

The Complexity of Node Blocking for Dags

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Abstract: We consider the following modification of annihilation game called node blocking. Given a directed graph, each vertex can be occupied by at most one token. There are two types of tokens, each player can move his type of tokens. The players alternate their moves and the current player i selects one token of type i and moves the token along a directed edge to an unoccupied vertex. If a player cannot make a move then he loses. We consider the problem of determining the complexity of the game: given an arbitrary configuration of tokens in a directed acyclic graph, does the current player has a winning strategy? We prove that the problem is PSPACE-complete.

Keywords: annihilation game, node blocking, PSPACE-completeness

1 Introduction

The study of annihilation games has been suggested by John Conway and the first papers were published by Fraenkel and Yesha [7, 9]. They considered a 2-player game played on an underlying directed graph G (possibly with cycles). The current player selects a token and moves it along an arc outgoing from a vertex containing the token. If a vertex contains two tokens then they are removed from G (*annihilation*). Authors in [9] gave a polynomial-time algorithm for computing a winning strategy. In this paper, including all the mentioned here results, we assume the normal play, where the first player unable to make a move loses (misère annihilation games have been considered in [2]).

Fraenkel considered in [4] a generalization of cellular-automata games to two-player games and provided a strategy for such cases. In particular, if for each vertex there is at most one outgoing arc then it is possible to derive a polynomial-time strategy [4]. Since the formulation of the game is equivalent to the one mentioned above, this result can be directly applied for the annihilation game.

Fraenkel in [3] studied the connections between annihilation games and error-correcting codes. The authors in [6] gave an algorithm for computing error-correcting

codes. The algorithm is polynomial in the size of the code and uses the theory of two-player cellular-automata games.

In the following we are interested in generalizations of the annihilation game, where there is more than one type of token and/or there is a different interaction between the tokens. Assume that $r \geq 2$ types of tokens are given and each type of token can be moved along a subset of the edges. Given a configuration of tokens in a graph, deciding whether the current player has a winning strategy is PSPACE-complete for acyclic graphs [5].

A modification called *hit*, where $r \geq 2$ types of tokens and edges are distinguished was considered in [5]. A move consists of selecting a token of type i and moving along an arc of type $i \in \{1, \dots, r\}$. The target vertex v cannot be occupied by a token of type i , but if v contains token of other type then it is removed (so, when the move ends v is occupied by the token of type i). The complexity of determining the outcome of this game is PSPACE-complete for acyclic graphs and $r = 2$ [5]. A modification of hit called *capture* has the same rules except that each token can travel along any edge. Capture is PSPACE-complete for acyclic and EXPTIME-complete for general graphs [10].

In a *node blocking* each token is of one of the two types. Each vertex can contain at most one token. Player i can move the tokens of type i , $i = 1, 2$. All tokens can move along all arcs. A player i makes a move, by selecting one token of type i (occupying a vertex $v \in V$) and an unoccupied vertex $u \in V$ such that $(v, u) \in E$ and moving the token from v to u . The first player unable to make a move loses and his opponent wins the game. There is a tie if there is no last move. First, the game was proved to be NP-hard [8], then PSPACE-hard for general graphs [5]. The complexity for general graphs has been finally proved in [10] to be EXPTIME-complete.

In an *edge blocking* all tokens are identical, i.e. each player can move any token, while each arc is of type 1 or 2 and a player i makes his move by moving a token along an arc of type i , $i = 1, 2$. Similarly as before, the first player who cannot make a move loses. A tie occurs if there is no last move. This game is PSPACE-complete for dags.

The following table summarizes the complexity of all the mentioned two-player annihilation games. We list only the strongest known results.

Game:	dag	general
Annihilation	PSPACE-complete [5]	?*
Hit	PSPACE-complete [5]	?*
Capture	PSPACE-complete [10]	EXPTIME-complete [10]
Node blocking	?	EXPTIME-complete [10]
Edge blocking	PSPACE-complete [5]	?*

Note that for the entries labeled as “?” can be replaced by “PSPACE-hard” (which can be concluded from the corresponding results for acyclic graphs), but the question remains whether the games are in PSPACE. In this paper we are interested in the problem marked by “?”, listed also in [1] as one of the open problems. In Section 3 we prove PSPACE-completeness of this game.

2 Definitions

In the following a token of type 1 (respectively 2) will be called a *white token* (*black token*, resp.) and denoted by symbol W_t (B_t , resp.). The player moving the white (black) tokens will be denoted by W (B , respectively).

Let $G = (V(G), E(G))$ be a directed graph. For $v \in V(G)$ define $\deg_G^+(v) = |\{u \in V(G) : (u, v) \in E(G)\}|$, $\deg_G^-(v) = |\{u \in V(G) : (v, u) \in E(G)\}|$. A notation $u \rightarrow_p v$ is used to denote a move made by player, $p \in \{W, B\}$, in which the token has been removed from u and placed at the vertex v . Given the positions of tokens, define $f(v)$ for $v \in V(G)$ to be one of three possible values W_t, B_t, \emptyset indicating that a white or black token is at the vertex v or there is no token at v , respectively. In the latter case we say that v is *empty*. Note that if $f(u) = \emptyset$ or $f(v) \neq \emptyset$ then the move $u \rightarrow_p v$ is incorrect.

Let us recall a PSPACE-complete Quantified Boolean Formula (QBF) problem [11]. The input for the problem is a formula Q in the form

$$Q_1 x_1 \dots Q_n x_n F(x_1, \dots, x_n),$$

where $Q_i \in \{\exists, \forall\}$ for $i = 1, \dots, n$. Decide whether Q is true. In our case we use a restricted case of this problem where $Q_1 = \exists$, $Q_{i+1} \neq Q_i$ for $i = 1, \dots, n-1$, n is even, and F is a 3CNF formula, i.e. $F = F_1 \wedge F_2 \wedge \dots \wedge F_m$, where $F_i = (l_{i,1} \vee l_{i,2} \vee l_{i,3})$ and each literal $l_{i,j}$ is a variable or the negation of a variable, $i = 1, \dots, m$, $j = 1, 2, 3$.

3 PSPACE-hardness of node blocking

Define a *variable component* G_i corresponding to x_i as follows:

$$V(G_i) = \{s, t, x, y\} \cup \{v_1, \dots, v_4\},$$

$$E(G_i) = \{(s, v_1), (v_1, v_2), (v_2, v_3), (v_3, t), (v_4, t), (v_4, v_2), (x, v_4), (y, v_4)\}$$

for $i = 2j - 1$, and

$$V(G_i) = \{s, t, x, y\} \cup \{v_1, \dots, v_8\},$$

$$\begin{aligned} E(G_i) = & \{(s, v_1), (v_1, v_2), (v_2, v_3), (v_3, t), (v_4, t), (v_4, v_2), \\ & (v_5, v_4), (v_6, v_4), (v_7, v_5), (v_8, v_6), (x, v_7), (y, v_8)\} \end{aligned}$$

for $i = 2j$, where $j = 1, \dots, n/2$. Fig. 1 depicts these subgraphs. If i is odd then G_i is called a *white component* and in this case an initial placement of tokens in G_i is $f(s) = f(v_4) = f(x) = f(y) = W_t$, $f(v_3) = \emptyset$ and $f(v_1) = f(v_2) = f(t) = B_t$ (see also Fig. 1(a)). In a *black component* G_i , where i is even, we have $f(s) = f(v_4) = \dots = f(v_8) = B_t$, $f(v_3) = \emptyset$ and $f(v_1) = f(v_2) = f(x) = f(y) = f(t) = W_t$ (see also Fig. 1(b)). In both cases the above configuration of tokens will be called the *initial state* of G_i .

Removing a token from a graph without placing it on another vertex is an invalid operation. However, assume for now that, given an initial state of G_i , the first move is a deletion of a token occupying the vertex t (we will assume in Lemma 1 that the game starts in this way). Then, W (respectively B) becomes the current player in the white (black, resp.) component G_i . Furthermore, we assume that the game in G_i ends when $f(s)$ becomes \emptyset .

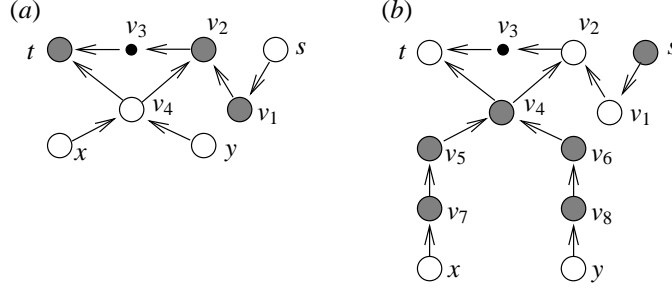


Figure 1: The graphs G_i for (a) $i = 2j - 1$ and (b) $i = 2j$, $j = 1, \dots, n/2$

Lemma 1 *If G_i is a white (respectively black) component then W (B , resp.) has a winning strategy. At the end of the game we have that if G_i is a white component then exactly one of the vertices x, y is empty, and if G_i is a black component then exactly one of the vertices x, y, v_5, v_6 is empty.*

Proof: First assume that G_i is a white component. Let $f(t) = \emptyset$ and W is the current player. The first two moves are $v_4 \rightarrow_W t$, $v_2 \rightarrow_B v_3$. Then there are two possibilities:

$$x \rightarrow_W v_4 \text{ or } y \rightarrow_W v_4. \quad (1)$$

In both cases the game continues as follows: $v_1 \rightarrow_B v_2$, $s \rightarrow_W v_1$. The thesis follows.

Let G_i be a black component with $f(t) = \emptyset$ and B is the current player. Similarly as before we have $v_4 \rightarrow_B t$, $v_2 \rightarrow_W v_3$. The third move is $v_5 \rightarrow_B v_4$ or $v_6 \rightarrow_B v_4$. Since they are symmetrical, assume in the following that the first case occurred. We have $v_1 \rightarrow_W v_2$. Then B has a choice:

$$v_7 \rightarrow_B v_5 \text{ or } s \rightarrow_B v_2. \quad (2)$$

If the first move occurred then we have $x \rightarrow_W v_7$. Then, $s \rightarrow_B v_2$, which ends the game and the vertex x is empty among the vertices listed in the lemma. If B selected the second move in (2) then the game ends with $f(v_5) = \emptyset$. \square

Now we define a graph G_F , corresponding to the Boolean formula F . In order to distinguish a vertex $v \in V(G_i)$ from the vertices of the other variable components we will write $v(G_i)$. G_F contains disjoint white components G_{2i-1} for $i = 1, \dots, n/2$ and disjoint black components G_{2i} , $i = 1, \dots, n/2$, connected in such a way that $s(G_i) = t(G_{i+1})$ for $i = 1, \dots, n - 1$. The graph G_F contains additionally the vertices $w, v(F_1), \dots, v(F_m)$, an arc $(w, t(G_n))$, the arcs $(v(F_i), w)$ for $i = 1, \dots, m$, and $(x(G_i), v(F_j)) \in E(G_F)$ iff F_j contains x_i , while $(y(G_i), v(F_j)) \in E(G_F)$ iff F_j contains \bar{x}_i , a negation of the variable x_i . Initially, all the subgraphs G_i are in the initial state, except that $f(t(G_1)) = \emptyset$. Let $f(w) = W_t$, $f(v(F_j)) = B_t$ for $j = 1, \dots, m$. Before we prove the main theorem, let us demonstrate the above reduction by giving an example

$$Q = \exists_{x_1} \forall_{x_2} \exists_{x_3} \forall_{x_4} (x_2 \vee \bar{x}_3 \vee x_4) \wedge (x_1 \vee x_2 \vee \bar{x}_4) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_4). \quad (3)$$

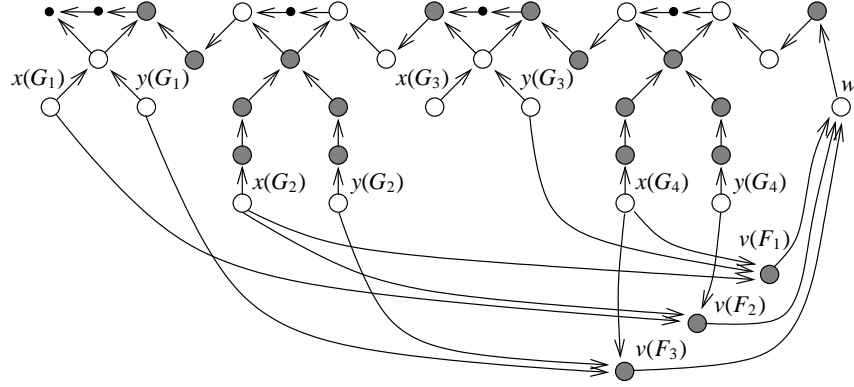


Figure 2: A complete instance of the graph G_F corresponding to (3)

Fig. 2 shows the corresponding graph G_F .

For brevity we introduce a notation: we say that the game *arrives at* a component G_i (and *leaves* the component G_{i-1} , $i > 1$) if $f(t(G_i)) = \emptyset$ (note that for $i > 1$ this is equivalent to $f(s(G_{i-1})) = \emptyset$ in the graph G_F). The game *is in* G_i if it arrived at G_i but did not leave G_i .

Theorem 1 *Node blocking is PSPACE-complete for directed acyclic graphs.*

Proof: First we prove by an induction on $i = 1, \dots, n$ that we may without loss of generality assume that if the game arrives at the component G_i then

- (i) for each $j < i$ exactly one of the vertices $x(G_j), y(G_j)$ (if G_j is a white component) or exactly one of the vertices $x(G_j), y(G_j), v_5(G_j), v_6(G_j)$ (if G_j is a black component) is empty,
- (ii) all tokens in components G_j , for $j = i, \dots, n$ are in the initial state, except that $f(t(G_i)) = \emptyset$.

The cases for $i = 1$ and $i > 1$ are analogous. If the game is in G_i then (by the induction hypothesis) all possible moves are the ones along the arcs in G_i , $v_2(G_j) \rightarrow_p v_3(G_j)$ for $j > i$ and $v_7(G_j) \rightarrow_B v_5(G_j)$ or $v_8(G_j) \rightarrow_B v_6(G_j)$ for a black component G_j , $j < i$. In the latter case W responds $x(G_j) \rightarrow_W v_7(G_j)$ or $y(G_j) \rightarrow_W v_8(G_j)$, respectively, so we consider the first two cases. Let G_j be a white component (the other case is analogous) and B moves a token along an arc which does not belong to $E(G_i)$, i.e.

$$v_2(G_j) \rightarrow_B v_3(G_j), j > i. \quad (4)$$

For each move (4) W responds

$$v_4(G_j) \rightarrow_W v_2(G_j). \quad (5)$$

For other moves of B , W responds as in the proof of Lemma 1. Consider the case when the game arrives at the component which is not in the initial state, because the moves (4)

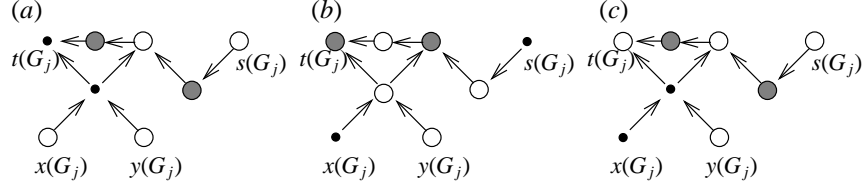


Figure 3: (a) the game arrives at G_j , (b) the game leaves G_j , (c) W wins the game

and (5) have been performed. This situation is given in Fig. 3(a). Since W is the current player, the first move in G_j is $x(G_j) \rightarrow_W v_4(G_j)$ or $y(G_j) \rightarrow_W v_4(G_j)$. In both cases the remaining sequence of moves is identical: $v_3(G_j) \rightarrow_B t(G_j)$, $v_2(G_j) \rightarrow_W v_3(G_j)$, $v_1(G_j) \rightarrow_B v_2(G_j)$, $s(G_j) \rightarrow_W v_1(G_j)$. The result is shown in Fig. 3(b). This proves that if B performs a move along an arc which is not in G_i when the game is in G_i then W decides among one of the moves $x(G_j) \rightarrow_W v_4(G_j)$ or $y(G_j) \rightarrow_W v_4(G_j)$ when the game is in G_j . This, however is only true under the assumption that after (4) and (5) W plays according to the schema given in the proof of Lemma 1. If the white player managed to place a token at the vertex $v_4(G_j)$ before the game arrived at G_j then the move $v_4(G_j) \rightarrow_W t(G_j)$ gives a situation depicted in Fig. 3(c) — the black player cannot make a move in G_j . So, if the game is in G_i and a move (4) occurred, then either the game creates the same configuration of tokens in variable components (restricted to the vertices $x(G_k), y(G_k), k = 1, \dots, n$), or B loses the game. Thus, w.l.o.g. we may assume that if the game is in G_i then the components $G_j, j > i$ are in the initial state, i.e. (ii) is true.

Assuming the players make only moves along the arcs of G_i , if the game arrives at G_{i+1} then Lemma 1 implies that (i) is satisfied.

Now we can prove the theorem. Assume that Q is true and we show that W has a winning strategy. If x_i is true (respectively false), $i = 2k - 1, k = 1, \dots, n/2$, then W plays in G_i in such a way that if the game leaves G_i then $f(x(G_i)) = W_t$ ($f(y(G_i)) = W_t$, respectively). Assume that the game leaves G_n . Then we have $w \rightarrow_W s(G_n)$ and $v(F_j) \rightarrow_B w$, for some $j \in \{1, \dots, m\}$. Since Q is true, there is a true literal $l_{j,k}$ in F_j , $k \in \{1, 2, 3\}$. If $l_{j,k} = x_i$ then $f(x(G_i)) = W_t$ and W can make the move $x(G_i) \rightarrow_W v(F_j)$. If $l_{j,k} = \bar{x}_i$ then $f(y(G_i)) = W_t$ and the move $y(G_i) \rightarrow_W v(F_j)$ is possible. Note that if $x(G_i)$ or $y(G_i)$ belongs to a black component, then (because Q is true) W always has a possibility to make the above move in such a way that it holds $f(v_5(G_i)) = B_t$ or $f(v_6(G_i)) = B_t$, respectively. If B can make a move then it must be $v_7(G_j) \rightarrow_B v_5(G_j)$ or $v_8(G_j) \rightarrow_B v_6(G_j)$, but then W responds $x(G_j) \rightarrow_B v_7(G_j)$ or $y(G_j) \rightarrow_B v_8(G_j)$. No other moves are possible, so W wins the game. The above holds for each index j .

Let now W have a winning strategy. If the values of $x_1, \dots, x_i, i = 2k$ have been set then let $x_{i+1} = \text{true}$ if we have the move $y(G_{i+1}) \rightarrow_W v_4(G_{i+1})$ during the game in G_{i+1} , and let $x_{i+1} = \text{false}$ if there is a move $x(G_{i+1}) \rightarrow_W v_4(G_{i+1})$ during the game in G_{i+1} . The game leaves G_n and we have the moves $w \rightarrow_W s(G_n)$, $v(F_j) \rightarrow_W w$ for some $j \in \{1, \dots, m\}$. The black player chooses j arbitrarily and since W has a winning strategy there is possible a move $x(G_k) \rightarrow_W v(F_j)$ or $y(G_k) \rightarrow_W v(F_j)$. From the

construction of the strategy for W we have that there is the literal $x_k = \text{true}$ in F_j or the literal $\overline{x_k} = \text{true}$ in F_j , respectively.

Observe that $|V(G_F)| = 7n/2 + 11n/2 + m + 2$, so this is a polynomial reduction. This proves PSPACE-hardness of node blocking. One can argue that G_F is acyclic which implies that the game is in PSPACE. \square

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