## Abstract

We prove a formal power series identity, relating the arithmetic sum-of-divisors function to commuting triples of permutations. This establishes a conjecture of Franklin T. Adams-Watters.

## A formal identity involving commuting triples of permutations

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The object of this note is to establish the following formal identity:

$$\prod_{j=1}^{\infty} (1 - u^j)^{-\sigma(j)} = \sum_{n=0}^{\infty} \frac{T(n)}{n!} u^n,$$
 (1)

where  $\sigma$  is the arithmetic sum-of-divisors function, and T(n) is the number of triples of pairwise-commuting elements of the symmetric group  $S_n$ . (Here  $S_0$  is the trivial group.) This is a surprising fact, as there seems no obvious reason for any connection between the function  $\sigma$  and commuting permutations.

The power series expansion of the left-hand side of this identity has coefficients which are listed on the Online Encyclopedia of Integer Sequences (OEIS) [3] as sequence A061256. The coefficients on the right-hand side are listed as sequence A079860. The identity of the two sequences has been stated conjecturally on OEIS. This conjecture (from 2006) is due to Franklin T. Adams-Watters; he informs me that it was based empirically on the numerical evidence.

For a finite group G, we shall write k(G) for the number of conjugacy classes of G. The following simple fact seems first to have been stated by Erdős and Turán [1].

LEMMA 1. The number of pairs of commuting elements of G is |G| k(G).

Let  $g \in G$ . It follows from Lemma 1 that the number of commuting triples of G whose first element is g, is given by  $|\operatorname{Cent}_G(g)| k(\operatorname{Cent}_G(g))$ . So if T(G) is the total number of commuting triples, then

$$\frac{T(G)}{|G|} = \sum_{g \in G} \frac{|\operatorname{Cent}_G(g)|}{|G|} k(\operatorname{Cent}_G(g)) = \sum_{i=1}^r k(\operatorname{Cent}_G(g_i)), \tag{2}$$

where  $\{g_1, \ldots, g_r\}$  is a set of conjugacy class representatives for G.

In the case that G is the symmetric group  $S_n$ , the conjugacy classes are parameterized by partitions of n, whose parts correspond to cycle lengths. Let  $g \in S_n$  have  $m_t$  cycles of length t

for all t. Then the centralizer of g in  $S_n$  is given (up to isomorphism) by

$$\operatorname{Cent}_{S_n}(g) \cong \prod_{t=1}^n W(t, m_t),$$

where W(t,m) is the wreath product  $\mathbf{Z}_t \wr S_m$ . (Here  $\mathbf{Z}_t$  is used as a shorthand for  $\mathbf{Z}/t\mathbf{Z}$ , the integers modulo t.) It follows that

$$k(\operatorname{Cent}_{S_n}(g)) = \prod_{t=1}^n k(W(t, m_t)).$$
(3)

We may regard an element of W(t,m) as a pair (A,e), where  $A \in \mathbf{Z}_t^m$  and  $e \in S_m$ . There is a natural action of  $S_m$  on the coordinates of  $\mathbf{Z}_t^m$  given by  $(B^e)_i = B_{ie^{-1}}$ . The group multiplication \* in W(t,m) is defined by

$$(A, e) * (B, f) = (A + B^e, ef).$$

Conjugacy in groups of the form  $H \wr S_m$  is described in [2, Section 4.2]; the case that  $H = \mathbf{Z}_t$  is relatively straightforward. Let (A, e) be an element of W(t, m), where  $A = (a_1, \ldots, a_m)$ . Let c be a cycle of the permutation e, and let  $\operatorname{supp}(c)$  be the support of c (i.e. the elements of  $\{1, \ldots, m\}$  moved by c). We shall write |c| for  $|\operatorname{supp}(c)|$ , the length of the cycle. Define the cycle  $\operatorname{sum} A[c] \in \mathbf{Z}_t$  by

$$A[c] = \sum_{i \in \text{supp}(c)} a_i.$$

The cycle sum invariant of (A, e) corresponding to the cycle c is defined to be the pair (A[c], |c|). The element (A, e) has one such invariant for each cycle of e.

LEMMA 2. Two elements (A, e) and (B, f) of W(t, m) are conjugate in W(t, m) if and only if they have the same cycle sum invariants—that is, if and only if there is a bijection  $\tau$  between the cycles of e and the cycles of f, such that for any cycle e of e we have  $(A[e], |e|) = (B[e\tau], |e\tau|)$ .

*Proof.* See [2, Theorem 4.2.8], of which this is a particular case. 
$$\Box$$

Let (A, e) be an element of W(t, m). For each  $z \in \mathbf{Z}_t$  we define  $\lambda_z$  to be the partition such that the multiplicity of  $\ell$  as a part of  $\lambda_z$  is equal to the multiplicity of  $(z, \ell)$  as a cycle sum invariant of (A, e). Lemma 2 tells us that the partitions  $\lambda_z$  for  $z \in \mathbf{Z}_t$  determine the conjugacy class of (A, e) in W(t, m). Conversely, a collection of t arbitrary partitions  $\{\lambda_z \mid z \in \mathbf{Z}_t\}$  determines a conjugacy class of W(t, m) if and only if the total sum of the sizes of the partitions  $\lambda_z$  is equal to m.

Let p(d) denote the number of partitions of d, and let P(u) be the power series

$$P(u) = \sum_{d=0}^{\infty} p(d)u^d.$$

Consider the formal series

$$Q(u) = \prod_{t=1}^{\infty} P(u^t)^t.$$

From the discussion above, it is easily seen that each monomial term of degree tm in the expansion of  $P(u^t)^t$  corresponds to a conjugacy class of W(t, m), and that we therefore have

$$P(u^t)^t = \sum_{m=0}^{\infty} k(W(t,m))u^{tm}.$$

Now any single term in the expansion of Q(u) corresponds to a choice, firstly of parameters  $m_t$  such that  $\sum_t t m_t$  is finite, and secondly of a conjugacy class of  $W(t, m_t)$  for each t. It follows from (3) that each term of degree n in this expansion corresponds to a conjugacy class of  $\operatorname{Cent}_{S_n}(g)$ , where g is an element of  $S_n$  with  $m_t$  cycles of length t. Now by (2) we have the formal identity

$$Q(u) = \sum_{n=0}^{\infty} \frac{T(n)}{n!} u^n.$$

Thus Q(u) is equal to the right-hand side of (1), and it remains only to show that Q(u) is also equal to the left-hand side.

We use the Eulerian expansion of P(u),

$$P(u) = \prod_{s=1}^{\infty} (1 - u^s)^{-1}.$$

From this it follows that

$$Q(u) = \prod_{t=1}^{\infty} \prod_{s=1}^{\infty} (1 - u^{st})^{-t} = \prod_{j=1}^{\infty} \prod_{t|j} (1 - u^j)^{-t} = \prod_{j=1}^{\infty} (1 - u^j)^{-\sigma(j)},$$

as required.

Finally, I am indebted to Mark Wildon for the observation that both sides of (1) are convergent in the open unit disc |u| < 1, and that they therefore represent a complex function which is analytic in this disc. This can be seen by expressing the formal logarithm of the left-hand side of (1) as

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\sigma(j)}{k} u^{jk} = \sum_{d=1}^{\infty} \left( \sum_{a|d} \frac{a\sigma(a)}{d} \right) u^d,$$

which has radius of convergence 1, since clearly

$$\sum_{a|d} a\sigma(a) < d^4.$$

Thus the left-hand side of (1) represents an analytic function on the disc |u| < 1, and it follows that the right-hand side is the Taylor series of that function. An immediate consequence of this observation is that the growth of T(n)/n! is subexponential; I do not know of an easy combinatorial proof of this fact.

## References

- P. Erdős and P. Turán, On some problems of a statistical group theory IV, Acta Mathematica Academiae Scientiarum Hungaricae 19 3–4 (1968), 413–435.
- Gordon James and Adalbert Kerber, The representation theory of the symmetric group, Encyclopedia of Mathematics and its Applications 16, Addison-Wesley, 1981.
- 3. The Online Encyclopedia of Integer Sequences, http://oeis.org/.

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