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THE CHROMATIC NUMBER OF ALMOST STABLE KNESER HYPERGRAPHS

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ABSTRACT. Let V(n, k, s) be the set of k-subsets S of [n] such that for all $i, j \in S$, we have $|i-j| \ge s$. We define almost s-stable Kneser hypergraph $KG^r {[n] \atop k}_{s-\text{stab}} \sim to$ be the r-uniform hypergraph whose vertex set is V(n, k, s) and whose edges are the r-uples of disjoint elements of V(n, k, s).

With the help of a Z_p -Tucker lemma, we prove that, for p prime and for any $n \geq kp$, the chromatic number of almost 2-stable Kneser hypergraphs $KG^p\binom{[n]}{k}_{2-\text{stab}}^{2-\text{stab}}$ is equal to the chromatic number of the usual Kneser hypergraphs $KG^p\binom{[n]}{k}$, namely that it is equal to $\left\lfloor \frac{n-(k-1)p}{p-1} \right\rfloor$.

Defining $\mu(r)$ to be the number of prime divisors of r, counted with multiplicities, this result implies that the chromatic number of almost $2^{\mu(r)}$ -stable Kneser hypergraphs $KG^r {\binom{[n]}{k}}_{2^{\mu(r)}-\text{stab}}^{\sim}$ is equal to the chromatic number of the usual Kneser hypergraphs $KG^r {\binom{[n]}{k}}$ for any $n \ge kr$, namely that it is equal to $\left\lceil \frac{n-(k-1)r}{r-1} \right\rceil$.

1. INTRODUCTION AND MAIN RESULTS

Let [a] denote the set $\{1, \ldots, a\}$. The Kneser graph $KG^2\binom{[n]}{k}$ for integers $n \ge 2k$ is defined as follows: its vertex set is the set of k-subsets of [n] and two vertices are connected by an edge if they have an empty intersection.

Kneser conjectured [6] in 1955 that its chromatic number $\chi\left(KG^2\binom{[n]}{k}\right)$ is equal to n-2k+2. It was proved to be true by Lovász in 1979 in a famous paper [7], which is the first and one of the most spectacular application of algebraic topology in combinatorics.

Soon after this result, Schrijver [11] proved that the chromatic number remains the same when we consider the subgraph $KG^2\binom{[n]}{k}_{2-\text{stab}}$ of $KG^2\binom{[n]}{k}$ obtained by restricting the vertex set to the *k*-subsets that are 2-stable, that is, that do not contain two consecutive elements of [n] (where 1 and *n* are considered to be also consecutive).

Let us recall that an hypergraph \mathcal{H} is a set family $\mathcal{H} \subseteq 2^V$, with vertex set V. An hypergraph is said to be *r*-uniform if all its edges $S \in \mathcal{H}$ have the same cardinality *r*. A proper coloring with *t* colors of \mathcal{H} is a map $c: V \to [t]$ such that there is no monochromatic edge, that is such that in each edge there are two vertices *i* and *j* with $c(i) \neq c(j)$. The smallest number *t* such that there exists such a proper coloring is called the chromatic number of \mathcal{H} and denoted by $\chi(\mathcal{H})$.

In 1986, solving a conjecture of Erdős [4], Alon, Frankl and Lovász [2] found the chromatic number of *Kneser hypergraphs*. The Kneser hypergraph $KG^r\binom{[n]}{k}$ is a *r*-uniform hypergraph which has the *k*subsets of [*n*] as vertex set and whose edges are formed by the *r*-uple of disjoint *k*-subsets of [*n*]. Let n, k, r, t be positive integers such that $n \ge (t-1)(r-1)+rk$. Then $\chi\left(KG^r\binom{[n]}{k}\right) > t$. Combined with a lemma by Erdős giving an explicit proper coloring, it implies that $\chi\left(KG^r\binom{[n]}{k}\right) = \left\lceil \frac{n-(k-1)r}{r-1} \right\rceil$. The proof found by Alon, Frankl and Lovász used tools from algebraic topology.

In 2001, Ziegler gave a combinatorial proof of this theorem [13], which makes no use of homology, simplicial approximation,... He was inspired by a combinatorial proof of the Lovász theorem found by Matoušek [9]. A subset $S \subseteq [n]$ is *s*-stable if any two of its elements are at least "at distance *s*

apart" on the *n*-cycle, that is, if $s \leq |i - j| \leq n - s$ for distinct $i, j \in S$. Define then $KG^r {\binom{[n]}{k}}_{s-\text{stab}}$ as the hypergraph obtained by restricting the vertex set of $KG^r {\binom{[n]}{k}}$ to the *s*-stable *k*-subsets. At the end of his paper, Ziegler made the supposition that the chromatic number of $KG^r {\binom{[n]}{k}}_{r-\text{stab}}$ is equal to the chromatic number of $KG^r {\binom{[n]}{k}}$ for any $n \geq kr$. This supposition generalizes both Schrijver's theorem and the Alon-Frankl-Lovász theorem. Alon, Drewnowski and Łucsak make this supposition an explicit conjecture in [1].

Conjecture 1. Let n, k, r be non-negative integers such that $n \ge rk$. Then

$$\chi\left(KG^r\binom{[n]}{k}_{r-stab}\right) = \left\lceil \frac{n-(k-1)r}{r-1} \right\rceil.$$

We prove a weaker form of this statement, but which strengthes the Alon-Frankl-Lovász theorem. Let V(n, k, s) be the set of k-subsets S of [n] such that for all $i, j \in S$, we have $|i - j| \ge s$ We define the almost s-stable Kneser hypergraphs $KG^r {[n] \choose k}_{s-\text{stab}}^{\sim}$ to be the r-uniform hypergraph whose vertex set is V(n, k, s) and whose edges are the r-uples of disjoint elements of V(n, k, s).

Theorem 1. Let p be a prime number and n, k be non negative integers such that $n \ge pk$. We have

$$\chi\left(KG^p\binom{[n]}{k}_{2-\mathrm{stab}}^{\sim}\right) \ge \left\lceil \frac{n-(k-1)p}{p-1} \right\rceil.$$

Combined with the lemma by Erdős, we get that

$$\chi \left(KG^p \binom{[n]}{k}_{2-\text{stab}}^{\infty} \right) = \left\lceil \frac{n - (k-1)p}{p-1} \right\rceil$$

Moreover, we will see that it is then possible to derive the following corollary. Denote by $\mu(r)$ the number of prime divisors of r counted with multiplicities. For instance, $\mu(6) = 2$ and $\mu(12) = 3$. We have

Corollary 1. Let n, k, r be non-negative integers such that $n \ge rk$. We have

$$KG^r {[n] \choose k}_{2^{\mu(r)}\text{-stab}}^{\sim} = \left\lceil \frac{n - (k-1)r}{r-1} \right\rceil.$$

2. Notations and tools

 $Z_p = \{\omega, \omega^2, \dots, \omega^p\}$ is the cyclic group of order p, with generator ω .

We write σ^{n-1} for the (n-1)-dimensional simplex with vertex set [n] and by σ_{k-1}^{n-1} the (k-1)-skeleton of this simplex, that is the set of faces of σ^{n-1} having k or less vertices.

If A and B are two sets, we write $A \uplus B$ for the set $(A \times \{1\}) \cup (B \times \{2\})$. For two simplicial complexes, K and L, with vertex sets V(K) and V(L), we denote by K * L the *join* of these two complexes, which is the simplicial complex having $V(K) \uplus V(L)$ as vertex set and

$$\{F \uplus G : F \in \mathsf{K}, G \in \mathsf{L}\}$$

as set of faces. We define also K^{*n} to be the join of n disjoint copies of K.

Let $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n$. We denote by $\operatorname{alt}(X)$ the size of the longest alternating subsequence of non-zero terms in X. A sequence (j_1, j_2, \ldots, j_m) of elements of Z_p is said to be *alternating* if any two consecutive terms are different. For instance (assume p = 5) $\operatorname{alt}(\omega^2, \omega^3, 0, \omega^3, \omega^5, 0, 0, \omega^2) = 4$ and $\operatorname{alt}(\omega^1, \omega^4, \omega^4, \omega^4, 0, 0, \omega^4) = 2$.

Any element element $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n$ can alternatively and without further mention be denoted by a *p*-uple (X_1, \ldots, X_p) where $X_j := \{i \in [n] : x_i = \omega^j\}$. Note that the X_j are then necessarily disjoint. For two elements $X, Y \in (Z_p \cup \{0\})^n$, we denote by $X \subseteq Y$ the fact that for all $j \in [p]$ we have $X_j \subseteq Y_j$. When $X \subseteq Y$, note that the sequence of non-zero terms in (x_1,\ldots,x_n) is a subsequence of (y_1,\ldots,y_n) .

The proof of Theorem 1 makes use of a variant of the Z_p -Tucker lemma by Ziegler [13].

Lemma 1 (Z_p -Tucker lemma). Let p be a prime, $n, m \ge 1, \alpha \le m$ and let

$$\begin{array}{rcl} \lambda: & (Z_p \cup \{0\})^n \setminus \{(0, \dots, 0)\} & \longrightarrow & Z_p \times [m] \\ & X & \longmapsto & (\lambda_1(X), \lambda_2(X)) \end{array}$$

be a Z_p -equivariant map satisfying the following properties:

- for all $X^{(1)} \subseteq X^{(2)} \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$, if $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) \le \alpha$, then $\lambda_1(X^{(1)}) = \lambda_2(X^{(2)}) \le \alpha$. $\lambda_1(X^{(2)}):$
- for all $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \in (Z_p \cup \{0\})^n \setminus \{(0,\ldots,0)\}, \text{ if } \lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \ldots = \lambda_2(X^{(p)}) \ge \alpha + 1, \text{ then the } \lambda_1(X^{(i)}) \text{ are not pairwise distinct for } i = 1,\ldots,p.$

Then
$$\alpha + (m - \alpha)(p - 1) \ge n$$
.

We can alternatively say that $X \mapsto \lambda(X) = (\lambda_1(X), \lambda_2(X))$ is a Z_p -equivariant simplicial map from sd (Z_p^{*n}) to $(Z_p^{*\alpha}) * ((\sigma_{p-2}^{p-1})^{*(m-\alpha)})$, where sd(K) denotes the first barycentric subdivision of a simplicial complex K.

Proof of the Z_p -Tucker lemma. According to Dold's theorem [3, 8], if such a map λ exists, the dimension of $(Z_p^{*\alpha}) * ((\sigma_{p-2}^{p-1})^{*(m-\alpha)})$ is strictly larger than the connectivity of Z_p^{*n} , that is $\alpha + (m-\alpha)(p-1) - 1 > n-2$.

It is also possible to give a purely combinatorial proof of this lemma through the generalized Ky Fan theorem from [5].

3. Proof of the main results

Proof of Theorem 1. We follow the scheme used by Ziegler in [13]. We endow $2^{[n]}$ with an arbitrary linear order \leq .

Assume that $KG^p\binom{[n]}{k}_{2-\text{stab}}^{\sim}$ is properly colored with C colors $\{1,\ldots,C\}$. For $S \in V(n,k,2)$, we denote by c(S) its color. Let $\alpha = p(k-1)$ and m = p(k-1) + C.

Let $X = (x_1, \ldots, x_n) \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$. We can write alternatively $X = (X_1, \ldots, X_p)$.

• if $alt(X) \leq p(k-1)$, let j be the index of the X_j containing the smallest integer (ω^j is then the first non-zero term in (x_1, \ldots, x_n)), and define

$$\lambda(X) := (j, \operatorname{alt}(X)).$$

• if $alt(X) \ge p(k-1) + 1$: in the longest alternating subsequence of non-zero terms of X, at least one of the elements of Z_p appears at least k times; hence, in at least one of the X_j there is an element S of V(n, k, 2); choose the smallest such S (according to \preceq). Let j be such that $S \subseteq X_j$ and define

$$\lambda(X) := (j, c(S) + p(k-1)).$$

 λ is Z_p -equivariant map from $(Z_p \cup \{0\})^n \setminus \{(0, \dots, 0)\}$ to $Z_p \times [m]$. Let $X^{(1)} \subseteq X^{(2)} \in (Z_p \cup \{0\})^n \setminus \{(0, \dots, 0)\}$. If $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) \leq \alpha$, then the longest alternating subsequences of non-zero terms of $X^{(1)}$ and $X^{(2)}$ have same size. Clearly, the first non-zero terms of $X^{(1)}$ and $X^{(2)}$ are equal.

Let $X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X^{(p)} \in (Z_p \cup \{0\})^n \setminus \{(0, \ldots, 0)\}$. If $\lambda_2(X^{(1)}) = \lambda_2(X^{(2)}) = \ldots = (X^{(p)})^n$ $\lambda_2(X^{(p)}) \geq \alpha + 1$, then for each $i \in [p]$ there is $S_i \in V(n,k,2)$ and $j_i \in [p]$ such that we have $S_i \subseteq X_{j_i}^{(i)}$ and $\lambda_2(X^{(i)}) = c(S_i)$. If all $\lambda_1(X^{(i)})$ would be distinct, then it would mean that all j_i would be distinct, which implies that the S_i would be disjoint but colored with the same color, which is impossible since c is a proper coloring.

We can thus apply the Z_p -Tucker lemma (Lemma 1) and conclude that $n \leq p(k-1) + C(p-1)$, that is

$$C \ge \left\lceil \frac{n - (k - 1)p}{p - 1} \right\rceil.$$

To prove Corollary 1, we prove the following lemma, both statement and proof of which are inspired by Lemma 3.3 of [1].

Lemma 2. Let r_1, r_2, s_1, s_2 be non-negative integers ≥ 1 , and define $r = r_1 r_2$ and $s = s_1 s_2$. Assume that for i = 1, 2 we have $\chi\left(KG^{r_i}\binom{[n]}{k}_{s_i\text{-stab}}^{\sim}\right) = \left\lceil \frac{n-(k-1)r_i}{r_i-1} \right\rceil$ for all integers n and k such that $n \geq r_i k$.

Then we have $\chi\left(KG^r\binom{[n]}{k}_{s-\text{stab}}^{\sim}\right) = \left\lceil \frac{n-(k-1)r}{r-1} \right\rceil$ for all integers n and k such that $n \ge rk$.

Proof. Let $n \ge (t-1)(r-1) + rk$. We have to prove that $\chi\left(KG^r\binom{[n]}{k}_{s-\text{stab}}\right) > t$. For a contradiction, assume that $KG^r\binom{[n]}{k}_{s-\text{stab}}$ is properly colored with $C \leq t$ colors. For $S \in V(n, k, p)$, we denote by c(S) its color. We wish to prove that there are S_1, \ldots, S_r disjoint elements of V(n, k, s) with $c(S_1) = \ldots = c(S_r).$

Take $A \in V(n, n_1, s_1)$, where $n_1 := r_1 k + (t-1)(r_1-1)$. Denote $a_1 < ... < a_{n_1}$ the elements of A and define $h: V(n_1, k, s_2) \to [t]$ as follows: let $B \in V(n_1, k, s_2)$; the k-subset $S = \{a_i : i \in B\} \subseteq [n]$ is an element of V(n,k,s), and gets as such a color c(S); define h(B) to be this c(S). Since $n_1 = r_1 k + (t-1)(r_1-1)$, there are B_1, \ldots, B_{r_1} disjoint elements of $V(n_1, k, s_2)$ having the same color by h. Define h(A) to be this common color.

Make the same definition for all $A \in V(n, n_1, s_1)$. The map \tilde{h} is a coloring of $KG^{r_2}\binom{[n]}{n_1}_{s_1\text{-stab}}^{\sim}$ with t colors. Now, note that

$$(t-1)(r-1) + rk = (t-1)(r_1r_2 - r_2 + r_2 - 1) + r_1r_2k = (t-1)(r_2 - 1) + r_2((t-1)(r_1 - 1) + r_1k)$$

and thus that $n \ge (t-1)(r_2 - 1) + r_2n_1$. Hence, there are A_1, \ldots, A_{r_2} disjoint elements of $V(n, n_1, s_1)$

with the same color. Each of the A_i gets its color from r_1 disjoint elements of V(n, k, s), whence

Proof of Corollary 1. Direct consequence of Theorem 1 and Lemma 2.

there are r_1r_2 disjoint elements of V(n, k, s) having the same color by the map c.

4. Short combinatorial proof of Schrijver's theorem

Recall that Schrijver's theorem is

Theorem 2. Let
$$n \ge 2k$$
. $\chi \left(KG \binom{[n]}{k}_{2-\text{stab}} \right) = n - 2k + 2$.

When specialized for p = 2, Theorem 1 does not imply Schrijver's theorem since the vertex set is allowed to contain subsets with 1 and n together. Anyway, by a slight modification of the proof, we can get a short combinatorial proof of Schrijver's theorem. Alternative proofs of this kind – but not that short - have been proposed in [10, 13]

For a positive integer n, we write $\{+, -, 0\}^n$ for the set of all signed subsets of [n], that is, the family of all pairs (X^+, X^-) of disjoint subsets of [n]. Indeed, for $X \in \{+, -, 0\}^n$, we can define $X^+ := \{i \in [n] : X_i = +\}$ and analogously X^- .

We define $X \subseteq Y$ if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$.

By alt(X) we denote the length of the longest alternating subsequence of non-zero signs in X. For instance: alt(+0 - - + 0 -) = 4, while alt(- - + - + 0 + -) = 5.

The proof makes use of the following well-known lemma see [8, 12, 13] (which is a special case of Lemma 1 for p = 2).

Lemma 3 (Tucker's lemma). Let $\lambda : \{-,0,+\}^n \setminus \{(0,0,\ldots,0)\} \rightarrow \{-1,+1,\ldots,-n,+n\}$ be a map such that $\lambda(-X) = -\lambda(X)$. Then there exist A, B in $\{-,0,+\}^n$ such that $A \subseteq B$ and $\lambda(A) = -\lambda(B)$.

Proof of Schrijver's theorem. The inequality $\chi\left(KG^2\binom{[n]}{k}_{2-\text{stab}}\right) \leq n-2k+2$ is easy to prove (with an explicit coloring) and well-known. So, to obtain a combinatorial proof, it is sufficient to prove the reverse inequality.

Let us assume that there is a proper coloring c of $KG^2\binom{[n]}{k}_{2-\text{stab}}$ with n-2k+1 colors. We define the following map λ on $\{-, 0, +\}^n \setminus \{(0, 0, \dots, 0)\}$.

- if $\operatorname{alt}(X) \leq 2k 1$, we define $\lambda(X) = \pm \operatorname{alt}(X)$, where the sign is determined by the first sign of the longest alternating subsequence of X (which is actually the first non zero term of X).
- if $\operatorname{alt}(X) \ge 2k$, then X^+ and X^- both contain a stable subset of [n] of size k. Among all stable subsets of size k included in X^- and X^+ , select the one having the smallest color. Call it S. Then define $\lambda(X) = \pm (c(S) + 2k 1)$ where the sign indicates which of X^- or X^+ the subset S has been taken from. Note that $c(S) \le n 2k$.

The fact that for any $X \in \{-, 0, +\}^n \setminus \{(0, 0, \dots, 0)\}$ we have $\lambda(-X) = -\lambda(X)$ is obvious. λ takes its values in $\{-1, +1, \dots, -n, +n\}$. Now let us take A and B as in Tucker's lemma, with $A \subseteq B$ and $\lambda(A) = -\lambda(B)$. We cannot have $\operatorname{alt}(A) \leq 2k - 1$ since otherwise we will have a longest alternating in B containg the one of A, of same length but with a different sign. Hence $\operatorname{alt}(A) \geq 2k$. Assume w.l.o.g. that $\lambda(A)$ is defined by a stable subset $S_A \subseteq A^-$. Then the stable subset S_B defining $\lambda(B)$ is such that $S_B \subseteq B^+$, which implies that $S_A \cap S_B = \emptyset$. We have moreover $c(S_A) = |\lambda(A)| = |\lambda(B)| = c(S_B)$, but this contradicts the fact that c is proper coloring of $KG^2\binom{[n]}{k}_{2-\operatorname{stab}}$.

5. Concluding Remarks

We have seen that one of the main ingredients is the notion of alternating sequence of elements in Z_p . Here, our notion only requires that such an alternating sequence must have $x_i \neq x_{i+1}$. To prove Conjecture 1, we need probably something stronger. For example, a sequence is said to be alternating if any p consecutive terms are all distinct. Anyway, all our attempts to get something through this approach have failed.

Recall that Alon, Drewnowski and Łucsak [1] proved Conjecture 1 when r is a power of 2. With the help of a computer and lpsolve, we check that Conjecture 1 is moreover true for

- $n \le 9, k = 2, r = 3.$
- $n \le 12, k = 3, r = 3.$
- $n \le 14, k = 4, r = 3.$
- $n \le 13, k = 2, r = 5.$
- $n \le 16, k = 3, r = 5.$
- $n \le 21, k = 4, r = 5.$

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