycle type $2^{\frac{n}{2}}$ when n is even and $2^{\frac{n-1}{2}} 1$ when*

FIGURE 7.1. Conjugacy classes of reflections in dihedral groups

n is odd. The permutation $w_o c_n$ has cycle type $2^{(\frac{n}{2}-1)}1^2$ when n is even and $2^{\frac{n-1}{2}}1$ when n is odd.

Proof. This is straightforward from the definition of permutation multiplication. Looking at the dihedral group generated by c_n and w_o, this essentially asserts that if n is odd, any element of the unique conjugacy class of reflections within the symmetry group of an n-gon fixes one vertex and if n is even there are two conjugacy classes of reflections fixing zero or two vertices. See Figure 7.1. □

This simple lemma about permutation multiplication allows us to pin down, including sign, the action of the operators w_o and $w_o c_n$ on the cellular representation. As in Section 3, given a rectangular partition $\lambda = b^a$, identify the basis of the KL cellular representation of S_n corresponding to λ with $SYT(\lambda)$.

Proposition 7.2. *Let $\lambda = b^a$ be a rectangle and let ρ be the KL cellular representation corresponding to λ. We have that*

(7.1)
$$\rho(w_o)P = \begin{cases} e(P) & \text{if } b \text{ is even} \\ (-1)^{\lfloor \frac{a}{2} \rfloor}e(P) & \text{if } b \text{ is odd} \end{cases}$$

(7.2)
$$\rho(w_o c_n)P = \begin{cases} e(j(P)) & \text{if } b \text{ is even} \\ (-1)^{(\frac{a}{2}-1)}e(j(P)) & \text{if } b \text{ is odd and } a \text{ is even} \\ (-1)^{\lfloor \frac{a}{2} \rfloor}e(j(P)) & \text{if } b, a \text{ are odd.} \end{cases}$$

Proof. Recall that the KL cellular representation corresponding to a shape λ is an irreducible S_n module of shape λ. By Theorem 2.10 and Proposition 3.5, we need only verify that the signs in these equations are correct. To do this we will use the

Murnaghan-Nakayama rule (see, for example, [25]). For any of the 3 cycle types which appear in Lemma 7.1, there exists a corresponding removal of rim hooks from the corresponding rectangle with the appropriate sign.

In particular, if b is even, we can fill the diagram of λ with $\frac{n}{2}$ rim hooks of shape 2×1 in the case of w_o and $(\frac{n}{2} - 1)$ rim hooks of shape 2×1 together with two rim hooks of shape 1×1 in the case of $w_o c$. In either case, the product $(-1)^{h(\nu)}$ over all the rim hooks ν in either of these fillings, where $h(\nu)$ is the height of ν, is equal to 1.

If b is odd and a is even for the case of w_o, we can fill the diagram of λ with $\frac{n-a}{2}$ rim hooks of shape 2×1 in the left $(b-1) \times a$ box and $\frac{a}{2}$ rim hooks of shape 1×2 in the first column. The sign of this filling is equal to $(-1)^{\frac{a}{2}}$. In the case of $w_o c$ with the same parity conditions on a and b, we can replace the upper left 1×2 rectangle in the filling with two 1×1 rim hooks, and the resulting filling has sign $(-1)^{(\frac{a}{2}-1)}$.

If b and a are both odd, in the case of w_o and $w_o c_n$ we can fill the diagram of λ with $\frac{n-a}{2}$ rim hooks of shape 2×1 in the left $(b-1) \times a$ box, $\lfloor \frac{a}{2} \rfloor$ rim hooks of shape 1×2 in the lower portion of the first column of λ, and one 1×1 rim hook in the upper left hand corner of λ. The sign of this filling is $(-1)^{\lfloor \frac{a}{2} \rfloor}$.

It is possible to show that the sign of all rim hook fillings of λ corresponding to the conjugacy class of either w_o or $w_o c_n$ is the same, either 1 or -1. The desired result now follows from the Murnaghan-Nakayama rule. $\qquad\square$

Observe that, in this case, we were able to use the Murnaghan-Nakayama rule to deduce the sign of our permutation matrices relatively easily. This is in contrast to the proof of Lemma 3.4, where we needed to use special facts about the KL cellular representation. In fact, it is possible to use the Murnaghan-Nakayama rule in a similar way to derive Lemma 3.4 in the cases where both a and b are not odd, but for this particular case more specific facts are needed.

We already know that the number of points fixed by the action of a given power of j on standard tableaux is given by the evaluation of the q-hook length formula at an appropriate root of unity. The previous lemma allows us to give the number of fixed points of the operators e and ej on standard tableaux as a character evaluation.

Proposition 7.3. *Let $\lambda = b^a$ be a rectangular partition of n, let c be the long cycle in S_n, and let $\chi^\lambda : S_n \to \mathbb{C}$ be the irreducible character of S_n corresponding to λ. Let $\langle e, j \rangle$ be the dihedral subgroup of $S_{SYT(\lambda)}$ generated by e and j and let $\phi : \langle w_o, c_n \rangle \to \langle e, j \rangle$ be the group epimorphism defined by $\phi(w_o) = e$ and $\phi(c) = j$. For any $g \in \langle e, j \rangle$ and any w such that $\phi(w) = g$ one has*

$$|SYT(\lambda)^g| = \pm \chi^\lambda(w).$$

More precisely, we have that

$$(7.3) \qquad |SYT(\lambda)^e| = \begin{cases} \chi^\lambda(w_o) & \text{if } b \text{ is even} \\ (-1)^{\lfloor \frac{a}{2} \rfloor} \chi^\lambda(w_o) & \text{if } b \text{ is odd} \end{cases}$$

$$(7.4) \qquad |SYT(\lambda)^{ej}| = \begin{cases} \chi^\lambda(w_o c_n) & \text{if } b \text{ is even} \\ (-1)^{(\frac{a}{2}-1)} \chi^\lambda(w_o c_n) & \text{if } b \text{ is odd and } a \text{ is even} \\ (-1)^{\lfloor \frac{a}{2} \rfloor} \chi^\lambda(w_o c_n) & \text{if } b \text{ and } a \text{ are odd.} \end{cases}$$

Proof. This follows easily from the above lemma. □

Our next task is to extend these results to facts about the dual canonical basis. Let λ be a rectangular partition and let $k \geq 0$. By the results of Section 4, there exists a basis $\{I_\alpha(P)\}$ for the $GL_k(\mathbb{C})$-module $V_{T,k}$ for $T \in SYT(\lambda)$ fixed, where α ranges over all compositions of n of length k and P ranges over all elements of $RST(\lambda, k, \alpha)$. For any choice of α and P and any diagonal matrix $g = \text{diag}(x_1, \ldots, x_k) \in GL(\mathbb{C}^k)$, the vector $I_\alpha(P)$ is an eigenvector for g with eigenvalue $x_1^{-\alpha_1} x_2^{-\alpha_2} \cdots$. We first show how multiplication by the long elements interacts with the projection maps π_α.

Lemma 7.4. *Let λ be a rectangular partition and fix $T \in SYT(\lambda)$. Let w_{o_n} be the long permutation in S_n and let w_{o_k} be the long permutation in S_k. We have the following commutative diagram, where $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$ and $w_{o_k} \cdot \alpha$ is the composition of n of length k given by $(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$.*

$$(7.5) \qquad \begin{array}{ccc} & w_{o_n} & \\ V_{T,n,1^n} & \longrightarrow & V_{T,n,1^n} \\ \pi_\alpha \downarrow & & \downarrow \pi_{w_{o_k} \cdot \alpha} \\ V_{T,k,\alpha} & \longrightarrow & V_{T,k,w_{o_k} \cdot \alpha} \\ & w_{o_k} & \end{array}$$

Proof. The commutativity of the corresponding diagram of polynomial rings is easily verified by computing the image under the two possible compositions of the generators x_{ij}, as before. The commutativity of the above diagram is induced from this other commutative diagram. □

Using this lemma, we obtain the action of the long element in S_k and its product with the long cycle S_k on the basis elements $I_\alpha(P)$, $P \in RST(\lambda, k, \alpha)$.

Lemma 7.5. *Let w_{o_k} be the long element in S_k, let c_k be the long cycle $(1, 2, \ldots, k)$ in S_k, and let $\lambda = b^a$ be a rectangular partition, and let $P \in RST(\lambda, k, \alpha)$. We have*

that

$$w_{o_k} I_\alpha(P) = \begin{cases} I_{(\alpha_k,\ldots,\alpha_1)}(e(P)) & \text{if } a \text{ is even} \\ (-1)^{\lfloor \frac{b}{2} \rfloor} I_{(\alpha_k,\ldots,\alpha_1)}(e(P)) & \text{if } a \text{ is odd} \end{cases}$$

$$w_{o_k} c_k I_\alpha(P) = \begin{cases} (-1)^{\alpha_k(b-1)} I_{(\alpha_{k-1},\alpha_{k-2},\ldots,\alpha_1,\alpha_k)}(ej(P)) & \text{if } a \text{ is even} \\ (-1)^{(\frac{b}{2}-1+\alpha_k(b-1))} I_{(\alpha_{k-1},\alpha_{k-2},\ldots,\alpha_1,\alpha_k)}(ej(P)) & \text{if } a \text{ is odd and } b \text{ is even} \\ (-1)^{(\lfloor \frac{b}{2} \rfloor + \alpha_k(b-1))} I_{(\alpha_{k-1},\alpha_{k-2},\ldots,\alpha_1,\alpha_k)}(ej(P)) & \text{if } b, a \text{ are odd.} \end{cases}$$

Proof. This combines previously proven results about the actions of the long element and the long cycle. $\square$

Our results about fixed points have two flavors. When k is odd, j and ej are conjugate in the dihedral group $\langle e, j \rangle$ and hence have the same number of fixed points as operators on $CST(\lambda, k)$. However, when k is odd, e and ej are not conjugate, so different numbers of fixed points are possible. Parts of the next theorem come from conjectures and results of Abuzzahab, Korson, Li, and Meyer [1] and a theorem of Stembridge [35].

Theorem 7.6. *Let $\lambda = b^a$ be a rectangular partition of n and let $k \geq 0$. Assume that at k is odd. We have that*

$$(7.6) \qquad\qquad |CST(\lambda, k)^e| = (-1)^{\kappa(\lambda)} s_\lambda(1, -1, 1, \ldots, |(-1)^{k-1})$$

$$(7.7) \qquad\qquad\qquad = |CST(\lambda, k)^{ej}|.$$

On the other hand, if k is even, we have that

$$(7.8)$$

$$|CST(\lambda, k)^e| = (-1)^{\kappa(\lambda)} s_\lambda(1, -1, 1, \ldots, |(-1)^{k-1})$$

$$(7.9)$$

$$|CST(\lambda, k)^{ej}| = \begin{cases} (-1)^{\kappa(\lambda)} s_\lambda(1, -1, \ldots, (-1)^{k-3}, (-1)^{k-2}, (-1)^{k-2}) & a \text{ and } b \text{ even} \\ (-1)^{\kappa(\lambda)} s_\lambda(1, -1, \ldots, (-1)^{k-1}) & a \text{ even, } b \text{ odd} \\ (-1)^{\frac{b}{2}-1+\kappa(\lambda)} s_\lambda(1, -1, \ldots, (-1)^{k-1}) & a \text{ odd, } b \text{ even} \\ (-1)^{\lfloor \frac{b}{2} \rfloor + \kappa(\lambda)} s_\lambda(1, -1, \ldots, (-1)^{k-3}, (-1)^{k-2}, (-1)^{k-2}) & a \text{ and } b \text{ odd.} \end{cases}$$

Proof. The claims about the operator e are Theorem 3.1 of [35]. The claim about the operator ej in the case where k is odd follows from the claim about the operator j by conjugacy, as mentioned above. We are reduced to the claim about the operator ej in the case where k is even. As before, we prove the equivalent statement for row strict tableaux. This can be obtained by copying the above claim *verbatim*, replacing every RST with a CST, every λ with a λ', and swapping every b for every a.

Since k is even, we know that the operator $w_o c_k$ as an element of $GL(\mathbb{C}^k)$ has eigenvalues 1 with multiplicity $\frac{k}{2} + 1$ and -1 with multiplicity $\frac{k}{2} - 1$. The claim now follows by examining the signs in Lemma 5.4 with the appropriate parity conditions on a and b in mind. In particular, multiplication by terms of the form $(-1)^{\alpha_k}$ corresponds to swapping a sign of one of the arguments in the appropriate Schur function. $\square$

Observe that in the above 'column strict' result one does *not* have that for any g in the group of operators generated by evacuation and promotion the fixed point set $|CST(\lambda, k)^g|$ is equal to plus or minus the character of a general linear group element corresponding to g. Roughly speaking, the reason for this is that Equation (5.4) implies that the long cycle acts as promotion on column strict tableaux up to a sign which is *not* in general independent of weight space.

8. Applications to other Combinatorial Actions

For a positive integer n, let B_n denote the group of signed permutations of $[n]$ having order $2^n n!$. The group B_n is a Coxeter group with generators $s_0', s_1', s_2', \ldots, s_{n-1}'$, where s_i' for $i > 0$ is the adjacent transposition switching positions i and $i + 1$ but preserving signs and s_0' switches the sign of the element in the first position. Under this identification, the long element of B_n is the signed permutation which sends 1 to -1, 2 to -2, $\ldots$, and n to $-n$. Since the long element $w_o^{B_n}$ in Type B_n is the scalar transformation -1, we have in particular that $w_o^{B_n} s_i' = s_i' w_o^{B_n}$ for all i, so that reduced expressions for the long element are mapped to other reduced expressions for the long element under cyclic rotation. Our cyclic sieving results specialize in the case of standard tableaux of the square shape n^n to analyze this cyclic action.

Theorem 8.1. *Let X be the set of reduced words for the long element in B_n. Let $C = \mathbb{Z}/n^2\mathbb{Z}$ act on X by cyclically rotating words. Let $X(q) = f^{n^n}(q)$ be the $q-$hook length formula. We have that the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon.*

Proof. By Theorem 1.3, we have a CSP involving the cyclic action of promotion on the set $SYT(n^n)$ and the same polynomial $X(q)$. Let Y be the set of standard *shifted* tableaux with n^2 boxes and staircase shape (see, for example, [10], [26], or [38]). Shifted jeu-de-taquin promotion gives a cyclic action on Y.

By a result of Haiman ([10], Proposition 8.11) we have a bijection between the sets $SYT(n^n)$ and Y under which ordinary promotion gets sent to shifted promotion. Moreover, the type B Edelman-Greene correspondence gives a bijection between Y and the set X under which shifted jeu-de-taquin promotion maps to cyclic rotation of reduced words ([11], Theorem 5.12). $\square$

We can also apply our enumeration results for standard tableaux to prove cyclic sieving phenomena for handshake patterns and noncrossing partitions. This gives

a new proof of a result of White [37], as well as a new proof of results of Heitsch [12] which have biological applications related to RNA secondary structure. The case of handshake patterns gives a very explicit realization of the dihedral actions investigated in Section 7.

Given $n \in \mathbb{N}$, a *handshake pattern of size 2n* consists of a circle around which the points $1, 2, \ldots, 2n$ are written clockwise and a perfect matching on the set $[2n]$ such that, when drawn on the circle, none of the arcs in this matching intersect. This can be thought of as a way in which the people labelled $1, 2, \ldots, 2n$ can all shake hands so that no one crosses arms. Let H_n denote the set of all handshake patterns of size $2n$. Handshake patterns can be identified with the basis elements for the Temperley-Lieb algebra $T_n(\zeta)$ (see, for example, [23]).

For $n \in \mathbb{N}$, a *noncrossing partition of [n]* is a set partition $(P_1|P_2|\ldots|P_k)$ of the set $[n]$ so that whenever there are integers a, b, c, d, i, and j with $1 \leq a < b < c < d \leq n$ and $a, c \in P_i$ and $b, d \in P_j$, we must necessarily have that $i = j$. Noncrossing partitions have a pictorial interpretation. Drawing the numbers $1, 2, \ldots, n$ clockwise around a circle, a partition ı of $[n]$ is noncrossing if and only if, when the blocks of π are drawn on the circle, none of the regions intersect.

Recall that a poset L is a lattice if and only if given any pair of elements $x, y \in L$, there exist unique least upper bounds and greatest lower bounds for x and y (denoted $x \vee y$ and $x \wedge y$, respectively). If L is a finite lattice and $x \in L$, then an element $y \in L$ is said to be a *complement to x* if we have $x \vee y = \hat{1}$ and also $x \wedge y = \hat{0}$. Here $\hat{1}$ and $\hat{0}$ denote the unique maximal and minimal elements of L. A finite lattice L is said to be *complemented* if every element has at least one complement. The set $NC(n)$ of all noncrossing partitions of $[n]$ is a lattice with respect to the partial order given by refinement (however, this lattice is *not* a sublattice of the lattice of all partitions since the formulas for the least upper bounds and greatest lower bounds do not agree). The lattice $NC(n)$ is complemented, and given a noncrossing partition π of $[n]$, Kreweras complementation [15] gives a way to produce a complement of π. Write the numbers $1, 1', 2, 2', \ldots, n, n'$ clockwise around a circle. Draw the blocks of π on the numbers $1, 2, \ldots, n$. Now draw the unique maximal noncrossing partition of the numbers $1', 2', \ldots, n'$ which does not intersect the blocks of π. Call this new partition π'. Kreweras showed that π' is a complement to π in $NC(n)$. Complements in $NC(n)$ are not in general unique and it is an open problem to determine how many complements there are in $NC(n)$ for a fixed noncrossing partition π of $[n]$.

Both of the sets H_n and $NC(n)$ have cardinality given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Moreover, these sets both carry an action of the cyclic group - in the case of H_n given by the action of rotating the table clockwise by one position and in the case of $NC(n)$ given by Kreweras complementation. As a corollary of our earlier work, we get results of White [37] and Heitsch [12].

Theorem 8.2. *Let $n \in \mathbb{N}$ and let $C = \mathbb{Z}/(2n)\mathbb{Z}$. Let $X(q)$ be the q-Catalan number*

$$X(q) = C_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q.$$

Let X be either H_n equipped with the C-action of rotation or $NC(n)$ equipped with the C-action of Kreweras complementation.

Then, the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon.

Proof. It is well known (see, e.g., Heitsch [12]) that there exists a bijection $NC(n) \to H_n$ under which Kreweras complementation corresponds to rotation, so we need only prove this theorem in the case where $X = H_n$. To do this, we use a bijection $H_n \to SYT((n, n))$ which maps rotation to jeu-de-taquin promotion. It is well known that the q-Catalan numbers are equal to the q-hook length formula in the special case of a 2 by n partition, so then the result will follow from Theorem 1.3.

To construct this bijection, given a handshake pattern $h \in H_n$, fill the top row of a 2 by n tableau T with the smaller members of each of the n handshake pairs in h in increasing order left to right. Fill the bottom row of T with the smaller members of each handshake pair in increasing order from left to right. Since h is a handshake pattern, it follows that T is a standard tableau. It is easy to see that the correspondence $h \mapsto T$ defines a bijection $H_n \to SYT((n, n))$.

White proved that, under the above bijection, rotation maps to promotion [37]. For completeness, we prove this again here.

Given a tableau $T \in SYT(n, n)$, define the *ascent set* $A(T)$ of T to be the subset of $[2n-1]$ defined by $i \in A(T)$ if and only if $i + 1$ occurs strictly north and weakly east of i in T. It is easy to check that for $i = 1, 2, \ldots, 2n - 2$, i is contained in $A(T)$ if and only if i is contained in $A(j(T))$. Moreover, by Lemma 3.3, j acts cyclically on the extended descent set. It is easy to check that the ascent set and extended descent set of a two-row rectangular standard tableau T together determine the associated handshake pattern. Moreover, it is easy to check that table rotation acts cyclically on the ascent set and extended descent set of the associated tableau. Therefore, the bijection between H_n and $SYT((n, n))$ maps table rotation to promotion, as desired. $\square$

White proved the portion of Theorem 8.2 involving handshake patterns directly in the investigation of the rectangular standard tableaux CSP conjecture, thus proving the conjecture in the case of tableaux with 2 rows. Having proven this more general conjecture in Theorem 8.2 we get White's result on handshake patterns as a corollary.

The set H_n also carries an action of the dihedral group D_{4n} of order $4n$. Namely, we think of the points $1, 2, \ldots 2n$ as the vertices of a regular $2n$-gon and let D_{4n} act

naturally. Let r denote the reflection in D_{4n} across the line bisecting the pairs of vertices $(1, 2n)$ and $(n, n+1)$. Let s denote rotation in D_{4n} by one clockwise unit.

Proposition 8.3. *Let X be either the set H_n or the set $NC(n)$. X carries an action of D_{4n}, where r and s have the action described above in the case of H_n and act by reflection about the line through 1 bisecting the circle and Kreweras complementation in the case of $NC(n)$. Let w_o denote the long element of S_{2n} and let c denote the long cycle $(1, 2, \ldots, 2n)$ in S_{2n}.*

Then, the number of fixed points of the operators r and rs on X are given by the formulas:

$$(8.1) \qquad |X^r| = \begin{cases} \chi^{(n,n)}(w_o) & \text{if } n \text{ is even} \\ (-1)\chi^{(n,n)}(w_o) & \text{if } n \text{ is odd} \end{cases}$$

$$(8.2) \qquad |X^{rs}| = \chi^{(n,n)}(w_o c)$$

Proof. It is easy to see, given the simplified algorithm for evacuation in the case of rectangular tableaux, that under White's bijection $H_n \to SYT((n,n))$ the action of r corresponds to evacuation. Moreover, it is simple to see that under the bijection of Heitsch [12] $H_n \to NC(n)$, the action of r corresponds to reflection across the line bisecting the circle and going through the vertex labelled 1. Therefore, the result follows from the specialization of Proposition 7.2 to the case $a = 2$ and $b = n$. $\qquad \square$

9. OPEN PROBLEMS

Most of the results in this paper have enumerated some fixed point set via some representation theoretic interpretation of the associated action. These enumerative results imply a collection of bijections involving standard tableaux.

Corollary 9.1. *Let $\lambda \vdash n$ be a rectangular partition and suppose $d|n$. Then, there is a bijective correspondence between the set of tableaux in $SYT(\lambda)$ fixed by j^d and the set of column strict $\frac{n}{d}$-ribbon tableaux of shape λ with content $1^{\frac{n}{d}}$.*

Corollary 9.2. *Let $\lambda \vdash n$ be a rectangular partition and suppose $d|k$ are positive integers. Then, there is a bijective correspondence between the set of tableaux in $CST(\lambda, k)$ fixed by j^d and the set of column strict $\frac{k}{d}$-ribbon tableaux of shape λ with labels drawn from $[\frac{k}{d}]$.*

Corollary 9.3. *Let $\lambda \vdash n$ be a rectangular partition and let $\alpha \models n$ be a composition with $\ell(\alpha) = k$. Suppose that $d|k$ and $c_k^d \cdot \alpha = \alpha$. Then, there is a bijective correspondence between the set of tableaux in $CST(\lambda, k, \alpha)$ which are fixed by j^d and the set of column strict $\frac{k}{d}$-ribbon tableaux of shape λ and content $(\alpha_1, \alpha_2, \ldots, \alpha_d)$.*

These combinatorial results naturally lead to the following problem.

Problem 9.4. *Write down an explicit bijection between any of the above three pairs of sets.*

As we mentioned, one of the reasons that we required λ to be rectangular in Theorems 1.3, 1.4, and 1.5 was that j^n does not in general fix every element of $SYT(\lambda)$ for arbitrary partitions $\lambda \vdash n$. This leads to the following question.

Question 9.5. *Let $\lambda \vdash n$ be an arbitrary partition. What is the order of the promotion operator j on $SYT(\lambda)$? What about on $CST(\lambda, k)$ for some fixed $k \geq 0$? What about on $CST(\lambda, k, \alpha)$ for some fixed $k \geq 0$ and composition $\alpha \models n$ with $\ell(\alpha) = k$?*

Related to determining the order of promotion on nonrectangular tableaux is the problem of determining the cycle structure of its action.

Question 9.6. *Let $\lambda \vdash n$ be an arbitrary partition and let the cyclic group C act on $SYT(\lambda)$ where the action is generated by jeu-de-taquin promotion. Find an explicit polynomial $X(q)$ such that $(SYT(\lambda), C, X(q))$ exhibits the CSP. Do the same for column strict tableaux, with or without fixed content.*

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