On a generalization of distance sets

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Abstract

A subset X in the d-dimensional Euclidean space is called a k-distance set if there are exactly k distinct distances between two distinct points in X and a subset X is called a locally k-distance set if for any point x in X, there are at most k distinct distances between x and other points in X.

Delsarte, Goethals, and Seidel gave the Fisher type upper bound for the cardinalities of k-distance sets on a sphere in 1977. In the same way, we are able to give the same bound for locally k-distance sets on a sphere. In the first part of this paper, we prove that if X is a locally k-distance set attaining the Fisher type upper bound, then determining a weight function w, (X, w) is a tight weighted spherical $2k$-design. This result implies that locally k-distance sets attaining the Fisher type upper bound are k-distance sets. In the second part, we give a new absolute bound for the cardinalities of k-distance sets on a sphere. This upper bound is useful for k-distance sets for which the linear programming bound is not applicable. In the third part, we discuss about locally two-distance sets in Euclidean spaces. We give an upper bound for the cardinalities of locally two-distance sets in Euclidean spaces. Moreover, we prove that the existence of a spherical two-distance set in $(d-1)$-space which attains the Fisher type upper bound is equivalent to the existence of a locally two-distance set but not a two-distance set in d-space with more than $d(d+1)/2$ points. We also classify optimal (largest possible) locally two-distance sets for dimensions less than eight. In addition, we determine the maximum cardinalities of locally two-distance sets on a sphere for dimensions less than forty.

1 Introduction

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. For $X \subset \mathbb{R}^d$, let $A(X) = \{d(x,y) | x, y \in X, x \neq y\}$ where $d(x,y)$ is the Euclidean distance between x and y in $\mathbb{R}^d$. We call X a *k-distance set* if $|A(X)| = k$. Moreover for any $x \in X$, define $A_X(x) = \{d(x,y) | y \in X, x \neq y\}$. We will abbreviate $A(x) = A_X(x)$ whenever there is no risk of confusion. A subset $X \subset \mathbb{R}^d$ is called a *locally k-distance set* if $|A_X(x)| \leq k$ for all $x \in X$. Clearly every k-distance set is a locally k-distance set. A locally k-distance set is said to be *proper* if it is not a k-distance set. Two subsets in $\mathbb{R}^d$ are said to be isomorphic if there exists a similar transformation from one to the other. An interesting problem for k-distance sets (resp. locally k-distance set) is to determine the largest possible cardinality of k-distance sets (resp. locally k-distance set) in $\mathbb{R}^d$. We denote this number by $DS_d(k)$ (resp. $LDS_d(k)$) and a k-distance set X (resp. locally k-distance set X) in $\mathbb{R}^d$ is said to be *optimal* if $|X| = DS_d(k)$ (resp. $LDS_d(k)$). Moreover we denote the maximum cardinality of a k-distance set (resp. locally k-distance set) in the unit sphere $S^{d-1} \subset \mathbb{R}^d$ by $DS_d^*(k)$ (resp. $LDS_d^*(k)$).

For upper bounds on the cardinalities of distance sets in $\mathbb{R}^d$, Bannai-Bannai-Stanton [4] and Blokhuis [8] gave $DS_d(k) \leq \binom{d+k}{k}$. For $k = 2$, the numbers $DS_d(2)$ are known for $d \leq 8$ (Kelly [17], Croft [9] and Lisoněk [19]). For $d = 2$, the numbers $DS_2(k)$ are known and optimal k-distance sets are classified for $k \leq 5$ (Erdős-Fishburn [14], Shinohara [21], [22]). Moreover we have $DS_3(3) = 12$ and every optimal three-distance set is isomorphic to the set of vertices of a regular icosahedron (Shinohara [23]).

d	1	2	3	4	5	6	7	8
$DS_d(2)$	3	5	6	10	16	27	29	45

k	1	2	3	4	5
$DS_2(k)$	3	5	7	9	12

Table: Maximum cardinalities for two-distance sets and planar k-distance sets

We have an lower bound for $DS_d^*(2)$ of $d(d+1)/2$ since the set of all midpoints of the edges of a d-dimensional regular simplex is a two-distance set on a sphere with $d(d+1)/2$ points. Musin determined that $DS_d^*(2) = d(d+1)/2$ for $7 \leq d \leq 21$, $24 \leq d \leq 39$ [20]. For $2 \leq d \leq 6$, we have $DS_d^*(2) = DS_d(2)$ and for $d = 22$, we have $DS_d^*(2) = 275$. For $d = 23$, $DS_d^*(2) = 276$ or 277 [20].

Delsarte, Goethals, and Seidel gave the Fisher type upper bound for the cardinalities of k-distance sets on a sphere [10]. This upper bound also applies to locally k-distance sets on a sphere.

Theorem 1.1 (Fisher type inequality [10]). (i) *Let X be a locally k-distance set on S^{d-1}. Then, $|X| \leq \binom{d+k-1}{k} + \binom{d+k-2}{k-1} (=: N_d(k))$.*
(ii) *Let X be an antipodal (i.e. for any $x \in X$, $-x \in X$) locally k-distance set on S^{d-1}. Then, $|X| \leq 2\binom{d+k-2}{k-1} (=: N'_d(k))$.*

It is well known that if a k-distance set X attains this upper bound, then X is a tight spherical design. We will give the definition of spherical designs in the next section. Of course, k-distance sets which attain this upper bound are optimal. This optimal k-distance set is very interesting because of its relationship with the design theory. Classification of tight spherical t-designs have been well studied in [5, 6, 7]. Classifications of tight spherical t-designs are complete, except for $t = 4, 5, 7$. This implies that classifications of k-distance sets (resp. antipodal k-distance sets) which attain this upper bound are complete, except for $k = 2$ (resp. $k = 3, 4$). For $t = 4$, a tight spherical four-design in S^{d-1} exists only if $d = 2$ or $d = (2l+1)^2 - 3$ for a positive integer l and the existence of a tight spherical four-design in S^{d-1} is known only for $d = 2, 6$ or 22.

In Section 2, we prove the following theorem.

Theorem 1.2. (i) *Let X be a locally k-distance set on S^{d-1}. If $|X| = N_d(k)$, then for some determined weight function w, (X, w) is a tight weighted spherical $2k$-design. Conversely, if (X, w) is a tight weighted spherical $2k$-design, then X is a locally k-distance set (indeed, X is a k-distance set).*
(ii) *Let X be an antipodal locally k-distance set on S^{d-1}. If $|X| = N'_d(k)$, then for some determined weight function w, (X, w) is a tight weighted spherical $(2k-1)$-design. Conversely, if (X, w) is a tight weighted spherical $(2k-1)$-design, then X is an antipodal locally k-distance set (indeed, X is an antipodal k-distance set).*

This theorem implies that the concept of locally distance sets is a natural generalization of distance sets, because this theorem is a generalization of the relationship between tight spherical designs and distance sets.

Indeed, Theorem 1.2 implies the following.

Theorem 1.3. (i) *Let X be a locally k-distance set on S^{d-1}. If $|X| = N_d(k)$, then X is a k-distance set.*
(ii) *Let X be an antipodal locally k-distance set on S^{d-1}. If $|X| = N'_d(k)$, then X is a k-distance set.*

In Section 3, we give a new upper bound for k-distance sets on S^{d-1}. This upper bound is useful for k-distance sets to which the linear programming bound is not applicable.

In Section 4, we discuss locally two-distance sets in $\mathbb{R}^d$. We first give an upper bound for the cardinalities of locally two-distance sets. Moreover, we mention that every proper locally two-distance set in $\mathbb{R}^d$ with more than $d(d+1)/2$ points contains a two-distance set in S^{d-2} which attains the Fisher type upper bound. Note that a two-distance set in $\mathbb{R}^d$ with $d(d+1)/2$ points exists. We also classify optimal locally two-distance sets in $\mathbb{R}^d$ for $d < 8$. In addition, we determine $LDS_2^*(d)$ for $d < 40$ by using the value of $DS_d^*(2)$ for $d < 40$. In particular, we do not know $DS_{23}^*(2)$ but can determine $LDS_{23}^*(2)$.

2 Locally distance sets and weighted spherical designs

We prove Theorem 1.2 in this section. First, we give the definition of weighted spherical designs.

Definition 2.1 (Weighted spherical designs)**.** Let X be a finite set on S^{d-1}. Let w be a weight function: $w : X \to \mathbb{R}_{>0}$, such that $\sum_{x\in X} w(x) = 1$. (X, w) is called a weighted spherical t-design if the following equality holds for any polynomial f in d variables and of degree at most t:

$$\frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(x)d\sigma(x) = \sum_{x\in X} w(x)f(x),$$

where the left hand side involves the integral of f on the sphere. X is called a spherical t-design if $w(x) = 1/|X|$ for all $x \in X$.

We have the following lower bound for the cardinalities of weighted spherical t-designs.

Theorem 2.1 (Fisher type inequality [10, 11])**.** (i) *Let X be a weighted spherical $2e$-design. Then, $|X| \geq \binom{d+e-1}{e} + \binom{d+e-2}{e-1} = N_d(e)$.*
(ii) *Let X be a weighted spherical $(2e-1)$-design. Then, $|X| \geq 2\binom{d+e-2}{e-1} = N'_d(e)$.*

If equality holds, X is said to be tight. The following theorem shows a strong relationship between tight spherical t-designs and k-distance sets.

Theorem 2.2 (Delsarte, Goethals and Seidel [10])**.** (i) *X is a k-distance set on S^{d-1} with $N_d(k)$ points if and only if X is a tight spherical $2k$-design.*
(ii) *X is an antipodal k-distance set on S^{d-1} with $N'_d(k)$ points if and only if X is a tight spherical $(2k-1)$-design.*

Remark 2.1. In particular, X is a two-distance set on S^{d-1} with $N_d(2)$ points if and only if X is a tight spherical four-design. X is an antipodal three-distance set on S^{d-1} with $N'_d(2)$ points if and only if X is a tight spherical five-design. Note that the existence of a tight spherical four-design on S^{d-2} is equivalent to the existence of a tight spherical five-design on S^{d-1}. Let X be a tight spherical five-design on S^{d-1}. Then, we can put $A(X) = \{\alpha, \beta, 2\}$ $(\alpha < \beta)$. For a fixed $x \in X$, we define $X_\alpha := \{y \in X \mid d(x, y) = \alpha\}$. Then, we can regard X_α as a tight spherical four-design on S^{d-2}. This relationship between tight four-designs and five-designs is important in Section 4.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set on S^{d-1}. Let $\mathrm{Harm}_l(\mathbb{R}^d)$ be the linear space of all real harmonic homogeneous polynomials of degree l, in d variables. We put $h_l := \dim(\mathrm{Harm}_l(\mathbb{R}^d))$. Let H_l be the characteristic matrix of degree l. Namely, H_l is indexed by X and an orthonormal basis $\{\varphi_{l,i}\}_{i=0,1,\ldots,h_l}$ of $\mathrm{Harm}_l(\mathbb{R}^{d-1})$ with respect to the inner product $\langle f, g\rangle = \frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(x)g(x)d\sigma(x)$ and that its (i, j)th element is $\varphi_{l,j}(x_i)$. The following gives the definition of Gegenbauer polynomials and discusses the Addition Formula which will be used in the succeeding discussion.

Definition 2.2. Gegenbauer polynomials are a set of orthogonal polynomials $\{G_l^{(d)}(t) \mid l = 1, 2, \ldots\}$ of one variable t. For each l, $G_l^{(d)}(t)$ is a polynomial of degree l, defined in the following manner.

1. $G_0^{(d)}(t) \equiv 1$, $G_1^{(d)}(t) = dt$.
2. $tG_l^{(d)}(t) = \lambda_{l+1}G_{l+1}^{(d)}(t) + (1-\lambda_{l-1})G_{l-1}^{(d)}(t)$ for $l \geq 1$, where $\lambda_l = \frac{l}{d+2l-2}$.

Note that $G_l^{(d)}(1) = \dim(\mathrm{Harm}_l(\mathbb{R}^d)) = h_l$.

Theorem 2.3 (Addition formula [10, 1])**.** *For any x, y on S^{d-1}, we have*

$$\sum_{k=1}^{h_l} \varphi_{l,k}(x)\varphi_{l,k}(y) = G_l^{(d)}((x, y)).$$

The following is a key theorem to prove Theorem 1.3.

Theorem 2.4. *The following are equivalent:*
(i) (X, w) *is a weighted spherical t-design.*
(ii) ${}^tH_eWH_e = I$ *and* ${}^tH_eWH_r = 0$ *for* $e = \lfloor \frac{t}{2} \rfloor$ *and* $r = e-(-1)^t$. *Here,* $W = \mathrm{Diag}\{w(x_1), w(x_2), \ldots, w(x_n)\}$.

We require the two following two lemmas in order to prove Theorem 2.4.

Lemma 2.1 (Lemma 3.2.8 in [1] or [10])**.** *We have the Gegenbauer expansion* $G_k^{(d)}G_l^{(d)} = \sum_{i=0}^{k+l} q_i(k,l)G_i^{(d)}$. *Then, the following hold.*
(i) *For any i, k and l,* $q_i(k,l) \geq 0$.
(ii) *For any k and l,* $q_0(k,l) = h_k\delta_{k,l}$, *where* $\delta_{k,l} = 1$ *if* $k = l$ *and* $\delta_{k,l} = 0$ *if* $k \neq l$.
(iii) $q_i(k,l) \neq 0$ *if and only if* $|k-l| \leq i \leq k+l$ *and* $i \equiv k+l \mod 2$.

For an $m \times n$ matrix M, we define $||M||^2 := \sum_{i=1}^{m}\sum_{j=1}^{n} M(i,j)^2$, namely the sum of squares of all matrix entries.

Lemma 2.2. *For* $k + l \geq 1$,

$$\left|\left|{}^tH_kWH_l - \Delta_{k,l}\right|\right|^2 = \sum_{i=1}^{k+l} q_i(k,l)\left|\left|{}^tH_iWH_0\right|\right|^2 \tag{1}$$

where

$$\Delta_{k,l} = \begin{cases} I, & \text{if } k = l \\ 0, & \text{if } k \neq l \end{cases}.$$

Proof. Note that

$$||{}^tH_kWH_l||^2 = \sum_{i=1}^{h_k}\sum_{j=1}^{h_l}\left(\sum_{x \in X} w(x)\varphi_{k,i}(x)\varphi_{l,j}(x)\right)^2 \tag{2}$$

$$= \sum_{x\in X}\sum_{y \in X} w(x)w(y)\sum_{i=1}^{h_k}\varphi_{k,i}(x)\varphi_{k,i}(y)\sum_{j=1}^{h_l}\varphi_{l,j}(x)\varphi_{l,j}(y) \tag{3}$$

$$= \sum_{x\in X}\sum_{y \in X} w(x)w(y)G_k^{(d)}\left((x,y)\right)G_l^{(d)}\left((x,y)\right).$$

When $l = 0$, we have

$$||{}^tH_kWH_0||^2 = \sum_{x\in X}\sum_{y \in X} w(x)w(y)G_k^{(d)}\left((x,y)\right). \tag{4}$$

If $k \neq l$, then

$$\begin{aligned}
||{}^tH_kWH_l||^2 &= \sum_{x\in X}\sum_{y \in X} w(x)w(y)G_k^{(d)}\left((x,y)\right)G_l^{(d)}\left((x,y)\right) \\
&= \sum_{x\in X}\sum_{y \in X} w(x)w(y)\sum_{i=0}^{k+l} q_i(k,l)G_i^{(d)}\left((x,y)\right) \\
&= \sum_{i=0}^{k+l} q_i(k,l)||{}^tH_iWH_0||^2 \qquad (\because \text{equality } (4)) \\
&= \sum_{i=1}^{k+l} q_i(k,l)||{}^tH_iWH_0||^2 \qquad (\because \text{Lemma } 2.1).
\end{aligned}$$

If $k=l$, then the summation of the squares of the diagonal entries is

$$\begin{aligned}
&\sum_{i=1}^{h_k}\left(\left({}^tH_kWH_k-I\right)(i,i)\right)^2=\sum_{i=1}^{h_k}\left(\sum_{x\in X}w(x)\varphi_{k,i}(x)\varphi_{k,i}(x)-1\right)^2\\
=&\sum_{i=1}^{h_k}\left(\left(\sum_{x\in X}w(x)\varphi_{k,i}(x)\varphi_{k,i}(x)\right)^2-2\sum_{x\in X}w(x)\varphi_{k,i}(x)\varphi_{k,i}(x)+1\right)\\
=&\sum_{i=1}^{h_k}\left(\sum_{x\in X}w(x)\varphi_{k,i}(x)\varphi_{k,i}(x)\right)^2-2\sum_{x\in X}w(x)\sum_{i=1}^{h_k}\varphi_{k,i}(x)\,\varphi_{k,i}(x)+h_k\\
=&\sum_{i=1}^{h_k}\left(\sum_{x\in X}w(x)\varphi_{k,i}(x)\varphi_{k,i}(x)\right)^2-2\sum_{x\in X}w(x)G_k^{(d)}(1)+h_k\\
=&\sum_{i=1}^{h_k}\left(\sum_{x\in X}w(x)\varphi_{k,i}(x)\varphi_{k,i}(x)\right)^2-h_k
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\left|{}^tH_kWH_k-I\right|\right|^2 &= ||{}^tH_kWH_k||^2-h_k\\
&=\sum_{i=0}^{2k}q_i(k,k)||{}^tH_iWH_0||^2-h_k\\
&=\sum_{i=1}^{2k}q_i(k,k)||{}^tH_iWH_0||^2. \qquad (5)
\end{aligned}$$

□

Proof of Theorem 2.4. (i) ⇒ (ii) is clear. We prove (ii) ⇒ (i). By Lemma 2.4,

$$\left|\left|{}^tH_eWH_e-I\right|\right|^2=\sum_{i=1}^{2e}q_i(e,e)\left|\left|{}^tH_iWH_0\right|\right|^2=0. \qquad (6)$$

We have ${}^tH_iWH_0=0$ for even $i\le t$, because $q_i(e,e)>0$ for even i, and $q_i(e,e)=0$ for odd i. On the other hand,

$$\left|\left|{}^tH_eWH_r\right|\right|^2=\sum_{i=1}^{2e-(-1)^t}q_i(e,r)\left|\left|{}^tH_iWH_0\right|\right|^2=0. \qquad (7)$$

We have ${}^tH_iWH_0=0$ for odd $i\le t$, because $q_i(e,r)>0$ for odd i, and $q_i(e,r)=0$ for even i. Therefore, these imply that for any $f\in \mathrm{P}_t(S^{d-1})$, the following equality holds:

$$\frac{1}{|S^{d-1}|}\int_{S^{d-1}}f(x)d\sigma(x)=\sum_{x\in X}w(x)f(x).$$

□

Proof of Theorem 1.2. Let $X=\{x_1,x_2,\ldots,x_n\}$ be a locally k-distance set on S^{d-1}. Suppose $|X|=N_d(k)$. Let $(,)$ be the standard inner product in $\mathbb{R}^d$. For each $x\in X$, we define $A_{inn}(x):=\{(x,y)\mid x\ne y\in X\}$. For each $x\in X$, we define the polynomial in d variables:

$$F_x(\xi):=(x,\xi)^{k-|A_{inn}(x)|}\prod_{\alpha\in A_{inn}(x)}\frac{(x,\xi)-\alpha}{1-\alpha},$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_d)$. $F_x(\xi)$ is of degree k for all $x \in X$. For all $x_i, x_j \in X$,

$$F_{x_i}(x_j) = \delta_{i,j},$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$. We have the Gegenbauer expansion:

$$F_x(\xi) = \sum_{i=0}^{k} f_i^{(x)} G_i^{(d)}((x, \xi)),$$

where $f_i^{(x)}$ are real numbers and $G_i^{(d)}$ is the Gegenbauer polynomial of degree i normalized by $G_i^{(d)}(1) = h_i = \dim(\mathrm{Harm}_i(\mathbb{R}^d))$. In particular, we remark that $f_k^{(x)} > 0$ for every $x \in X$. By the addition formula,

$$F_x(\xi) = \sum_{i=0}^{k} f_i^{(x)} G_i^{(d)}((x, \xi)) = \sum_{i=0}^{k} f_i^{(x)} \sum_{j=1}^{h_i} \varphi_{i,j}(x)\varphi_{i,j}(\xi) \tag{8}$$

for $\xi \in S^{d-1}$. We define the diagonal matrices $C_i := \mathrm{Diag}\{f_i^{(x_1)}, f_i^{(x_2)}, \ldots, f_i^{(x_n)}\}$ for $0 \leq i \leq k$. $[C_0H_0, C_1H_1, \ldots, C_kH_k]$ and $[H_0, H_1, \ldots H_k]$ are $n \times n$ matrices. By the equality (8), we have the equality:

$$[C_0H_0, C_1H_1, \ldots, C_kH_k] \begin{bmatrix} {}^tH_0 \\ {}^tH_1 \\ \vdots \\ {}^tH_k \end{bmatrix} = [F_{x_i}(x_j)]_{i,j} = I. \tag{9}$$

Therefore, $[C_0H_0, C_1H_1, \ldots, C_kH_k]$ and $[H_0, H_1, \ldots H_k]$ are non-singular matrices. Thus,

$$\begin{bmatrix} {}^tH_0 \\ {}^tH_1 \\ \vdots \\ {}^tH_k \end{bmatrix} [C_0H_0, C_1H_1, \ldots, C_kH_k] = I \tag{10}$$

$$\begin{bmatrix} {}^tH_0C_0H_0 & {}^tH_0C_1H_1 & \cdots & {}^tH_0C_kH_k \\ {}^tH_1C_0H_0 & {}^tH_1C_1H_1 & \cdots & {}^tH_1C_kH_k \\ \vdots & \vdots & \ddots & \vdots \\ {}^tH_kC_0H_0 & {}^tH_kC_1H_1 & \cdots & {}^tH_kC_kH_k \end{bmatrix} = I. \tag{11}$$

Therefore, ${}^tH_kC_kH_k = I$ and ${}^tH_{k-1}C_kH_k = 0$. If we define the weight function $w(x) := f_k^{(x)}$ for $x \in X$, then X is a tight weighted spherical $2k$-design on S^{d-1} by Theorem 2.4.

Antipodal case Let X be an antipodal k-distance set with $N'_d(k)$. There exist a subset Y such that $X = Y \cup (-Y)$ and $|X| = 2|Y|$. We define $A^2_{inn}(x) := \{(x, y)^2 \mid y \in X, y \neq \pm x\}$ and

$$\varepsilon = \begin{cases} 1, & \text{if } k \text{ is even}, \\ 0, & \text{if } k \text{ is odd}. \end{cases}$$

For each $y \in Y$, we define the polynomial in d variables

$$F_y(\xi) := (y, \xi)^{k-1-2|A^2_{inn}(y)\setminus\{0\}|} \prod_{0 \neq \alpha^2 \in A^2_{inn}(y)} \frac{(y, \xi)^2 - \alpha^2}{1 - \alpha^2}.$$

$F_y(\xi)$ is of degree $k-1$ for all $y \in Y$. For all $y_i, y_j \in Y$,

$$F_{y_i}(y_j) = \delta_{i,j}$$

We have the Gegenbauer expansion:

$$F_y(\xi) = \sum_{i=0}^{k-1} f_i^{(y)} G_i^{(d)}((y,\xi)).$$

Note that $f_i = 0$ for $i \equiv k \mod 2$. In particular, we remark that $f_{k-1}^{(y)} > 0$ for every $y \in Y$. We define the diagonal matrices $C_i := \mathrm{Diag}\{f_i^{(y_1)}, f_i^{(y_2)}, \dots, f_i^{(y_{n/2})}\}$ for $0 \le i \le k-1$. Let $H_l^{(Y)}$ be the characteristic matrix with respect to Y. $[C_\varepsilon H_\varepsilon^{(Y)}, C_{\varepsilon+2} H_{\varepsilon+2}^{(Y)}, \dots, C_{k-1} H_{k-1}^{(Y)}]$ and $[H_\varepsilon^{(Y)}, H_{\varepsilon+2}^{(Y)}, \dots, H_{k-1}^{(Y)}]$ are $n/2 \times n/2$ matrices. By the addition formula, we have the equality:

$$[C_\varepsilon H_\varepsilon^{(Y)}, C_{\varepsilon+2} H_{\varepsilon+2}^{(Y)}, \dots, C_{k-1} H_{k-1}^{(Y)}] \begin{bmatrix} {}^t H_\varepsilon^{(Y)} \\ {}^t H_{\varepsilon+2}^{(Y)} \\ \vdots \\ {}^t H_{k-1}^{(Y)} \end{bmatrix} = I. \tag{12}$$

Therefore, $[C_\varepsilon H_\varepsilon^{(Y)}, C_{\varepsilon+2} H_{\varepsilon+2}^{(Y)}, \dots, C_{k-1} H_{k-1}^{(Y)}]$ and $[H_\varepsilon^{(Y)}, H_{\varepsilon+2}^{(Y)}, \dots, H_{k-1}^{(Y)}]$ are non-singular matrices. Thus,

$$\begin{bmatrix} {}^t H_\varepsilon^{(Y)} \\ {}^t H_{\varepsilon+2}^{(Y)} \\ \vdots \\ {}^t H_{k-1}^{(Y)} \end{bmatrix} [C_\varepsilon H_\varepsilon^{(Y)}, C_{\varepsilon+2} H_{\varepsilon+2}^{(Y)}, \dots, C_{k-1} H_{k-1}^{(Y)}] = I \tag{13}$$

$$\begin{bmatrix} {}^t H_\varepsilon^{(Y)} C_\varepsilon H_\varepsilon^{(Y)} & {}^t H_\varepsilon^{(Y)} C_{\varepsilon+2} H_{\varepsilon+2}^{(Y)} & \cdots & {}^t H_\varepsilon^{(Y)} C_{k-1} H_{k-1}^{(Y)} \\ {}^t H_{\varepsilon+2}^{(Y)} C_\varepsilon H_\varepsilon^{(Y)} & {}^t H_{\varepsilon+2}^{(Y)} C_{\varepsilon+2} H_{\varepsilon+2}^{(Y)} & \cdots & {}^t H_{\varepsilon+2}^{(Y)} C_{k-1} H_{k-1}^{(Y)} \\ \vdots & \vdots & \ddots & \vdots \\ {}^t H_{k-1}^{(Y)} C_\varepsilon H_\varepsilon^{(Y)} & {}^t H_{k-1}^{(Y)} C_{\varepsilon+2} H_{\varepsilon+2}^{(Y)} & \cdots & {}^t H_{k-1}^{(Y)} C_{k-1} H_{k-1}^{(Y)} \end{bmatrix} = I. \tag{14}$$

Therefore, ${}^t H_{k-1}^{(Y)} C_{k-1} H_{k-1}^{(Y)} = I$. Let H_l be a characteristic matrix with respect to X. We select the weight function $w(x) := f_{k-1}^{(x)}/2$ and $w(-x) = w(x)$ for $x \in X$. Since X is antipodal, this implies ${}^t H_{k-1} W H_{k-1} = I$ and ${}^t H_{k-1} W H_k = 0$. Therefore, X is a tight weighted spherical $(2k-1)$-design by Theorem 2.4.

($\Leftarrow$) It is known that tight weighted spherical $2k$-designs (resp. $(2k-1)$-design) are tight spherical $2k$-design (resp. $(2k-1)$-design) [24, 2, 3]. Therefore, a tight weighted spherical $2k$-design (resp. $(2k-1)$-design) is an k-distance set (resp. antipodal k-distance set). ☐

Theorem 1.2 implies that (antipodal) locally k-distance sets attaining their Fisher type upper bound are (antipodal) k-distance sets .

3 A new upper bound for k-distance sets on S^{d-1}

The following upper bound for the cardinalities of k-distance sets is well known.

Theorem 3.1 (Linear programming bound [10])**.** *Let X be a k-distance set on S^{d-1}. We define the polynomial $F_X(t) := \prod_{\alpha \in A_{inn}(X)} (t-\alpha)$ for X where $A_{inn}(X) := \{(x,y) \mid x, y \in X, x \neq y\}$. We have the Gegenbauer expansion*

$$F_X(t) = \prod_{\alpha \in A_{inn}(X)} (t-\alpha) = \sum_{i=0}^{k} f_i G_i^{(d)}(t),$$

where f_i are real numbers. If $f_0 > 0$ and $f_i \ge 0$ for all $1 \le i \le k$, then

$$|X| \le \frac{F_X(1)}{f_0}.$$

This upper bound is very useful when $A_{inn}(X)$ is given. However, if some f_i happens to be negative, then we have no useful upper bound for the cardinalities of k-distance sets. In this section, we give a useful upper bound for this case. Namely, we prove the following theorem in this section.

Theorem 3.2. *Let X be a k-distance set on S^{d-1}. We define the polynomial $F_X(t)$ of degree k:*

$$F_X(t) := \prod_{\alpha \in A_{inn}(X)} (t-\alpha) = \sum_{i=0}^{k} f_i G_i^{(d)}(t),$$

where f_i are real number. Then,

$$|X| \leq \sum_{i \text{ with } f_i>0} h_i, \tag{15}$$

where the summation is over i with $0 \leq i \leq k$ satisfying $f_i > 0$ and $h_i = \dim(\mathrm{Harm}_i(\mathbb{R}^d))$.

If $f_i > 0$ for all $0 \leq i \leq k$, then this upper bound is the same as the Fisher type inequality. The following lemma is a key lemma required in order to prove Theorem 3.2.

Lemma 3.1. *Let M be a symmetric matrix in $M_n(\mathbb{R})$ and N be an $m \times n$ matrix. N^T denotes the transpose matrix of N. $D_{u,v}$ is an $m \times m$ diagonal matrix such that the number of positive entries is u and the number of negative entries is v. If the equality*

$$NMN^T = D_{u,v}$$

holds, then the number of positive (resp. negative) eigenvalues of M is bounded below by u (resp. v).

Proof. Let $\{p_i\}_{i=1,2,\dots u}$ be row vectors in N satisfying $p_iM{p_i}^T > 0$. Since for $i \neq j$, $p_iM{p_j}^T = 0$, we have

$$\left(\sum_{i=1}^{u} a_i p_i\right) M \left(\sum_{i=1}^{u} a_i p_i\right)^T = \sum_{i=1}^{u} {a_i}^2 p_i M {p_i}^T \tag{16}$$

for real numbers a_i. If $\sum_{i=1}^{u} a_i p_i = 0$, then all a_i are zero. Therefore, $\{p_i\}_{i=1,2,\dots,u}$ are linearly independent, and $u \leq \min\{n, m\}$. In particular, $u \leq n$. There exist n-length row vectors $\{q_i\}_{i=u+1,u+2,\dots,n}$, such that

$$P = \left[{p_1}^T/\sqrt{p_1M{p_1}^T}, {p_2}^T/\sqrt{p_2M{p_2}^T}, \dots, {p_u}^T/\sqrt{p_uM{p_u}^T}, {q_{u+1}}^T, {q_{u+2}}^T, \dots, {q_n}^T\right]^T$$

is a non-singular matrix. Then,

$$PMP^T = \begin{bmatrix} I_u & S \\ S^T & M' \end{bmatrix}, \tag{17}$$

where I_u is the identity matrix of degree u, M' is an $(n-u) \times (n-u)$ symmetric matrix, and S is a $u \times (n-u)$ matrix. We put

$$R_1 = \begin{bmatrix} I_u & \mathbb{O} \\ -S^T & I_{n-u} \end{bmatrix}.$$

Then,

$$R_1PMP^T{R_1}^T = \begin{bmatrix} I_u & \mathbb{O} \\ \mathbb{O} & M' - S^TS \end{bmatrix} \tag{18}$$

Since $M' - S^TS$ is a symmetric matrix, there exists an orthogonal matrix Q, such that $Q(M'-S^TS)Q^T$ is a diagonal matrix. We put

$$R_2 = \begin{bmatrix} I_v & \mathbb{O} \\ \mathbb{O} & Q \end{bmatrix}.$$

Then,

$$R_2R_1PM(R_2R_1P)^T = R_2R_1PMP^T{R_1}^T{R_2}^T = \begin{bmatrix} I_u & \mathbb{O} \\ \mathbb{O} & Q(M'-S^TS)Q^T \end{bmatrix}. \tag{19}$$

Since R_2R_1P is a non-singular matrix, the number of positive eigenvalues of M is equal to the number of positive diagonal entries of $R_2R_1PM(R_2R_1P)^T$. Therefore, the number of positive eigenvalues of M is at least u. This fact implies this lemma. The proof of the result for negative eigenvalues is a similar as above method. □

Proof of Theorem 3.2. Let $X := \{x_1, x_2, \ldots, x_{|X|}\}$ be a k-distance set on S^{d-1}. Let $\{\varphi_{l,k}\}_{1\le k\le h_l}$ be an orthonormal basis of $\mathrm{Harm}_l(\mathbb{R}^d)$. H_l is the characteristic matrix. We have the Gegenbauer expansion $F_X(t) = \prod_{\alpha\in A_{inn}(X)} \frac{t-\alpha}{1-\alpha} = \sum_{i=0}^{s} f_i G_i^{(d)}(t)$. Then,

$$[f_0H_0, f_1H_1, \ldots, f_kH_k] \quad \text{and} \quad [H_0, H_1, \ldots, H_s]$$

are $|X| \times \sum_{i=0}^{k} h_i$ matrices. By the addition formula,

$$I_{|X|} = [f_0H_0, f_1H_1, \ldots, f_kH_k]\begin{bmatrix} {}^tH_0 \\ {}^tH_1 \\ \vdots \\ {}^tH_k \end{bmatrix} = [H_0, H_1, \ldots, H_k]\mathrm{Diag}\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_1 \\ \vdots \\ f_k \\ \vdots \\ f_k \end{bmatrix}\begin{bmatrix} {}^tH_0 \\ {}^tH_1 \\ \vdots \\ {}^tH_k \end{bmatrix},$$

where $I_{|X|}$ is the identity matrix of degree $|X|$, $\mathrm{Diag}[*]$ denotes a diagonal matrix, and the number of entries f_i is h_i. By Lemma 3.1,

$$|X| \le \sum_{i \text{ with } f_i>0} h_i.$$

□

By using a similar method, we prove a similar upper bound for the antipodal case.

Theorem 3.3 (Antipodal case)**.** *Let X be an antipodal k-distance set on S^{d-1}. We define the polynomial $F_X(t)$ of degree $k-1$:*

$$F_X(t) := \prod_{\alpha\in A_{inn}(X)\setminus\{-1\}} (t-\alpha) = \sum_{i=0}^{k-1} f_i G_i^{(d)}(t),$$

where the f_i are real and $f_i = 0$ for $i \equiv k \mod 2$. Then,

$$|X| \le 2 \sum_{i \text{ with } f_i>0} h_i, \tag{20}$$

where the summation is over i with $0 \le i \le k$ satisfying $f_i > 0$.

Corollary 3.1. *Let X be a two-distance set and $A_{inn}(X) = \{\alpha, \beta\}$. Then, $F_X(t) := (t-\alpha)(t-\beta) = \sum_{i=0}^{2} f_i G_i^{(d)}(t)$ where $f_0 = \alpha\beta + 1/d$, $f_1 = -(\alpha+\beta)/d$ and $f_2 = 2/(d(d+2))$. If $\alpha+\beta \ge 0$, then*

$$|X| \le h_0 + h_2 = \binom{d+1}{2}.$$

Musin proved this corollary by using a polynomial method in [20]. This corollary is used in proof of Theorem 4.2. The following examples attain this upper bound in Corollary 3.1.

Example 3.1. Let U_d be a d-dimensional regular simplex. We define

$$X := \left\{ \frac{x+y}{2} \middle| x, y \in U_d, x \neq y \right\}$$

for $d \geq 7$. Then, X is a two-distance set on S^{d-1}, $|X| = d(d+1)/2$, $f_0 > 0$, $f_1 \leq 0$ and $f_2 > 0$.

Let us introduce some examples which attain the upper bound in Theorem 3.2 and 3.3.

Corollary 3.2 (The case $k = 1$, $f_1 > 0$ and $f_0 \leq 0$)**.** *Let X be a 1-distance set and $A_{inn}(X) = \{\alpha\}$. Then, $F_X(t) := t - \alpha = \sum_{i=0}^{1} f_i G_i^{(d)}(t)$ where $f_1 = 1/d$ and $f_0 = -\alpha$. If $\alpha \geq 0$, then*

$$|X| \leq h_1 = d.$$

Clearly, a d-point $(d-1)$-dimensional regular simplex with a nonnegative inner product on S^{d-1} attains this upper bound.

Corollary 3.3. *Let X be an k-distance set on S^{d-1}. We have the Gegenbauer expansion $F_X(t) = \prod_{\alpha \in A_{inn}(X)}(t - \alpha) = \sum_{i=0}^{k} f_i G_i^{(d)}(t)$. If $f_i > 0$ for all $i \equiv k \mod 2$ and $f_i \leq 0$ for all $i \equiv k-1 \mod 2$, then*

$$|X| \leq \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} h_{k-2i} = \binom{d+k-1}{k}.$$

The following examples attain their upper bounds.

Example 3.2. Let X be a tight spherical $(2k-1)$-design, that is, X is an antipodal k-distance set with $N'_d(k)$ points. There exist a subset Y such that $X = Y \cup (-Y)$ and $|X| = 2|Y|$. Y is an $(k-1)$-distance set and $F_Y(t) := \sum_{i=0}^{k-1} f_i G_i^{(d)}(t)$. Then, $f_i = 0$ for all $i \equiv k-2 \mod 2$ and $f_i > 0$ for all $i \equiv k-1 \mod 2$ and $|Y| = \binom{d+k-2}{k-1}$.

4 Locally two-distance sets

In this section, we will consider locally two-distance sets. Recall that a locally two-distance set is said to be *proper* if it is not a two-distance set. The following examples imply that there are infinitely many proper locally two-distance sets when their cardinalities are small for their dimensions.

Example 4.1. Let U_d be the vertex set of a regular simplex in $\mathbb{R}^d$ and O be the center of the regular simplex. Let y be a point on the line passing through $x \in U_d$ and O. Then $U_d \cup \{y\}$ is a locally two-distance set. Except for finitely many exceptions, such locally two-distance sets are proper.

Example 4.2. Let $\{e_1, e_2, \ldots, e_d\}$ be an orthonormal basis of $\mathbb{R}^d$. Let

$$X = \{x_1, y_1, x_2, y_2, \ldots, x_{k-1}, y_{k-1}\}$$

where

$$x_1 = e_1, \quad y_1 = -e_1$$

and

$$x_j = \frac{1}{j} e_{2j-2} + \frac{\sqrt{j^2-1}}{j} e_{2j-1}, \quad y_j = \frac{1}{j} e_{2j-2} - \frac{\sqrt{j^2-1}}{j} e_{2j-1}$$

for $2 \leq j \leq k-1$. Then X is a locally two-distance set and a k-distance set in $\mathbb{R}^{2k-3}$.

4.1 An upper bound for the cardinalities of locally two-distance sets

Lemma 4.1. (i) *Let $X \subset \mathbb{R}^d$ be a locally two-distance set with at least $d+2$ points. If $d \geq 2$, then there exist points $x, x' \in X$ ($x \neq x'$) such that $A(x) = A(x') = \{\alpha, \alpha'\}$ for some $\alpha, \alpha' \in \mathbb{R}_{>0}$ ($\alpha \neq \alpha'$).*
(ii) *Let X be a locally two-distance set in $\mathbb{R}^d$ with $n \geq d+2$ points. Then there exists $Y \subset X$ with $|Y| = n - d$ and $|A(x)| = 2$ for any $x \in Y$.*

Proof. (i) Let X be a locally two-distance set in $\mathbb{R}^d$ with more than $d+1$ points. Let $B(\alpha; x) = \{y \in X | d(x, y) = \alpha\}$ for any $x \in X$ and $\alpha \in A(x)$. Since $DS_d(1) = d+1$, there exists $x \in X$ such that $|A(x)| = 2$. Let $A(x) = \{\alpha_1, \alpha_2\}$, $Y_1 = B(\alpha_1; x)$ and $Y_2 = B(\alpha_2; x)$. For $y_1 \in Y_1$ and $y_2 \in Y_2$, if $d(y_1, y_2) \in \{\alpha_1, \alpha_2\}$, then we have $A(x) = A(y_1)$ or $A(x) = A(y_2)$ and this lemma holds. Otherwise, there exists $\beta \notin \{\alpha_1, \alpha_2\}$ such that $d(y_1, y_2) = \beta$ for all $y_1 \in Y_1$ and $y_2 \in Y_2$. Thus $A(y_i) = \{\alpha_i, \beta\}$ for any $y_i \in Y_i$ ($i = 1, 2$). Moreover, $|Y_1| \geq 2$ or $|Y_2| \geq 2$ since $|X| \geq 4$.
(ii) Let X be a locally two-distance set in $\mathbb{R}^d$ with $n \geq d+2$ points. Let Y' be the set of all points in X with $|A(x)| = 1$. Then clearly $A(x) = A(x')$ for any $x, x' \in Y'$. Therefore Y' is a one-distance set and $|Y'| \leq d+1$. Moreover if $|Y'| = d+1$, then $Y' \cup \{y\}$ must be a one-distance set for any $y \in X \setminus Y'$, which is a contradiction. Thus $|Y'| \leq d$ and $|X \setminus Y'| \geq n - d$. ☐

Remark 4.1. When we consider optimal locally two-distance sets, the condition $|X| \geq d+2$ in Lemma 4.1 is not so important because there is a lower bound $d(d+1)/2 \leq DS_d(2) \leq LDS_d(2)$ (cf. Example 3.1).

Let X be a locally two-distance set. A subset $Y \subset X$ is called a *saturated subset* if $|Y| \geq 2$ and Y is a maximal subset such that there exists α, β ($\alpha \neq \beta$) with $A_X(y) = \{\alpha, \beta\}$ for any $y \in Y$. Lemma 4.1 assures us that every locally two-distance set in $\mathbb{R}^d$ with at least $d+2$ points contains a saturated subset. Let $Y = \{y_1, y_2, \ldots y_m\} \subset X$ be a saturated subset. Then Y is a two-distance set and $X \setminus Y$ is a locally two-distance set in the space $\{x \in \mathbb{R}^d | d(y_1, x) = d(y_2, x) = \cdots = d(y_m, x)\}$ by maximality. If $X \setminus Y \neq \emptyset$, then all points in Y are on a common sphere. Moreover $Y \cup \{x\}$ is a two-distance set for any $x \in X \setminus Y$.

Lemma 4.2. *Let $Y = \{y_0, y_1, \ldots, y_{m-1}\} \subset \mathbb{R}^d$. Without loss of generality, we may assume that y_0 is the origin of $\mathbb{R}^d$. Let $\dim(Y)$ be the dimension of the space spanned by Y and $Sol(Y) = \{x \in \mathbb{R}^d | d(y_0, x) = d(y_1, x) = \cdots = d(y_{m-1}, x)\}$. Then $Sol(Y)$ is contained in a $(d - \dim(Y))$-dimensional affine subspace if $Sol(Y) \neq \emptyset$.*

Proof. Let $y_i = (y_{i1}, y_{i2}, \ldots, y_{id})$ for $1 \leq i \leq m-1$ and let $x = (x_1, x_2, \ldots, x_d)$. For $1 \leq i \leq m-1$, $d(y_i, x) = d(y_0, x)$ implies

$$\sum_{k=1}^{d} y_{i\,k} x_k = \frac{1}{2} \sum_{k=1}^{d} {y_{i\,k}}^2.$$

Therefore

$$Sol(Y) = \left\{ x \in \mathbb{R}^d | \begin{pmatrix} y_{1\,1} & y_{1\,2} & \cdots & y_{1\,d} \\ y_{2\,1} & y_{2\,2} & \cdots & y_{2\,d} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m-1\,1} & y_{m-1\,2} & \cdots & y_{m-1\,d} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix} \right\}$$

where

$$c_i = \frac{1}{2} \sum_{k=1}^{d} {y_{i\,k}}^2.$$

Since the rank of the above matrix is $\dim(Y)$, $Sol(Y)$ is contained in a $(d - \dim(Y))$-dimensional subspace if $Sol(Y) \neq \emptyset$. ☐

By Lemma 4.2, the following lemma holds.

Lemma 4.3. *Let X be a locally two-distance set in $\mathbb{R}^d$. Let $Y \subset X$ be a saturated subset and $\dim(Y) = i$. Then $X \setminus Y$ is a locally two-distance set with $\dim(X \setminus Y) \leq d - i$.*

Remark 4.2. Let X be a locally two-distance set and Y be a saturated subset of X in $\mathbb{R}^d$. Then we have $\dim(Y) \neq 0$ by Lemma 4.1. Moreover, if $\dim(Y) = d$, then $\dim(X \setminus Y) = 0$ by Lemma 4.3. In this case, $|X \setminus Y| \leq 1$ and X is a two-distance set. Therefore $1 \leq \dim(Y) \leq d-1$ for every saturated subset Y of a proper locally two-distance set X in $\mathbb{R}^d$. Moreover all points in Y are on a common sphere since $X \setminus Y \neq \emptyset$.

From the above remark, we have an upper bound for the cardinality of a proper locally two-distance set.

Theorem 4.1. *Let X be a proper locally two-distance set in $\mathbb{R}^d$. Then*

$$|X| \leq f(d)$$

where

$$f(d) = \max_{1 \leq i \leq d-1} \{DS_i^*(2) + LDS_{d-i}(2)\}.$$

In particular,

$$LDS_d(2) \leq \max\{DS_d(2), f(d)\}$$

Proof. Let X be a proper locally two-distance set in $\mathbb{R}^d$ and Y be a saturated subset of X and $i = \dim(Y)$. Then $1 \leq i \leq d-1$ and all points in Y are on a common sphere by Remark 4.2, so $|Y| \leq DS_i^*(2)$. On the other hand, $|X \setminus Y| \leq LDS_{d-i}(2)$ by Lemma 4.3. Therefore $|X| \leq DS_i^*(2) + LDS_{d-i}(2) \leq f(d)$. □

Corollary 4.1. *Every locally two-distance set in $\mathbb{R}^d$ with at least $d(d+1)/2+3$ points is a two-distance set. In particular $LDS_d(2) \leq \binom{d+2}{2}$.*

Proof. Let X be a proper locally two-distance set in $\mathbb{R}^d$. As we will see in Proposition 4.1, $LDS_d(2) \leq \binom{d+2}{2}$ for small d. Assume $LDS_i(2) \leq \binom{i+2}{2}$ for any $i \leq d-1$. By Theorem 4.1,

$$\begin{aligned}
|X| &\leq \max_{1 \leq i \leq d-1} \{DS_i^*(2) + LDS_{d-i}(2)\} \\
&\leq \max_{1 \leq i \leq d-1} \left\{ \frac{i^2+3i}{2} + \frac{(d-i+2)(d-i+1)}{2} \right\} \\
&= \frac{1}{2} \max_{1 \leq i \leq d-1} \{2i^2 - 2di + d^2 + 3d + 2\} \\
&= \frac{d(d+1)}{2} + 2
\end{aligned}$$

Therefore this corollary holds. □

Remark 4.3. (i) Since the set of midpoints of a regular simplex in $\mathbb{R}^d$ is a two-distance set with $d(d+1)/2$ points, Corollary 4.1 implies $DS_d(2) \leq LDS_d(2) \leq DS_d(2) + 2$. For $d \leq 8$, $d \neq 3$, we will see that $DS_d(2) = LDS_d(2)$ in Proposition 4.1.
(ii) For spherical cases, similarly we have $DS_d^*(2) \leq LDS_d^*(2) \leq DS_d^*(2) + 1$.

Problem 4.1. When does $DS_d(2) < LDS_d(2)$ (resp. $DS_d^*(2) < LDS_d^*(2)$) hold?

We will give partial results for general cases in Section 4.2 and give an answer for $d \leq 8$ in Section 4.4.

4.2 Partial answer of Problem 4.1

Lemma 4.4. (i) *Let X be a proper locally two-distance set in $\mathbb{R}^d$ for $d \geq 3$. If $d(d+1)/2 < |X|$, then there exist $N_{d-1}(2)$-point two-distance set in S^{d-2} or $(N_{d-1}(2)-1)$-point two-distance set Y in S^{d-2} with $A(Y) = \{1, 2/\sqrt{3}\}$.*
(ii) *Let X be a proper locally two-distance set in S^{d-1} for $d \geq 3$. If $d(d+1)/2 < |X|$, then there exist $N_{d-1}(2)$-point two-distance set Y in S^{d-2} with $\sqrt{2} \in A(Y)$ or $A(Y) = \{\alpha, \alpha/\sqrt{\alpha^2-1}\}$.*

Proof. (i) For the case where $d \in \{3,4\}$, we will prove this proposition directly in Proposition 4.1. Therefore we assume that $d \geq 5$ in this proof. Let X be a proper locally two-distance set in $\mathbb{R}^d$ with more than $d(d+1)/2$ points and let Y be a saturated subset of X. We may assume that Y has maximum cardinality among saturated subsets of X. Let $i = \dim(Y)$. Then $1 \leq i \leq d-1$ since Y is a saturated subset and X is not a two-distance set. If $2 \leq i \leq d-2$, then $d(d+1)/2 \geq |X|$ for $d \geq 5$ by Theorem 4.1. Moreover if $i = 1$, then $|Y| \leq 2$ and $|X \setminus Y| \geq d(d+1)/2 - 2 > d(d-1)+3$ for $d \geq 3$. Since $X \setminus Y$ is a locally two-distance set in $\mathbb{R}^{d-1}$, $X \setminus Y$ is a two-distance set by Corollary 4.1. By Lemma 4.1, $X \setminus Y$ contains a saturated subset Y' and $|Y'| > |Y|$. This is a contradiction to the assumption. Therefore $i = d-1$. Since $|X| \geq d(d+1)/2+1 = N_{d-1}(2)+2$ and $|X \setminus Y| \leq LDS_1(2) = 3$, $|Y| \geq N_{d-1}(2)-1$. It is enough to consider the case $|Y| = N_{d-1}(2)-1$, otherwise $|Y| = N_{d-1}(2)$ and this proposition holds. In this case, $|X \setminus Y| = 3$. Let $A(Y) = \{\alpha, \beta\}$ and $X \setminus Y = \{x_1, x_2, x_3\}$. For any $i \in \{1,2,3\}$, $A(x_i) \neq \{\alpha, \beta\}$ since Y is a saturated subset. Moreover $d(x_i, y) = \alpha$ for all $y \in Y$ or $d(x_i, y) = \beta$ for all $y \in Y$. Since $\dim(X \setminus Y) = 1$, there are four possibilities for the x_i. Without loss of generality, we may assume $d(x_1, y) = d(x_2, y) = \alpha$ for all $y \in Y$ and $d(x_3, y) = \beta$ for all $y \in Y$. Then $d(x_1, x_3) = d(x_2, x_3) = \gamma$ for $\gamma \notin \{\alpha, \beta\}$ and $d(x_1, x_2) = \alpha$. It follows from these conditions that Y is an $(N_{d-1}(2)-1)$-point two-distance set Y in S^{d-2} with $A(Y) = \{1, 2/\sqrt{3}\}$.
(ii) Let X be a proper locally two-distance set in S^{d-1} with more than $d(d+1)/2$ points and let Y be a saturated subset of X. Similar to the above case, we may assume $i = \dim(Y) = d-1$. Since $|X| \geq N_{d-1}(2)+2$ and $|X \setminus Y| \leq LDS_1^*(2) = 2$, $|Y| \geq N_{d-1}(2)$. Therefore, $|Y| = N_{d-1}(2)$. □

Theorem 4.2. (i) *If there exists a proper locally two-distance set X in $\mathbb{R}^d$ with more than $d(d+1)/2$ points, then there exists an $N_{d-1}(2)$-point two-distance set in S^{d-2}.*
(ii) *If there exists a proper locally two-distance set X in S^{d-1} with more than $d(d+1)/2$ points, then there exists an $N_{d-1}(2)$-point two-distance set in S^{d-2}. In particular, a locally two-distance set in S^{d-1} with more than $d(d+1)/2$ points is a subset of a tight spherical five-design.*

Proof. (i) Let X be a proper locally two-distance set in $\mathbb{R}^d$ with more than $d(d+1)/2$ points. We assume that X does not contain $N_{d-1}(2)$-point two-distance set in S^{d-2}. Then X contains $(N_{d-1}(2)-1)$-point two-distance set $Y \subset S^{d-2}$ with $A(Y) = \{1, 2/\sqrt{3}\}$ by Lemma 4.4(i) . However there does not exist such a two-distance set Y by Corollary 3.1.
(ii) This is clear by Lemma 4.4 (ii) and Remark 2.1. □

Remark 4.4. Since $d(d+1)/2 \leq DS_d(2)$ (resp. $d(d+1)/2 \leq DS_d^*(2)$), the assumption in Theorem 4.2 (i) (resp. (ii)) can be replaced by $DS_d(2) < LDS_d(2)$ (resp. $DS_d^*(2) < LDS_d^*(2)$).

4.3 Classifications of optimal two-distance sets

Euclidean cases $DS_d(2)$ is determined for $d \leq 8$ and optimal two-distance sets are classified for $d \leq 7$ (Kelly [17], Croft [9], Einhorn-Schoenberg [13] and Lisoněk [19]). We introduce the results in this subsection.

$d = 2$: $DS_2(2)$ and the optimal planar two-distance set is isomorphic to the set of vertices of a regular pentagon (Kelly [17], Einhorn-Schoenberg [13]). We denote the set of vertices of the regular pentagon with side length 1 by R_5. Then $A(R_5) = \{1, \tau\}$ where $\tau = (1+\sqrt{5})/2$.

$d = 3$: $DS_3(2)$ and there are exactly six optimal two distance sets in $\mathbb{R}^3$ (Croft [9], Einhorn-Schoenberg [13]). They are the set of vertices of a regular octahedron, a right prism which has a equilateral triangle base and square sides and the remaining four sets are subsets of a regular icosahedron.

$d = 4$: $DS_4(2) = 10$ and the optimal two-distance set in $\mathbb{R}^4$ is isomorphic to the set of midpoints of the edges of a regular simplex in $\mathbb{R}^4$. This set corresponds to the Petersen graph.

$d = 5$: $DS_5(2) = 16$ and the optimal two-distance set in $\mathbb{R}^5$ is isomorphic to the set given by the Clebsch graph. Points of the set are given by the following.

$$-e_i + \sum_{k=1}^{5} e_k \quad (1 \le i \le 5),$$

$$e_i + e_j \quad (1 \le i < j \le 5)$$

and the origin O of $\mathbb{R}^5$.

$d = 6$: $DS_6(2) = 27$ and the optimal two-distance set in $\mathbb{R}^6$ is isomorphic to the set obtained from the Schläfli graph.

$d = 7$: $DS_7(2) = 29$ and the optimal two-distance set in $\mathbb{R}^7$ is isomorphic to the set which is given by the following points.

$$-e_i + \frac{1}{7}(3+\sqrt{2}) \sum_{k=1}^{7} e_k \quad (1 \le i \le 7),$$

$$e_i + e_j \quad (1 \le i < j \le 7)$$

and

$$\frac{1}{7}(2+3\sqrt{2}) \sum_{k=1}^{7} e_k.$$

$d = 8$: A two-distance set in $\mathbb{R}^8$ with $\binom{10}{2} = 45$ points is known. Let

$$X_1 = \{e_i - \frac{1}{12} \sum_{k=1}^{8} e_k | i = 1, 2, \dots 8\} \cup \{-\frac{1}{3} \sum_{k=1}^{8} e_k\}$$

and

$$X_2 = \{-(x+y) | x, y \in X_1, x \neq y\}$$

Then X_1 is the vertex set of a regular simplex and $X_1 \cup X_2$ is a two-distance set with $A(X_1 \cup X_2) = \{\sqrt{2}, 2\}$

Spherical cases For $2 \le d \le 6$, every optimal two-distance set in $\mathbb{R}^d$ is on a sphere. Optimal two-distance sets in S^6 are given from three Chang graphs or the set of midpoints of edges of a regular simplex in $\mathbb{R}^7$. Moreover, Musin [20] determined $DS^*_d(2)$ for $7 \le d < 40$.

Theorem 4.3. *$DS^*_d(2) = d(d+1)/2$ for the cases where $7 \le d \le 21, 24 \le d < 40$. When $d = 22, 23$, $DS^*_{22}(2) = 275$ and $DS^*_{23}(2) = 276$ or 277.*

4.4 Optimal locally two-distance sets

Euclidean cases By using classifications of optimal two-distance sets and Theorem 4.1, we have the following proposition.

Proposition 4.1. *Every optimal locally two-distance set in $\mathbb{R}^d$ is a two-distance set for $d = 2, 4, 5, 6, 8$. Moreover there are four seven-point locally two-distance set in $\mathbb{R}^3$ up to isomorphism and five 29-point locally two-distance set in $\mathbb{R}^7$ up to isomorphism. In particular $DS_d(2) = LDS_d(2)$ for $d = 1, 2, 4 \le d \le 8$ and $LDS_3(2) = 7$.*

Proof. $d = 1$: It is clear that every three-point set in $\mathbb{R}^1$ which is not a one-distance set is a locally two-distance set and that there is no four-point locally two-distance set in $\mathbb{R}^1$.

For $2 \le d \le 7$, we classify optimal locally two-distance sets in $\mathbb{R}^d$. For each case, we pick a saturated subset Y of X and we let $Y' = X \setminus Y$. Note that if X is not a two-distance set, then $1 \le \dim(Y) \le d-1$.

$d=2$: We will classify five-point locally two-distance sets X in $\mathbb{R}^2$. We may assume that $\dim(Y)=1$ and $|Y|=2$, otherwise X is a two-distance set. Let $Y=\{y_1,y_2\}$, $Y'=\{x_1,x_2,x_3\}$ and $A(y_1)=A(y_2)=\{\alpha,\beta\}$. Without of generality, we may assume $d(x_1,y_i)=d(x_2,y_i)=\alpha$ and $d(x_3,y_i)=\beta$ for $i\in\{1,2\}$ since there are exactly four possibilities for the x_j. If $d(x_1,x_3)\in\{\alpha,\beta\}$, then $A(x_1)=\{\alpha,\beta\}$ or $A(x_3)=\{\alpha,\beta\}$. This is a contradiction to the maximality of the saturated subset Y. So $d(x_1,x_3)=\gamma\notin\{\alpha,\beta\}$. Similarly $d(x_2,x_3)=\gamma$. Therefore x_3 is a midpoint of both the segment y_1y_2 and the segment x_1x_2. It is easy to check that such a locally two-distance set does not exist. Therefore $\dim(Y)\neq 1$ and X is a two-distance set. By the classification of five-point two-distance sets in $\mathbb{R}^2$, $X=R_5$.

$d=3$: We will classify seven-point locally two-distance sets X in $\mathbb{R}^3$. We may assume $1\le\dim(Y)\le 2$, otherwise X is a two-distance set. We need to consider two cases (a) $\dim(Y)=1$ and (b) $\dim(Y)=2$.
(a) In this case, $|Y|=2$ and $Y'=R_5$ by the above classification. Let $Y=\{y_1,y_2\}$ and $Y'=\{x_1,x_2,\dots,x_5\}$. Then $d(x_j,y_i)=1$ for any $j\in\{1,2\}$ and $i\in\{1,2,\dots,5\}$ or $d(x_j,y_i)=\tau$ for any $j\in\{1,2\}$ and $i\in\{1,2,\dots,5\}$. In this case, there are two seven-point locally two-distance sets up to isomorphism.
(b) In this case, $|Y|\in\{4,5\}$. If $|Y|=4$, then $|Y'|=3$. Similar to the case where $d=2$, there exists a point $x\in Y'$ which is the midpoint of the other two points. Then $Y\cup\{x\}$ is a five-point locally two-distance set in $\mathbb{R}^2$ and x is a center of the circle passing through other four points. By the classification of five-point locally two-distance sets in $\mathbb{R}^2$, such a locally two-distance set does not exist. If $|Y|=5$, then $|Y'|=2$. In this case, $Y=R_5$ and there are four locally two-distance sets up to isomorphism. These sets contains the sets in case (a).

$d=4$: We will classify ten-point locally two-distance sets X in $\mathbb{R}^4$. If $\dim(Y)\neq 2$, then X is a two-distance set or $|X|<10$. Therefore we assume $\dim(Y)=2$. Then $|Y|=|Y'|=5$ and both Y and Y' are sets of vertices of a regular pentagon. Let

$$Y=\{(\cos\frac{2\pi j}{5},\sin\frac{2\pi j}{5},0,0)|j=0,1,\dots 4\}$$

and

$$Y'=\{(0,0,r\cos\frac{2\pi j}{5},r\sin\frac{2\pi j}{5})|j=0,1,\dots 4\}.$$

Then $d(x,y)=\sqrt{1+r^2}>1$ for any $y\in Y$ and $x\in Y'$. Therefore we may assume $d(x,y)=\tau$ where $\tau=(1+\sqrt{5})/2$. Then $r=\sqrt{\tau}$ and $A(x)=\{\tau^{1/2},\tau,\tau^{3/2}\}$ for $x\in Y'$. This is not a locally two-distance set. Therefore a ten-point locally two-distance set is a two-distance set.

$d=5$: We will classify sixteen-point locally two-distance sets X in $\mathbb{R}^5$. Since $DS_i^*(2)+LDS_{d-i}(2)<16$ for $1\le i\le 4$, X is a two-distance set.

$d=6$: We will classify 27-point locally two-distance sets X in $\mathbb{R}^6$. By Corollary 4.1, every 27-point locally two-distance set in $\mathbb{R}^6$ is a two-distance set.

$d=7$: We will classify 29-point locally two-distance sets X in $\mathbb{R}^7$. If $\dim(Y)\notin\{1,6\}$, then X is a two-distance set or $|X|<29$. We divide into two cases: (a) $\dim(Y)=1$ and (b) $\dim(Y)=6$.
(a) In this case, similar to the classification of case (a) for $d=3$, we prove that there are two 29-point locally two-distance sets up to isomorphism.
(b) In this case, similar to the classification of case (b) for $d=3$, we can prove that there are four locally two-distance sets which contain the sets in case (a).

$d=8$: We will consider 45-point locally two-distance sets in $\mathbb{R}^8$. By Corollary 4.1, every 45-point locally two-distance set in $\mathbb{R}^8$ is a two-distance set. □

Spherical cases For spherical cases, we have the following proposition by Theorem 4.2 and Theorem 4.3.

Proposition 4.2. *$LDS_d^*(2)=DS_d^*(2)$ for $2\le d<40$ and $d\notin\{3,7,23\}$. When $d\in\{3,7,23\}$, $LDS_3^*(2)=7$, $LDS_7^*(2)=29$ and $LDS_{23}^*(2)=277$. In particular, there is a unique optimal locally* *two-distance set in S^{d-1} if $d \in \{3,7\}$ and there is a unique optimal locally two-distance set in S^{23} if $DS^*_{23}(2) = 276$.*

4.5 Optimal locally three-distance sets

It seems difficult to determine $LDS_d(k)$ and classify the optimal configurations for $k \geq 3$. However there is a result for $k = 3$ and $d = 2$ by Erdős-Fishburn [15] and Fishburn [16].

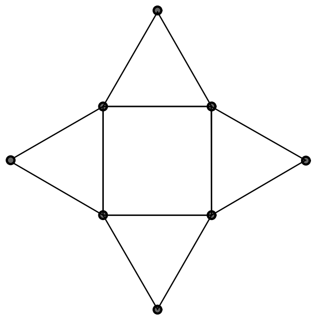

Figure 1.

Proposition 4.3. (i) *Let X be an eight-point planar set. Then $\sum_{P\in X} |A_X(P)| \geq 24$.*
(ii) *Every eight-point planar set X with $\sum_{P\in X} |A_X(P)| = 24$ is similar to Figure 1.*
(iii) *Every eight-point locally three-distance set in $\mathbb{R}^2$ is similar to Figure 1. In particular, $LDS_3(3) = 8$.*

Proof. (i), (ii) See [15], [16].
(iii) This is immediate from (i), (ii). □

The second author proved that $DS_3(3) = 12$ and that every twelve-point three-distance set in $\mathbb{R}^3$ is similar to the set of vertices of a regular icosahedron ([23]).

Problem 4.2. Is every locally three-distance set in $\mathbb{R}^3$ with twelve points similar to the set of vertices of a regular icosahedron?

In fact, there are many differences between k-distance sets and locally k-distance sets when cardinalities are small. Moreover we saw that $DS_d(k) < LDS_d(k)$ for some cases. However no known optimal k-distance sets are locally $(k-1)$-distance sets.

Problem 4.3. Are there any optimal k-distance sets which are locally $(k-1)$-distance sets?

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