# Vertex Turán problems in the hypercube 

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#### Abstract

Let $\mathcal{Q}_{n}$ be the $n$-dimensional hypercube: the graph with vertex set $\{0,1\}^{n}$ and edges between vertices that differ in exactly one coordinate. For $1 \leq d \leq n$ and $F \subseteq\{0,1\}^{d}$ we say that $S \subseteq\{0,1\}^{n}$ is $F$-free if every embedding $i:\{0,1\}^{d} \rightarrow\{0,1\}^{n}$ satisfies $i(F) \nsubseteq S$. We consider the question of how large $S \subseteq\{0,1\}^{n}$ can be if it is $F$-free. In particular we generalise the main prior result in this area, for $F=\{0,1\}^{2}$, due to E.A. Kostochka and prove a local stability result for the structure of near-extremal sets.

We also show that the density required to guarantee an embedded copy of at least one of a family of forbidden configurations may be significantly lower than that required to ensure an embedded copy of any individual member of the family.

Finally we show that any subset of the $n$-dimensional hypercube of positive density will contain exponentially many points from some embedded $d$-dimensional subcube if $n$ is sufficiently large.


## 1 Introduction

For $n \geq 1$ let $V_{n}=\{0,1\}^{n}$. The $n$-dimensional hypercube, $\mathcal{Q}_{n}$, is the graph with vertex set $V_{n}$ and edges between vertices that differ in exactly one coordinate.

An embedding of $\mathcal{Q}_{d}$ into $\mathcal{Q}_{n}$ is an injective map $i: V_{d} \rightarrow V_{n}$ that preserves the edges of $\mathcal{Q}_{d}$. (Note that the image of $V_{d}$ under any such embedding consists of $2^{d}$ elements of $V_{n}$ given by fixing $n-d$ coordinates and allowing the other $d$ coordinates to vary.)

Given $F \subseteq V_{d}$, where $1 \leq d \leq n$, we say that $S \subseteq V_{n}$ is $F$-free if every embedding $i: V_{d} \rightarrow V_{n}$ satisfies $i(F) \nsubseteq S$. For a family $\mathcal{F}$ of subsets of $V_{d}$ we say that $S$ is $\mathcal{F}$-free if $S$ is $F$-free for all $F \in \mathcal{F}$. We define

$$
\operatorname{exc}(n, \mathcal{F})=\max \left\{|S|: S \subseteq V_{n} \text { is } \mathcal{F} \text {-free }\right\}
$$

[^0]It is easy to see (via averaging) that for any family $\mathcal{F}$ of subsets of $V_{d}$ the ratio $\operatorname{exc}(n, \mathcal{F}) / 2^{n}$ is non-increasing and bounded below (by zero). Hence we can define the vertex Turán density by

$$
\lambda(\mathcal{F})=\lim _{n \rightarrow \infty} \frac{\operatorname{exc}(n, \mathcal{F})}{2^{n}}
$$

We write $\lambda(F)$ instead of $\lambda(\{F\})$.
If $x \in V_{n}$ and $i \in[n]=\{1,2, \ldots, n\}$ then $x_{i} \in\{0,1\}$ denotes the $i$ th coordinate of $x$. The support of $x \in V_{n}$ is $\operatorname{supp}(x)=\left\{i \in[n]: x_{i}=1\right\}$ and the weight of $x$ is $|x|=|\operatorname{supp}(x)|$. The set of all $x \in V_{n}$ of weight $r$ is called the $r$ th layer.

The quantity $\operatorname{exc}\left(n, V_{2}\right)$ was determined by Kostochka [13] (and independently by Johnson and Entringer [11]). They also showed that the unique largest $V_{2}$-free subset of $V_{n}$ can be obtained by deleting every third layer of $V_{n}$.

Theorem 1 (Kostochka [13]) For $n \geq 2$ we have exc $\left(n, V_{2}\right)=\left\lceil 2^{n+1} / 3\right\rceil$. If $S \subseteq V_{n}$ is $V_{2}$-free and $|S|=\operatorname{exc}\left(n, V_{2}\right)$ then, up to automorphisms of $\mathcal{Q}_{n}, S$ is

$$
S_{i}=\left\{x \in V_{n}:|x| \not \equiv i \bmod 3\right\}
$$

for some $i \in\{0,1,2\}$.
The problem of determining $\operatorname{exc}\left(n, V_{d}\right)$ has been considered by various authors but mainly when $d$ is close to $n$ (see [12, [17, [19]) or when $n$ is small [10.

Recently, Alon, Krech and Szabó [1] described the problem of finding $\lambda\left(V_{d}\right)$. Their motivation was a related question of Erdős [8] who asked for the largest number of edges in a $\mathcal{Q}_{2}$-free subgraph of $\mathcal{Q}_{n}$. The conjectured answer to this is $(1 / 2+o(1)) e\left(\mathcal{Q}_{n}\right)$ while the best upper bound is around $0.62256 e\left(\mathcal{Q}_{n}\right)$ due to Thomason and Wagner [20] extending earlier work of Chung [5]. A result relating the maximum density of edges of a $\mathcal{Q}_{d}$-free subgraph of $\mathcal{Q}_{n}$ to the analagous density for certain other forbidden subgraphs was proved by Offner [16] using the supersaturation method. He also proved a vertex version of this result although our notion of containment as an embedded copy is slightly different from his. The general vertex version of the problem which we consider here is extremely natural but does not seem to have received attention.

## 2 Results

Our first result is a generalisation of Theorem 1 We show that asymptotically the density required to guarantee a copy of $V_{2}$ is sufficient to ensure copies of other larger configurations.

The configuration $G_{d}$ will be the set of all vertices of $V_{d}$ of weight zero or one together with a set of vertices of weight two whose supports form the edge set of a complete bipartite graph $K(\lceil d / 2\rceil,\lfloor d / 2\rfloor)$ (since all such configurations


Figure 1: Forbidden configurations $G_{2}, G_{3}$ and $G_{4}$ (black points)
are isomorphic the precise choice of bipartition is unimportant, we will take the one given by parity of coordinates). Formally for $d \geq 2$ we define

$$
G_{d}=\left\{x \in V_{d}:|x|=0,1 \text { or }(|x|=2, \operatorname{supp}(x)=\{i, j\}, i \not \equiv j \bmod 2)\right\} .
$$

For example

$$
G_{2}=V_{2}, \quad G_{3}=\{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(0,1,1)\}
$$

The relationship of our result to Theorem 1 is rather like that of the ErdősStone theorem for 3-chromatic graphs to Mantel's theorem in extremal graph theory.

We also show that if $S \subseteq V_{n}$ is $G_{d}$-free and near-extremal in size then $S$ must locally resemble the "two-out-of three-layers" extremal construction given in Theorem 1

We require some notation. Denote the Hamming distance in $\mathcal{Q}_{n}$ by dist $(x, y)$, this is the number of coordinates in which $x, y \in V_{n}$ differ. For $l \geq 0$ and $x \in V_{n}$ let $\Gamma_{l}(x)=\left\{y \in V_{n}: \operatorname{dist}(x, y)=l\right\}$. Given $S \subseteq V_{n}$ we let $h_{l}(x)=\left|S \cap \Gamma_{l}(x)\right|$ denote the number of elements of $S$ at distance $l$ from $x$.

Although $h_{l}(x)$ depends on the set $S$, for ease of notation we will suppress this. It will always be clear from the context what subset of $V_{n}$ we are considering.

For $x \in V_{n}, S \subseteq V_{n}$ and $l \geq 1$ it is natural to view $S \cap \Gamma_{l}(x)$ as an $l$-uniform hypergraph. Formally we define

$$
S^{l}(x)=\left\{\operatorname{supp}(x) \Delta \operatorname{supp}(y): y \in S \cap \Gamma_{l}(x)\right\}
$$

For a set $X$ and integer $r \geq 0$ we write $\binom{X}{r}$ for the family of all subsets of $X$ of size $r$. So $A \in\binom{[n]}{l}$ belongs to $S^{l}(x)$ if flipping all of the coordinates of $x$ indexed by $A$ yields an element $y \in S$.

The precise definition of local stability is given in Theorem 2 below but the three conditions may be paraphrased as follows.
(a) For most $x \in V_{n} \backslash S$, most of the neighbours of $x$ (in $\mathcal{Q}_{n}$ ) belong to $S$.
(b) For most $x \in S$, approximately half of the neighbours of $x$ belong to $S$.
(c) For most $x \in S$ the graph $S^{(2)}(x)$ (corresponding to points at distance two from $x$ in $S$ ) is almost a clique on $[n]$ with a clique on $S^{(1)}(x)$ removed.

Theorem 2 If $d \geq 2$ and $G_{d}$ is as defined above then
(i) (Vertex Turán density)

$$
\lambda\left(G_{d}\right)=\frac{2}{3}
$$

(ii) (Local stability) If $\epsilon>0$ there exists $\delta=\delta(\epsilon, d)$ satisfying $\lim _{\epsilon \rightarrow 0^{+}} \delta(\epsilon, d)=$ 0 and $n_{0}=n_{0}(\epsilon, d)$ such that if $n \geq n_{0}$ and $S \subseteq V_{n}$ is $G_{d}$-free with $|S| \geq$ $(2 / 3-\epsilon) 2^{n}$ then locally $S$ resembles the set $S_{0}=\left\{x \in V_{n}:|x| \not \equiv 0 \bmod 3\right\}$ in the following sense. There exists $T \subseteq V_{n}$ with $|T| \leq \delta 2^{n}$ and
(a) $h_{1}(x) \geq(1-\delta) n$ for all $x \in V_{n} \backslash(S \cup T)$.
(b) $\left|h_{1}(x)-n / 2\right| \leq \delta n$ for all $x \in S \backslash T$.
(c) $\left|S^{2}(x) \Delta\left(\binom{[n]}{2} \backslash\binom{S^{1}(x)}{2}\right)\right| \leq \delta\binom{n}{2}$ for all $x \in S \backslash T$.

Note that a "global" stability result cannot hold for this problem in the sense that there exist near-extremal size $G_{d}$-free subsets of $V_{n}$ that cannot be obtained from the "two-out-of-three-layers" construction by deleting/adding a small number of points and taking an automorphism of the hypercube. For example

$$
\begin{aligned}
& S=\left\{x \in V_{n}:|x| \leq n / 2,|x| \not \equiv 0 \bmod 3\right\} \cup \\
& \qquad\left\{x \in V_{n}:|x| \geq n / 2+3,|x \Delta[n / 2]| \not \equiv 0 \bmod 3\right\} .
\end{aligned}
$$

Our second result (Theorem 3) shows that the density required to ensure a copy of at least one of a family of forbidden configurations may be significantly lower than that required to ensure a copy of any individual member of the family. This is in contrast to ordinary graph Turán densities where the Erdős-StoneSimonovits theorem implies that for any family $\mathcal{F}$ of graphs $\pi(\mathcal{F})=\min \{\pi(F)$ : $F \in \mathcal{F}\}$ (where $\pi(\mathcal{F})$ is the classical Turán density of the family of graphs $\mathcal{F})$. This "non-principality" of the vertex Turán density is analogous to that previously observed for $r$-uniform hypergraph Turán densities by Balogh [2] and Mubayi and Pikhurko [15] when $r \geq 3$.

Let

$$
\begin{gathered}
F_{1}=\{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\} \\
F_{2}=\{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}, \quad F_{3}=\{(0,0,0),(1,1,1)\}
\end{gathered}
$$

Theorem 3 determines $\lambda(\mathcal{F})$ for all families $\mathcal{F} \subseteq\left\{F_{1}, F_{2}, F_{3}\right\}$. In particular $\lambda\left(\left\{F_{2}, F_{3}\right\}\right)<\lambda\left(\left\{F_{1}, F_{2}\right\}\right)<\min \left\{\lambda\left(F_{1}\right), \lambda\left(F_{2}\right), \lambda\left(F_{3}\right)\right\}$.


Figure 2: Forbidden configurations $F_{1}, F_{2}$ and $F_{3}$

Theorem 3 If $F_{1}, F_{2}$ and $F_{3}$ are as defined above then

$$
\begin{gathered}
\lambda\left(F_{1}\right)=\lambda\left(F_{2}\right)=\lambda\left(F_{3}\right)=\lambda\left(\left\{F_{1}, F_{3}\right\}\right)=\frac{1}{2}, \\
\lambda\left(\left\{F_{1}, F_{2}\right\}\right)=\frac{1}{3}, \quad \lambda\left(\left\{F_{2}, F_{3}\right\}\right)=\lambda\left(\left\{F_{1}, F_{2}, F_{3}\right\}\right)=\frac{1}{4} .
\end{gathered}
$$

Finally we consider the following question: given a subset of $V_{n}$ of positive density how many vertices must it contain from some $d$-dimensional subcube? For $1 \leq t \leq d$ let $\mathcal{F}_{d, t}=\left\{F \subseteq V_{d}:|F|=t\right\}$. If $\lambda\left(\mathcal{F}_{d, t}\right)=0$ then for any $\epsilon>0$ and $n \geq n_{0}(\epsilon, d)$ sufficiently large, any subset of $V_{n}$ of density at least $\epsilon$ will contain at least $t$ vertices from some $d$-dimensional subcube. We would like to determine the value of

$$
\mu(d)=\max \left\{t: \lambda\left(\mathcal{F}_{d, t}\right)=0\right\} .
$$

The following construction does not contain vertices from more than one layer of any $d$-dimensional subcube of $\mathcal{Q}_{n}$ and has density approximately $1 /(d+1)$

$$
S_{d+1}=\left\{x \in V_{n}:|x| \equiv 0 \bmod d+1\right\} .
$$

Since the largest layer of $V_{d}$ has size $\binom{d}{\lfloor d / 2\rfloor}$ this proves the upper bound in Theorem 4. The lower bound tells us that $\mu(d)$ is exponential in $d$.

Theorem 4 If $d \geq 2$ then

$$
t_{2}(d)+t_{3}(d) \leq \mu(d) \leq\binom{ d}{\lfloor d / 2\rfloor}
$$

where

$$
t_{2}(d)=\left\{\begin{array}{lll}
0, & \text { if }\lceil d / 3\rceil \text { is odd }, \\
1, & \text { otherwise. }
\end{array} \quad t_{3}(d)=\left\{\begin{array}{lll}
3^{d / 3}, & d \equiv 0 & \bmod 3 \\
4 \cdot 3^{(d-4) / 3}, & d \equiv 1 & \bmod 3 \\
2 \cdot 3^{(d-2) / 3}, & d \equiv 2 & \bmod 3
\end{array}\right.\right.
$$

## 3 Proofs

We will make use of a number of classical results from extremal graph and hypergraph theory.

Theorem 5 (Mantel [14]) If $G=(V, E)$ is a triangle-free graph with $|V|=n$ then $|E| \leq n^{2} / 4$.

For $s \geq t \geq 1$ let $K(s, t)$ denote the complete bipartite graph with vertex classes of size $s$ and $t$. For $r \geq 3$ and $t_{1} \geq t_{2} \geq \cdots \geq t_{r} \geq 1$ let $K^{(r)}\left(t_{1}, \ldots, t_{r}\right)$ denote the complete $r$-partite $r$-graph with vertex classes of size $t_{1}, \ldots, t_{r}$.

Theorem 6 (Erdős-Stone [9]) If $G=(V, E)$ is a $K(s, t)$-free graph and $|V|=n$ then $|E|=O\left(n^{2-1 / t}\right)$.

Theorem 7 (Erdős [7]) If the r-graph $G=(V, E)$ is $K^{(r)}\left(t_{1}, t_{2}, \ldots, t_{r}\right)$-free and $|V|=n$ then $|E|=O\left(n^{r-1 / t_{1}}\right)$.

We will make repeated use of the following special case of the CauchySchwarz inequality.

Lemma 8 If $a_{1}, \ldots, a_{s} \in \mathbb{R}$ and $\frac{1}{s} \sum_{i=1}^{s} a_{i} \geq A$ then

$$
\frac{1}{s} \sum_{i=1}^{s} a_{i}^{2} \geq A^{2} .
$$

For $x, y \in V_{n}$ let $h_{l}(x, y)$ denote the number of elements of $S$ at distance $l$ from both $x$ and $y$, i.e.

$$
h_{l}(x, y)=\left|S \cap \Gamma_{l}(x) \cap \Gamma_{l}(y)\right| .
$$

The following simple lemma underpins all our results.
Lemma 9 If $S \subseteq V_{n}$ and $l \geq 1$ then

$$
\begin{equation*}
\sum_{v \in V_{n}} h_{l}^{2}(v)=\sum_{x \in S}\binom{2 l}{l} h_{2 l}(x)+O\left(n^{2 l-1} 2^{n}\right) \tag{i}
\end{equation*}
$$

(ii)

$$
\sum_{v \in S} h_{l}^{2}(v)=\sum_{x \in S} \sum_{z \in S \cap \Gamma_{2 l}(x)} h_{l}(x, z)+O\left(n^{2 l-1} 2^{n}\right) .
$$

Proof: For (i) consider the sum

$$
\sum_{x \in S} \sum_{y \in \Gamma_{l}(x)} h_{l}(y)
$$

For each $v \in V_{n}$ the term $h_{l}(v)$ occurs once for each element of $S$ at distance $l$ from $v$, i.e. $h_{l}(v)$ times. Hence

$$
\sum_{x \in S} \sum_{y \in \Gamma_{l}(x)} h_{l}(y)=\sum_{v \in V_{n}} h_{l}^{2}(v)
$$

Moreover for a fixed choice of $x \in S$ the inner sum counts elements of $S$ that can be reached from $x$ by flipping $l$ coordinates of $x$ and then flipping $l$ coordinates of the resulting point. Hence if $0 \leq k \leq l$ then the sum counts those elements of $S$ at distance $2 k$ from $x$ precisely $\binom{2 \bar{k}}{k}\binom{n-2 k}{l-k}$ times. Thus

$$
\begin{aligned}
\sum_{x \in S} \sum_{y \in \Gamma_{l}(x)} h_{l}(y) & =\sum_{x \in S} \sum_{k=0}^{l}\binom{2 k}{k}\binom{n-2 k}{l-k} h_{2 k}(x) \\
& =\sum_{x \in S}\binom{2 l}{l} h_{2 l}(x)+O\left(n^{2 l-1} 2^{n}\right),
\end{aligned}
$$

since $h_{2 k}(x)=O\left(n^{2 k}\right)$ for any $0 \leq k \leq l$ and $|S| \leq 2^{n}$. Hence (i) holds.
For (ii) consider the sum

$$
\sum_{x \in S} \sum_{y \in S \cap \Gamma_{l}(x)} h_{l}(y)
$$

For each $v \in S$ the term $h_{l}(v)$ occurs once for each element of $S$ at distance $l$ from $v$, i.e. $h_{l}(v)$ times. Hence

$$
\sum_{x \in S} \sum_{y \in S \cap \Gamma_{l}(x)} h_{l}(y)=\sum_{v \in S} h_{l}^{2}(v) .
$$

The same argument as used for (i) implies that the contribution to the LHS of this sum from $z \in S$ satisfying $\operatorname{dist}(x, z)<2 l$ is at most $O\left(n^{2 l-1} 2^{n}\right)$. Finally $z \in S \cap \Gamma_{2 l}(z)$ contributes one to this sum for each choice of $y \in S \cap \Gamma_{l}(x)$ such that $\operatorname{dist}(y, z)=l$, i.e. $h_{l}(x, z)$ times. The result follows.

Proof of Theorem [3: For the lower bounds note that the following sets are $F_{1}, F_{2},\left\{F_{1}, F_{2}\right\}$ and $\left\{F_{2}, F_{3}\right\}$-free respectively:

$$
\begin{aligned}
& S_{1}=\left\{x \in V_{n}:|x| \equiv 0 \bmod 2\right\}, \quad S_{2}=\left\{x \in V_{n}:|x| \equiv 0,1 \bmod 4\right\}, \\
& S_{1,2}=\left\{x \in V_{n}:|x| \equiv 0 \bmod 3\right\} \quad S_{2,3}=\left\{x \in V_{n}:|x| \equiv 0 \bmod 4\right\} .
\end{aligned}
$$

Moreover these sets have asymptotic densities $1 / 2,1 / 2,1 / 3$ and $1 / 4$ respectively. Since $S_{1}$ is also $F_{3}$-free and $S_{2,3}$ is also $F_{1}$-free it is sufficient to prove that these values are also upper bounds for the vertex Turán densities.

Let $T_{1}$ be $F_{1}$-free with $\left|T_{1}\right|=\alpha_{1} 2^{n}$. If $x \in T_{1}$ then $h_{1}(x)=\left|T_{1} \cap \Gamma_{1}(x)\right| \leq 2$, hence

$$
\begin{aligned}
n \alpha_{1} 2^{n} & =\sum_{x \in V_{n}} h_{1}(x) \\
& \leq \sum_{x \in T_{1}} 2+\sum_{x \in V_{n} \backslash T_{1}} n \\
& =2 \alpha_{1} 2^{n}+n\left(1-\alpha_{1}\right) 2^{n} .
\end{aligned}
$$

So $\alpha_{1} \leq n /(2 n-2)$ and hence $\lambda\left(F_{1}\right)=1 / 2$.
Similarly if $T_{3}$ is $F_{3}$-free and $\left|T_{3}\right|=\alpha_{3} 2^{n}$ then $h_{3}(x)=\left|T_{3} \cap \Gamma_{3}(x)\right|=0$ for all $x \in T_{3}$. Hence

$$
\binom{n}{3} \alpha_{3} 2^{n}=\sum_{x \in V_{n}} h_{3}(x) \leq\binom{ n}{3}\left|V_{n} \backslash T_{3}\right|=\binom{n}{3}\left(1-\alpha_{3}\right) 2^{n}
$$

Thus $\alpha_{3} \leq 1 / 2$ and hence $\lambda\left(F_{3}\right)=1 / 2$.
Let $T_{2}$ be $F_{2}$-free with $\left|T_{2}\right|=\alpha_{2} 2^{n}$. For $x \in T_{2}$ let

$$
T_{2}^{2}(x)=\left\{\operatorname{supp}(x) \Delta \operatorname{supp}(y): y \in T_{2} \cap \Gamma_{2}(x)\right\}
$$

Consider the graph with vertex set $[n]$ and edge set $T_{2}^{2}(x)$. Since $T_{2}$ is $F_{2}$-free this graph is triangle-free (a triangle would correspond to $a, b, c \in T_{2}$ such that $\{x, a, b, c\}$ forms a copy of $F_{2}$ in $T_{2}$ ). Hence, by Mantel's theorem, $\left|T_{2}^{2}(x)\right| \leq$ $n^{2} / 4$. Thus for $x \in T_{2}$ we have $h_{2}(x)=\left|T_{2}^{2}(x)\right| \leq n^{2} / 4$, so Lemma 9 (i) with $l=1$ implies that

$$
\begin{equation*}
\sum_{x \in V_{n}} h_{1}^{2}(x) \leq n^{2} \alpha_{2} 2^{n-1}+O\left(n 2^{n}\right) \tag{1}
\end{equation*}
$$

Since $\sum_{x \in V_{n}} h_{1}(x)=n \alpha_{2} 2^{n}$, Cauchy-Schwarz implies that

$$
n^{2} \alpha_{2}^{2} 2^{n} \leq n^{2} \alpha_{2} 2^{n-1}+O\left(n 2^{n}\right)
$$

Hence $\alpha_{2} \leq 1 / 2+o(1)$ and so $\lambda\left(F_{2}\right)=1 / 2$.
Now let $T_{1,2}$ be $\left\{F_{1}, F_{2}\right\}$-free with $\left|T_{1,2}\right|=\alpha_{1,2} 2^{n}$. For each $x \in T_{1,2}$ we have both $h_{1}(x) \leq 2$ and $h_{2}(x) \leq n^{2} / 4$. So (11) holds with $\alpha_{2}$ replaced by $\alpha_{1,2}$ and

$$
\sum_{x \in V_{n} \backslash T_{1,2}} h_{1}(x) \geq(n-2) \alpha_{1,2} 2^{n} .
$$

Hence Cauchy-Schwarz implies that

$$
\left(\frac{(n-2) \alpha_{1,2}}{1-\alpha_{1,2}}\right)^{2}\left(1-\alpha_{1,2}\right) 2^{n} \leq n^{2} \alpha_{1,2} 2^{n-1}+O\left(n 2^{n}\right)
$$

So $\alpha_{1,2} \leq n^{2} /\left(2(n-2)^{2}+n^{2}\right)+o(1)$ and $\lambda\left(\left\{F_{1}, F_{2}\right\}\right)=1 / 3$.
Finally let $T_{2,3}$ be $\left\{F_{2}, F_{3}\right\}$-free and $\left|T_{2,3}\right|=\alpha_{2,3} 2^{n}$. Since $T_{2,3}$ is $F_{2}$-free, (1) holds with $\alpha_{2}$ replaced by $\alpha_{2,3}$.

Let $Y=\left\{x \in V_{n}: h_{1}(x) \geq 3\right\}$ and $|Y|=\beta 2^{n}$. Since $\sum_{x \in V_{n}} h_{1}(x)=n \alpha_{2,3} 2^{n}$ and $h_{1}(x) \leq 2$ for $x \in V_{n} \backslash Y$ we have

$$
\sum_{x \in Y} h_{1}(x) \geq n \alpha_{2,3} 2^{n}-2(1-\beta) 2^{n}
$$

Hence, using Cauchy-Schwarz,

$$
\sum_{x \in V_{n}} h_{1}^{2}(x) \geq \sum_{x \in Y} h_{1}^{2}(x) \geq \frac{\left(\alpha_{2,3} n-2+2 \beta\right)^{2} 2^{n}}{\beta}
$$

Thus

$$
\alpha_{2,3} \leq \frac{\beta}{2}+o(1)
$$

If we show that $\beta \leq 1 / 2$ we will be done. If $\beta>1 / 2$ then $|Y|>2^{n-1}$ and so there exist $y, z \in Y$ such that $\operatorname{dist}(y, z)=1$. Let $a, b, c \in T_{2,3} \backslash\{y, z\}$ satisfy $\operatorname{dist}(a, y)=\operatorname{dist}(b, z)=\operatorname{dist}(c, z)=1$ (such points exist by the definition of $Y$ ). Now either $\operatorname{dist}(a, b)=3$ or $\operatorname{dist}(a, c)=3$ and so $T_{2,3}$ contains a copy of $F_{3}$. Hence $\beta \leq 1 / 2$ and so $\alpha_{2,3} \leq 1 / 4+o(1)$ and $\lambda\left(\left\{F_{2}, F_{3}\right\}\right)=1 / 4$.

For $d \geq 3$ define

$$
\begin{gathered}
F_{1}^{d}=\left\{x \in V_{d}:|x|=0,1\right\}, \quad F_{d}^{d}=\left\{x \in V_{d}:|x|=0, d\right\} \\
F_{2}^{d}=\left\{x \in V_{d}:|x|=0 \text { or }(|x|=2, \operatorname{supp}(x)=\{i, j\}, i \not \equiv j \bmod 2)\right\}
\end{gathered}
$$

The proof of Theorem 3 is easily extended to give the following result.
Theorem 10 If $d_{1}, d_{2}, d_{3} \geq 3$ with $d_{3}$ odd then

$$
\begin{gathered}
\lambda\left(F_{1}^{d_{1}}\right)=\lambda\left(F_{2}^{d_{2}}\right)=\lambda\left(F_{d_{3}}^{d_{3}}\right)=\lambda\left(\left\{F_{1}^{d_{1}}, F_{d_{3}}^{d_{3}}\right\}\right)=\frac{1}{2} \\
\lambda\left(\left\{F_{1}^{d_{1}}, F_{2}^{d_{2}}\right\}\right)=\frac{1}{3}, \quad \lambda\left(\left\{F_{2}^{d_{2}}, F_{d_{3}}^{d_{3}}\right\}\right)=\lambda\left(\left\{F_{1}^{d_{1}}, F_{2}^{d_{2}}, F_{d_{3}}^{d_{3}}\right\}\right)=\frac{1}{4} .
\end{gathered}
$$

In fact it is an immediate consequence of a result of Chung, Füredi, Graham and Seymour [6] that $\operatorname{exc}\left(n, F_{1}^{d}\right)=2^{n-1}$ for $n$ sufficiently large.

Proof of Theorem 圆: We start by proving $\lambda\left(G_{d}\right)=2 / 3$. Since

$$
S_{0}=\left\{x \in V_{n}:|x| \not \equiv 0 \bmod 3\right\}
$$

is $G_{d}$-free we have $\lambda\left(G_{d}\right) \geq 2 / 3$.
Let $2 \leq d \leq n$ and $S \subseteq V_{n}$ be $G_{d}$-free. If $|S|=\alpha 2^{n}$ then we need to show that $\alpha \leq 2 / 3+o(1)$.

For $x \in S$ and $l \geq 1$ recall that

$$
S^{l}(x)=\left\{\operatorname{supp}(x) \Delta \operatorname{supp}(y): y \in S \cap \Gamma_{l}(x)\right\}
$$

The fact that $S$ is $G_{d^{-}}$-free implies that for any $x \in S$ the graph with vertex set $S^{1}(x)$ and edge set $S^{2}(x)$ is $K(\lceil d / 2\rceil,\lfloor d / 2\rfloor)$-free and hence $K(d, d)$-free. Thus the Erdős-Stone theorem implies that it contains at most $O\left(n^{2-1 / d}\right)$ edges. Hence

$$
\begin{equation*}
h_{2}(x) \leq\binom{ n}{2}-\binom{h_{1}(x)}{2}+O\left(n^{2-1 / d}\right) \tag{2}
\end{equation*}
$$

Applying Lemma 9 (i) with $l=1$ we obtain

$$
\begin{equation*}
\sum_{x \in V_{n}} h_{1}^{2}(x) \leq \sum_{x \in S}\left(n^{2}-h_{1}^{2}(x)\right)+O\left(n^{2-1 / d} 2^{n}\right) \tag{3}
\end{equation*}
$$

Now let $\beta$ be defined by $\sum_{x \in S} h_{1}(x)=\beta \alpha n 2^{n}$, so $0 \leq \beta \leq 1$. Using (3) and applying Cauchy-Schwarz to the sums $\sum_{x \in S} h_{1}^{2}(x)$ and $\sum_{x \in V_{n} \backslash S} h_{1}^{2}(x)$ we obtain

$$
2 \beta^{2} n^{2} \alpha 2^{n}+\left(\frac{(1-\beta) \alpha n}{1-\alpha}\right)^{2}(1-\alpha) 2^{n} \leq \alpha n^{2} 2^{n}+O\left(n^{2-1 / d} 2^{n}\right)
$$

So

$$
\alpha\left(2-2 \beta-\beta^{2}\right) \leq 1-2 \beta^{2}+o(1)
$$

If $2-2 \beta-\beta^{2}>0$ then

$$
\begin{equation*}
\alpha \leq \frac{1-2 \beta^{2}}{2-2 \beta-\beta^{2}}+o(1) \tag{4}
\end{equation*}
$$

and the RHS is maximised at $\beta=1 / 2$ when it equals $2 / 3+o(1)$. So suppose that $2-2 \beta-\beta^{2} \leq 0$. Since $0 \leq \beta \leq 1$ this implies that $\beta \geq \sqrt{3}-1$.

If $x \in S$ and $z \in S \cap \Gamma_{2}(x)$ then $h_{1}(x, z)=\left|S \cap \Gamma_{1}(x) \cap \Gamma_{1}(z)\right| \leq 2$. Moreover since $S$ is $G_{d}$-free the Erdős-Stone theorem implies that

$$
\left|\left\{z \in S \cap \Gamma_{2}(x): h_{1}(x, z)=2\right\}\right|=O\left(n^{2-1 / d}\right)
$$

Finally

$$
\left|\left\{z \in S \cap \Gamma_{2}(x): h_{1}(x, z)=1\right\}\right| \leq\binom{ n}{2}-\binom{h_{1}(x)}{2}
$$

Hence Lemma 9 (ii) with $l=1$ implies that

$$
\sum_{x \in S} h_{1}^{2}(x) \leq \sum_{x \in S}\left(\binom{n}{2}-\binom{h_{1}(x)}{2}\right)+O\left(n^{2-1 / d} 2^{n}\right)
$$

Using $\sum_{x \in S} h_{1}(v)=\beta \alpha n 2^{n}$ and Cauchy-Schwarz we obtain

$$
3 \beta^{2} n^{2} \alpha 2^{n-1} \leq n^{2} \alpha 2^{n-1}+O\left(n^{2-1 / d} 2^{n}\right)
$$

Hence $\beta \leq 1 / \sqrt{3}+o(1)<\sqrt{3}-1$ for $n$ large, and so $\lambda\left(G_{d}\right)=2 / 3$.
We now need to show that the local stability conditions hold. Suppose that $S \subseteq V_{n}$ is $G_{d}$-free and has size $|S| \geq(2 / 3-\epsilon) 2^{n}$ for some $\epsilon>0$. For $n$ large (4)
implies that $|S| \leq(2 / 3+\epsilon) 2^{n}$. If $\epsilon \geq 1 / 100$ then we may take $\delta=1$ and the conditions hold trivially so suppose that $\epsilon \leq 1 / 100$.

Since $\alpha \geq 2 / 3-\epsilon$, (4) implies that $|\beta-1 / 2| \leq \sqrt{\epsilon}$ for $n$ large. Let $\delta_{1}=2 \epsilon^{1 / 4}$ and suppose there exists $W \subset V_{n} \backslash S$ such that $h_{1}(x)<\left(1-\delta_{1}\right) n$ for all $x \in W$ and $|W| \geq \delta_{1} 2^{n}$. Then

$$
\begin{aligned}
\sum_{x \in V_{n} \backslash S} h_{1}(x) & \leq\left|V_{n} \backslash(S \cup W)\right| n+\left(1-\delta_{1}\right) n|W| \\
& \leq\left(1-\alpha-2 \epsilon^{1 / 4}\right) n 2^{n}+2 \epsilon^{1 / 4}\left(1-2 \epsilon^{1 / 4}\right) n 2^{n} \\
& \leq\left(\frac{1}{3}-3 \sqrt{\epsilon}\right) n 2^{n}
\end{aligned}
$$

But

$$
\begin{equation*}
\sum_{x \in V_{n} \backslash S} h_{1}(x)=(1-\beta) \alpha n 2^{n} \geq\left(\frac{1}{3}-\frac{7 \sqrt{\epsilon}}{6}\right) n 2^{n} \tag{5}
\end{equation*}
$$

Hence (a) holds for any $\delta \geq 2 \epsilon^{1 / 4}$.
For (b) we will require the following defect form of the Cauchy-Schwarz inequality (cf. Bollobás [3] page 125).
Lemma 11 If $a_{1}, \ldots, a_{s} \in \mathbb{R}, 1 \leq t \leq s, \frac{1}{s} \sum_{i=1}^{s} a_{i}=A, \frac{1}{t} \sum_{i=1}^{t} a_{i} \geq A^{\prime}$, $t \geq \gamma s$ and $A^{\prime} \geq A+\eta$ then

$$
\frac{1}{s} \sum_{i=1}^{s} a_{i}^{2} \geq A^{2}+\gamma \eta^{2}
$$

Let $\delta_{2}=2 \epsilon^{1 / 6}$ and suppose there exists $W \subset S$ satisfying $|W| \geq \delta_{2} 2^{n}$ and $\left|h_{1}(x)-n / 2\right| \geq \delta_{2} n$ for all $x \in W$. Let

$$
W^{+}=\left\{x \in W: h_{1}(x) \geq n / 2+\delta_{2} n\right\}
$$

and $W^{-}=W \backslash W^{+}$. Suppose that $\left|W^{+}\right| \geq \delta_{2} 2^{n-1}$ (the case $\left|W^{-}\right| \geq \delta_{2} 2^{n-1}$ is similar so we omit it).

Now

$$
\frac{1}{|S|} \sum_{x \in S} h_{1}(x)=\beta n, \quad \frac{1}{\left|W^{+}\right|} \sum_{x \in W^{+}} h_{1}(x) \geq \frac{n}{2}+\delta_{2} n
$$

so using $|\beta-1 / 2|<\sqrt{\epsilon}$ we have

$$
\frac{1}{\left|W^{+}\right|} \sum_{x \in W^{+}} h_{1}(x) \geq \frac{1}{|S|} \sum_{x \in S} h_{1}(x)+n\left(\delta_{2}-\sqrt{\epsilon}\right)
$$

Since $\left|W^{+}\right| \geq \delta_{2} 2^{n-1} \geq \delta_{2}|S| / 2$, Lemma (with $A=\beta n, \eta=n\left(\delta_{2}-\sqrt{\epsilon}\right.$ ) and $\gamma=\delta_{2} / 2$ ) implies that

$$
\begin{equation*}
\frac{1}{|S|} \sum_{x \in S} h_{1}^{2}(x) \geq \beta^{2} n^{2}+\frac{\delta_{2} n^{2}\left(\delta_{2}-\sqrt{\epsilon}\right)^{2}}{2} \tag{6}
\end{equation*}
$$

Using (5) and Cauchy-Schwarz (Lemma 8) we also have

$$
\begin{equation*}
\sum_{x \in V_{n} \backslash S} h_{1}^{2}(x) \geq \frac{1}{3}(1-7 \sqrt{\epsilon}) n^{2} 2^{n} \tag{7}
\end{equation*}
$$

Now (3) implies that for $n$ large

$$
\sum_{x \in S} 2 h_{1}^{2}(x)+\sum_{x \in V_{n} \backslash S} h_{1}^{2}(x) \leq n^{2}|S|+\epsilon n^{2} 2^{n} .
$$

Using (6) and (7) this yields

$$
\left(\frac{2}{3}-\epsilon\right)\left(2 \beta^{2}+\delta_{2}\left(\delta_{2}-\sqrt{\epsilon}\right)^{2}\right)+\frac{1}{3}(1-7 \sqrt{\epsilon}) \leq \frac{2}{3}+2 \epsilon
$$

Substituting $\delta_{2}=2 \epsilon^{1 / 6}$ and using $\beta \geq 1 / 2-\sqrt{\epsilon}, \epsilon \leq 1 / 100$ we obtain a contradiction. Hence (b) holds for any $\delta \geq 2 \epsilon^{1 / 6}$.

Finally for (c) let $\delta_{3}=4 \epsilon^{1 / 4}$. Since $S$ is $G_{d}$-free the Erdős-Stone theorem implies that for any $x \in S, S^{2}(x)$ contains at most $O\left(n^{2-1 / d}\right)$ edges from $\binom{S^{1}(x)}{2}$. Hence

$$
\left|S^{2}(x) \cap\binom{S^{1}(x)}{2}\right|=O\left(n^{2-1 / d}\right) \leq \frac{\delta_{3}}{2}\binom{n}{2}
$$

for $n$ large. So suppose there exists $W \subseteq S$ such that $|W| \geq \delta_{3} 2^{n}$ and for all $x \in W$

$$
\left|\left(\binom{[n]}{2} \backslash\binom{S^{1}(x)}{2}\right) \backslash S^{2}(x)\right| \geq \frac{\delta_{3}}{2}\binom{n}{2}
$$

Since (2) holds for all $x \in S$, Lemma 9 (i) with $l=1$ implies that

$$
\sum_{x \in V_{n}} h_{1}^{2}(x) \leq \sum_{x \in S}\left(n^{2}-h_{1}^{2}(x)+O\left(n^{2-1 / d}\right)\right)-\delta_{3}^{2}\binom{n}{2} 2^{n}
$$

So for $n$ large

$$
2 \sum_{x \in S} h_{1}^{2}(x)+\sum_{x \in V_{n} \backslash S} h_{1}^{2}(x) \leq n^{2}|S|-\frac{\delta_{3}^{2}}{3} n^{2} 2^{n} .
$$

Applying Cauchy-Schwarz to $\sum_{x \in S} h_{1}^{2}(x)$ and using (7) we obtain

$$
2 \beta^{2} \alpha+\frac{1}{3}(1-7 \sqrt{\epsilon}) \leq \alpha-\frac{\delta_{3}^{2}}{3}
$$

However this gives a contradiction using $|\beta-1 / 2| \leq \sqrt{\epsilon}, \alpha \leq 2 / 3+\epsilon$ and $\delta_{3}=4 \epsilon^{1 / 4}$. Hence (c) holds for any $\delta \geq 4 \epsilon^{1 / 4}$. Thus we can satisfy all of the local stability conditions by taking $\delta=4 \epsilon^{1 / 6}$, which clearly also satisfies the condition $\lim _{\epsilon \rightarrow 0^{+}} \delta(\epsilon)=0$.

Proof of Theorem 4; Let $d \geq 2$ and $\epsilon>0$ be given. Let $n$ be large and $S \subseteq V_{n}$ satisfy $|S| \geq \epsilon 2^{n}$. For $r \geq 1$ we have

$$
\sum_{x \in V_{n}} h_{r}(x)=\binom{n}{r}|S| .
$$

Hence $|S| \geq \epsilon 2^{n}$ implies that for any $r \geq 1$ there exists $x \in V_{n}$ such that $h_{r}(x) \geq$ $\epsilon\binom{n}{r}$. Let $r=\lceil d / 3\rceil$ and $3 \geq p_{1} \geq p_{2} \geq \cdots \geq p_{r} \geq 2$ satisfy $\sum_{i=1}^{r} p_{i}=d$. If $n$ is sufficiently large relative to $\epsilon$ and $d$ then Theorem 7 implies that the $r$-graph

$$
S^{r}(x)=\left\{\operatorname{supp}(x) \Delta \operatorname{supp}(y): y \in S \cap \Gamma_{r}(x)\right\}
$$

contains a copy of $K:=K^{(r)}\left(p_{1}, p_{2}, \ldots, p_{r}\right)$. It is easy to check that this $r$-graph has $t_{3}$ edges (where $t_{3}$ is defined in the statement of Theorem (4). Moreover, since $K$ is an $r$-graph with $d$ vertices, the corresponding subset of $S$ lies in a copy of $V_{d}$. Thus $\lambda\left(\mathcal{F}_{d, t_{3}}\right)=0$.

If $r=\lceil d / 3\rceil$ is odd then $t_{2}=0$ and the proof is complete so suppose that $\lceil d / 3\rceil$ is even. Now Lemma 9 (i) (with $l=r / 2$ ) followed by an application of Cauchy-Schwarz implies that

$$
\begin{aligned}
\binom{r}{r / 2} \sum_{x \in S} h_{r}(x) & =\sum_{x \in V_{n}} h_{r / 2}^{2}(x)+O\left(n^{r-1} 2^{n}\right) \\
& \geq\binom{ n}{r / 2}^{2} \epsilon^{2} 2^{n}+O\left(n^{r-1} 2^{n}\right)
\end{aligned}
$$

Hence there is $x \in S$ such that

$$
h_{r}(x) \geq \epsilon\binom{n}{r}+O\left(n^{r-1}\right) .
$$

The same argument as above implies that $S^{r}(x)$ contains a copy of $K$. Hence there is a copy of $V_{d}$ containing $t_{3}+1$ points from $S$, so $\lambda\left(\mathcal{F}_{d, t_{3}+t_{2}}\right)=0$.

## 4 Questions

There are many open problems concerning the vertex Turán density. We collect what seem to be the most appealing ones here.

All of the constructions we have considered are of the form $\left\{x \in V_{n}:|x| \in I\right\}$ for some $I \subseteq[n]$.

Question 12 Is it true that for any family $\mathcal{F}=\left\{F_{1}, \ldots, F_{k}\right\}$ of subsets of $V_{d}$ there are sets $I_{n} \subseteq[n]$ with

$$
S_{n}=\left\{x \in V_{n}:|x| \in I_{n}\right\}
$$

$\mathcal{F}$-free for all $n$ and

$$
\lim _{n \rightarrow \infty} \frac{\left|S_{n}\right|}{2^{n}}=\lambda(\mathcal{F}) ?
$$

All of our results are for the vertex Turán density, one could also ask for the exact value of $\operatorname{exc}(n, \mathcal{F})$ and what the extremal examples are.

As we mentioned above the question of determining (or improving bounds on) $\lambda\left(V_{d}\right)$ was posed by Alon, Krech and Szabó [1]. This is perhaps the most natural forbidden configuration to consider.

Recall that $\mathcal{F}_{d, t}=\left\{F \subseteq V_{d}:|F|=t\right\}$ and $\mu(d)=\max \left\{t: \lambda\left(\mathcal{F}_{d, t}\right)=0\right\}$. By Theorem 4 we have $\mu(d) \leq\binom{ d}{\lfloor d / 2\rfloor}$.

Question 13 Is it true that for all $d \geq 2, \mu(d)=\binom{d}{\lfloor d / 2\rfloor}$ ?
Theorem 4 tells us that $\mu(3)=3$ and $\mu(4) \geq 5$. Whether or not $\mu(4)=6$ is unresolved.

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