# THE CHROMATIC NUMBER OF ALMOST STABLE KNESER HYPERGRAPHS 

FRÉDÉRIC MEUNIER


#### Abstract

Let $V(n, k, s)$ be the set of $k$-subsets $S$ of $[n]$ such that for all $i, j \in S$, we have $|i-j| \geq s$ We define almost $s$-stable Kneser hypergraph $K G^{r}\binom{[n]}{k} \sim$-stab ${ }^{\sim}$ to be the $r$-uniform hypergraph whose vertex set is $V(n, k, s)$ and whose edges are the $r$-uples of disjoint elements of $V(n, k, s)$.

With the help of a $Z_{p}$-Tucker lemma, we prove that, for $p$ prime and for any $n \geq k p$, the chromatic number of almost 2-stable Kneser hypergraphs $K G^{p}\binom{[n]}{k} 2_{2 \text {-stab }}^{\sim}$ is equal to the chromatic number of the usual Kneser hypergraphs $K G^{p}\binom{[n]}{k}$, namely that it is equal to $\left\lceil\frac{n-(k-1) p}{p-1}\right\rceil$.

Defining $\mu(r)$ to be the number of prime divisors of $r$, counted with multiplicities, this result implies that the chromatic number of almost $2^{\mu(r)}$-stable Kneser hypergraphs $K G^{r}\binom{[n]}{k} 2^{\mu(r)}$-stab is equal to the chromatic number of the usual Kneser hypergraphs $K G^{r}\binom{[n]}{k}$ for any $n \geq k r$, namely that it is equal to $\left\lceil\frac{n-(k-1) r}{r-1}\right\rceil$.


## 1. Introduction and main results

Let $[a]$ denote the set $\{1, \ldots, a\}$. The Kneser graph $K G^{2}\binom{[n]}{k}$ for integers $n \geq 2 k$ is defined as follows: its vertex set is the set of $k$-subsets of $[n]$ and two vertices are connected by an edge if they have an empty intersection.

Kneser conjectured [6] in 1955 that its chromatic number $\chi\left(K G^{2}\binom{[n]}{k}\right)$ is equal to $n-2 k+2$. It was proved to be true by Lovász in 1979 in a famous paper [7], which is the first and one of the most spectacular application of algebraic topology in combinatorics.

Soon after this result, Schrijver [11] proved that the chromatic number remains the same when we consider the subgraph $K G^{2}\binom{[n]}{k}_{2 \text {-stab }}$ of $K G^{2}\binom{[n]}{k}$ obtained by restricting the vertex set to the $k$-subsets that are 2 -stable, that is, that do not contain two consecutive elements of [ $n$ ] (where 1 and $n$ are considered to be also consecutive).

Let us recall that an hypergraph $\mathcal{H}$ is a set family $\mathcal{H} \subseteq 2^{V}$, with vertex set $V$. An hypergraph is said to be $r$-uniform if all its edges $S \in \mathcal{H}$ have the same cardinality $r$. A proper coloring with $t$ colors of $\mathcal{H}$ is a map $c: V \rightarrow[t]$ such that there is no monochromatic edge, that is such that in each edge there are two vertices $i$ and $j$ with $c(i) \neq c(j)$. The smallest number $t$ such that there exists such a proper coloring is called the chromatic number of $\mathcal{H}$ and denoted by $\chi(\mathcal{H})$.

In 1986, solving a conjecture of Erdős 4], Alon, Frankl and Lovász [2] found the chromatic number of Kneser hypergraphs. The Kneser hypergraph $K G^{r}\binom{[n]}{k}$ is a $r$-uniform hypergraph which has the $k$ subsets of $[n]$ as vertex set and whose edges are formed by the $r$-uple of disjoint $k$-subsets of $[n]$. Let $n, k, r, t$ be positive integers such that $n \geq(t-1)(r-1)+r k$. Then $\chi\left(K G^{r}\binom{[n]}{k}\right)>t$. Combined with a lemma by Erdős giving an explicit proper coloring, it implies that $\chi\left(K G^{r}\binom{[n]}{k}\right)=\left\lceil\frac{n-(k-1) r}{r-1}\right\rceil$. The proof found by Alon, Frankl and Lovász used tools from algebraic topology.

In 2001, Ziegler gave a combinatorial proof of this theorem [13], which makes no use of homology, simplicial approximation,... He was inspired by a combinatorial proof of the Lovász theorem found by Matoušek [9]. A subset $S \subseteq[n]$ is $s$-stable if any two of its elements are at least "at distance $s$
apart" on the $n$-cycle, that is, if $s \leq|i-j| \leq n-s$ for distinct $i, j \in S$. Define then $K G^{r}\binom{[n]}{k}_{s-\text { stab }}$ as the hypergraph obtained by restricting the vertex set of $K G^{r}\binom{[n]}{k}$ to the $s$-stable $k$-subsets. At the end of his paper, Ziegler made the supposition that the chromatic number of $K G^{r}\binom{[n]}{k}_{r \text {-stab }}$ is equal to the chromatic number of $K G^{r}\binom{[n]}{k}$ for any $n \geq k r$. This supposition generalizes both Schrijver's theorem and the Alon-Frankl-Lovász theorem. Alon, Drewnowski and Łucsak make this supposition an explicit conjecture in [1].

Conjecture 1. Let $n, k, r$ be non-negative integers such that $n \geq r k$. Then

$$
\chi\left(K G^{r}\binom{[n]}{k}_{r-s t a b}\right)=\left\lceil\frac{n-(k-1) r}{r-1}\right\rceil .
$$

We prove a weaker form of this statement, but which strengthes the Alon-Frankl-Lovász theorem. Let $V(n, k, s)$ be the set of $k$-subsets $S$ of $[n]$ such that for all $i, j \in S$, we have $|i-j| \geq s$ We define the almost $s$-stable Kneser hypergraphs $K G^{r}\binom{[n]}{k}_{s \text {-stab }}^{\sim}$ to be the $r$-uniform hypergraph whose vertex set is $V(n, k, s)$ and whose edges are the $r$-uples of disjoint elements of $V(n, k, s)$.
Theorem 1. Let $p$ be a prime number and $n, k$ be non negative integers such that $n \geq p k$. We have

$$
\chi\left(K G^{p}\binom{[n]}{k}_{2 \text {-stab }}^{\sim}\right) \geq\left\lceil\frac{n-(k-1) p}{p-1}\right\rceil .
$$

Combined with the lemma by Erdős, we get that

$$
\chi\left(K G^{p}\binom{[n]}{k}_{2 \text {-stab }}^{\sim}\right)=\left\lceil\frac{n-(k-1) p}{p-1}\right\rceil .
$$

Moreover, we will see that it is then possible to derive the following corollary. Denote by $\mu(r)$ the number of prime divisors of $r$ counted with multiplicities. For instance, $\mu(6)=2$ and $\mu(12)=3$. We have

Corollary 1. Let $n, k, r$ be non-negative integers such that $n \geq r k$. We have

$$
K G^{r}\binom{[n]}{k}_{2^{\mu(r)} \text {-stab }}^{\sim}=\left\lceil\frac{n-(k-1) r}{r-1}\right\rceil
$$

## 2. Notations and tools

$Z_{p}=\left\{\omega, \omega^{2}, \ldots, \omega^{p}\right\}$ is the cyclic group of order $p$, with generator $\omega$.
We write $\sigma^{n-1}$ for the $(n-1)$-dimensional simplex with vertex set $[n]$ and by $\sigma_{k-1}^{n-1}$ the ( $k-1$ )skeleton of this simplex, that is the set of faces of $\sigma^{n-1}$ having $k$ or less vertices.

If $A$ and $B$ are two sets, we write $A \uplus B$ for the set $(A \times\{1\}) \cup(B \times\{2\})$. For two simplicial complexes, K and L , with vertex sets $V(\mathrm{~K})$ and $V(\mathrm{~L})$, we denote by $\mathrm{K} * \mathrm{~L}$ the join of these two complexes, which is the simplicial complex having $V(\mathrm{~K}) \uplus V(\mathrm{~L})$ as vertex set and

$$
\{F \uplus G: F \in \mathrm{~K}, G \in \mathrm{~L}\}
$$

as set of faces. We define also $\mathrm{K}^{* n}$ to be the join of $n$ disjoint copies of K .
Let $X=\left(x_{1}, \ldots, x_{n}\right) \in\left(Z_{p} \cup\{0\}\right)^{n}$. We denote by alt $(X)$ the size of the longest alternating subsequence of non-zero terms in $X$. A sequence $\left(j_{1}, j_{2}, \ldots, j_{m}\right)$ of elements of $Z_{p}$ is said to be alternating if any two consecutive terms are different. For instance (assume $p=5$ ) $\operatorname{alt}\left(\omega^{2}, \omega^{3}, 0, \omega^{3}, \omega^{5}, 0,0, \omega^{2}\right)=4$ and $\operatorname{alt}\left(\omega^{1}, \omega^{4}, \omega^{4}, \omega^{4}, 0,0, \omega^{4}\right)=2$.

Any element element $X=\left(x_{1}, \ldots, x_{n}\right) \in\left(Z_{p} \cup\{0\}\right)^{n}$ can alternatively and without further mention be denoted by a $p$-uple $\left(X_{1}, \ldots, X_{p}\right)$ where $X_{j}:=\left\{i \in[n]: x_{i}=\omega^{j}\right\}$. Note that the $X_{j}$ are then necessarily disjoint. For two elements $X, Y \in\left(Z_{p} \cup\{0\}\right)^{n}$, we denote by $X \subseteq Y$ the fact
that for all $j \in[p]$ we have $X_{j} \subseteq Y_{j}$. When $X \subseteq Y$, note that the sequence of non-zero terms in $\left(x_{1}, \ldots, x_{n}\right)$ is a subsequence of $\left(y_{1}, \ldots, y_{n}\right)$.

The proof of Theorem 1 makes use of a variant of the $Z_{p}$-Tucker lemma by Ziegler [13].
Lemma 1 ( $Z_{p}$-Tucker lemma). Let $p$ be a prime, $n, m \geq 1, \alpha \leq m$ and let

$$
\begin{array}{ccc}
\lambda: \quad\left(Z_{p} \cup\{0\}\right)^{n} \backslash\{(0, \ldots, 0)\} & \longrightarrow & Z_{p} \times[m] \\
X & \longmapsto & \left(\lambda_{1}(X), \lambda_{2}(X)\right)
\end{array}
$$

be a $Z_{p}$-equivariant map satisfying the following properties:

- for all $X^{(1)} \subseteq X^{(2)} \in\left(Z_{p} \cup\{0\}\right)^{n} \backslash\{(0, \ldots, 0)\}$, if $\lambda_{2}\left(X^{(1)}\right)=\lambda_{2}\left(X^{(2)}\right) \leq \alpha$, then $\lambda_{1}\left(X^{(1)}\right)=$ $\lambda_{1}\left(X^{(2)}\right)$;
- for all $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \in\left(Z_{p} \cup\{0\}\right)^{n} \backslash\{(0, \ldots, 0)\}$, if $\lambda_{2}\left(X^{(1)}\right)=\lambda_{2}\left(X^{(2)}\right)=$ $\ldots=\lambda_{2}\left(X^{(p)}\right) \geq \alpha+1$, then the $\lambda_{1}\left(X^{(i)}\right)$ are not pairwise distinct for $i=1, \ldots, p$.
Then $\alpha+(m-\alpha)(p-1) \geq n$.
We can alternatively say that $X \mapsto \lambda(X)=\left(\lambda_{1}(X), \lambda_{2}(X)\right)$ is a $Z_{p}$-equivariant simplicial map from sd $\left(Z_{p}^{* n}\right)$ to $\left(Z_{p}^{* \alpha}\right) *\left(\left(\sigma_{p-2}^{p-1}\right)^{*(m-\alpha)}\right)$, where $\operatorname{sd}(\mathrm{K})$ denotes the fist barycentric subdivision of a simplicial complex K .

Proof of the $Z_{p}$-Tucker lemma. According to Dold's theorem [3, 8, if such a map $\lambda$ exists, the dimension of $\left(Z_{p}^{* \alpha}\right) *\left(\left(\sigma_{p-2}^{p-1}\right)^{*(m-\alpha)}\right)$ is strictly larger than the connectivity of $Z_{p}^{* n}$, that is $\alpha+$ $(m-\alpha)(p-1)-1>n-2$.

It is also possible to give a purely combinatorial proof of this lemma through the generalized Ky Fan theorem from [5].

## 3. Proof of the main results

Proof of Theorem 1. We follow the scheme used by Ziegler in [13]. We endow $2^{[n]}$ with an arbitrary linear order $\preceq$.

Assume that $K G^{p}\binom{[n]}{k}_{2 \text {-stab }}^{\sim}$ is properly colored with $C$ colors $\{1, \ldots, C\}$. For $S \in V(n, k, 2)$, we denote by $c(S)$ its color. Let $\alpha=p(k-1)$ and $m=p(k-1)+C$.

Let $X=\left(x_{1}, \ldots, x_{n}\right) \in\left(Z_{p} \cup\{0\}\right)^{n} \backslash\{(0, \ldots, 0)\}$. We can write alternatively $X=\left(X_{1}, \ldots, X_{p}\right)$.

- if alt $(X) \leq p(k-1)$, let $j$ be the index of the $X_{j}$ containing the smallest integer ( $\omega^{j}$ is then the first non-zero term in $\left.\left(x_{1}, \ldots, x_{n}\right)\right)$, and define

$$
\lambda(X):=(j, \operatorname{alt}(X)) .
$$

- if alt $(X) \geq p(k-1)+1$ : in the longest alternating subsequence of non-zero terms of $X$, at least one of the elements of $Z_{p}$ appears at least $k$ times; hence, in at least one of the $X_{j}$ there is an element $S$ of $V(n, k, 2)$; choose the smallest such $S$ (according to $\preceq$ ). Let $j$ be such that $S \subseteq X_{j}$ and define

$$
\lambda(X):=(j, c(S)+p(k-1)) .
$$

$\lambda$ is $Z_{p}$-equivariant map from $\left(Z_{p} \cup\{0\}\right)^{n} \backslash\{(0, \ldots, 0)\}$ to $Z_{p} \times[m]$.
Let $X^{(1)} \subseteq X^{(2)} \in\left(Z_{p} \cup\{0\}\right)^{n} \backslash\{(0, \ldots, 0)\}$. If $\lambda_{2}\left(X^{(1)}\right)=\lambda_{2}\left(X^{(2)}\right) \leq \alpha$, then the longest alternating subsequences of non-zero terms of $X^{(1)}$ and $X^{(2)}$ have same size. Clearly, the first non-zero terms of $X^{(1)}$ and $X^{(2)}$ are equal.

Let $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \in\left(Z_{p} \cup\{0\}\right)^{n} \backslash\{(0, \ldots, 0)\}$. If $\lambda_{2}\left(X^{(1)}\right)=\lambda_{2}\left(X^{(2)}\right)=\ldots=$ $\lambda_{2}\left(X^{(p)}\right) \geq \alpha+1$, then for each $i \in[p]$ there is $S_{i} \in V(n, k, 2)$ and $j_{i} \in[p]$ such that we have $S_{i} \subseteq X_{j_{i}}^{(i)}$ and $\lambda_{2}\left(X^{(i)}\right)=c\left(S_{i}\right)$. If all $\lambda_{1}\left(X^{(i)}\right)$ would be distinct, then it would mean that all $j_{i}$
would be distinct, which implies that the $S_{i}$ would be disjoint but colored with the same color, which is impossible since $c$ is a proper coloring.

We can thus apply the $Z_{p}$-Tucker lemma (Lemma (1) and conclude that $n \leq p(k-1)+C(p-1)$, that is

$$
C \geq\left\lceil\frac{n-(k-1) p}{p-1}\right\rceil .
$$

To prove Corollary 1 we prove the following lemma, both statement and proof of which are inspired by Lemma 3.3 of [1].

Lemma 2. Let $r_{1}, r_{2}, s_{1}, s_{2}$ be non-negative integers $\geq 1$, and define $r=r_{1} r_{2}$ and $s=s_{1} s_{2}$.
Assume that for $i=1,2$ we have $\chi\left(K G^{r_{i}\binom{[n]}{k} s_{i} \text {-stab }} \sim\left\lceil\frac{n-(k-1) r_{i}}{r_{i}-1}\right\rceil\right.$ for all integers $n$ and $k$ such that $n \geq r_{i} k$.

Then we have $\chi\left(K G^{r}\binom{[n]}{k}_{s \text { stab }}^{\sim}\right)=\left\lceil\frac{n-(k-1) r}{r-1}\right\rceil$ for all integers $n$ and $k$ such that $n \geq r k$.
Proof. Let $n \geq(t-1)(r-1)+r k$. We have to prove that $\chi\left(K G^{r}\binom{[n]}{k}_{s-\text { stab }}^{\sim}\right)>t$. For a contradiction, assume that $K G^{r}\binom{[n]}{k}_{s \text {-stab }}$ is properly colored with $C \leq t$ colors. For $S \in V(n, k, p)$, we denote by $c(S)$ its color. We wish to prove that there are $S_{1}, \ldots, S_{r}$ disjoint elements of $V(n, k, s)$ with $c\left(S_{1}\right)=\ldots=c\left(S_{r}\right)$.

Take $A \in V\left(n, n_{1}, s_{1}\right)$, where $n_{1}:=r_{1} k+(t-1)\left(r_{1}-1\right)$. Denote $a_{1}<\ldots<a_{n_{1}}$ the elements of $A$ and define $h: V\left(n_{1}, k, s_{2}\right) \rightarrow[t]$ as follows: let $B \in V\left(n_{1}, k, s_{2}\right)$; the $k$-subset $S=\left\{a_{i}: i \in B\right\} \subseteq[n]$ is an element of $V(n, k, s)$, and gets as such a color $c(S)$; define $h(B)$ to be this $c(S)$. Since $n_{1}=r_{1} k+(t-1)\left(r_{1}-1\right)$, there are $B_{1}, \ldots, B_{r_{1}}$ disjoint elements of $V\left(n_{1}, k, s_{2}\right)$ having the same color by $h$. Define $\tilde{h}(A)$ to be this common color.

Make the same definition for all $A \in V\left(n, n_{1}, s_{1}\right)$. The map $\tilde{h}$ is a coloring of $K G^{r_{2}}\binom{[n]}{n_{1}}_{s_{1} \text {-stab }}^{\sim}$ with $t$ colors. Now, note that
$(t-1)(r-1)+r k=(t-1)\left(r_{1} r_{2}-r_{2}+r_{2}-1\right)+r_{1} r_{2} k=(t-1)\left(r_{2}-1\right)+r_{2}\left((t-1)\left(r_{1}-1\right)+r_{1} k\right)$ and thus that $n \geq(t-1)\left(r_{2}-1\right)+r_{2} n_{1}$. Hence, there are $A_{1}, \ldots, A_{r_{2}}$ disjoint elements of $V\left(n, n_{1}, s_{1}\right)$ with the same color. Each of the $A_{i}$ gets its color from $r_{1}$ disjoint elements of $V(n, k, s)$, whence there are $r_{1} r_{2}$ disjoint elements of $V(n, k, s)$ having the same color by the map $c$.
Proof of Corollary 1. Direct consequence of Theorem 1 and Lemma 2,

## 4. Short combinatorial proof of Schrijver's theorem

Recall that Schrijver's theorem is
Theorem 2. Let $n \geq 2 k . \chi\left(K G\binom{[n]}{k}_{2 \text {-stab }}\right)=n-2 k+2$.
When specialized for $p=2$, Theorem 1 does not imply Schrijver's theorem since the vertex set is allowed to contain subsets with 1 and $n$ together. Anyway, by a slight modification of the proof, we can get a short combinatorial proof of Schrijver's theorem. Alternative proofs of this kind - but not that short - have been proposed in [10, 13]

For a positive integer $n$, we write $\{+,-, 0\}^{n}$ for the set of all signed subsets of $[n]$, that is, the family of all pairs $\left(X^{+}, X^{-}\right)$of disjoint subsets of $[n]$. Indeed, for $X \in\{+,-, 0\}^{n}$, we can define $X^{+}:=\left\{i \in[n]: X_{i}=+\right\}$ and analogously $X^{-}$.

We define $X \subseteq Y$ if and only if $X^{+} \subseteq Y^{+}$and $X^{-} \subseteq Y^{-}$.
By alt $(X)$ we denote the length of the longest alternating subsequence of non-zero signs in $X$. For instance: $\operatorname{alt}(+0--+0-)=4$, while alt $(--++-+0+-)=5$.

The proof makes use of the following well-known lemma see [8, 12, 13] (which is a special case of Lemma 1 for $p=2$ ).

Lemma 3 (Tucker's lemma). Let $\lambda:\{-, 0,+\}^{n} \backslash\{(0,0, \ldots, 0)\} \rightarrow\{-1,+1, \ldots,-n,+n\}$ be $a$ map such that $\lambda(-X)=-\lambda(X)$. Then there exist $A, B$ in $\{-, 0,+\}^{n}$ such that $A \subseteq B$ and $\lambda(A)=-\lambda(B)$.

Proof of Schrijver's theorem. The inequality $\chi\left(K G^{2}\binom{[n]}{k}_{2 \text {-stab }}\right) \leq n-2 k+2$ is easy to prove (with an explicit coloring) and well-known. So, to obtain a combinatorial proof, it is sufficient to prove the reverse inequality.

Let us assume that there is a proper coloring $c$ of $K G^{2}\binom{[n]}{k}_{2 \text {-stab }}$ with $n-2 k+1$ colors. We define the following map $\lambda$ on $\{-, 0,+\}^{n} \backslash\{(0,0, \ldots, 0)\}$.

- if $\operatorname{alt}(X) \leq 2 k-1$, we define $\lambda(X)= \pm \operatorname{alt}(X)$, where the sign is determined by the first sign of the longest alternating subsequence of $X$ (which is actually the first non zero term of $X$ ).
- if $\operatorname{alt}(X) \geq 2 k$, then $X^{+}$and $X^{-}$both contain a stable subset of $[n]$ of size $k$. Among all stable subsets of size $k$ included in $X^{-}$and $X^{+}$, select the one having the smallest color. Call it $S$. Then define $\lambda(X)= \pm(c(S)+2 k-1)$ where the sign indicates which of $X^{-}$or $X^{+}$the subset $S$ has been taken from. Note that $c(S) \leq n-2 k$.
The fact that for any $X \in\{-, 0,+\}^{n} \backslash\{(0,0, \ldots, 0)\}$ we have $\lambda(-X)=-\lambda(X)$ is obvious. $\lambda$ takes its values in $\{-1,+1, \ldots,-n,+n\}$. Now let us take $A$ and $B$ as in Tucker's lemma, with $A \subseteq B$ and $\lambda(A)=-\lambda(B)$. We cannot have alt $(A) \leq 2 k-1$ since otherwise we will have a longest alternating in $B$ containg the one of $A$, of same length but with a different sign. Hence $\operatorname{alt}(A) \geq 2 k$. Assume w.l.o.g. that $\lambda(A)$ is defined by a stable subset $S_{A} \subseteq A^{-}$. Then the stable subset $S_{B}$ defining $\lambda(B)$ is such that $S_{B} \subseteq B^{+}$, which implies that $S_{A} \cap S_{B}=\emptyset$. We have moreover $c\left(S_{A}\right)=|\lambda(A)|=|\lambda(B)|=c\left(S_{B}\right)$, but this contradicts the fact that $c$ is proper coloring of $K G^{2}\binom{[n]}{k}_{2 \text {-stab }}$.


## 5. Concluding remarks

We have seen that one of the main ingredients is the notion of alternating sequence of elements in $Z_{p}$. Here, our notion only requires that such an alternating sequence must have $x_{i} \neq x_{i+1}$. To prove Conjecture [1, we need probably something stronger. For example, a sequence is said to be alternating if any $p$ consecutive terms are all distinct. Anyway, all our attempts to get something through this approach have failed.

Recall that Alon, Drewnowski and Eucsak [1] proved Conjecture 1 when $r$ is a power of 2 . With the help of a computer and lpsolve, we check that Conjecture 1 is moreover true for

- $n \leq 9, k=2, r=3$.
- $n \leq 12, k=3, r=3$.
- $n \leq 14, k=4, r=3$.
- $n \leq 13, k=2, r=5$.
- $n \leq 16, k=3, r=5$.
- $n \leq 21, k=4, r=5$.


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Université Paris Est, LVMT, EnPC, 6-8 avenue Blaise Pascal, Cité Descartes Champs-sur-Marne, 77455 Marne-La-Vallée cedex 2, France.

E-mail address: frederic.meunier@enpc.fr

