A REGULARITY LEMMA AND TWINS IN WORDS

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ABSTRACT. For a word S, let f(S) be the largest integer m such that there are two disjoints identical (scattered) subwords of length m. Let $f(n, \Sigma) = \min\{f(S) : S \text{ is of length } n, \text{ over alphabet } \Sigma\}$. Here, it is shown that

$$2f(n, \{0, 1\}) = n - o(n)$$

using the regularity lemma for words. I.e., any binary word of length n can be split into two identical subwords (referred to as twins) and, perhaps, a remaining subword of length o(n). A similar result is proven for k identical subwords of a word over an alphabet with at most k letters.

Keywords: sequence, subword, identical subwords, twins in sequences.

1. INTRODUCTION

Let $S = s_1 \dots s_n$ be a word of length n, i.e., a sequence s_1, s_2, \dots, s_n . A (scattered) subword of S is a word $S' = s_{i_1}s_{i_2}\dots s_{i_s}$, where $i_1 < i_2 < \dots < i_s$. This notion was largely investigated in combinatorics on words and formal languages theory with special attention given to counting subword occurrences, different complexity questions, the problem of reconstructing a word from its subwords (see, e.g., [5, 10, 11]). For a word S, let f(S) be the largest integer m such that there are two disjoints identical subwords of S, each of length m. We call such subwords *twins*. For example, if $S = s_1s_2s_3s_4s_5s_6 = 001011$, then $S' = s_1s_5$ and $S_2 = s_4s_6$ are two identical subwords equal to 01. The question we are concerned with is "How large could the twins be in any word over a given alphabet?" One of the classical problems related to this question is the problem of finding longest subsequence common to two given sequences, see for example [4, 7, 13]. Indeed, if we split a given word S into two subwords word, it would correspond to disjoint identical subwords in S. Optimizing over all partitions gives largest twins.

Denoting Σ^n the set of words of length n over the alphabet Σ , let

$$f(n, \Sigma) = \min\{f(S) : S \in \Sigma^n\}.$$

Observe first, that $f(n, \{0, 1\}) \ge \lfloor (1/3)n \rfloor$. Indeed, consider any $S \in \Sigma^n$ and split it into consecutive triples. Each triple has either two zeros or two ones, so we can build a subword S_1 by choosing a repeated element from each triple, and similarly build a subword S_2 by choosing the second repeated element from each triple. For example, if $S = 001 \ 101 \ 111 \ 010$ then there are twins S_1, S_2 , each equal to $0 \ 1 \ 10 \ S = 001 \ 101 \ 111 \ 010$, here one word is marked bold, and the other marked red.

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In fact, we can find much larger identical subwords in any binary word. Our main result is

Theorem 1. There exists an absolute constant C such that

$$\left(1 - C\left(\frac{\log n}{\log\log n}\right)^{-1/4}\right)n \le 2f(n, \{0, 1\}) \le n - \log n.$$

In the proof we shall employ a classical density increment argument successfully applied in combinatorics and number theory, see e.g. the survey of Komlós and Simonovits [8] and some important applications [6] and [12]. We first show that we can partition any word S into consecutive factors that look as if they were random in a certain weak sense (we call them ε -regular). These ε -regular words can be partitioned (with the exception of ε proportion of letters) into two identical subwords. By appending these together for every ε -regular word, we eventually obtain identical subwords of roughly half the length of S.

We generalize the notion of two identical subwords in words to a notion of k identical subwords. For a given word S, let f(S, k) be the largest m so that S contains k pairwise disjoint identical subwords of length m each. Finally, let

$$f(n,k,\Sigma) = \min\{f(S,k): S \in \Sigma^n\}.$$

Theorem 2. For any integer $k \ge 2$, and alphabet Σ , $|\Sigma| \le k$,

$$\left(1 - C|\Sigma| \left(\frac{\log n}{\log \log n}\right)^{-1/4}\right) n \le k f(n, k, \Sigma).$$

In case when k is smaller than the size of the alphabet, we have the following bounds.

Theorem 3. For any integer $k \ge 2$, and alphabet Σ , $|\Sigma| > k$,

$$\left(\frac{k}{|\Sigma|} - C|\Sigma| \left(\frac{\log n}{\log\log n}\right)^{-1/4}\right) n \le k f(n,k,\Sigma) \le n - \max\{\alpha n, \log n\},$$

where $\alpha \in [0, 1/k]$ is the solution of the equation $\ell^{-(k-1)\alpha} \alpha^{-k\alpha} (1-k\alpha)^{k\alpha-1} = 1$, whenever such solution exists and 0 otherwise.

We shall sometimes refer to two disjoint identical subwords as *twins*, three disjoint identical subwords as *triplets*, k disjoint identical subwords as *k*-tuplets. We shall prove the regularity lemma for binary words in Section 2 and will prove the Theorem 1 in Section 3. We shall prove Theorems 2, 3 in Section 4. We shall ignore any divisibility issues as these will not affect our arguments.

2. Definitions and Regularity Lemma for Words

First, we shall introduce some notations (for more detail, see for instance [2, 9]). An alphabet Σ is a finite non-empty set of symbols called *letters*. For a (scattered) subword $S' = s_{i_1}s_{i_2}\ldots s_{i_s}$, of a word S, we call the set $\{i_1, i_2, \ldots, i_s\}$ a support of S' in S, and write $\sup(S')$, so the length of S', $|S'| = |\sup(S')|$. Denoting $I = \{i_1, \ldots, i_s\}$, we write S' = S[I]. A factor of S is a subword with consecutive elements of S, i.e., $s_i s_{i+1} \ldots s_{i+m}$, for some $1 \le i \le n$ and $0 \le m \le n-i$, we denote it S[i, i+m]. If S is a word over alphabet Σ and $q \in \Sigma$, we denote $|S|_q$ the number of elements of S equal to q. The density $d_q(S)$ is defined to be $|S|_q/|S|$. For two subwords S' and S'' of S, we say that S' is contained in S'' if $\operatorname{supp}(S') \subseteq \operatorname{supp}(S'')$, we also denote by $S' \cap S''$ a subword of S, $S[\operatorname{supp}(S') \cap \operatorname{supp}(S'')]$. If $S = s_1 \dots s_n$ and S[1, i] = A, S[i + 1, n] = B, then we write S = AB and call S a concatenation of A and B.

Definition 4 (ε -regular word). Call a word S of length n over an alphabet Σ ε -regular if for every i, $\varepsilon n + 1 \le i \le n - 2\varepsilon n + 1$ and every $q \in \Sigma$ it holds that

$$|d_q(S) - d_q(S[i, i + \varepsilon n - 1])| < \varepsilon.$$
(1)

Notice that in the case $|\Sigma| = |\{0,1\}| = 2$, $d_0(S) = 1 - d_1(S)$ and thus $|d_0(S) - d_0(S[i, i + \varepsilon n - 1])| < \varepsilon \iff |d_1(S) - d_1(S[i, i + \varepsilon n - 1])| < \varepsilon$. When $\Sigma = \{0, 1\}$, we shall denote $d(S) = d_1(S)$.

The notion of ε -regular words resembles the notion of pseudorandom (quasirandom) word, see [3]. However, these two notions are quite different. A word that consists of alternating 0s and 1s is ε -regular but not pseudorandom. Also, unlike in the case of stronger notions of pseudorandomness, one can check in a linear time whether a word is ε -regular, cf. [1] in the graph case.

Definition 5. We call $S := (S_1, \ldots, S_t)$ a partition of S if $S = S_1 S_2 \ldots S_t$, (S is concatenation of consecutive $S_i s$). A partition S is an ε -regular partition of a word $S \in \Sigma^n$ if

$$\sum_{\substack{i \in [t] \\ S_i \text{ is not } \varepsilon - \text{regular}}} |S_i| \le \varepsilon n,$$

i.e., the total length of ε -irregular subwords is at most εn .

The decomposition lemma we are going to show states the following:

Theorem 6 (Regularity Lemma for Words). For every $\varepsilon > 0$ and t_0 there is an n_0 and T_0 such that any word $S \in \Sigma^n$, for $n \ge n_0$ admits an ε -regular partition of S into S_1, \ldots, S_t with $t_0 \le t \le T_0$. In fact, $T_0 \le t_0 3^{1/\varepsilon^4}$ and $n_0 = t_0 \varepsilon^{-\varepsilon^{-4}}$.

To prove the regularity lemma, we introduce the notion of an index and a refinement and prove a few basic facts.

Definition 7 (Index of a partition). Let $S := (S_1, \ldots, S_t)$ be a partition of $S \in \Sigma^n$ into consecutive factors. We define

$$\operatorname{ind}(\mathcal{S}) = \sum_{q \in \Sigma} \sum_{i \in [t]} d_q(S_i)^2 \frac{|S_i|}{n}.$$

Further, for convenience we set $\operatorname{ind}_q(\mathcal{S}) = \sum_{i \in [t]} d_q(S_i)^2 \frac{|S_i|}{n}$.

Observe that $\operatorname{ind}(\mathcal{S})$ is bounded by 1 from above.

Definition 8 (Refinement of S). Let $S = (S_1, \ldots, S_t)$ and

$$\mathcal{S}' = (S'_{1,1}, S'_{1,2}, \dots, S'_{1,s_1}, S'_{2,1}, S'_{2,2}, \dots, S'_{2,s_2}, \dots, S'_{t,1}, S'_{t,2}, \dots, S'_{t,s_t})$$

be partitions of $S \in \Sigma^n$. We say that S' refines S and write $S' \preccurlyeq S$, if for every $i = 1, \ldots, t, S_i = S'_{i,1}S'_{i,2}\cdots S'_{i,s_i}$.

Lemma 9. Let S and S' be partitions of $S \in \Sigma^n$ If $S' \preccurlyeq S$ then

$$\operatorname{ind}(\mathcal{S}') \ge \operatorname{ind}(\mathcal{S})$$

Proof. Let $\mathcal{S} = (S_1, \ldots, S_t)$ and

$$\mathcal{S}' = (S'_{1,1}, S'_{1,2}, \dots, S'_{1,s_1}, S'_{2,1}, S'_{2,2}, \dots, S'_{2,s_2}, \dots, S'_{t,1}, S'_{t,2}, \dots, S'_{t,s_t}).$$

We proceed for each $q \in \Sigma$ as follows:

$$\begin{aligned} \operatorname{ind}_{q}(\mathcal{S}') &= \sum_{S' \in \mathcal{S}'} d_{q}(S')^{2} \frac{|S'|}{n} \\ &= \sum_{i=1}^{t} \sum_{j=1}^{s_{i}} d_{q}(S'_{i,j})^{2} \frac{|S'_{i,j}|}{n} \\ &= \sum_{i=1}^{t} \frac{|S_{i}|}{n} \sum_{j=1}^{s_{i}} d_{q}(S_{i,j})^{2} \frac{|S'_{i,j}|}{|S_{i}|} \\ &\operatorname{Jensen's inequality} \sum_{i=1}^{t} \frac{|S_{i}|}{n} \left(\sum_{j=1}^{s_{i}} d_{q}(S'_{i,j}) \frac{|S'_{i,j}|}{|S_{i}|} \right)^{2} \\ &= \sum_{i=1}^{t} \frac{|S_{i}|}{n} \left(\sum_{j=1}^{s_{i}} \frac{|S'_{i,j}|_{q}}{|S'_{i,j}|} \frac{|S_{i,j}|}{|S_{i}|} \right)^{2} \\ &= \sum_{i=1}^{t} \frac{|S_{i}|}{n} d_{q}(S_{i})^{2} \\ &= \operatorname{ind}_{q}(\mathcal{S}). \end{aligned}$$

Now, building the sum over all $q \in \Sigma$ yields:

$$\operatorname{ind}(\mathcal{S}') \ge \operatorname{ind}(\mathcal{S}).$$

The next lemma shows that if a word S is not ε -regular, then there is a refinement of (S) whose index exceeds the index of (S) by at least ε^3 .

Lemma 10. Let $S \in \Sigma^m$ be an ε -irregular word. Then there is a partition (A, B, C) of S such that $|A|, |B|, |C| \ge \varepsilon m$ and

$$\operatorname{ind}((A, B, C)) \ge \operatorname{ind}((S)) + \varepsilon^3 = \left(\sum_{q \in \Sigma} d_q(S)^2\right) + \varepsilon^3.$$
 (2)

Proof. Since S is not ε -regular, there exists an element $q \in \Sigma$ and an i with $\varepsilon m+1 \leq i \leq m-2\varepsilon m+1$ such that $|d-d(S[i,i+\varepsilon m-1])| \geq \varepsilon$, where $d := d_q(S)$ and $d(T) := d_q(T)$ for any factor T of S. Assume w.l.o.g. that $d-d(S[i,i+\varepsilon m-1]) \geq \varepsilon$ and set $\gamma := d - d(S[i,i+\varepsilon m-1])$, A := S[1,i-1], $B := S[i,i+\varepsilon m-1]$ and $C := S[i+\varepsilon m,m]$, a := |A|, $b := |B| = \varepsilon m$ and c := |C|.

Observe further that

$$|S|_q = d(A)a + d(B)b + d(C)c = dm, \quad d((A,C)) = \frac{dm - (d-\gamma)b}{a+c}, \quad d(B) = d - \gamma.$$

Since a + c = m - b and $\operatorname{ind}_q((A, B, C)) = \operatorname{ind}_q((A, C, B))$,

$$\operatorname{ind}_{q}((A, B, C)) \geq d((A, C))^{2} \frac{a+c}{m} + d(B)^{2} \frac{b}{m}$$

$$= \left(\frac{dm - (d-\gamma)b}{a+c}\right)^{2} \frac{a+c}{m} + (d-\gamma)^{2} \frac{b}{m}$$

$$= \frac{(dm - (d-\gamma)b)^{2}}{(m-b)m} + (d-\gamma)^{2} \frac{b}{m}$$

$$= \frac{1}{(m-b)m} \left[d^{2}(m^{2} - mb) + \gamma^{2}(mb)\right]$$

$$= d^{2} + \frac{\gamma^{2}b}{m-b} \geq d^{2} + \frac{\varepsilon^{3}m}{(1-\varepsilon)m} \geq d^{2} + \varepsilon^{3}.$$

The case when $d - d(S[i, i + \varepsilon n - 1]) \leq -\varepsilon$ works out similarly. Indeed, set $\gamma := d - d(S[i, i + \varepsilon m - 1])$ as before and notice that $|\gamma| \geq \varepsilon$ and all the computations above are exactly the same.

So, $\operatorname{ind}_q((A, B, C)) \geq d_q^2 + \varepsilon^3$. For all other $q' \in \Sigma$, Lemma 9 gives that $\operatorname{ind}_{q'}((A, B, C)) \geq \operatorname{ind}_{q'}((S)) = d_{q'}^2(S)$. Thus

$$\operatorname{ind}((A, B, C)) = \operatorname{ind}_q((A, B, C)) + \sum_{q' \in \Sigma - \{q\}} \operatorname{ind}_{q'}((A, B, C)) \ge \sum_{q' \in \Sigma} d_{q'}(S)^2 + \varepsilon^3.$$

Finally we are in position to finish the argument.

Proof of the Regularity Lemma for Words. Take $\varepsilon > 0$ and t_0 as given. We will give a bound on n_0 later. Suppose that we have a word $S \in \Sigma^n$. Split it into t_0 consecutive factors S_1, \ldots, S_{t_0} of the same length $\frac{n}{t_0}$. If $S := (S_1, \ldots, S_{t_0})$ is not an ε -regular partition, then let $I \subseteq [t_0]$ be the set of all indices such that, for every $i \in I$, S_i is not ε -regular (thus, $\sum_{i \in I} |S_i| \ge \varepsilon n$). Then, by Lemma 10, we can refine each $S_i, i \in I$, into factors A_i, B_i and C_i such that $\operatorname{ind}((A_i, B_i, C_i)) \ge \sum_{q \in \Sigma} d_q(S_i)^2 + \varepsilon^3$ (in the case that (1) is violated for several $q \in \Sigma$, choose an arbitrary such q). We perform such refinement for each $S_i, i \in I$, obtaining a partition $S' \preccurlyeq S$, noticing that

$$\operatorname{ind}(\mathcal{S}') = \sum_{q \in \Sigma} \sum_{j \in [t_0] \setminus I} d_q(S_j)^2 \frac{|S_j|}{n} + \sum_{q \in \Sigma} \sum_{i \in I} \left(d_q(A_i)^2 \frac{|A_i|}{n} + d_q(B_i)^2 \frac{|B_i|}{n} + d_q(C_i)^2 \frac{|C_i|}{n} \right)$$
$$= \sum_{q \in \Sigma} \sum_{j \in [t_0] \setminus I} d_q(S_j)^2 \frac{|S_j|}{n} + \sum_{i \in I} \operatorname{ind}((A_i, B_i, C_i)) \frac{|S_i|}{n}$$
$$\stackrel{(2)}{\geq} \sum_{q \in \Sigma} \sum_{j \in [t_0] \setminus I} d_q(S_j)^2 \frac{|S_j|}{n} + \sum_{i \in I} (\operatorname{ind}((S)) + \varepsilon^3) \frac{|S_i|}{n}$$
$$= \operatorname{ind}(\mathcal{S}) + \varepsilon^3 \frac{\sum_{i \in I} |S_i|}{n}$$
$$\geq \operatorname{ind}(\mathcal{S}) + \varepsilon^4.$$

Thus, \mathcal{S}' refines \mathcal{S} and has higher index. If \mathcal{S}' is not an ε -regular partition of S, then we can repeat the procedure above by refining \mathcal{S}' etc. Recall that an index of any partition \mathcal{S} is bounded from above by 1. Thus, since the increment of the index that we get at each step is at least ε^4 and each word in the partition decreases in length by a factor of at most ε at each step, it follows that we can perform at most ε^{-4} many steps so that the resulting factors are non-trivial, and therefore we will eventually find an ε -regular partition of S. Notice that such a partition consists of at most $3^{1/\varepsilon^4}t_0$ words, since at each iteration each of the words is partitioned into at most 3 new ones. Therefore, $T_0 \leq 3^{1/\varepsilon^4}t_0$ and each factor in the partition has length at least $t_0^{-1}\varepsilon^{1/\varepsilon^4}n$.

3. Proof of Theorem 1.

Before we prove our main theorem about binary words, we show a useful claim about twins in ε -regular words.

Claim 11. If S is an ε -regular word, then $2f(S) \ge |S| - 5\varepsilon |S|$.

Proof. Let |S| = m. We partition S into $t = 1/\varepsilon$ consecutive factors $S_1, \ldots, S_{1/\varepsilon}$, each of length εm . Since S is ε -regular, $|d(S_i) - d(S)| < \varepsilon$, for every $i \in \{2, \ldots, 1/\varepsilon - 1\}$. Thus each S_i has at least $(d(S) - \varepsilon)\varepsilon m$ occurrences of 1s and at least $(1 - d(S) - \varepsilon)\varepsilon m$ occurrences of 0s. Let $S_i(1)$ be a subword of S_i consisting of exactly $(d(S) - \varepsilon)\varepsilon m$ letters 1 and $S_i(0)$ be a subword of S_i consisting of exactly $(1 - d(S) - \varepsilon)\varepsilon m$ letters 0. Consider the following two disjoint subwords of S: $A = S_2(1)S_3(0)S_4(1)\cdots S_{t-2}(1)$ and $B = S_3(1)S_4(0)S_5(1)\cdots S_{t-2}(0)S_{t-1}(1)$. When t is odd, A and B are constructed similarly.

We see that A and B together have at least $m - 2\varepsilon^2 m(1/\varepsilon - 3) - 3\varepsilon m$ elements, where $2\varepsilon^2 m(1/\varepsilon - 3)$ is an upper bound on the number of 0s and 1s which we had to "throw away" to obtain *exactly* $(d(S) - \varepsilon)\varepsilon m$ letters 1 and $(1 - d(S) - \varepsilon)\varepsilon m$ letters 0 in each S_i , $2\varepsilon m$ is the number of elements in S_1 and S_t , and εm is the upper bound on $|S_2(0)| + |S_{t-1}(1)|$. Thus, $2f(S) \ge m - 5\varepsilon m$. This concludes the proof of the claim.

Notice that we could slightly improve on $5\varepsilon m$ above by finding in an already mentioned way twins of size $\varepsilon m/3$ each in S_1 and S_t , but this does not give great improvement.

Proof of Theorem 1. Let n be at least n_0 , which is as asserted by the Regularity Lemma for words for given $\varepsilon > 0$ and $t_0 := \lceil \frac{1}{\varepsilon} \rceil$. Furthermore, let S be a binary word of length n. Again, Theorem 6 asserts an ε -regular partition of S into S_1, \ldots, S_t with $1/\varepsilon \le t \le T_0$. We apply Claim 11 to every ε -regular factor S_i . Furthermore, since S_i s appear consecutively in S, we can put the twins from each of S_i s together obtaining twins for the whole word S. This way we see:

$$2f(S) \ge \sum_{\substack{i \in [t] \\ S_i \text{ is } \varepsilon - \text{regular}}} (|S_i| - 5\varepsilon |S_i|) \ge n - 5\varepsilon n - \varepsilon n = n - 6\varepsilon n$$

here εn corresponds to the total lengths of not ε -regular factors. Choosing $\varepsilon = C(\frac{\log n}{\log \log n})^{-1/4}$, and an appropriate C, we see that $n \ge \varepsilon^{-\varepsilon^{-4}}$. Therefore, by Theorem 6 $2f(n, \{0, 1\}) \ge (1 - C(\log n)^{-1/4})n$.

Next we shall prove the upper bound on $f(n, \{0, 1\})$ by constructing a binary word S such that $2f(S) \leq |S| - \log |S|$. Let $S = S_k S_{k-1} \dots S_0$, where $|S_i| = 3^i, S_i$ consists only of 1s for even *i*, and it consists only of 0s for odd *is*. I.e., *S* is built of iterated 1- or 0-blocks exponentially decreasing in size. Let *A* and *B* be twins in *S*. Assume first that *A* and *B* have the same number of elements in S_k . Since S_k has odd number of elements, and *A*, *B* restricted to $S' = S_{k-1}S_{k-2}\cdots S_0$ are twins, by induction we have that $|A|+|B| \leq (|S_k|-1)+(|S'|-\log(|S'|)) = |S|-1-\log(|S'|) \leq |S| - \log |S|$. That is true since $|S_k| = 3^k$, $|S| = (3^{k+1} - 1)/2$.

Now assume, w.l.o.g. that A has more elements in S_k than B in S_k . Then B has no element in S_{k-1} . We have that $|A \cap S_{k-1}| \ge |S_{k-1}|/2$, otherwise $|A| + |B| \le |S| - |S_{k-1}|/2 \le |S| - \log |S|$. So, $s = |A \cap S_{k-1}| \ge |S_{k-1}|/2 \ge 3^{k-1}/2$, and s elements of B must be in $S_{k-3} \cup S_{k-5} \cdots$. But $|S_{k-3}| + |S_{k-5}| + \cdots \le 3^{k-2}/2$, a contradiction proving Theorem 1.

Remark 12. One can find words of length n/2 - o(n) as described above by an algorithm with $O(\varepsilon^{-4}|Q|n)$ steps.

4. k-tuplets over alphabet of at most k letters

Proof of Theorem 2. As before, we concentrate first on ε -regular words. Let S be an ε -regular word of length m over alphabet $\Sigma = \{0, \ldots, \ell - 1\}$ and recall the assumption $\ell \leq k$. We partition S in $t = 1/\varepsilon$ consecutive factors $S_1, \ldots, S_{1/\varepsilon}$, each of length εm . Since S is ε -regular, $|d_q(S_i) - d_q(S)| < \varepsilon$, for every $i \in \{2, \ldots, 1/\varepsilon - 1\}$, and every $q \in \Sigma$. Thus S_i has at least $(d_q(S) - \varepsilon)\varepsilon m$ letters q, for each $q \in \Sigma$.

We construct k-tuplets A_1, \ldots, A_k as follows. Each of A_j s consists of consecutive blocks, with first block consisting of $(d_0(S) - \varepsilon)\varepsilon m$ letters 0, followed by a block of $(d_1(S) - \varepsilon)\varepsilon m$ letters 1, ..., followed by a block of $(d_{\ell-1}(S) - \varepsilon)\varepsilon m$ letters $\ell - 1$, followed by a block of $(d_0(S) - \varepsilon)\varepsilon m$ letters 0, and so on.

Since $k \geq |\Sigma|$, we will use all but at most $\frac{1}{\varepsilon}\varepsilon^2 m |\Sigma| + (2|\Sigma|)\varepsilon m = 3|\Sigma|\varepsilon m$ elements, where the first summand accounts for the number of elements that we did not use when choosing exactly $(d_q(S) - \varepsilon)\varepsilon m$ elements q from each S_i and each $q \in \Sigma$ and the second summand for the number of elements in S_1, \ldots, S_ℓ , and from $S_{1/\varepsilon - \ell + 1}, \ldots, S_{1/\varepsilon}$.

Below are the examples in the special cases when $|\Sigma| = \ell = k$ and when $|\Sigma| = 2$ and k = 4.

Example 1.

$$A_{1} = S_{2}(0)S_{3}(1)S_{4}(2)\cdots S_{\ell+1}(\ell-1)S_{\ell+2}(0)S_{\ell+3}(1)\cdots S_{2\ell+1}(\ell-1)\cdots,$$

$$A_{2} = S_{3}(0)S_{4}(1)S_{5}(2)\cdots S_{\ell+2}(\ell-1)S_{\ell+3}(0)S_{\ell+4}(1)\cdots S_{2\ell+2}(\ell-1)\cdots,$$

$$\vdots$$

$$A_{i} = S_{i+1}(0)S_{i+2}(1)S_{i+3}(2)\cdots S_{i+\ell}(\ell-1)S_{i+\ell+1}(0)S_{i+\ell+2}(1)\cdots S_{i+2\ell}(\ell-1)\cdots,$$

$$\vdots$$

$$A_{k} = S_{\ell+1}(0)S_{\ell+2}(1)S_{\ell+3}(2)\cdots S_{2\ell}(\ell-1)S_{2\ell+1}(0)\cdots S_{3\ell}(\ell-1)\cdots$$
Example 2.
$$A_{i} = S_{2}(0)S_{2}(1) \qquad S_{2}(0)S_{2}(1)\cdots$$

$$A_{1} = S_{2}(0)S_{3}(1) \qquad S_{6}(0)S_{7}(1)\cdots$$

$$A_{2} = S_{3}(0)S_{4}(1) \qquad S_{7}(0)S_{8}(1)\cdots$$

$$A_{3} = S_{4}(0)S_{5}(1) \qquad S_{8}(0)S_{9}(1)\cdots$$

$$A_{4} = S_{5}(0)S_{6}(1) \qquad S_{9}(0)S_{10}(1)\cdots$$

Here $S_i(j)$ is the block of $(d_j(S) - \varepsilon)\varepsilon m$ letters j taken from S_i . So, in general, the total number of elements in A_1, \ldots, A_k is at least $m - 3|\Sigma|\varepsilon m$. Thus, $kf(S) \ge m - 3|\Sigma|\varepsilon m$.

To provide the lower bound on $f(n, k, \Sigma)$ we proceed as in the proof of Theorem 1 by first finding a regular partition of a given word and then applying the above construction to regular factors with an appropriate choice of ε .

5. Large Alphabets and small k-tuplets

Proof of Theorem 3. The proof of the lower bound proceeds by considering a scattered word W consisting of the k most frequent letters. Clearly, $|W| \ge \frac{k}{|\Sigma|}n$, which together with Theorem 2 yields the lower bound.

The upper bound we obtain is either immediate from Theorem 1 or from computing the expected number of k-tuplets of length m each in a random word of length n over an alphabet Σ of size ℓ . If the expectation if less than 1, this means that there is a word S with f(S, k) < m. Indeed, there are

$$\frac{1}{k!} \prod_{i=0}^{k-1} \binom{n-im}{m}$$

distinct sets of k disjoint subwords each of length m in a word of length n. The probability that such a set corresponds to a k-tuplet, when each letter is chosen with probability $1/\ell$ independently, is $\ell^{(1-k)m}$. Thus, the expected number of k-tuplets is at most

$$\ell^{(1-k)m} \prod_{i=0}^{k-1} \binom{n-im}{m} = \ell^{-(k-1)m} \frac{n!}{(m!)^k (n-km)!} \le \ell^{-(k-1)m} \frac{n^n}{m^{km} (n-km)^{n-km}},$$

that is, for $m = \alpha n$, is at most

$$\ell^{-(k-1)\alpha n} \frac{n^n}{(\alpha n)^{k\alpha n} (n-k\alpha n)^{n-k\alpha n}} = \left(\ell^{-(k-1)\alpha} \alpha^{-k\alpha} (1-k\alpha)^{k\alpha - 1}\right)^n.$$

Thus, if $\ell^{-(k-1)\alpha} \alpha^{-k\alpha} (1-k\alpha)^{k\alpha-1}$ is less than 1 then $f(S,k) \leq \alpha n$. In particular, for k = 2 and $\ell = 5$ one can compute that $\alpha < 0.49$.

6. Concluding Remarks

$\Sigma \backslash n$	6	7	8	9	10	11	12	13	14	15	16	17
$\{0,1\}$	2	2	2	3	3	4	4	5	5	5	6	6
$\{0, 1, 2\}$	1	1	2	2	2	3	3	3	4	4	4	4

	$\Sigma \backslash n$	18	19	20	21	22	23	24		
	$\{0,1\}$	7	7	8						
	$\{0, 1, 2\}$	≤ 5	≤ 6	≤ 6	≤ 7	≤ 7	≤ 8	≤ 8		
Т	TABLE 1. Values for small t of $f(t, 2, 2)$ and $f(t, 2, 3)$.									

6.1. Small values of $f(n, k, \Sigma)$. We will slightly abuse notation and denote by $f(n, k, \ell)$ the value of $f(n, k, \Sigma)$ with $|\Sigma| = \ell$. In the introductory section it was

observed that f(3,2,2) = 1 yielding immediately a weak lower bound on f(n,2,2) to be $\lfloor n/3 \rfloor$. In general, it holds clearly

$$f(n,k,\ell) \ge \left\lfloor \frac{n}{m} \right\rfloor f(m,k,\ell).$$

For example, we determined (Theorem 3) a lower bound on f(n, 2, 3) to be $\frac{1}{3}n-o(n)$. We do not know whether it is tight and, more sadly, whether one can achieve it, without o(n) term, by finding a (reasonable) number t such that $f(t, 2, 3) \ge \frac{t}{3}$. If one could find such t this would immediately give another proof of $f(n, 2, 3) \ge \frac{1}{3}n - t$. However, the smallest value for such possible t could be 21, which already presents a computationally challenging task. In the tables above we summarize estimates on the values on $f(n, k, \ell)$, which were determined with the help of a computer. Thus, the first "open" case which might improve lower bound on f(n, 2, 3) is f(22, 2, 3).

6.2. Improving the $O\left(|\Sigma|\left(\frac{\log \log n}{\log n}\right)^{1/4}\right)n$ term. Further we remark, that a more careful analysis below of the increment argument in the proof of Theorem 6 leads to the bound $T_0 \leq t_0 3^{(-2\log \varepsilon)/\varepsilon^3}$, which in turn improves the bounds in Theorems 1 and 2 to

$$\left(1 - C|\Sigma| \left(\frac{(\log \log n)^2}{\log n}\right)^{1/3}\right) n \le k f(n, k, \Sigma).$$

Recall that in the proof of Theorem 6 we set up an index and refining a corresponding partition each time we increase it by at least ε^4 . Let's reconsider *j*th refinement step at which the partition $\mathcal{S} = (S_1, \ldots, S_{t_0})$ is to be refined. Further recall that *I* consists of the indices *i* such that S_i is not ε -regular. Let α_j be such that

$$\sum_{i \in I} |S_i| = \alpha_j n. \tag{3}$$

In the original proof we iterate as long as $\alpha_j \geq \varepsilon$ holds. And by performing an iteration step we merely use the fact that $\alpha_j \geq \varepsilon$ which leads to ε^4 increase of the index during one iteration step. Recall that $\operatorname{ind}(\mathcal{S})$ was defined as follows:

$$\operatorname{ind}(\mathcal{S}) = \sum_{q \in \Sigma} \sum_{j \in [|\mathcal{S}|]} d_q(S_j)^2 \frac{|S_j|}{n},$$

and for each further refinement $\mathcal{S}' \preccurlyeq \mathcal{S}$ it holds:

$$\operatorname{ind}(\mathcal{S}) \le \operatorname{ind}(\mathcal{S}') = \frac{(1 - \alpha_j)n}{n} \operatorname{ind}(\mathcal{S}_1) + \frac{\alpha_j n}{n} \operatorname{ind}(\mathcal{S}_2) \le \sum_{q \in \Sigma} \sum_{j \in [|\mathcal{S}|] \setminus I} d_q(S_j)^2 \frac{|S_j|}{n} + \alpha_j,$$
(4)

where S_1 consists of ε -regular words from S (these words are not partitioned/refined anymore) and S_2 consists of not ε -regular words from S (and their lengths sum up to $\alpha_j n$).

Let ℓ be the total number of iteration steps until we arrive at an ε -regular partition. Let $\alpha_1, \ldots, \alpha_\ell$ be the numbers, where $\alpha_j n$ is the sum over the lengths of not ε -regular words in the partition at step $j, j \in [\ell]$ (cf.(3)).

By the discussion above

$$1 \ge \alpha_1 \ge \alpha_2 \ge \ldots \ge \alpha_\ell \ge \varepsilon.$$

Next, we partition $(\varepsilon, 1]$ into $\log_2 \frac{1}{\varepsilon}$ consecutive intervals $(y_{i+1}, y_i]$ where $y_1 = 1$ and $y_{i+1} = y_i/2$. We claim that each interval $(y_{i+1}, y_i]$ contains at most $\frac{2}{\varepsilon^3} \alpha_j$ s. Indeed, the increase of the index during step j where $\alpha_j \in (y_{i+1}, y_i]$ is at least

$$\alpha_j \varepsilon^3 > y_{i+1} \varepsilon^3.$$

Further, let j' be the smallest index such that $\alpha_{j'} \leq y_i$ and j'' be the largest index such that $\alpha_{j''} > y_{i+1}$. Let ind_j be the index before the *j*th refinement step. Then by (4) the following holds for $j' + 1 \leq j \leq j''$:

$$\operatorname{ind}_{j'+1} \leq \operatorname{ind}_{j} \leq \operatorname{ind}_{j''} \leq \operatorname{ind}_{j'+1} + y_i.$$

This implies that the number of α_i s in the interval $(y_{i+1}, y_i]$ cannot be bigger than

$$\frac{y_i}{y_{i+1}\varepsilon^3} = \frac{2}{\varepsilon^3}$$

Thus, we obtain the following upper bound on ℓ

$$\ell \le \frac{2\log_2 \frac{1}{\varepsilon}}{\varepsilon^3},$$

which leads to $T_0 \leq t_0 3^{(-2\log \varepsilon)/\varepsilon^3}$, $n_0 = t_0 \varepsilon^{-(2\log 1/\varepsilon)/\varepsilon^3}$ and thus we can regularize with $\varepsilon = \left(\frac{(\log \log n)^2}{\log n}\right)^{1/3}$.

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