

# A characterization of $Q$ -polynomial association schemes

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## Abstract

We prove a necessary and sufficient condition for a symmetric association scheme to be a  $Q$ -polynomial scheme.

**Key words:**  $Q$ -polynomial association scheme,  $s$ -distance set.

## 1 Introduction

A *symmetric association scheme* of class  $d$  is a pair  $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ , where  $X$  is a finite set and each  $R_i$  is a nonempty subset of  $X \times X$  satisfying the following:

- (1)  $R_0 = \{(x, x) \mid x \in X\}$ ,
- (2)  $X \times X = \bigcup_{i=0}^d R_i$  and  $R_i \cap R_j$  is empty if  $i \neq j$ ,
- (3)  ${}^t R_i = R_i$  for any  $i \in \{0, 1, \dots, d\}$ , where  ${}^t R_i = \{(y, x) \mid (x, y) \in R_i\}$ ,
- (4) for all  $i, j, k \in \{0, 1, \dots, d\}$ , there exist integers  $p_{ij}^k$  such that for all  $x, y \in X$  with  $(x, y) \in R_k$ ,

$$p_{ij}^k = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|.$$

The integers  $p_{ij}^k$  are called the *intersection numbers*.

Let  $\mathfrak{X}$  be a symmetric association scheme. The  $i$ -th *adjacency matrix*  $A_i$  of  $\mathfrak{X}$  is the matrix with rows and columns indexed by  $X$  such that the  $(x, y)$ -entry is 1 if  $(x, y) \in R_i$  or 0 otherwise. The *Bose–Mesner algebra* of  $\mathfrak{X}$  is the algebra generated by the adjacency matrices  $\{A_i\}_{i=0}^d$  over the complex field  $\mathbb{C}$ . Then  $\{A_i\}_{i=0}^d$  is a natural basis of the Bose–Mesner algebra. By [2, page 59], the Bose–Mesner algebra has a second basis  $\{E_i\}_{i=0}^d$  such that

- (1)  $E_0 = |X|^{-1}J$ , where  $J$  is the all-ones matrix,
- (2)  $I = \sum_{i=0}^d E_i$ , where  $I$  is the identity matrix,
- (3)  $E_i E_j = \delta_{ij} E_i$ , where  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  if  $i \neq j$ .

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The basis  $\{E_i\}_{i=0}^d$  is called the *primitive idempotents* of  $\mathfrak{X}$ . We have the following equations:

$$A_i = \sum_{j=0}^d p_i(j) E_j, \quad (1.1)$$

$$E_i = \frac{1}{|X|} \sum_{j=0}^d q_i(j) A_j, \quad (1.2)$$

$$A_i A_j = \sum_{k=0}^d p_{ij}^k A_k, \quad (1.3)$$

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^d q_{ij}^k E_k, \quad (1.4)$$

where  $\circ$  denotes the *Hadamard product*, that is, the entry-wise matrix product. The matrices  $P = (p_j(i))_{i,j=0}^d$  and  $Q = (q_j(i))_{i,j=0}^d$  are called the first and second *eigenmatrices*, respectively. The numbers  $q_{ij}^k$  are called the *Krein parameters*. The Krein parameters are nonnegative real numbers (the Krein condition) [11] [2, page 69].

A symmetric association scheme is called a *P-polynomial scheme* (or a *metric scheme*) with respect to the ordering  $\{A_i\}_{i=0}^d$  if for each  $i \in \{0, 1, \dots, d\}$ , there exists a polynomial  $v_i$  of degree  $i$  such that  $p_i(j) = v_i(p_1(j))$  for any  $j \in \{0, 1, \dots, d\}$ . We say a symmetric association scheme is a *P-polynomial scheme* with respect to  $A_1$  if it has the *P-polynomial property* with respect to some ordering  $A_0, A_1, A_{i_2}, A_{i_3}, \dots, A_{i_d}$ . Dually a symmetric association scheme is called a *Q-polynomial scheme* (or a *cometric scheme*) with respect to the ordering  $\{E_i\}_{i=0}^d$  if for each  $i \in \{0, 1, \dots, d\}$ , there exists a polynomial  $v_i^*$  of degree  $i$  such that  $q_i(j) = v_i^*(q_1(j))$  for any  $j \in \{0, 1, \dots, d\}$ . Moreover a symmetric association scheme is called a *Q-polynomial scheme* with respect to  $E_1$  if it has the *Q-polynomial property* with respect to some ordering  $E_0, E_1, E_{i_2}, E_{i_3}, \dots, E_{i_d}$ . Note that both  $\{v_i\}_{i=0}^d$  and  $\{v_i^*\}_{i=0}^d$  form systems of orthogonal polynomials.

Throughout this paper, we use the notation  $m_i = q_i(0)$  and  $\theta_i^* = q_1(i)$  for  $0 \leq i \leq d$ . If an association scheme is *Q-polynomial*, then  $\{\theta_i^*\}_{i=0}^d$  are mutually distinct because the second eigenmatrix  $Q = (v_i^*(\theta_j^*))_{j,i=0}^d$  is non-singular. For a univariate polynomial  $f$  and a matrix  $M$ , we denote by  $f(M^\circ)$  the matrix obtained by substituting  $M$  into  $f$  with multiplication the Hadamard product. We introduce known equivalent conditions of the *Q-polynomial property* of symmetric association schemes [2, page 193]. The following are equivalent:

- (1)  $\mathfrak{X}$  is a *Q-polynomial scheme* with respect to the ordering  $\{E_i\}_{i=0}^d$ .
- (2)  $(q_{1,i}^j)_{i,j=0}^d$  is an irreducible tridiagonal matrix.
- (3) For each  $i \in \{0, 1, \dots, d\}$ , there exists a polynomial  $f_i$  of degree  $i$  such that  $E_i = f_i(E_1^\circ)$ .

In the present paper, we prove a new necessary and sufficient condition for a symmetric association scheme to be *Q-polynomial*. Since the *Q-polynomial property* of a symmetric association scheme of class 1 is trivial, we assume that  $d$  is greater than 1.

**Theorem 1.1.** *Let  $\mathfrak{X}$  be a symmetric association scheme of class  $d \geq 2$ . Suppose that  $\{\theta_j^*\}_{j=0}^d$  are mutually distinct. Then the following are equivalent:*

- (1)  $\mathfrak{X}$  is a  $Q$ -polynomial scheme with respect to  $E_1$ .
- (2) There exists  $l \in \{2, 3, \dots, d\}$  such that for any  $i \in \{1, 2, \dots, d\}$ ,

$$\prod_{\substack{j=1 \\ j \neq i}}^d \frac{\theta_0^* - \theta_j^*}{\theta_i^* - \theta_j^*} = -p_i(l).$$

Moreover if (2) holds, then  $l = i_d$ .

**Remark 1.2.** We call a finite set  $X$  in  $\mathbb{R}^m$  a  $d$ -distance set if the number of the Euclidean distances between distinct two points in  $X$  is equal to  $d$ . Larman–Rogers–Seidel [7] proved that if the size of a two-distance set with the distances  $a, b$  ( $a < b$ ) is greater than  $2m + 3$ , then there exists a positive integer  $k$  such that  $a^2/b^2 = (k-1)/k$ , i.e.  $k = b^2/(b^2 - a^2)$ . Bannai–Bannai [1] proved that the ratio  $k$  of the spherical embedding of a primitive association scheme of class 2 coincides with  $-p_i(2)$ . The research of the present paper is motivated by [1]. For a symmetric association scheme satisfying that  $\{\theta_j^*\}_{j=0}^d$  are mutually distinct, the values  $K_i := \prod_{j=1, j \neq i}^d (\theta_0^* - \theta_j^*)(\theta_i^* - \theta_j^*)^{-1}$  ( $1 \leq i \leq d$ ) are the generalized Larman–Rogers–Seidel’s ratios [10] of the spherical embedding of this association scheme with respect to  $E_1$ . Theorem 1.1 is an extension of Bannai–Bannai’s result to  $Q$ -polynomial schemes of any class. Furthermore Theorem 1.1 is a new characterization of the  $Q$ -polynomial property on the spherical embedding of a symmetric association scheme.

At the end of this paper, we give some sufficient conditions for the integrality of  $K_i$ .

## 2 Proof of Theorem 1.1

First we give several lemmas that will be needed to prove Theorem 1.1.

**Lemma 2.1.** *For any mutually distinct real numbers  $\beta_1, \beta_2, \dots, \beta_s$ , the following identity holds.*

$$\sum_{i=1}^s \beta_i^j \prod_{\substack{k=1 \\ k \neq i}}^s \frac{x - \beta_k}{\beta_i - \beta_k} = x^j$$

for any  $j \in \{0, 1, \dots, s-1\}$ , where  $x$  is a variable.

*Proof.* For each  $j \in \{0, 1, \dots, s-1\}$ , the polynomial

$$L_j(x) := \sum_{i=1}^s \beta_i^j \prod_{\substack{k=1 \\ k \neq i}}^s \frac{x - \beta_k}{\beta_i - \beta_k}$$

of degree at most  $s-1$  is known as the interpolation polynomial in the Lagrange form (see [3]). Namely, the property  $L_j(\beta_i) = \beta_i^j$  holds for any  $i \in \{1, 2, \dots, s\}$ . Therefore  $L_j(x) = x^j$ , and the lemma follows.  $\square$

We say  $E_j$  is a *component* of an element  $M$  of the Bose–Mesner algebra if  $E_j M \neq 0$ . Let  $N_h$  denote the set of indices  $j$  such that  $E_j$  is a component of  $E_1^{\circ h}$  but not of  $E_1^{\circ l}$  ( $0 \leq l \leq h-1$ ). Note that  $N_0 = \{0\}$  and  $N_1 = \{1\}$ .

**Lemma 2.2.** *Suppose  $\mathfrak{X}$  is a symmetric association scheme of class  $d \geq 2$ . Then the following are equivalent.*

(1)  $\mathfrak{X}$  is a  $Q$ -polynomial scheme with respect to  $E_1$ .

(2) The cardinality of  $N_d$  is equal to 1.

(3)  $N_d$  is nonempty.

*Proof.* (2)  $\Rightarrow$  (3): Clear.

(1)  $\Rightarrow$  (2): Without loss of generality, we assume that  $\mathfrak{X}$  is a  $Q$ -polynomial scheme with respect to  $\{E_i\}_{i=0}^d$ . By noting that  $\{\theta_i^*\}_{i=0}^d$  are mutually distinct,  $\{E_1^{\circ i}\}_{i=0}^d$  are linearly independent, and a basis of the Bose–Mesner algebra. We have

$$E_i = f_i(E_1^\circ) = \sum_{j=0}^i \alpha_{i,j} E_1^{\circ j},$$

where  $\alpha_{i,j} \in \mathbb{R}$  are the coefficients of a polynomial  $f_i$  of degree  $i$ . The upper triangular matrix  $(\alpha_{i,j})_{i,j=0}^d$  is non-singular because  $\alpha_{i,i} \neq 0$  for each  $i$ . Since the inverse matrix  $(\alpha'_{i,j})_{i,j=0}^d$  of  $(\alpha_{i,j})_{i,j=0}^d$  is also an upper triangular matrix with  $\alpha'_{i,i} \neq 0$  for each  $i$ , we can express

$$E_1^{\circ i} = \sum_{j=0}^i \alpha'_{i,j} E_j.$$

Therefore (2) follows.

(3)  $\Rightarrow$  (1): First we prove that if  $N_i$  is empty for some  $i \in \{1, 2, \dots, d-1\}$ , then  $N_{i+1}$  is also empty. Let  $\mathcal{I} = \cup_{j=0}^{i-1} N_j$ . We consider the expression  $\sum_{j=0}^{i-1} E_1^{\circ j} = \sum_{j \in \mathcal{I}} \beta_j E_j$ . Note that  $\beta_j > 0$  for any  $j \in \mathcal{I}$  by the Krein condition. Then we have

$$E_1 \circ \left( \sum_{h=0}^{i-1} E_1^{\circ h} \right) = \sum_{j \in \mathcal{I}} \beta_j \sum_{k=0}^d q_{1,j}^k E_k = \sum_{k=0}^d \sum_{j \in \mathcal{I}} \beta_j q_{1,j}^k E_k.$$

If  $N_i$  is empty, then

$$q_{1,j}^k = 0 \text{ for any } j \in \mathcal{I} \text{ and any } k \notin \mathcal{I} \quad (2.1)$$

because  $\beta_j > 0$  holds for any  $j \in \mathcal{I}$ . We can express  $E_1^{\circ i} = \sum_{j \in \mathcal{I}} \beta'_j E_j$ , where  $\beta'_j$  are non-negative integers for any  $j \in \mathcal{I}$ . By (2.1) and the equalities

$$E_1^{\circ(i+1)} = E_1 \circ E_1^{\circ i} = E_1 \circ \sum_{j \in \mathcal{I}} \beta'_j E_j = \sum_{k=0}^d \sum_{j \in \mathcal{I}} \beta'_j q_{1,j}^k E_k,$$

we obtain  $\sum_{j \in \mathcal{I}} \beta'_j q_{1,j}^k = 0$  for  $k \notin \mathcal{I}$ . Hence  $N_{i+1}$  is also empty. This means that if  $N_d$  is not empty, then the cardinalities of  $N_h$  is equal to 1 for any  $h \in \{0, 1, \dots, d\}$ . Put  $N_h = \{i_h\}$  and order  $E_0, E_1, E_{i_2}, E_{i_3}, \dots, E_{i_d}$ . Then we can construct polynomials  $f_h$  of degree  $h$  such that  $f_h(E_1^\circ) = E_{i_h}$  for any  $h \in \{0, 1, \dots, d\}$ . Hence (1) follows.  $\square$

Now we prove Theorem 1.1.

*Proof of Theorem 1.1.* (1)  $\Rightarrow$  (2): Without loss of generality, we assume that  $\mathfrak{X}$  is a  $Q$ -polynomial scheme with respect to  $\{E_i\}_{i=0}^d$ . For each  $i \in \{1, 2, \dots, d\}$ , we define the polynomial

$$F_i(t) := \prod_{\substack{j=1 \\ j \neq i}}^d \frac{|X|t - \theta_j^*}{\theta_i^* - \theta_j^*}$$

of degree  $d - 1$ . Set  $M_i = F_i(E_1^\circ)$ . Then  $|X|E_1 = \sum_{j=0}^d \theta_j^* A_j$  yields that the  $(x, y)$ -entries of  $M_i$  are

$$M_i(x, y) = \begin{cases} K_i & \text{if } (x, y) \in R_0, \\ 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise,} \end{cases}$$

where  $K_i := \prod_{j=1, j \neq i}^d (\theta_0^* - \theta_j^*)(\theta_i^* - \theta_j^*)^{-1}$ . Since  $F_i$  is a polynomial of degree  $d - 1$ , the matrix  $M_i$  is a linear combination of  $\{E_i\}_{i=0}^{d-1}$ . This means that  $M_i E_d = 0$ . By (1.1),

$$0 = M_i E_d = (K_i I + A_i) E_d = (K_i + p_i(d)) E_d$$

for any  $i \in \{1, 2, \dots, d\}$ . Therefore the desired result follows.

(2)  $\Rightarrow$  (1): From the equation  $A_i = \sum_{j=0}^d p_i(j) E_j$  and our assumptions, we have

$$A_i E_l = p_i(l) E_l = -K_i E_l.$$

By Lemma 2.1,

$$(|X|E_1)^{\circ j} E_l = ((\theta_0^*)^j I + \sum_{i=1}^d (\theta_i^*)^j A_i) E_l = ((\theta_0^*)^j - \sum_{i=1}^d (\theta_i^*)^j K_i) E_l = 0$$

for any  $j \leq d - 1$ . This means that  $l$  is not an element of  $N_j$  for any  $j \leq d - 1$ . Note that the following equality holds:

$$\prod_{j=1}^d \frac{|X|E_1 - \theta_j^* J}{\theta_0^* - \theta_j^*} = I,$$

where the multiplication is the Hadamard product. Obviously,  $I$  has  $E_l$  as a component. Since  $l \notin N_i$  for any  $i \in \{0, 1, \dots, d - 1\}$ , we have  $l \in N_d$ . By Lemma 2.2, the desired result follows.  $\square$

### 3 Integrality of $K_i$

In this section, we consider when  $K_i = -p_i(d)$  is an integer for each  $i \in \{1, 2, \dots, d\}$  for a  $Q$ -polynomial scheme. The following theorem is important in this section.

**Theorem 3.1** (Suzuki [12]). *Let  $\mathfrak{X}$  with  $m_1 > 2$  be a  $Q$ -polynomial scheme with respect to the ordering  $\{E_i\}_{i=0}^d$ . Suppose  $\mathfrak{X}$  is  $Q$ -polynomial with respect to another ordering. Then the new ordering is one of the following:*

- (1)  $E_0, E_2, E_4, E_6, \dots, E_5, E_3, E_1,$
- (2)  $E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, E_3, E_{d-3}, \dots,$
- (3)  $E_0, E_d, E_2, E_{d-2}, E_4, E_{d-4}, \dots, E_{d-5}, E_5, E_{d-3}, E_3, E_{d-1}, E_1,$
- (4)  $E_0, E_{d-1}, E_2, E_{d-3}, E_4, E_{d-5}, \dots, E_5, E_{d-4}, E_3, E_{d-2}, E_1, E_d,$  or
- (5)  $d = 5$  and  $E_0, E_5, E_3, E_2, E_4, E_1.$

Note that  $Q$ -polynomial schemes with  $m_1 = 2$  are the ordinary  $n$ -gons as distance-regular graphs.

**Proposition 3.2.** *Let  $\mathfrak{X}$  with  $m_1 > 2$  be a  $Q$ -polynomial association scheme with respect to the ordering  $\{E_i\}_{i=0}^d$ . If there exists  $t$  such that  $t \leq d/2$ ,  $t \equiv 1 \pmod{2}$  and  $m_t \neq m_{d-t+1}$ , then  $K_j$  is an integer for any  $j$ .*

*Proof.* Let  $\mathbb{F}$  be the splitting field of the scheme, generated by the entries of the first eigenmatrix  $P$ . Then  $\mathbb{F}$  is a Galois extension of the rational field. Let  $G$  be the Galois group  $\text{Gal}(\mathbb{F}/\mathbb{Q})$ . We consider the action of  $G$  on the primitive idempotents  $E_i$ , where elements of  $G$  are applied entry-wise. Then the action of  $G$  on  $\{E_i\}_{i=0}^d$  is faithful and  $|G| \leq 2$  [9].

Suppose  $K_j$  is not an integer for some  $j$ . Since  $-K_j = p_j(d)$  is an eigenvalue of  $A_j$ ,  $K_j$  is an algebraic integer. By the basic number theory,  $K_j$  is irrational. Therefore  $|G| \neq 1$  and hence  $|G| = 2$ . Let  $\sigma$  be the non-identity element of  $G$ . From the definition of  $K_j$ ,  $E_1$  must have an irrational entry, and  $E_1^\sigma \neq E_1$ . Therefore  $\{E_i^\sigma\}_{i=0}^d$  is another  $Q$ -polynomial ordering with the same polynomials  $f_i$ . Hence  $\{E_i^\sigma\}_{i=0}^d$  coincides with one of (1)–(5) in Theorem 3.1.

For  $d = 2$ , it is known that  $K_i$  is an integer for each  $i = 1, 2$  if  $m_1 \neq m_2$  [1]. For (1) and (2) with  $d > 2$ ,  $(E_1^\sigma)^\sigma \neq E_1$ , this contradicts that  $\sigma^2$  is the identity. Since  $p_j(d)$  is irrational and  $A_j E_d = p_j(d) E_d$ ,  $E_d$  has an irrational entry. Therefore  $E_d^\sigma \neq E_d$ . For (4),  $\sigma$  fixes  $E_d$ , a contradiction. Therefore the ordering  $\{E_i^\sigma\}_{i=0}^d$  coincides with (3) or (5).

Suppose that there exists  $t$  such that  $t \leq d/2$ ,  $t \equiv 1 \pmod{2}$  and  $m_t \neq m_{d-t+1}$ . Since  $E_t \circ I = (m_t/|X|)I$ , we have  $E_t^\sigma \circ I^\sigma = (m_t/|X|)I^\sigma$  and hence  $E_t^\sigma \circ I = (m_t/|X|)I \neq (m_{d-t+1}/|X|)I$ . Therefore  $E_t^\sigma \neq E_{d-t+1}$ . Thus, the ordering  $\{E_i^\sigma\}_{i=0}^d$  does not coincide with (3) for  $d \geq 2$ . If  $d = 5$ , then  $m_1 \neq m_5$  and hence  $E_1^\sigma \neq E_5$ . Therefore  $\{E_i^\sigma\}_{i=0}^5$  does not coincide with (5). Thus the proposition follows.  $\square$

Remark that the known  $Q$ -polynomial schemes with some irrational  $K_i$  and  $d > 2$  are the ordinary  $n$ -gons and the association scheme obtained from the icosahedron [5, 8]. We can give a similar equivalent condition of the  $P$ -polynomial property of symmetric association schemes [6]. Let  $\theta_i = p_1(i)$  for  $0 \leq i \leq d$ .

**Theorem 3.3.** *Let  $\mathfrak{X}$  be a symmetric association scheme of class  $d \geq 2$ . Suppose  $\{\theta_j\}_{j=0}^d$  are mutually distinct. Then the following are equivalent:*

- (1)  $\mathfrak{X}$  is a  $P$ -polynomial association scheme with respect to  $A_1$ .

(2) There exists  $l \in \{2, 3, \dots, d\}$  such that for any  $i \in \{1, 2, \dots, d\}$ ,

$$\prod_{\substack{j=1 \\ j \neq i}}^d \frac{\theta_0 - \theta_j}{\theta_i - \theta_j} = -q_i(l).$$

Moreover if (2) holds, then  $l = i_d$ .

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