# A REGULARITY LEMMA AND TWINS IN WORDS 

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#### Abstract

For a word $S$, let $f(S)$ be the largest integer $m$ such that there are two disjoints identical (scattered) subwords of length $m$. Let $f(n, \Sigma)=$ $\min \{f(S): S$ is of length $n$, over alphabet $\Sigma\}$. Here, it is shown that $$
2 f(n,\{0,1\})=n-o(n)
$$ using the regularity lemma for words. I.e., any binary word of length $n$ can be split into two identical subwords (referred to as twins) and, perhaps, a remaining subword of length $o(n)$. A similar result is proven for $k$ identical subwords of a word over an alphabet with at most $k$ letters.


Keywords: sequence, subword, identical subwords, twins in sequences.

## 1. Introduction

Let $S=s_{1} \ldots s_{n}$ be a word of length $n$, i.e., a sequence $s_{1}, s_{2}, \ldots, s_{n}$. A (scattered) subword of $S$ is a word $S^{\prime}=s_{i_{1}} s_{i_{2}} \ldots s_{i_{s}}$, where $i_{1}<i_{2}<\cdots<i_{s}$. This notion was largely investigated in combinatorics on words and formal languages theory with special attention given to counting subword occurrences, different complexity questions, the problem of reconstructing a word from its subwords (see, e.g., $[5,10,11])$. For a word $S$, let $f(S)$ be the largest integer $m$ such that there are two disjoints identical subwords of $S$, each of length $m$. We call such subwords twins. For example, if $S=s_{1} s_{2} s_{3} s_{4} s_{5} s_{6}=001011$, then $S^{\prime}=s_{1} s_{5}$ and $S_{2}=s_{4} s_{6}$ are two identical subwords equal to 01. The question we are concerned with is "How large could the twins be in any word over a given alphabet?" One of the classical problems related to this question is the problem of finding longest subsequence common to two given sequences, see for example [4, 7, 13]. Indeed, if we split a given word $S$ into two subwords with the same number of elements and find a common to these two subwords word, it would correspond to disjoint identical subwords in $S$. Optimizing over all partitions gives largest twins.

Denoting $\Sigma^{n}$ the set of words of length $n$ over the alphabet $\Sigma$, let

$$
f(n, \Sigma)=\min \left\{f(S): S \in \Sigma^{n}\right\}
$$

Observe first, that $f(n,\{0,1\}) \geq\lfloor(1 / 3) n\rfloor$. Indeed, consider any $S \in \Sigma^{n}$ and split it into consecutive triples. Each triple has either two zeros or two ones, so we can build a subword $S_{1}$ by choosing a repeated element from each triple, and similarly build a subword $S_{2}$ by choosing the second repeated element from each triple. For example, if $S=001101111010$ then there are twins $S_{1}, S_{2}$, each equal to 0110 : $S=001101111010$, here one word is marked bold, and the other marked red.

[^0]In fact, we can find much larger identical subwords in any binary word. Our main result is

Theorem 1. There exists an absolute constant $C$ such that

$$
\left(1-C\left(\frac{\log n}{\log \log n}\right)^{-1 / 4}\right) n \leq 2 f(n,\{0,1\}) \leq n-\log n
$$

In the proof we shall employ a classical density increment argument successfully applied in combinatorics and number theory, see e.g. the survey of Komlós and Simonovits [8] and some important applications [6] and [12]. We first show that we can partition any word $S$ into consecutive factors that look as if they were random in a certain weak sense (we call them $\varepsilon$-regular). These $\varepsilon$-regular words can be partitioned (with the exception of $\varepsilon$ proportion of letters) into two identical subwords. By appending these together for every $\varepsilon$-regular word, we eventually obtain identical subwords of roughly half the length of $S$.

We generalize the notion of two identical subwords in words to a notion of $k$ identical subwords. For a given word $S$, let $f(S, k)$ be the largest $m$ so that $S$ contains $k$ pairwise disjoint identical subwords of length $m$ each. Finally, let

$$
f(n, k, \Sigma)=\min \left\{f(S, k): S \in \Sigma^{n}\right\}
$$

Theorem 2. For any integer $k \geq 2$, and alphabet $\Sigma,|\Sigma| \leq k$,

$$
\left(1-C|\Sigma|\left(\frac{\log n}{\log \log n}\right)^{-1 / 4}\right) n \leq k f(n, k, \Sigma)
$$

In case when $k$ is smaller than the size of the alphabet, we have the following bounds.

Theorem 3. For any integer $k \geq 2$, and alphabet $\Sigma,|\Sigma|>k$,

$$
\left(\frac{k}{|\Sigma|}-C|\Sigma|\left(\frac{\log n}{\log \log n}\right)^{-1 / 4}\right) n \leq k f(n, k, \Sigma) \leq n-\max \{\alpha n, \log n\}
$$

where $\alpha \in[0,1 / k]$ is the solution of the equation $\ell^{-(k-1) \alpha} \alpha^{-k \alpha}(1-k \alpha)^{k \alpha-1}=1$, whenever such solution exists and 0 otherwise.

We shall sometimes refer to two disjoint identical subwords as twins, three disjoint identical subwords as triplets, $k$ disjoint identical subwords as $k$-tuplets. We shall prove the regularity lemma for binary words in Section 2 and will prove the Theorem 1 in Section 3. We shall prove Theorems 2, 3 in Section 4. We shall ignore any divisibility issues as these will not affect our arguments.

## 2. Definitions and Regularity Lemma for Words

First, we shall introduce some notations (for more detail, see for instance [2, 9]). An alphabet $\Sigma$ is a finite non-empty set of symbols called letters. For a (scattered) subword $S^{\prime}=s_{i_{1}} s_{i_{2}} \ldots s_{i_{s}}$, of a word $S$, we call the set $\left\{i_{1}, i_{2}, \ldots, i_{s}\right\}$ a support of $S^{\prime}$ in $S$, and write $\operatorname{supp}\left(S^{\prime}\right)$, so the length of $S^{\prime},\left|S^{\prime}\right|=\left|\operatorname{supp}\left(S^{\prime}\right)\right|$. Denoting $I=\left\{i_{1}, \ldots, i_{s}\right\}$, we write $S^{\prime}=S[I]$. A factor of $S$ is a subword with consecutive elements of $S$, i.e., $s_{i} s_{i+1} \ldots s_{i+m}$, for some $1 \leq i \leq n$ and $0 \leq m \leq n-i$, we denote it $S[i, i+m]$. If $S$ is a word over alphabet $\Sigma$ and $q \in \Sigma$, we denote $|S|_{q}$ the number of elements of $S$ equal to $q$. The density $d_{q}(S)$ is defined to be $|S|_{q} /|S|$.

For two subwords $S^{\prime}$ and $S^{\prime \prime}$ of $S$, we say that $S^{\prime}$ is contained in $S^{\prime \prime}$ if $\operatorname{supp}\left(S^{\prime}\right) \subseteq$ $\operatorname{supp}\left(S^{\prime \prime}\right)$, we also denote by $S^{\prime} \cap S^{\prime \prime}$ a subword of $S, S\left[\operatorname{supp}\left(S^{\prime}\right) \cap \operatorname{supp}\left(S^{\prime \prime}\right)\right]$. If $S=s_{1} \ldots s_{n}$ and $S[1, i]=A, S[i+1, n]=B$, then we write $S=A B$ and call $S$ a concatenation of $A$ and $B$.
Definition 4 ( $\varepsilon$-regular word). Call a word $S$ of length $n$ over an alphabet $\Sigma$ $\varepsilon$-regular if for every $i, \varepsilon n+1 \leq i \leq n-2 \varepsilon n+1$ and every $q \in \Sigma$ it holds that

$$
\begin{equation*}
\left|d_{q}(S)-d_{q}(S[i, i+\varepsilon n-1])\right|<\varepsilon . \tag{1}
\end{equation*}
$$

Notice that in the case $|\Sigma|=|\{0,1\}|=2, d_{0}(S)=1-d_{1}(S)$ and thus $\mid d_{0}(S)-$ $d_{0}(S[i, i+\varepsilon n-1])|<\varepsilon \Longleftrightarrow| d_{1}(S)-d_{1}(S[i, i+\varepsilon n-1]) \mid<\varepsilon$. When $\Sigma=\{0,1\}$, we shall denote $d(S)=d_{1}(S)$.

The notion of $\varepsilon$-regular words resembles the notion of pseudorandom (quasirandom) word, see [3]. However, these two notions are quite different. A word that consists of alternating 0 s and 1 s is $\varepsilon$-regular but not pseudorandom. Also, unlike in the case of stronger notions of pseudorandomness, one can check in a linear time whether a word is $\varepsilon$-regular, cf. [1] in the graph case.
Definition 5. We call $\mathcal{S}:=\left(S_{1}, \ldots, S_{t}\right)$ a partition of $S$ if $S=S_{1} S_{2} \ldots S_{t}$, ( $S$ is concatenation of consecutive $S_{i} s$ ). A partition $\mathcal{S}$ is an $\varepsilon$-regular partition of a word $S \in \Sigma^{n}$ if

$$
\sum_{\substack{i \in[t] \\ \text { not } \varepsilon-\text { regular }}}\left|S_{i}\right| \leq \varepsilon n
$$

i.e., the total length of $\varepsilon$-irregular subwords is at most $\varepsilon n$.

The decomposition lemma we are going to show states the following:
Theorem 6 (Regularity Lemma for Words). For every $\varepsilon>0$ and $t_{0}$ there is an $n_{0}$ and $T_{0}$ such that any word $S \in \Sigma^{n}$, for $n \geq n_{0}$ admits an $\varepsilon$-regular partition of $S$ into $S_{1}, \ldots, S_{t}$ with $t_{0} \leq t \leq T_{0}$. In fact, $T_{0} \leq t_{0} 3^{1 / \varepsilon^{4}}$ and $n_{0}=t_{0} \varepsilon^{-\varepsilon^{-4}}$.

To prove the regularity lemma, we introduce the notion of an index and a refinement and prove a few basic facts.
Definition 7 (Index of a partition). Let $\mathcal{S}:=\left(S_{1}, \ldots, S_{t}\right)$ be a partition of $S \in \Sigma^{n}$ into consecutive factors. We define

$$
\operatorname{ind}(\mathcal{S})=\sum_{q \in \Sigma} \sum_{i \in[t]} d_{q}\left(S_{i}\right)^{2} \frac{\left|S_{i}\right|}{n}
$$

Further, for convenience we set $\operatorname{ind}_{q}(\mathcal{S})=\sum_{i \in[t]} d_{q}\left(S_{i}\right)^{2} \frac{\left|S_{i}\right|}{n}$.
Observe that $\operatorname{ind}(\mathcal{S})$ is bounded by 1 from above.
Definition 8 (Refinement of $\mathcal{S}$ ). Let $\mathcal{S}=\left(S_{1}, \ldots, S_{t}\right)$ and

$$
\mathcal{S}^{\prime}=\left(S_{1,1}^{\prime}, S_{1,2}^{\prime}, \ldots, S_{1, s_{1}}^{\prime}, \quad S_{2,1}^{\prime}, S_{2,2}^{\prime}, \ldots, S_{2, s_{2}}^{\prime}, \quad \ldots, \quad S_{t, 1}^{\prime}, S_{t, 2}^{\prime}, \ldots, S_{t, s_{t}}^{\prime}\right)
$$

be partitions of $S \in \Sigma^{n}$. We say that $\mathcal{S}^{\prime}$ refines $\mathcal{S}$ and write $\mathcal{S}^{\prime} \preccurlyeq \mathcal{S}$, if for every $i=1, \ldots, t, S_{i}=S_{i, 1}^{\prime} S_{i, 2}^{\prime} \cdots S_{i, s_{i}}^{\prime}$.

Lemma 9. Let $\mathcal{S}$ and $\mathcal{S}^{\prime}$ be partitions of $S \in \Sigma^{n}$ If $\mathcal{S}^{\prime} \preccurlyeq \mathcal{S}$ then

$$
\operatorname{ind}\left(\mathcal{S}^{\prime}\right) \geq \operatorname{ind}(\mathcal{S})
$$

Proof. Let $\mathcal{S}=\left(S_{1}, \ldots, S_{t}\right)$ and

$$
\mathcal{S}^{\prime}=\left(S_{1,1}^{\prime}, S_{1,2}^{\prime}, \ldots, S_{1, s_{1}}^{\prime}, \quad S_{2,1}^{\prime}, S_{2,2}^{\prime}, \ldots, S_{2, s_{2}}^{\prime}, \quad \ldots, \quad S_{t, 1}^{\prime}, S_{t, 2}^{\prime}, \ldots, S_{t, s_{t}}^{\prime}\right)
$$

We proceed for each $q \in \Sigma$ as follows:

$$
\begin{aligned}
\operatorname{ind}_{q}\left(\mathcal{S}^{\prime}\right)= & \sum_{S^{\prime} \in \mathcal{S}^{\prime}} d_{q}\left(S^{\prime}\right)^{2} \frac{\left|S^{\prime}\right|}{n} \\
= & \sum_{i=1}^{t} \sum_{j=1}^{s_{i}} d_{q}\left(S_{i, j}^{\prime}\right)^{2} \frac{\left|S_{i, j}^{\prime}\right|}{n} \\
= & \sum_{i=1}^{t} \frac{\left|S_{i}\right|}{n} \sum_{j=1}^{s_{i}} d_{q}\left(S_{i, j}\right)^{2} \frac{\left|S_{i, j}^{\prime}\right|}{\left|S_{i}\right|} \\
& \text { Jensen's inequality } \sum_{i=1}^{t} \frac{\left|S_{i}\right|}{n}\left(\sum_{j=1}^{s_{i}} d_{q}\left(S_{i, j}^{\prime}\right) \frac{\left|S_{i, j}^{\prime}\right|}{\left|S_{i}\right|}\right)^{2} \\
& \geq \sum_{i=1}^{t} \frac{\left|S_{i}\right|}{n}\left(\sum_{j=1}^{s_{i}} \frac{\left|S_{i, j}^{\prime}\right|}{\left|S_{i, j}^{\prime}\right|} \frac{\left|S_{i, j}\right|}{\left|S_{i}\right|}\right)^{2} \\
= & \sum_{i=1}^{t} \frac{\left|S_{i}\right|}{n} d_{q}\left(S_{i}\right)^{2} \\
= & \operatorname{ind}_{q}(\mathcal{S}) .
\end{aligned}
$$

Now, building the sum over all $q \in \Sigma$ yields:

$$
\operatorname{ind}\left(\mathcal{S}^{\prime}\right) \geq \operatorname{ind}(\mathcal{S})
$$

The next lemma shows that if a word $S$ is not $\varepsilon$-regular, then there is a refinement of $(S)$ whose index exceeds the index of $(S)$ by at least $\varepsilon^{3}$.

Lemma 10. Let $S \in \Sigma^{m}$ be an $\varepsilon$-irregular word. Then there is a partition $(A, B, C)$ of $S$ such that $|A|,|B|,|C| \geq \varepsilon m$ and

$$
\begin{equation*}
\operatorname{ind}((A, B, C)) \geq \operatorname{ind}((S))+\varepsilon^{3}=\left(\sum_{q \in \Sigma} d_{q}(S)^{2}\right)+\varepsilon^{3} \tag{2}
\end{equation*}
$$

Proof. Since $S$ is not $\varepsilon$-regular, there exists an element $q \in \Sigma$ and an $i$ with $\varepsilon m+1 \leq$ $i \leq m-2 \varepsilon m+1$ such that $|d-d(S[i, i+\varepsilon m-1])| \geq \varepsilon$, where $d:=d_{q}(S)$ and $d(T):=d_{q}(T)$ for any factor $T$ of $S$. Assume w.l.o.g. that $d-d(S[i, i+\varepsilon m-1]) \geq \varepsilon$ and set $\gamma:=d-d(S[i, i+\varepsilon m-1]), A:=S[1, i-1], B:=S[i, i+\varepsilon m-1]$ and $C:=S[i+\varepsilon m, m], a:=|A|, b:=|B|=\varepsilon m$ and $c:=|C|$.

Observe further that

$$
|S|_{q}=d(A) a+d(B) b+d(C) c=d m, \quad d((A, C))=\frac{d m-(d-\gamma) b}{a+c}, \quad d(B)=d-\gamma
$$

Since $a+c=m-b$ and $\operatorname{ind}_{q}((A, B, C))=\operatorname{ind}_{q}((A, C, B))$,

$$
\begin{aligned}
\operatorname{ind}_{q}((A, B, C)) & \geq d((A, C))^{2} \frac{a+c}{m}+d(B)^{2} \frac{b}{m} \\
& =\left(\frac{d m-(d-\gamma) b}{a+c}\right)^{2} \frac{a+c}{m}+(d-\gamma)^{2} \frac{b}{m} \\
& =\frac{(d m-(d-\gamma) b)^{2}}{(m-b) m}+(d-\gamma)^{2} \frac{b}{m} \\
& =\frac{1}{(m-b) m}\left[d^{2}\left(m^{2}-m b\right)+\gamma^{2}(m b)\right] \\
& =d^{2}+\frac{\gamma^{2} b}{m-b} \geq d^{2}+\frac{\varepsilon^{3} m}{(1-\varepsilon) m} \geq d^{2}+\varepsilon^{3}
\end{aligned}
$$

The case when $d-d(S[i, i+\varepsilon n-1]) \leq-\varepsilon$ works out similarly. Indeed, set $\gamma:=$ $d-d(S[i, i+\varepsilon m-1])$ as before and notice that $|\gamma| \geq \varepsilon$ and all the computations above are exactly the same.

So, $\operatorname{ind}_{q}((A, B, C)) \geq d_{q}^{2}+\varepsilon^{3}$. For all other $q^{\prime} \in \Sigma$, Lemma 9 gives that $\operatorname{ind}_{q^{\prime}}((A, B, C)) \geq \operatorname{ind}_{q^{\prime}}((S))=d_{q^{\prime}}^{2}(S)$. Thus

$$
\operatorname{ind}((A, B, C))=\operatorname{ind}_{q}((A, B, C))+\sum_{q^{\prime} \in \Sigma-\{q\}} \operatorname{ind}_{q^{\prime}}((A, B, C)) \geq \sum_{q^{\prime} \in \Sigma} d_{q^{\prime}}(S)^{2}+\varepsilon^{3}
$$

Finally we are in position to finish the argument.
Proof of the Regularity Lemma for Words. Take $\varepsilon>0$ and $t_{0}$ as given. We will give a bound on $n_{0}$ later. Suppose that we have a word $S \in \Sigma^{n}$. Split it into $t_{0}$ consecutive factors $S_{1}, \ldots, S_{t_{0}}$ of the same length $\frac{n}{t_{0}}$. If $\mathcal{S}:=\left(S_{1}, \ldots, S_{t_{0}}\right)$ is not an $\varepsilon$-regular partition, then let $I \subseteq\left[t_{0}\right]$ be the set of all indices such that, for every $i \in I$, $S_{i}$ is not $\varepsilon$-regular (thus, $\sum_{i \in I}\left|S_{i}\right| \geq \varepsilon n$ ). Then, by Lemma 10, we can refine each $S_{i}, i \in I$, into factors $A_{i}, B_{i}$ and $C_{i}$ such that $\operatorname{ind}\left(\left(A_{i}, B_{i}, C_{i}\right)\right) \geq \sum_{q \in \Sigma} d_{q}\left(S_{i}\right)^{2}+\varepsilon^{3}$ (in the case that (1) is violated for several $q \in \Sigma$, choose an arbitrary such $q$ ). We perform such refinement for each $S_{i}, i \in I$, obtaining a partition $\mathcal{S}^{\prime} \preccurlyeq \mathcal{S}$, noticing that

$$
\begin{aligned}
\operatorname{ind}\left(\mathcal{S}^{\prime}\right)= & \sum_{q \in \Sigma} \sum_{j \in\left[t_{0}\right] \backslash I} d_{q}\left(S_{j}\right)^{2} \frac{\left|S_{j}\right|}{n}+ \\
& \sum_{q \in \Sigma} \sum_{i \in I}\left(d_{q}\left(A_{i}\right)^{2} \frac{\left|A_{i}\right|}{n}+d_{q}\left(B_{i}\right)^{2} \frac{\left|B_{i}\right|}{n}+d_{q}\left(C_{i}\right)^{2} \frac{\left|C_{i}\right|}{n}\right) \\
= & \sum_{q \in \Sigma} \sum_{j \in\left[t_{0}\right] \backslash I} d_{q}\left(S_{j}\right)^{2} \frac{\left|S_{j}\right|}{n}+\sum_{i \in I} \operatorname{ind}\left(\left(A_{i}, B_{i}, C_{i}\right)\right) \frac{\left|S_{i}\right|}{n} \\
& \stackrel{(2)}{\geq} \sum_{q \in \Sigma} \sum_{j \in\left[t_{0}\right] \backslash I} d_{q}\left(S_{j}\right)^{2} \frac{\left|S_{j}\right|}{n}+\sum_{i \in I}\left(\operatorname{ind}((S))+\varepsilon^{3}\right) \frac{\left|S_{i}\right|}{n} \\
= & \operatorname{ind}(\mathcal{S})+\varepsilon^{3} \frac{\sum_{i \in I}}{n} \\
\geq & \operatorname{ind}(\mathcal{S})+\varepsilon^{4} .
\end{aligned}
$$

Thus, $\mathcal{S}^{\prime}$ refines $\mathcal{S}$ and has higher index. If $\mathcal{S}^{\prime}$ is not an $\varepsilon$-regular partition of $S$, then we can repeat the procedure above by refining $\mathcal{S}^{\prime}$ etc. Recall that an index of any partition $\mathcal{S}$ is bounded from above by 1 . Thus, since the increment of the index that we get at each step is at least $\varepsilon^{4}$ and each word in the partition decreases in length by a factor of at most $\varepsilon$ at each step, it follows that we can perform at most $\varepsilon^{-4}$ many steps so that the resulting factors are non-trivial, and therefore we will eventually find an $\varepsilon$-regular partition of $S$. Notice that such a partition consists of at most $3^{1 / \varepsilon^{4}} t_{0}$ words, since at each iteration each of the words is partitioned into at most 3 new ones. Therefore, $T_{0} \leq 3^{1 / \varepsilon^{4}} t_{0}$ and each factor in the partition has length at least $t_{0}^{-1} \varepsilon^{1 / \varepsilon^{4}} n$.

## 3. Proof of Theorem 1.

Before we prove our main theorem about binary words, we show a useful claim about twins in $\varepsilon$-regular words.

Claim 11. If $S$ is an $\varepsilon$-regular word, then $2 f(S) \geq|S|-5 \varepsilon|S|$.
Proof. Let $|S|=m$. We partition $S$ into $t=1 / \varepsilon$ consecutive factors $S_{1}, \ldots, S_{1 / \varepsilon}$, each of length $\varepsilon m$. Since $S$ is $\varepsilon$-regular, $\left|d\left(S_{i}\right)-d(S)\right|<\varepsilon$, for every $i \in\{2, \ldots, 1 / \varepsilon-$ $1\}$. Thus each $S_{i}$ has at least $(d(S)-\varepsilon) \varepsilon m$ occurrences of 1 s and at least $(1-$ $d(S)-\varepsilon) \varepsilon m$ occurrences of 0 s. Let $S_{i}(1)$ be a subword of $S_{i}$ consisting of exactly $(d(S)-\varepsilon) \varepsilon m$ letters 1 and $S_{i}(0)$ be a subword of $S_{i}$ consisting of exactly ( $1-$ $d(S)-\varepsilon) \varepsilon m$ letters 0 . Consider the following two disjoint subwords of $S: A=$ $S_{2}(1) S_{3}(0) S_{4}(1) \cdots S_{t-2}(1)$ and $B=S_{3}(1) S_{4}(0) S_{5}(1) \cdots S_{t-2}(0) S_{t-1}(1)$. When $t$ is odd, $A$ and $B$ are constructed similarly.

We see that $A$ and $B$ together have at least $m-2 \varepsilon^{2} m(1 / \varepsilon-3)-3 \varepsilon m$ elements, where $2 \varepsilon^{2} m(1 / \varepsilon-3)$ is an upper bound on the number of 0 s and 1 s which we had to "throw away" to obtain exactly $(d(S)-\varepsilon) \varepsilon m$ letters 1 and $(1-d(S)-\varepsilon) \varepsilon m$ letters 0 in each $S_{i}, 2 \varepsilon m$ is the number of elements in $S_{1}$ and $S_{t}$, and $\varepsilon m$ is the upper bound on $\left|S_{2}(0)\right|+\left|S_{t-1}(1)\right|$. Thus, $2 f(S) \geq m-5 \varepsilon m$. This concludes the proof of the claim.

Notice that we could slightly improve on $5 \varepsilon m$ above by finding in an already mentioned way twins of size $\varepsilon m / 3$ each in $S_{1}$ and $S_{t}$, but this does not give great improvement.

Proof of Theorem 1. Let $n$ be at least $n_{0}$, which is as asserted by the Regularity Lemma for words for given $\varepsilon>0$ and $t_{0}:=\left\lceil\frac{1}{\varepsilon}\right\rceil$. Furthermore, let $S$ be a binary word of length $n$. Again, Theorem 6 asserts an $\varepsilon$-regular partition of $S$ into $S_{1}, \ldots$, $S_{t}$ with $1 / \varepsilon \leq t \leq T_{0}$. We apply Claim 11 to every $\varepsilon$-regular factor $S_{i}$. Furthermore, since $S_{i} \mathrm{~S}$ appear consecutively in $S$, we can put the twins from each of $S_{i}$ s together obtaining twins for the whole word $S$. This way we see:

$$
2 f(S) \geq \sum_{\substack{i \in[t] \\ S_{i} \text { is } \varepsilon-\text { regular }}}\left(\left|S_{i}\right|-5 \varepsilon\left|S_{i}\right|\right) \geq n-5 \varepsilon n-\varepsilon n=n-6 \varepsilon n
$$

here $\varepsilon n$ corresponds to the total lengths of not $\varepsilon$-regular factors. Choosing $\varepsilon=$ $C\left(\frac{\log n}{\log \log n}\right)^{-1 / 4}$, and an appropriate $C$, we see that $n \geq \varepsilon^{-\varepsilon^{-4}}$. Therefore, by Theorem $\left.62 f(n,\{0,1\}) \geq\left(1-C(\log n)^{-1 / 4}\right)\right) n$.

Next we shall prove the upper bound on $f(n,\{0,1\})$ by constructing a binary word $S$ such that $2 f(S) \leq|S|-\log |S|$. Let $S=S_{k} S_{k-1} \ldots S_{0}$, where $\left|S_{i}\right|=3^{i}, S_{i}$
consists only of 1 s for even $i$, and it consists only of 0 s for odd $i$ s. I.e., $S$ is built of iterated 1- or 0-blocks exponentially decreasing in size. Let $A$ and $B$ be twins in $S$. Assume first that $A$ and $B$ have the same number of elements in $S_{k}$. Since $S_{k}$ has odd number of elements, and $A, B$ restricted to $S^{\prime}=S_{k-1} S_{k-2} \cdots S_{0}$ are twins, by induction we have that $|A|+|B| \leq\left(\left|S_{k}\right|-1\right)+\left(\left|S^{\prime}\right|-\log \left(\left|S^{\prime}\right|\right)\right)=|S|-1-\log \left(\left|S^{\prime}\right|\right) \leq$ $|S|-\log |S|$. That is true since $\left|S_{k}\right|=3^{k},|S|=\left(3^{k+1}-1\right) / 2$.
Now assume, w.l.o.g. that $A$ has more elements in $S_{k}$ than $B$ in $S_{k}$. Then $B$ has no element in $S_{k-1}$. We have that $\left|A \cap S_{k-1}\right| \geq\left|S_{k-1}\right| / 2$, otherwise $|A|+|B| \leq$ $|S|-\left|S_{k-1}\right| / 2 \leq|S|-\log |S|$. So, $s=\left|A \cap S_{k-1}\right| \geq\left|S_{k-1}\right| / 2 \geq 3^{k-1} / 2$, and $s$ elements of $B$ must be in $S_{k-3} \cup S_{k-5} \cdots$. But $\left|S_{k-3}\right|+\left|S_{k-5}\right|+\cdots \leq 3^{k-2} / 2$, a contradiction proving Theorem 1.

Remark 12. One can find words of length $n / 2-o(n)$ as described above by an algorithm with $O\left(\varepsilon^{-4}|Q| n\right)$ steps.

## 4. $k$-TUPLETS OVER ALPHABET OF AT MOST $k$ LETTERS

Proof of Theorem 2. As before, we concentrate first on $\varepsilon$-regular words. Let $S$ be an $\varepsilon$-regular word of length $m$ over alphabet $\Sigma=\{0, \ldots, \ell-1\}$ and recall the assumption $\ell \leq k$. We partition $S$ in $t=1 / \varepsilon$ consecutive factors $S_{1}, \ldots, S_{1 / \varepsilon}$, each of length $\varepsilon m$. Since $S$ is $\varepsilon$-regular, $\left|d_{q}\left(S_{i}\right)-d_{q}(S)\right|<\varepsilon$, for every $i \in\{2, \ldots, 1 / \varepsilon-1\}$, and every $q \in \Sigma$. Thus $S_{i}$ has at least $\left(d_{q}(S)-\varepsilon\right) \varepsilon m$ letters $q$, for each $q \in \Sigma$.

We construct $k$-tuplets $A_{1}, \ldots, A_{k}$ as follows. Each of $A_{j}$ s consists of consecutive blocks, with first block consisting of $\left(d_{0}(S)-\varepsilon\right) \varepsilon m$ letters 0 , followed by a block of $\left(d_{1}(S)-\varepsilon\right) \varepsilon m$ letters $1, \ldots$, followed by a block of $\left(d_{\ell-1}(S)-\varepsilon\right) \varepsilon m$ letters $\ell-1$, followed by a block of $\left(d_{0}(S)-\varepsilon\right) \varepsilon m$ letters 0 , and so on.

Since $k \geq|\Sigma|$, we will use all but at most $\frac{1}{\varepsilon} \varepsilon^{2} m|\Sigma|+(2|\Sigma|) \varepsilon m=3|\Sigma| \varepsilon m$ elements, where the first summand accounts for the number of elements that we did not use when choosing exactly $\left(d_{q}(S)-\varepsilon\right) \varepsilon m$ elements $q$ from each $S_{i}$ and each $q \in \Sigma$ and the second summand for the number of elements in $S_{1}, \ldots, S_{\ell}$, and from $S_{1 / \varepsilon-\ell+1}$, $\ldots, S_{1 / \varepsilon}$.

Below are the examples in the special cases when $|\Sigma|=\ell=k$ and when $|\Sigma|=2$ and $k=4$.
Example 1.

$$
\begin{array}{cc}
A_{1} & =S_{2}(0) S_{3}(1) S_{4}(2) \cdots S_{\ell+1}(\ell-1) S_{\ell+2}(0) S_{\ell+3}(1) \cdots S_{2 \ell+1}(\ell-1) \cdots \\
A_{2} & =S_{3}(0) S_{4}(1) S_{5}(2) \cdots S_{\ell+2}(\ell-1) S_{\ell+3}(0) S_{\ell+4}(1) \cdots S_{2 \ell+2}(\ell-1) \cdots \\
\vdots & \\
A_{i} & =S_{i+1}(0) S_{i+2}(1) S_{i+3}(2) \cdots S_{i+\ell}(\ell-1) S_{i+\ell+1}(0) S_{i+\ell+2}(1) \cdots S_{i+2 \ell}(\ell-1) \cdots \\
\vdots & \\
A_{k} & =r
\end{array}
$$

Example 2.

$$
\begin{aligned}
& A_{1}=S_{2}(0) S_{3}(1) \quad S_{6}(0) S_{7}(1) \cdots \\
& A_{2}=\quad S_{3}(0) S_{4}(1) \quad S_{7}(0) S_{8}(1) \cdots \\
& A_{3}=\quad S_{4}(0) S_{5}(1) \quad S_{8}(0) S_{9}(1) \cdots \\
& A_{4}=\quad S_{5}(0) S_{6}(1) \quad S_{9}(0) S_{10}(1) \cdots
\end{aligned}
$$

Here $S_{i}(j)$ is the block of $\left(d_{j}(S)-\varepsilon\right) \varepsilon m$ letters $j$ taken from $S_{i}$. So, in general, the total number of elements in $A_{1}, \ldots, A_{k}$ is at least $m-3|\Sigma| \varepsilon m$. Thus, $k f(S) \geq$ $m-3|\Sigma| \varepsilon m$.

To provide the lower bound on $f(n, k, \Sigma)$ we proceed as in the proof of Theorem 1 by first finding a regular partition of a given word and then applying the above construction to regular factors with an appropriate choice of $\varepsilon$.

## 5. Large alphabets and small $k$-Tuplets

Proof of Theorem 3. The proof of the lower bound proceeds by considering a scattered word $W$ consisting of the $k$ most frequent letters. Clearly, $|W| \geq \frac{k}{|\Sigma|} n$, which together with Theorem 2 yields the lower bound.

The upper bound we obtain is either immediate from Theorem 1 or from computing the expected number of $k$-tuplets of length $m$ each in a random word of length $n$ over an alphabet $\Sigma$ of size $\ell$. If the expectation if less than 1 , this means that there is a word $S$ with $f(S, k)<m$. Indeed, there are

$$
\frac{1}{k!} \prod_{i=0}^{k-1}\binom{n-i m}{m}
$$

distinct sets of $k$ disjoint subwords each of length $m$ in a word of length $n$. The probability that such a set corresponds to a $k$-tuplet, when each letter is chosen with probability $1 / \ell$ independently, is $\ell^{(1-k) m}$. Thus, the expected number of $k$ tuplets is at most

$$
\ell^{(1-k) m} \prod_{i=0}^{k-1}\binom{n-i m}{m}=\ell^{-(k-1) m} \frac{n!}{(m!)^{k}(n-k m)!} \leq \ell^{-(k-1) m} \frac{n^{n}}{m^{k m}(n-k m)^{n-k m}}
$$

that is, for $m=\alpha n$, is at most

$$
\ell^{-(k-1) \alpha n} \frac{n^{n}}{(\alpha n)^{k \alpha n}(n-k \alpha n)^{n-k \alpha n}}=\left(\ell^{-(k-1) \alpha} \alpha^{-k \alpha}(1-k \alpha)^{k \alpha-1}\right)^{n}
$$

Thus, if $\ell^{-(k-1) \alpha} \alpha^{-k \alpha}(1-k \alpha)^{k \alpha-1}$ is less than 1 then $f(S, k) \leq \alpha n$. In particular, for $k=2$ and $\ell=5$ one can compute that $\alpha<0.49$.

## 6. Concluding Remarks

| $\Sigma \backslash n$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\{0,1\}$ | 2 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 5 | 6 | 6 |
| $\{0,1,2\}$ | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 4 |


| $\Sigma \backslash n$ | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\{0,1\}$ | 7 | 7 | 8 |  |  |  |  |
| $\{0,1,2\}$ | $\leq 5$ | $\leq 6$ | $\leq 6$ | $\leq 7$ | $\leq 7$ | $\leq 8$ | $\leq 8$ |

TABLE 1. Values for small $t$ of $f(t, 2,2)$ and $f(t, 2,3)$.
6.1. Small values of $f(n, k, \Sigma)$. We will slightly abuse notation and denote by $f(n, k, \ell)$ the value of $f(n, k, \Sigma)$ with $|\Sigma|=\ell$. In the introductory section it was
observed that $f(3,2,2)=1$ yielding immediately a weak lower bound on $f(n, 2,2)$ to be $\lfloor n / 3\rfloor$. In general, it holds clearly

$$
f(n, k, \ell) \geq\left\lfloor\frac{n}{m}\right\rfloor f(m, k, \ell)
$$

For example, we determined (Theorem 3) a lower bound on $f(n, 2,3)$ to be $\frac{1}{3} n-o(n)$. We do not know whether it is tight and, more sadly, whether one can achieve it, without $o(n)$ term, by finding a (reasonable) number $t$ such that $f(t, 2,3) \geq \frac{t}{3}$. If one could find such $t$ this would immediately give another proof of $f(n, 2,3) \geq \frac{1}{3} n-$ $t$. However, the smallest value for such possible $t$ could be 21 , which already presents a computationally challenging task. In the tables above we summarize estimates on the values on $f(n, k, \ell)$, which were determined with the help of a computer. Thus, the first "open" case which might improve lower bound on $f(n, 2,3)$ is $f(22,2,3)$.
6.2. Improving the $O\left(|\Sigma|\left(\frac{\log \log n}{\log n}\right)^{1 / 4}\right) n$ term. Further we remark, that a more careful analysis below of the increment argument in the proof of Theorem 6 leads to the bound $T_{0} \leq t_{0} 3^{(-2 \log \varepsilon) / \varepsilon^{3}}$, which in turn improves the bounds in Theorems 1 and 2 to

$$
\left(1-C|\Sigma|\left(\frac{(\log \log n)^{2}}{\log n}\right)^{1 / 3}\right) n \leq k f(n, k, \Sigma)
$$

Recall that in the proof of Theorem 6 we set up an index and refining a corresponding partition each time we increase it by at least $\varepsilon^{4}$. Let's reconsider $j$ th refinement step at which the partition $\mathcal{S}=\left(S_{1}, \ldots, S_{t_{0}}\right)$ is to be refined. Further recall that $I$ consists of the indices $i$ such that $S_{i}$ is not $\varepsilon$-regular. Let $\alpha_{j}$ be such that

$$
\begin{equation*}
\sum_{i \in I}\left|S_{i}\right|=\alpha_{j} n \tag{3}
\end{equation*}
$$

In the original proof we iterate as long as $\alpha_{j} \geq \varepsilon$ holds. And by peforming an iteration step we merely use the fact that $\alpha_{j} \geq \varepsilon$ which leads to $\varepsilon^{4}$ increase of the index during one iteration step. Recall that $\operatorname{ind}(\mathcal{S})$ was defined as follows:

$$
\operatorname{ind}(\mathcal{S})=\sum_{q \in \Sigma} \sum_{j \in[|\mathcal{S}|]} d_{q}\left(S_{j}\right)^{2} \frac{\left|S_{j}\right|}{n}
$$

and for each further refinement $\mathcal{S}^{\prime} \preccurlyeq \mathcal{S}$ it holds:
$\operatorname{ind}(\mathcal{S}) \leq \operatorname{ind}\left(\mathcal{S}^{\prime}\right)=\frac{\left(1-\alpha_{j}\right) n}{n} \operatorname{ind}\left(\mathcal{S}_{1}\right)+\frac{\alpha_{j} n}{n} \operatorname{ind}\left(\mathcal{S}_{2}\right) \leq \sum_{q \in \Sigma} \sum_{j \in[|\mathcal{S}|] \backslash I} d_{q}\left(S_{j}\right)^{2} \frac{\left|S_{j}\right|}{n}+\alpha_{j}$,
where $\mathcal{S}_{1}$ consists of $\varepsilon$-regular words from $\mathcal{S}$ (these words are not partitioned/refined anymore) and $\mathcal{S}_{2}$ consists of not $\varepsilon$-regular words from $\mathcal{S}$ (and their lengths sum up to $\left.\alpha_{j} n\right)$.

Let $\ell$ be the total number of iteration steps until we arrive at an $\varepsilon$-regular partition. Let $\alpha_{1}, \ldots, \alpha_{\ell}$ be the numbers, where $\alpha_{j} n$ is the sum over the lengths of not $\varepsilon$-regular words in the partition at step $j, j \in[\ell]$ (cf.(3)).

By the discussion above

$$
1 \geq \alpha_{1} \geq \alpha_{2} \geq \ldots \geq \alpha_{\ell} \geq \varepsilon
$$

Next, we partition $(\varepsilon, 1]$ into $\log _{2} \frac{1}{\varepsilon}$ consecutive intervals $\left(y_{i+1}, y_{i}\right]$ where $y_{1}=1$ and $y_{i+1}=y_{i} / 2$. We claim that each interval $\left(y_{i+1}, y_{i}\right]$ contains at most $\frac{2}{\varepsilon^{3}} \alpha_{j} \mathrm{~s}$. Indeed, the increase of the index during step $j$ where $\alpha_{j} \in\left(y_{i+1}, y_{i}\right]$ is at least

$$
\alpha_{j} \varepsilon^{3}>y_{i+1} \varepsilon^{3}
$$

Further, let $j^{\prime}$ be the smallest index such that $\alpha_{j^{\prime}} \leq y_{i}$ and $j^{\prime \prime}$ be the largest index such that $\alpha_{j^{\prime \prime}}>y_{i+1}$. Let ind ${ }_{j}$ be the index before the $j$ th refinement step. Then by (4) the following holds for $j^{\prime}+1 \leq j \leq j^{\prime \prime}$ :

$$
\operatorname{ind}_{j^{\prime}+1} \leq \operatorname{ind}_{j} \leq \operatorname{ind}_{j^{\prime \prime}} \leq \operatorname{ind}_{j^{\prime}+1}+y_{i} .
$$

This implies that the number of $\alpha_{j}$ s in the interval $\left(y_{i+1}, y_{i}\right]$ cannot be bigger than

$$
\frac{y_{i}}{y_{i+1} \varepsilon^{3}}=\frac{2}{\varepsilon^{3}}
$$

Thus, we obtain the following upper bound on $\ell$

$$
\ell \leq \frac{2 \log _{2} \frac{1}{\varepsilon}}{\varepsilon^{3}}
$$

which leads to $T_{0} \leq t_{0} 3^{(-2 \log \varepsilon) / \varepsilon^{3}}, n_{0}=t_{0} \varepsilon^{-(2 \log 1 / \varepsilon) / \varepsilon^{3}}$ and thus we can regularize with $\varepsilon=\left(\frac{(\log \log n)^{2}}{\log n}\right)^{1 / 3}$.

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