# Edge-distance-regular graphs

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**Abstract**

Edge-distance-regularity is a concept recently introduced by the authors which is similar to that of distance-regularity, but now the graph is seen from each of its edges instead of from its vertices. More precisely, a graph $\Gamma$ with adjacency matrix $\boldsymbol{A}$ is edge-distance-regular when it is distance-regular around each of its edges and with the same intersection numbers for any edge taken as a root. In this paper we study this concept, give some of its properties, such as the regularity of $\Gamma$, and derive some characterizations. In particular, it is shown that a graph is edge-distance-regular if and only if its $k$-incidence matrix is a polynomial of degree $k$ in $\boldsymbol{A}$ multiplied by the (standard) incidence matrix. Also, the analogue of the spectral excess theorem for distance-regular graphs is proved, so giving a quasi-spectral characterization of edge-distance-regularity. Finally, it is shown that every nonbipartite graph which is both distance-regular and edge-distance-regular is a generalized odd graph.

*Keywords:* distance-regularity, local spectra, predistance polynomials.

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# 1 Introduction

Given a set of vertices of a simple connected graph $\Gamma = (V, E)$, $C$, with eccentricity $\varepsilon_C$, consider the partition of $V$ given by the distance to $C$: $V = C_0 \cup C_1 \cup \cdots \cup C_{\varepsilon_C}$, where $C_k = \{i \in V \mid \partial(i, C) = k\}$. We say that $\Gamma$ is *$C$-local pseudo-distance-regular* whenever this partition of the vertex set is pseudo-regular, that is, when the numbers

$$c_k(i) = \frac{1}{\nu_i} \sum_{j \in \Gamma(i) \cap C_{k-1}} \nu_j, \quad a_k(i) = \frac{1}{\nu_i} \sum_{j \in \Gamma(i) \cap C_k} \nu_j, \quad b_k(i) = \frac{1}{\nu_i} \sum_{j \in \Gamma(i) \cap C_{k+1}} \nu_j,$$

being $\nu_i$ the $i$-th component of the unique positive eigenvector of the adjacency matrix of $\Gamma$ with mininum component equal to one, $\boldsymbol{\nu}$, do not depend on the chosen vertex $i \in C_k$, but only on the value of $k$. If it is the case, we denote them simply by $c_k$, $a_k$ and $b_k$ and call them the *pseudo-intersection numbers*. When the considered graph $\Gamma$ is regular, these parameters coincide with the usual intersection numbers and in this case $\Gamma$ is $C$-local pseudo-distance-regular if and only if $C$ is a completely regular code. Notice that when a graph is $\{i\}$-local pseudo-distance regular for every vertex $i$ and with the same intersection numbers, it is distance-regular. By considering edges as sets of two vertices, we can also see the graph from a global point of view.

**Definition 1.1** A graph $\Gamma$ is edge-distance-regular when it is $e$-local pseudo-distance-regular with intersection numbers not depending on $e \in E$.

Several quasi-spectral characterizations are known for local pseudo-distance-regularity, most of them obtained through predistance polynomials [2,3]. In this paper we develop the study of edge-distance-regularity and prove similar results to those known for (vertex) distance-regularity.

# 2 Notation and preliminaires

Let $\Gamma$ be a graph with adjacency matrix $\boldsymbol{A}$. Its spectrum is denoted by $\operatorname{sp}\Gamma = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \ldots, \lambda_d^{m(\lambda_0)}\}$, where the eigenvalues are listed in decreasing order and $m(\lambda_l)$ is the multiplicity of $\lambda_l$ as an eigenvalue of $\boldsymbol{A}$. Let $\operatorname{ev}\Gamma$ for the set of different eigenvalues of $\Gamma$. The principal idempotents of $\boldsymbol{A}$ are denoted by $\boldsymbol{E}_l$, $l = 0, 1, \ldots, d$. The Perron-Frobenius Theorem ensures that $m(\lambda_0) = 1$ and guaranties the existence of a positive eigenvector $\boldsymbol{\nu} \in \ker(\boldsymbol{A} - \lambda_0 \boldsymbol{I})$ with minimum component equal to one. Given a nonempty set $C$ of vertices of $\Gamma$, we consider the map $\boldsymbol{\rho} : \mathcal{P}(V) \to \mathcal{V}$ defined by $\boldsymbol{\rho}\emptyset = \boldsymbol{0}$ and $\boldsymbol{\rho}C = \sum_{i \in C} \nu_i \boldsymbol{e}_i$

for $C \neq \emptyset$ and denote by $\boldsymbol{e}_C$ the normalized of the vector $\boldsymbol{\rho}C$. If $\boldsymbol{e}_C = \boldsymbol{z}_C(\lambda_0) + \boldsymbol{z}_C(\lambda_1) + \cdots + \boldsymbol{z}_C(\lambda_d)$ is the spectral decomposition of $\boldsymbol{e}_C$, that is $z_C(\lambda_l) = \boldsymbol{E}_l \boldsymbol{e}_C$, the $C$-*multiplicity* of the eigenvalue $\lambda_l$ is defined by $m_C(\lambda_l) = \|\boldsymbol{z}_C(\lambda_l)\|^2$. We denote by $\operatorname{ev}_C \Gamma = \{\mu_0, \mu_1, \ldots, \mu_{d_C}\}$ the set of different eigenvalues with non-zero $C$-multiplicity and write $\operatorname{sp}_C \Gamma = \{\mu_0^{m_C(\mu_0)}, \mu_1^{m_C(\mu_1)}, \ldots, \mu_{d_C}^{m_C(\mu_{d_C})}\}$ for the $C$-*spectrum* of $\Gamma$. Analogous to the relation between the diameter of a graph and its number of different eigenvalues, the eccentricity of $C$ is bounded by $\varepsilon_C \leq d_C$, and when equality is attained we say that $C$ is an *extremal* set. If $C$ is a single vertex $u$, the $u$-local multiplicities coincide with the diagonal entries of the idempotents, $m_i(\lambda_l) = (\boldsymbol{E}_l)_{uu}$. By analogy, for every pair of vertices $u, v \in V$, the $uv$-crossed multiplicity of $\lambda$ is $m_{uv}(\lambda_l) = (\boldsymbol{E}_l)_{uv}$.

Let $\mathcal{M} = \{\lambda_0 > \lambda_1 > \ldots > \lambda_d\}$ be a mesh of real numbers and $g : \mathcal{M} \to \mathbb{R}$ a weight function defined on it . In $\mathbb{R}[x]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the polynomial $Z(x) = \prod_{l=0}^{d}(x - \lambda_l)$, we define the scalar product associate to $(\mathcal{M}, g)$ by $\langle p, q \rangle := \sum_{l=0}^{d} g(\lambda_l) p(\mu_l) q(\mu_l)$. The canonical orthogonal system associated $(\mathcal{M}, g)$ is the unique family of polynomials $\{p_k\}_{0 \leq k \leq d}$ with $\deg p_k = k$ and $\|p_k\|^2 = p_k(\lambda_0)$. See [1] for a comprehensive study of this family. The $C$-local predistance polynomials $\{p_k^C\}_{0 \leq k \leq d_C}$ are the canonincal orthogonal system associated to the mesh $\operatorname{ev}_C \Gamma$, with weight function $m_C : \operatorname{ev}_C \Gamma \to \mathbb{R}$ given by the $C$-multiplicities. Similarly, the predistance polynomials $\{p_k\}_{0 \leq k \leq d}$ are the canonical orthogonal system associated to $\operatorname{ev} \Gamma$ and weight function given by $g(\lambda_l) = m(\lambda_l)/n$.

## 3 Edge-spectrum regularity

Formally, we do not distinguish between an edge $e \in E$ with vertices $u, v$ and the set $\{u, v\}$. Thus, we denote the (*local*) $e$-*multiplicities* of $\Gamma$ as $m_e(\lambda_i) = \|\boldsymbol{E}_i \boldsymbol{e}_e\|^2$, $i = 0, 1, \ldots, d$, where $\boldsymbol{e}_e = \frac{\boldsymbol{\rho}_e}{\|\boldsymbol{\rho}_e\|} = \frac{\nu_u \boldsymbol{e}_u + \nu_v \boldsymbol{e}_v}{\sqrt{\nu_u^2 + \nu_v^2}}$. From this, note that the relationship between the $e$-multiplicity and the local and crossed multiplicities of $u$ and $v$ is $m_e(\lambda_i) = \frac{1}{\nu_u^2 + \nu_v^2}(\nu_u^2 m_u(\lambda_i) + 2\nu_u \nu_v m_{uv}(\lambda_i) + \nu_v^2 m_v(\lambda_i))$.

If $|\operatorname{ev}_e \Gamma| = d_e + 1$, the eccentricity of $e$, seen as a set of two vertices, satisfies $\varepsilon_e \leq d_e$. We define the *edge-diameter* of $\Gamma$ by $\tilde{D} = \max_{e \in E} \varepsilon_e$. Notice that $\tilde{D}$ coincides with the diameter of the line graph $L\Gamma$ of $\Gamma$. Consequently, if $\Gamma$ have diameter $D$ we have $D - 1 \leq \tilde{D} \leq D$ and, if $\Gamma$ is bipartite, $\tilde{D} = D - 1$.

**Lemma 3.1** *The $e$-multiplicities of a graph $\Gamma = (V, E)$ with spectrum* $\operatorname{sp} \Gamma$ *satisfy the following properties:*

(a) $\displaystyle\sum_{i=0}^{d} m_e(\lambda_i) = 1$ *for every* $e \in E$.

(b) *If* $\Gamma$ *is regular, then* $\displaystyle\sum_{e\in E} m_e(\lambda_i) = \frac{\lambda_0+\lambda_i}{2} m(\lambda_i)$ *for every* $\lambda_i \in \operatorname{ev}\Gamma$.

For every eigenvalue $\lambda_i \in \operatorname{ev}\Gamma$, the *mean vertex-multiplicity* and *mean edge-multiplicity* are, respectively,

$$g(\lambda_i) = \frac{1}{|V|}\sum_{u\in V} m_u(\lambda_i) = \frac{m(\lambda_i)}{|V|}, \qquad \tilde{g}(\lambda_i) = \frac{1}{|E|}\sum_{e\in E} m_e(\lambda_i).$$

Inspired by the concept of (vertex) spectrum-regularity, we say that $\Gamma$ is *edge-spectrum-regular* if, for every $\lambda_i \in \operatorname{ev}\Gamma$, the edge-multiplicity $m_e(\lambda_i)$ does not depend on $e \in E$. Whereas spectrum-regularity implies regularity, in the case of edge-spectrum-regularity we have the following result.

**Proposition 3.2** *Let* $\Gamma$ *be a connected edge-spectrum-regular graph. Then,* $\Gamma$ *is either regular or bipartite biregular.*

We say that a graph $\Gamma$ is *bispectrum-regular* when it is both spectrum-regular and edge-spectrum-regular. This is the case, for instance, when $\Gamma$ is distance-regular. More generally, we have the following result.

**Proposition 3.3** $\Gamma$ *is bispectrum-regular if and only if it is 1-walk-regular.*

## 4 Edge-distance-regularity

Given a graph $\Gamma = (V, E)$ and an edge $e \in E$, consider the partition of $V$ induced by the distance from $e$, that is $V = e_0 \cup e_1 \cup \cdots \cup e_{\varepsilon_e}$, where $e_k = \Gamma_k(e)$. We say that $\Gamma$ is *e-local pseudo-distance-regular* if this partition is pseudo-regular. One of the advantages of considering edges is that we can see the graph from a global point of view, that is, from every edge, in the same way as we get distance-regularity by seeing the graph from every vertex.

**Definition 4.1** A graph $\Gamma$ is edge-distance-regular when it is $e$-local pseudo-distance-regular with intersection numbers not depending on $e \in E$.

**Proposition 4.2** *Let* $\Gamma$ *be an edge-distance-regular graph with diameter* $D$ *and* $d+1$ *distinct eigenvalues. Then,* $\Gamma$ *is regular and*

(a) $\Gamma$ *has spectrally maximum diameter* ($D = d$) *and its edge-diameter satisfies* $\tilde{D} = D$ *if* $\Gamma$ *is nonbipartite and* $\tilde{D} = D - 1$ *otherwise.*

(*b*) *$\Gamma$ is edge-spectrum regular and, for every $e \in E$, the e-spectrum satisfies:*

(*b*1) *If $\Gamma$ is nonbipartite, $\operatorname{ev}_e \Gamma = \operatorname{ev} \Gamma$ and $m_e(\lambda_i) = \left(1 + \frac{\lambda_i}{\lambda_0}\right) \frac{m(\lambda_i)}{|V|}$, $\lambda_i \in \operatorname{ev} \Gamma$.*

(*b*2) *If $\Gamma$ is bipartite, $\operatorname{ev}_e \Gamma = \operatorname{ev} \Gamma \setminus \{-\lambda_0\}$ and $m_e(\lambda_i) = \left(1 + \frac{\lambda_i}{\lambda_0}\right) \frac{m(\lambda_i)}{|V|}$, $\lambda_i \in \operatorname{ev} \Gamma \setminus \{-\lambda_0\}$.*

**Definition 4.3** The *$k$-incidence matrix* of $\Gamma = (V, E)$ is the $(|V| \times |E|)$-matrix $\boldsymbol{B}_k = (b_{ue})$ with entries $b_{ue} = 1$ if $\partial(u, e) = k$, and $b_{ue} = 0$ otherwise.

**Theorem 4.4** *A regular graph $\Gamma$ with edge-diameter $\tilde{D}$ is edge-distance-regular if and only if, for every $k = 0, 1, \ldots, \tilde{D}$, there exists a polynomial $\tilde{p}_k$ of degree $k$ such that $\tilde{p}_k(\boldsymbol{A})\boldsymbol{B}_0 = \boldsymbol{B}_k$.*

Godsil and Shawe-Taylor [4] defined a distance-regularised graph as that being distance-regular around each of its vertices (these graphs are a common generalisation of distance-regular graphs and generalised polygons.) They showed that distance-regularised graphs are either distance-regular or distance-biregular. Inspired by this, we introduce the following concept.

**Definition 4.5** A regular graph $\Gamma$ is said to be edge-distance-regularised when it is edge-distance-regular around each of its edges.

Let $\operatorname{ev}_E \Gamma = \bigcup_{e \in E} \operatorname{ev}_e \Gamma$ and denote by $\operatorname{ev}_E^\star \Gamma = \operatorname{ev}_E \Gamma \setminus \{\lambda_0\}$ and $\tilde{d} = |\operatorname{ev}_E^\star \Gamma|$. If $\Gamma$ is edge-distance-regular, Proposition 4.2 establishes that $\operatorname{ev}_E \Gamma = \operatorname{ev} \Gamma$ if $\Gamma$ is nonbipartite, and $\operatorname{ev}_E \Gamma = \operatorname{ev} \Gamma \setminus \{\lambda_0\}$ otherwise. Consider the canonical orthogonal system $\{\tilde{p}_k\}_{0 \leq k \leq \tilde{d}}$ associated to $(\operatorname{ev}_E \Gamma, \tilde{g})$, and their sum polynomials $\{\tilde{q}_k\}_{0 \leq k \leq \tilde{d}}$ defined by $\tilde{q}_k = \tilde{p}_0 + \tilde{p}_1 + \cdots + \tilde{p}_k$.

**Theorem 4.6** *Let $\Gamma = (V, E)$ be a regular graph with $\tilde{d} = |\operatorname{ev}_E \Gamma|$. Let $H_{\tilde{d}-1}$ be the harmonic mean of the numbers $|N_{\tilde{d}-1}(e)|$ for $e \in E$. Then, $\Gamma$ is edge-distance-regularised if and only if $H_{\tilde{d}-1} = 2\tilde{q}_{\tilde{d}-1}(\lambda_0)$.*

**Corollary 4.7** *Let $\Gamma = (V, E)$ be a regular graph with $\tilde{d} = |\operatorname{ev}_E \Gamma|$. Let $M_{\tilde{d}}$ be the (arithmetic) mean of the numbers $|e_{\tilde{d}}|$ for $e \in E$. Then, $\Gamma$ is edge-distance-regularised if and only if $M_{\tilde{d}} = 2\tilde{p}_{\tilde{d}}(\lambda_0)$.*

As a consequence we have the following theorem, which can be seen as an analogue for the Spectral Excess Theorem for (vertex) distance-reguarity [3].

**Theorem 4.8** *A regular graph $\Gamma = (V, E)$ with $\tilde{d} = |\operatorname{ev}_E \Gamma|$ is edge-distance-regular if and only if, for every edge $e \in E$, $|e_{\tilde{d}}| = 2\tilde{p}_{\tilde{d}}(\lambda_0)$.*

Remark that, as proved in [1], we can specify the value of $\tilde{p}_{\tilde{d}}$ in terms of the edge spectrum. In what follows, $\widehat{\pi}_0$, $\pi_i$ and $\overline{\pi}_i$, $0 \leq i \leq d$, are moment-like parameters computed from the spectrum.

**Theorem 4.9** *Let $\Gamma = (V, E)$ be a regular graph with $d+1$ distinct eigenvalues, and spectrally maximum edge-diameter $\tilde{D} = \tilde{d}$. Then, $\Gamma$ is edge-distance-regular if and only if, for every edge $e \in E$, $|e_{\tilde{D}}| = \frac{4|E|}{\overline{\pi}_0^2} \left( \sum_{i=0}^{d} \frac{\lambda_0+\lambda_i}{m(\lambda_i)\overline{\pi}_i^2} \right)^{-1}$.*

**Proposition 4.10** *Let $\Gamma$ be a $\lambda_0$-regular graph with edge-diameter $\tilde{D} = |\operatorname{ev}^\star \Gamma| = d$. Assume that, for every vertex $u \in V$ and every edge $e \in E$,*

$$\frac{|e_d|}{|u_d|} = \frac{\widehat{\pi}_0}{\pi_0} \frac{|V|}{(-1)^d \, p_d(-\lambda_0)},$$

*where $p_d$ is the $d$-th predistance polynomial of $\Gamma$. Then, $\Gamma$ is edge-distance-regular if and only if it is distance-regular.*

Van Damm and Haemers [5] showed that any connected regular graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ is a generalized odd graph. We show that the same result holds when $\Gamma$ is both distance-regular and edge-distance-regular.

**Proposition 4.11** *Let $\Gamma$ be a distance-regular graph with intersection numbers $c_k, a_k, b_k$, $0 \le k \le d$. Suppose that $a_d \neq 0$. Then, $\Gamma$ is edge-distance-regular if and only if it is a generalized odd graph.*

Remark that a nonbipartite graph that is both distance-regular and edge-distance-regular has intersection number $a_d > 0$.

## References


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