THREE-CLASS ASSOCIATION SCHEMES FROM CYCLOTOMY

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ABSTRACT. We give three constructions of three-class association schemes as fusion schemes of the cyclotomic scheme, two of which are primitive.

1. INTRODUCTION

Association schemes form a central part of algebraic combinatorics, and plays important roles in several branches of mathematics, such as coding theory and graph theory. Two-class symmetric association schemes are equivalent to strongly regular graphs, and are extensively studied. The natural graph theoretical extension of strongly regular graph is distance regular graph, whose distance relations form an association scheme. Distance regular graphs have attracted considerable attention, and important progress has been achieved on this topic. We refer the reader to the book [7] and the undergoing survey [12]. There are not so many papers about three-class association schemes, see the survey [11] by van Dam and the references therein. It is the purpose of this note to present new constructions of primitive three-class association schemes using cyclotomy in finite fields. As consequences, we obtain three new infinite families of three-class association schemes, two of which are primitive.

Quite recently, there have been several constructions of strongly regular graphs with new parameters and skew Hadamard difference sets from cyclotomy, the latter giving rise to two-class nonsymmetric association schemes, see [16, 18, 20, 23] for strongly regular graphs and [9, 17, 18, 24] for skew Hadamard difference sets. In [27], the authors discussed the problem when a Cayley graph on a finite field with a single cyclotomic class as its connection set can form a strongly regular graph. Such a strongly regular graph is called *cyclotomic*. They raised the following conjecture: if the Cayley graph on the finite field \mathbb{F}_q of order $q = p^f$ with a multiplicative subgroup C of index M of \mathbb{F}_q as its connection set is cyclotomic strongly regular, then either of the following holds:

- (1) (subfield case) C is the multiplicative group of a subfield of \mathbb{F}_q ,
- (2) (semi-primitive case) $-1 \in \langle p \rangle \leq \mathbb{Z}_M^*$,
- (3) (exceptional case) it is either of eleven sporadic examples of cyclotomic strongly regular graphs (see [27, Table 1]).

This conjecture is still open but the authors gave a proof in a partial case assuming the generalized Riemann hypothesis. On the other hand, in [16, 18, 20, 24], several of these sporadic examples have been generalized into infinite families by taking a union of several cyclotomic classes and doing detailed computations using Gauss sums. For other constructions of strongly regular graphs from cyclotomy, we refer the reader to the references in [18].

Also, recently, skew Hadamard difference sets are currently under intensive study. There was a major conjecture in this area: Up to equivalence the Paley (quadratic residue) difference sets are the only skew Hadamard difference sets in abelian groups. This conjecture turned out to be false: Ding

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and Yuan [14] gave two counterexamples of this conjecture in finite fields with characteristic three. Furthermore, Muzychuk [26] constructed infinitely many inequivalent skew Hadamard difference sets in elementary abelian groups of order q^3 . Recently, in [9, 17, 18, 24], the authors constructed further counterexamples of this conjecture by taking suitably a union of cyclotomic classes. See the introduction of [17] (or [9]) for a short survey on skew Hadamard difference sets.

Thus, a lot of strongly regular graphs and skew Hadamard difference sets have been obtained from cyclotomy. Therefore, we can say that the cyclotomy is a quite powerful tool to construct two-class association schemes. In this note, we shall try to construct three-class association schemes from cyclotomy involving computations of Gauss sums based on the Hasse-Davenport theorem.

This note is organized as follows: In Section 2, we review about association schemes and characters of finite fields. In Section 3, we introduce a partition of \mathbb{Z}_M , and compute some group ring elements in $\mathbb{Z}[\mathbb{Z}_M]$ based on the results in [1]. In Section 4, we give three constructions of three-class association schemes in finite fields with characteristic 2 as fusion schemes of the cyclotomic schemes. The parameters of association schemes obtained in Section 4 are listed in the appendix. We shall use the standard notations on group rings as can be found in the book [6].

2. Preliminaries

Let X be a nonempty finite set, and a set of symmetric relations R_0, R_1, \dots, R_d be a partition of $X \times X$ such that $R_0 = \{(x, x) | x \in X\}$. Denote by A_i the adjacency matrix of R_i for each *i*, whose (x, y)-th entry is 1 if $(x, y) \in R_i$ and 0 otherwise. We call $(X, \{R_i\}_{i=0}^d)$ a *d*-class association scheme if there exist numbers $p_{i,i}^k$ such that

$$A_i A_j = \sum_{k=0}^d p_{i,j}^k A_k.$$

These numbers are called the intersection numbers of the scheme. The \mathbb{C} -linear span of A_0, A_1, \dots, A_d forms a semisimple algebra of dimension d + 1, called the *Bose-Mesner algebra* of the scheme. With respect to the basis A_0, A_1, \dots, A_d , the matrix of the multiplication by A_i is denoted by B_i , namely

$$A_i(A_0, A_1, \cdots, A_d) = (A_0, A_1, \cdots, A_d)B_i, \ 0 \le i \le d.$$

Since each A_i is symmetric, this algebra is commutative. There exists a set of minimal idempotents E_0, E_1, \dots, E_d which also forms a basis of the algebra. The $(d+1) \times (d+1)$ matrix P such that

$$(A_0, A_1, \cdots, A_d) = (E_0, E_1, \cdots, E_d)P$$

is called the *first eigenmatrix* of the scheme. Dually, the $(d+1) \times (d+1)$ matrix Q such that

$$(E_0, E_1, \cdots, E_d) = \frac{1}{|X|} (A_0, A_1, \cdots, A_d) Q$$

is called the *second eigenmatrix* of the scheme. We clearly have PQ = |X|I.

We call an association scheme $(X, \{R_i\}_{i=0}^d)$ a translation association scheme or a Schur ring if X is a (additively written) finite abelian group and there exists a partition $S_0 = \{0\}, S_1, \dots, S_d$ of X such that

 $R_i = \{ (x, x + y) | x \in X, y \in S_i \}.$

For brevity, we will just say that $(X, \{S_i\}_{i=0}^d)$ is an association scheme.

Assume that $(X, \{S_i\}_{i=0}^d)$ is a translation association scheme. There is an equivalence relation defined on the character group \hat{X} of X as follows: $\chi \sim \chi'$ if and only if $\chi(S_i) = \chi'(S_i)$ for each $0 \leq i \leq d$. Here $\chi(S) = \sum_{g \in S} \chi(g)$, for any $\chi \in \hat{X}$, and $S \subseteq X$. Denote by D_0, D_1, \dots, D_d the equivalence classes, with D_0 consisting of only the principal character. Then $(\hat{X}, \{D_i\}_{i=0}^d)$ forms a translation association scheme, called the *dual* of $(X, \{S_i\}_{i=0}^d)$. The first eigenmatrix of the dual scheme is equal to the second eigenmatrix of the original scheme. Please refer to [4] and [7] for more details.

A classical example of translation schemes is the cyclotomic scheme which we describe now. Let p be a prime and $q = p^f (f \ge 1)$ be a prime power, M|q-1, and γ be a primitive element of the finite field $F = \mathbb{F}_q$. Define the multiplicative subgroup $C_0^{(M,F)} = \langle \gamma^M \rangle$. Its cosets $C_i^{(M,F)} = \gamma^i C_0^{(M,F)}$, $0 \le i \le M-1$, are called the *cyclotomic classes* of order M of F. Together with $\{0\}$, they form an M-class association scheme, which is called the *cyclotomic scheme*. To describe its first eigenmatrix, we define the *Gauss periods*

$$\eta_a = \sum_{x \in C_a^{(M,F)}} \psi(x), \ 0 \le a \le M - 1,$$

where ψ is the canonical additive character of F defined by $\psi(x) = e^{\frac{2\pi i}{p} \operatorname{Tr}(x)}, x \in F$. The first eigenmatrix P of the scheme is

$$P = \begin{pmatrix} 1 & \frac{q-1}{M} & \frac{q-1}{M} & \frac{q-1}{M} & \cdots & \frac{q-1}{M} \\ 1 & \eta_0 & \eta_1 & \eta_2 & \cdots & \eta_{M-1} \\ 1 & \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_0 \\ \vdots & & & & \\ 1 & \eta_{M-1} & \eta_0 & \eta_1 & \cdots & \eta_{M-2} \end{pmatrix}.$$
 (2.1)

For each multiplicative character χ of F_a^* , the multiplicative group of F, we define the Gauss sum

$$G_F(\chi) = \sum_{x \in F^*} \psi(x)\chi(x)$$

The following relation will be repeatedly used in this paper (cf. [22, P. 195]):

$$\psi(x) = \frac{1}{q-1} \sum_{\chi \in \widehat{F^*}} G_F(\chi) \chi^{-1}(x), \ \forall x \in F^*.$$

Then, the Gauss period can be expressed as a linear combination of Gauss sums as follows:

$$\eta_{i} = \psi(\gamma^{i}C_{0}^{(M,F)})$$

$$= \frac{1}{q-1} \sum_{\chi \in \widehat{F^{*}}} G_{F}(\chi)\chi^{-1}(\gamma^{i}) \sum_{x \in C_{0}^{(M,F)}} \chi^{-1}(x)$$

$$= \frac{1}{M} \sum_{i=0}^{M-1} G_{F}(\phi^{-i})\phi(\gamma^{i}),$$

where ϕ is a multiplicative character of order M of F^* .

In this note, we are interested in the fusion schemes of the cyclotomic scheme, namely schemes whose relations are unions of the relations in the cyclotomic scheme. We shall need the following well-known criterion due to Bannai [3] and Muzychuk [25], called the *Bannai-Muzychuk criterion*: Let P be the first eigenmatrix of an association scheme $(X, \{R_i\}_{0 \le i \le d})$, and $\Lambda_0 := \{0\}, \Lambda_1, \ldots, \Lambda_{d'}$ be a partition of $\{0, 1, \ldots, d\}$. Then $(X, \{R_{\Lambda_i}\}_{0 \le i \le d'})$ forms an association scheme if and only if there exists a partition $\{\Delta_i\}_{0 \le i \le d'}$ of $\{0, 1, 2, \ldots, d\}$ with $\Delta_0 = \{0\}$ such that each (Δ_i, Λ_j) -block of P has a constant row sum. Moreover, the constant row sum of the (Δ_i, Λ_j) -block is the (i, j)-th entry of the first eigenmatrix of the fusion scheme.

We close this section by recording the well-known Hasse-Davenport theorem on Gauss sums.

Theorem 1. ([2, Theorem 11.5.2]) Let χ be a nonprincipal multiplicative character of $\mathbb{F}_q = \mathbb{F}_{p^f}$ and let χ' be the lifted character of χ to the extension field $\mathbb{F}_{q'} = \mathbb{F}_{p^{fs}}$, i.e., $\chi'(\alpha) := \chi(\operatorname{Norm}_{\mathbb{F}_{q'}/\mathbb{F}_q}(\alpha))$ for any $\alpha \in \mathbb{F}_{q'}^*$. Then, it holds that

$$G_{\mathbb{F}_{q'}}(\chi') = (-1)^{s-1} (G_{\mathbb{F}_q}(\chi))^s.$$

3. A partition of \mathbb{Z}_M , $M = \frac{2^{3s}-1}{2^s-1}$

Let s be a positive integer, and set $M := \frac{2^{3s}-1}{2^s-1}$. Denote by $F := \mathbb{F}_{2^{3s}}, E := \mathbb{F}_{2^s}$ the finite field with 2^{3s} and 2^s elements respectively. Let

$$D := \{ u \in F^* : \operatorname{Tr}_{F/E}(u^{-1}) = 0 \}.$$
(3.1)

This set D is E^* -invariant, namely $Dg = \{dg : d \in D\} = D$ for any $g \in E^*$. Therefore $\psi(\omega^a D)$ depends only on $a \pmod{M}$. First, we show that $\psi(\omega^a D), 0 \le a \le M-1$, take exactly three values. Since D is a union of E^* cosets, we have

$$\psi(\omega^{a}D) = \frac{1}{2^{s}-1} \sum_{u \in D} \sum_{x \in E^{*}} \psi(x\omega^{a}u)$$

= $\#\{u : u \in D, \operatorname{Tr}_{F/E}(\omega^{a}u) = 0\} - \frac{1}{2^{s}-1} \#\{u : u \in D, \operatorname{Tr}_{F/E}(\omega^{a}u) \neq 0\}$
= $-(2^{s}+1) + \frac{2^{s}}{2^{s}-1} \#\{u : u \in D, \operatorname{Tr}_{F/E}(\omega^{a}u) = 0\}.$

It is clear that $u^{\frac{2^{3s}-1}{2^s-1}} \operatorname{Tr}_{F/E}(u^{-1}) = \operatorname{Tr}_{F/E}(u^{1+2^s})$, so

$$D = \{ u \in F^* : \operatorname{Tr}_{F/E}(u^{1+2^s}) = 0 \}.$$

Since $Q(x) = \text{Tr}_{L/F}(x^{1+2^s})$ is a nondegenerate quadratic form, the corresponding quadric \mathcal{Q} in $PG(2, 2^s)$ intersects $2^s + 1$ lines in 1 point, and $2^{2s-1} + 2^{s-1}$ lines in 2 points [19]. According to [1, p. 328], the tangent lines are given by

$$L_a = \{ [x] : x \in F^*, \operatorname{Tr}_{F/E}(ax) = 0 \}$$

with $\operatorname{Tr}_{F/E}(a) = 0$, $a \neq 0$. Here we use [x] for the projective point corresponding to the 1-dimensional subspace spanned by x for each $x \in F^*$.

Therefore, the set

$$S_a := \{ u : \operatorname{Tr}_{F/E}(u^{1+2^s}) = 0, \operatorname{Tr}_{F/E}(\omega^a u) = 0 \}$$

has size 0, $2^s - 1$ or $2(2^s - 1)$, depending on whether L_{w^a} is a passant line, a tangent line or a secant line. Denote by T_1 (resp. T_2) those a in \mathbb{Z}_M such that S_a has size $2^s - 1$ (resp. $2(2^s - 1)$). Then $|T_1| = 2^s + 1$, $|T_2| = 2^{2s-1} + 2^{s-1}$. Denote by T_3 the remaining elements of \mathbb{Z}_M other than T_1 and T_2 . We have $|T_3| = 2^{2s-1} - 2^{s-1}$. To sum up, we have the following result.

Lemma 2. The sums $\psi(\omega^a D)$, $0 \le a \le M - 1$, take exactly three values, which are

$$\psi(\omega^{a}D) = \begin{cases} -1 & \text{if } a \in T_{1}, \\ 2^{s} - 1 & \text{if } a \in T_{2}, \\ -2^{s} - 1 & \text{if } a \in T_{3}. \end{cases}$$

The sets T_1 , T_2 and T_3 form a partition of \mathbb{Z}_M . We now prove the following lemma, which is essential for our construction.

Lemma 3. With the above notations, we have

$$(T_2 - T_3)T_1^{(-1)} = 2^s T_1, (3.2)$$

$$(T_2 - T_3)T_2^{(-1)} = 2^{2s-1} + 2^{s-1}(\mathbb{Z}_M - T_1), \qquad (3.3)$$

$$(T_2 - T_3)T_3^{(-1)} = -2^{2s-1} + 2^{s-1}(\mathbb{Z}_M - T_1).$$
(3.4)

Proof: First we recall from [1, p. 327] that

$$T_1 = \{i \in \mathbb{Z}_M : \operatorname{Tr}_{F/E}(w^i) = 0\}$$

by examining the tangent lines of the quadric Q. It is clear that T_1 is the classical Singer difference set in \mathbb{Z}_M , so it holds in the group ring $\mathbb{Z}[\mathbb{Z}_M]$ that (see [6])

$$T_1 T_1^{(-1)} = 2^s + \mathbb{Z}_M. \tag{3.5}$$

Moreover, we observe that $\{2i : i \in T_1\}$ is equal to T_1 by the definition of T_1 .

We first show that

$$T_1^2 = T_1 + 2T_2.$$

For any $i, j \in T_1$, the line

$$L_{\omega^{i+j}} := \{ [x] : x \in F^*, \operatorname{Tr}_{F/E}(\omega^{i+j}x) = 0 \}$$

intersects the quadric \mathcal{Q} at the points $[\omega^{-i}]$ and $[\omega^{-j}]$, since the set of equations

$$\operatorname{Tr}_{F/E}(X^{-1}) = 0, \quad \operatorname{Tr}_{F/E}(\omega^{i+j}X) = 0$$

has the solutions $X = \omega^{-i}$, $X = \omega^{-j}$. It follows that if i, j are two distinct elements in \mathbb{Z}_M , then these two points are distinct and $L_{w^{i+j}}$ is a secant line. Together with $\{2i : i \in T_1\} = T_1$, we conclude that in T_1^2 , each element of T_1 has coefficient 1, each element of T_3 has coefficient 0, and each element of T_2 has even coefficient. Now write d_i for the coefficients of i in T_1^2 for each $i \in \mathbb{Z}_M$. By direct computation, we have $(T_1T_1^{(-1)})^2 = 2^{2s} + (2^{2s} + 3 \cdot 2^s + 1)\mathbb{Z}_M$. Examining the coefficient of the identity on both sides of the equation, we get

$$|T_1| \cdot 1^2 + \sum_{i \in T_2} d_i^2 = 2^{2s+1} + 3 \cdot 2^s + 1,$$

which yields $\sum_{i \in T_2} d_i^2 = 4|T_2|$. Also, we have $\sum_{i \in T_2} d_i = |T_1|^2 - |T_1| = 2|T_2|$. Since $d_i/2$ is a nonnegative integer for each $i \in T_2$, and

$$\sum_{i\in T_2} \left(\frac{d_i}{2}\right)^2 = |T_2|, \quad \sum_{i\in T_2} \left(\frac{d_i}{2}\right) = |T_2|,$$

we immediately get $d_i = 2$ for any $i \in T_2$.

We have $T_1 + 2T_2 = \mathbb{Z}_M + (T_2 - T_3), T_1^{(-1)} \mathbb{Z}_M = (2^s + 1)\mathbb{Z}_M$. Multiplying both sides of $T_1^2 = T_1 + 2T_2$ with $T_1^{(-1)}$, we get

$$T_1 \cdot (2^s + \mathbb{Z}_M) = (2^s + 1)\mathbb{Z}_M + (T_2 - T_3)T_1^{(-1)}.$$

The Eqn. (3.2) then follows.

Since $T_1 + T_2 + T_3 = \mathbb{Z}_M$, Eqn. (3.2) yields that $(T_2 - T_3)(\mathbb{Z}_M - T_2 - T_3)^{(-1)} = 2^s T_1$. On the other hand, $(T_2 - T_3)(T_2 - T_3)^{(-1)} = 2^{2s}$ by [1, p. 328]. Combining these equations, we get Eqn. (3.3) and Eqn. (3.4).

Remark 4. We deduce from Eqn. (3.2), Eqn. (3.5) and $T_1 + T_2 + T_3 = \mathbb{Z}_M$ that

$$T_1 T_2^{(-1)} = 2^{s-1} T_1^{(-1)} + 2^{s-1} \mathbb{Z}_M - 2^{s-1},$$
(3.6)

$$T_1 T_3^{(-1)} = -2^{s-1} T_1^{(-1)} + 2^{s-1} \mathbb{Z}_M - 2^{s-1}.$$
(3.7)

The following equations then follow from direct computations:

$$T_1^2 T_1^{(-1)} = 2^s T_1 + (2^s + 1) \mathbb{Z}_M, ag{3.8}$$

$$T_1^2 T_2^{(-1)} = 2^{2s-1} + (2^{s-1} + 2^{2s-1}) \mathbb{Z}_M - 2^{s-1} T_1,$$
(3.9)

$$T_1^2 T_3^{(-1)} = -2^{2s-1} + 2^{2s-1} \mathbb{Z}_M - 2^{s-1} T_1.$$
(3.10)

TABLE 1. The values of $\psi(\omega^a R_k)$'s

	R_0	R_1	R_2	R_3
$\omega^a = 0$	1	$2^{2s} - 1$	$2^{s-1}(2^{2s}-1)$	$2^{s-1}(2^s-1)^2$
a = 0	1	$2^{2s} - 1$	$-2^{s-1}(2^s+1)$	$-2^{s-1}(2^s-1)$
$a \in -T_1$	1	-1	$2^{s-1}(2^s-1)$	$-2^{s-1}(2^s-1)$
$a \not\in -T_1 \cup \{0\}$	1	-1	-2^{s-1}	2^{s-1}

4. Three-Class Association Schemes in $\mathbb{F}_{2^{3s}}$ and Their Extensions to $\mathbb{F}_{2^{6s}}$ and $\mathbb{F}_{2^{9s}}$

We fix the following notations throughout this section: Let s be a positive integer, $M = \frac{2^{3s}-1}{2^s-1}$, and let T_1, T_2, T_3 be as introduced in the previous section. We define

$$H := \mathbb{F}_{2^{9s}}, \ G := \mathbb{F}_{2^{6s}}, \ F := \mathbb{F}_{2^{3s}}, \ E := \mathbb{F}_{2^{s}}.$$

Let $C_i^{(M,F)}$, $C_i^{(M,G)}$, $C_i^{(M,H)}$, $0 \le i \le M - 1$, be the cyclotomic classes of order M in F, G, H respectively. Clearly $C_0^{(M,F)}$ is equal to E^* , the multiplicative group of E. Let ψ , $\psi' \psi''$ be the canonical additive character of H, G and F respectively. Also, write η_a , η'_a , η'_a , $0 \le a \le M - 1$ for their Gauss periods respectively. Fix a primitive element β of H and a primitive element γ of G such that Norm_{$H/F}(<math>\beta$) = Norm_{G/F}(γ), where Norm_{H/F} and Norm_{G/F} is the norm from H to F and from G to F respectively. Write</sub>

$$\omega := \operatorname{Norm}_{H/F}(\beta) = \operatorname{Norm}_{G/F}(\gamma)_{\mathfrak{f}}$$

which is a primitive element of F.

4.1. Imprimitive Association Schemes in $\mathbb{F}_{2^{3s}}$. In this section, we construct an imprimitive three-class association scheme in $\mathbb{F}_{2^{3s}}$. Now we prove the following theorem.

Theorem 5. Take the following partition of F:

$$R_0 = \{0\}, R_1 = \bigcup_{i \in T_1} C_i^{(M,F)}, R_2 = \bigcup_{i \in T_2} C_i^{(M,F)}, R_3 = \bigcup_{i \in T_3} C_i^{(M,F)}.$$

Then, $(F, \{R_i\}_{i=0}^3)$ is a three-class association scheme, whose parameters are listed in the appendix.

Proof of Theorem 5: As before, it is easily verified that $\psi(\omega^a C_0^{(M,F)}) = 2^s - 1$ or -1 according to $\operatorname{Tr}_{F/E}(\omega^a) = 0$ or not, i.e., $a \in T_1$ or $a \notin T_1$. Now, we compute that

$$\psi(\omega^{a}R_{k}) = \sum_{i \in T_{k}} \psi(\omega^{a+i}C_{0}^{(M,F)})$$

= $(2^{s}-1)|T_{1} \cap (a+T_{k})| - |(\mathbb{Z}_{M} \setminus T_{1}) \cap (a+T_{k})|$
= $2^{s}|T_{1} \cap (a+T_{k})| - |T_{k}|.$

The term $|T_1 \cap (a+T_k)|$ is the coefficient of a in the group ring element $T_1T_k^{(-1)}$. We have computed $T_1T_k^{(-1)}$, $1 \le k \le 3$, in Eqn. (3.5)-(3.7). For each k = 1, 2, 3, the sum $\psi(\omega^a R_k)$ is now computed directly and listed in Table 1. By the Bannai-Muzychuk criterion, $(F, \{R_i\}_{i=0}^3)$ is a three-class association scheme.

Remark 6. By the proof of Theorem 5, the dual scheme of the association scheme in Theorem 5 is given by

$$D_0 = \{0\}, \ D_1 = C_0^{(M,F)}, \ D_2 = \bigcup_{i \in -T_1} C_i^{(M,F)}, \ D_3 = \bigcup_{i \in \mathbb{Z}_M \setminus (-T_1 \cup \{0\})} C_i^{(M,F)}.$$

This scheme is imprimitive since $D_0 \cup D_1 = E$. Their character values are listed in Table 2, which we shall need later. Observe that $D_2 = D$.

TABLE 2. The values of $\psi(\omega^a D_k)$'s

	D_0	D_1	D_2	D_3
$\omega^a = 0$	1	$2^{s} - 1$	$2^{2s} - 1$	$2^{3s} - 2^{2s} - 2^s + 1$
$a \in T_1$	1	$2^{s} - 1$	-1	$-2^{s}+1$
$a \in T_2$	1	-1	$2^{s} - 1$	$-2^{s}+1$
$a \in T_3$	1	-1	$-2^{s}-1$	$2^{s} + 1$

4.2. Primitive Association Schemes in $\mathbb{F}_{2^{6s}}$ and $\mathbb{F}_{2^{9s}}$. In this subsection, we construct primitive association schemes in $G = \mathbb{F}_{2^{6s}}$ and $H = \mathbb{F}_{2^{9s}}$.

Theorem 7. (i) Take the following partition of G:

$$R'_0 = \{0\}, \ R'_1 = \bigcup_{i \in T_1} C_i^{(M,G)}, \ R'_2 = \bigcup_{i \in T_2} C_i^{(M,G)}, \ R'_3 = \bigcup_{i \in T_3} C_i^{(M,G)}.$$

Then, $(G, \{R'_i\}_{i=0}^3)$ is a three-class association scheme, whose parameters are listed in the appendix. (ii) Take the following partition of H:

$$R_0'' = \{0\}, \ R_1'' = \bigcup_{i \in T_1} C_i^{(M,H)}, \ R_2'' = \bigcup_{i \in T_2} C_i^{(M,H)}, \ R_3'' = \bigcup_{i \in T_3} C_i^{(M,H)}.$$

Then, $(H, \{R''_i\}_{i=0}^3)$ is a three-class association scheme, whose parameters are listed in the appendix. **Proof of Theorem 7 (i):** For any χ' of G such that $\chi'^M = 1$, there exists a character χ of F^* such

Proof of Theorem 7 (1): For any χ' of G such that $\chi''' = 1$, there exists a character χ of F^* such that

$$\chi|_{E^*} = 1, \ \chi' = \chi \circ \operatorname{Norm}_{G/F}.$$

We first compute the Gauss periods $\eta'_a = \psi'(\gamma^a C_0^{(M,G)}), 0 \le a \le M - 1$. By the Hasse-Davenport theorem and $G_F(\chi) = 2^s \sum_{x \in T_1} \chi(\gamma^x)$ (see [15, Theorem 2.1] or [2, Lemma 12.0.2] for a proof), we have

$$\begin{split} \eta'_{a} &= \frac{1}{M} \sum_{\ell=0}^{M-1} G_{G}(\chi'^{-\ell}) \chi'^{\ell}(\gamma^{a}) \\ &= -\frac{1}{M} + \frac{-1}{M} \sum_{\ell=1}^{M-1} G_{F}(\chi^{-\ell})^{2} \chi^{\ell}(\omega^{a}) \\ &= -\frac{1}{M} + \frac{-2^{s}}{M} \sum_{\ell=1}^{M-1} G_{F}(\chi^{-\ell}) \sum_{i \in T_{1}} \chi^{\ell}(\omega^{a-i}) \\ &= -\frac{1}{M} + \frac{-2^{s}}{M} \left(\sum_{\ell=0}^{M-1} G_{F}(\chi^{-\ell}) \sum_{i \in T_{1}} \chi^{\ell}(\omega^{a-i}) + 2^{s} + 1 \right) \\ &= -2^{s} \psi(\omega^{a}D) - 1, \end{split}$$

where D is defined in (3.1). By Lemma 2, we obtain

$$\eta'_{a} = \begin{cases} 2^{s} - 1 & \text{if } a \in T_{1}, \\ -2^{2s} + 2^{s} - 1 & \text{if } a \in T_{2}, \\ 2^{2s} + 2^{s} - 1 & \text{if } a \in T_{3}. \end{cases}$$

TABLE 3. The values of $\psi(\gamma^a R'_k)$'s

	R'_0	R'_1	R'_2	R'_3
$\gamma^a = 0$	1	$(2^{2s} - 1)(2^{3s} + 1)$	$2^{s-1}(2^{2s}-1)(2^{3s}+1)$	$2^{s-1}(2^s-1)^2(2^{3s}+1)$
a = 0	1	$2^{2s} - 1$	$2^{s-1}(2^s+1)(-2^{2s}+2^s-1)$	$2^{s-1}(2^s-1)(2^{2s}+2^s-1)$
$a \in T_1$	1	$-2^{3s}+2^{2s}-1$	$2^{s-1}(2^{2s}-1)$	$2^{s-1}(2^s-1)^2$
$a \notin T_1 \cup \{0\}$	1	$2^{2s} - 1$	-2^{s-1}	$2^{s-1}(-2^{s+1}+1)$

TABLE 4. The values of $\psi(\gamma^a D'_k)$'s

	D'_0	D'_1	D'_2	D'_3
$\gamma^a = 0$	1	$(2^s - 1)(2^{3s} + 1)$	$(2^{2s} - 1)(2^{3s} + 1)$	$(2^{2s}-1)^2(2^{2s}-2^s+1)$
$a \in T_1$	1	$2^{s} - 1$	$-2^{3s} + 2^{2s} - 1$	$(2^s - 1)^2(2^s + 1)$
$a \in T_2$	1	$-2^{2s}+2^s-1$	$2^{2s} - 1$	$-2^{s}+1$
$a \in T_3$	1	$2^{2s} + 2^s - 1$	$2^{2s} - 1$	$-(2^s+1)(2^{s+1}-1)$

Now, we compute that

$$\begin{aligned} \psi'(\gamma^a R'_k) &= \sum_{i \in T_k} \eta_{a+i} \\ &= (2^s - 1)|T_1 \cap a + T_k| + (-2^{2s} + 2^s - 1)|T_2 \cap a + T_k| + (2^{2s} + 2^s - 1)|T_3 \cap a + T_k| \\ &= (2^s - 1)|T_k| - 2^{2s}|T_2 \cap a + T_k| + 2^{2s}|T_3 \cap a + T_k|. \end{aligned}$$

Clearly, $-2^{2s}|T_2 \cap a + T_k| + 2^{2s}|T_3 \cap a + T_k|$ is the coefficient of a in the element $-2^{2s}(T_2 - T_3)T_k^{(-1)}$. The elements $(T_2 - T_3)T_k^{(-1)}$, k = 1, 2, 3, have been computed in Lemma 3. For each k = 1, 2, 3, the sum $\psi(\omega^a R'_k)$ follows directly and is listed in Table 3. By the Bannai-Muzychuk criterion, $(G, \{R'_i\}_{i=0}^3)$ is a three-class association scheme.

Remark 8. By the proof of Theorem 7, the dual scheme of the association scheme in Theorem 5 is given by

$$D'_{0} = \{0\}, \ D'_{1} = C_{0}^{(M,G)}, \ D'_{2} = \bigcup_{i \in T_{1}} C_{i}^{(M,G)}, \ D'_{3} = \bigcup_{i \in (T_{2} \cup T_{3}) \setminus \{0\}} C_{i}^{(M,G)}$$

Their character values are listed in Table 4.

Next, we give a proof of the second statement of Theorem 7 (ii). **Proof of Theorem 7 (ii):** For any multiplicative character χ'' of H such that $\chi''^M = 1$, there exists a character χ of F^* such that

$$\chi|_{E^*} = 1, \ \chi'' = \chi \circ \operatorname{Norm}_{H/F}.$$

We first compute the Gauss periods $\eta_a'' = \psi''(\gamma^a C_0^{(M,H)}), 0 \le a \le M-1$. By the Hasse-Davenport theorem and $G_F(\chi) = 2^s \sum_{i \in T_1} \chi(\omega^i)$, we have

$$\begin{split} \eta_a'' &= \psi''(\beta^a C_0^{(M,H)}) \\ &= \frac{1}{M} \sum_{\ell=0}^{M-1} G_H(\chi''^{-\ell}) \chi''^{\ell}(\beta^a) \\ &= -\frac{1}{M} + \frac{1}{M} \sum_{\ell=1}^{M-1} G_F(\chi^{-\ell})^3 \chi^{\ell}(\omega^a) \\ &= \frac{-1 + 2^{2s} |T_1|^2}{M} + \frac{2^{2s}}{M} \sum_{\ell=0}^{M-1} G_F(\chi^{-\ell}) \sum_{i,j \in T_1} \chi^{\ell}(\omega^{a-i-j}). \end{split}$$

TABLE 5. The values of $\psi(\beta^a R_k'')$'s

	R_0''	R_1''	R_2''	R_3''
$\beta^a = 0$	1	$(2^{2s} - 1)(2^{6s} + 2^{3s} + 1)$	$2^{s-1}(2^{2s}-1)(2^{6s}+2^{3s}+1)$	$2^{s-1}(2^s-1)^2(2^{6s}+2^{3s}+1)$
$a \in T_1$	1	$-2^{3s}+2^{2s}-1$	$2^{s-1}(2^{3s}+2^s+1)(2^s-1)$	$-2^{s-1}(2^s-1)(2^{3s}-2^s+1)$
$a \in T_2$	1	$(2^s - 1)(2^{3s} + 2^s + 1)$	$-2^{s-1}(2^{3s+1}-2^{2s}+1)$	$2^{s-1}(2^s-1)^2$
$a \in T_3$	1	$-(2^s+1)(2^{3s}-2^s+1)$	$2^{s-1}(2^{2s}-1)$	$2^{s-1}(2^{3s+1} + 2^{2s} - 2^{s+1} + 1)$

We have $\frac{-1+2^{2s}|T_1|^2}{M} = 2^{2s} + 2^s - 1$. In this case, η''_a , $0 \le a \le M - 1$, take more than three values. Therefore, using Eqn (3.8), we compute directly

$$\psi''(\beta^a R_1'') - (2^{2s} + 2^s - 1)|T_1| = \frac{2^{2s}}{M} \sum_{\ell=0}^{M-1} G_F(\chi^{-\ell}) \chi^\ell(\omega^a) \sum_{i,j,k\in T_1} \chi^{-\ell}(\omega^{i+j-k})$$
$$= \frac{2^{3s}}{M} \sum_{\ell=0}^{M-1} G_F(\chi^{-\ell}) \sum_{i\in T_1} \chi^\ell(\omega^{-i+a}) + \frac{2^{2s}(2^s+1)}{M} \sum_{\ell=0}^{M-1} G_F(\chi^{-\ell}) \sum_{i\in \mathbb{Z}_M} \chi^\ell(\omega^{i+a})$$
$$= 2^{3s} \psi(\omega^a D) - 2^{2s}(2^s+1).$$

Recall that $D = \bigcup_{i \in -T_1} C_i^{(M,F)}$. It follows that

$$\psi''(\beta^a R_1'') = 2^{2s} - 1 + 2^{3s} \psi(\omega^a D)$$

By the character values of Lemma 2, we obtain

$$\psi''(\beta^a R_1'') = \begin{cases} -2^{3s} + 2^{2s} - 1 & \text{if } a \in T_1, \\ 2^{3s}(2^s - 1) + 2^{2s} - 1 & \text{if } a \in T_2, \\ -(2^s + 1)(2^{3s} - 2^s + 1) & \text{if } a \in T_3. \end{cases}$$

In exactly the same way, using Eqn. (3.9) and (3.10) we can compute $\psi''(\beta^a R''_k)$ for k = 2, 3 directly. We record the result in Table 5. By the Bannai-Muzychuk criterion, $(H, \{R''_i\}_{i=0}^3)$ is a three-class association scheme, and the above is the first eigenmatrix of the association scheme; also, the scheme is self-dual.

5. Concluding Remarks

In this note, we gave three constructions of three-class association schemes from cyclotomy. In general, one can obtain a three-class association scheme from a two-class association scheme (a strongly regular Cayley graph) under a certain condition as follows: Let X be a (additively written) finite abelian group and S be a subset of $X \setminus \{0\}$ such that S = -S. Define $R_0 = \{0\}, R_1 = S, R_2 = X^* \setminus S$. Assume that $(X, \{R_i\}_{i=0}^2)$ forms a two-class association scheme, i.e., Cay(X, S) is strongly regular. Let $(X, \{D_i\}_{i=0}^2)$ be the dual of $(X, \{R_i\}_{i=0}^2)$, where we assume that R_1 is contained in D_1 . Define

$$R'_0 = \{0\}, R'_1 = R_1, R'_2 = D_1 \setminus R_1, R'_3 = D_2.$$

Then, $(X, \{R'_i\}_{i=0}^3)$ is a three-class association scheme. This construction is essentially given in [21, Corollary 3.2].

Our three constructions of three-class association schemes given in this note are not included in this construction. In fact, the association schemes of Theorems 5 and 7 (i) are not self-dual but the association scheme obtained from the above construction is self-dual. Furthermore, neither of the relations of the association scheme in Theorem 7 (ii) is strongly regular but two relations of the association scheme from the above construction are strongly regular.

An interesting question that is worth looking into is: what is the relations between the principal part of the first eigenmatrices of these three schemes we constructed? According to [5], the principal parts of the first eigenmatrices of the underlying cyclotomic schemes of the same order M satisfy the

Hasse-Davenport property, namely, that of $\mathbb{F}_{2^{6s}}$ (resp. $\mathbb{F}_{2^{9s}}$) is square (resp. cube) of that of $\mathbb{F}_{2^{3s}}$ up to a sign. This property seems to be lost after taking fusion using the index sets T_1 , T_2 and T_3 . Our schemes are interesting in that they can serve as examples to test such properties should there be any.

Finally, we comment that the Gauss periods in consideration takes three values in $\mathbb{F}_{2^{6s}}$ and more in $\mathbb{F}_{2^{9s}}$. It seems pretty hard to consider the fusions of the cyclotomic scheme in comparison with the case where the Gauss periods take only two values. Hopefully this will yield us more examples of primitive association schemes with new parameters. We will look into this problem in the future research.

APPENDIX: PARAMETERS OF THE SCHEMES

Throughout this appendix, we write $q = 2^s$. In this appendix, we give computational results (by Maple) for parameters of the three-class association schemes we constructed in the previous section. The first and second eigenmatrices of the schemes have been obtained during the proofs.

(1) We have the following computational result for the scheme $(F, \{R_i\}_{i=0}^3)$ of Theorem 5: Let A_i and A'_i denote the adjacency matrices of R_i and D_i respectively. With $A_i(A_0, A_1, A_2, A_3) = (A_0, A_1, A_2, A_3)B_i$, we have

$$B_{1} = \begin{pmatrix} 0 & q^{2} - 1 & 0 & 0 \\ 1 & q^{2} - 2 & 0 & 0 \\ 0 & 0 & \frac{1}{2}q^{2} + \frac{1}{2}q - 1 & \frac{1}{2}q(q - 1) \\ 0 & 0 & \frac{1}{2}q(q + 1) & \frac{1}{2}(q - 2)(q + 1) \end{pmatrix},$$

$$B_{2} = \begin{pmatrix} 0 & 0 & \frac{1}{2}q^{3} - \frac{1}{2}q & 0 \\ 0 & 0 & \frac{1}{4}q(q^{2} + q - 2) & \frac{1}{4}q^{2}(q - 1) \\ 1 & \frac{1}{2}q^{2} + \frac{1}{2}q - 1 & \frac{1}{4}(q^{2} + q - 6)q & \frac{1}{4}(q^{2} - 3q + 2)q \\ 0 & \frac{1}{2}q(q + 1) & \frac{1}{4}(q - 2)q(q + 1) & \frac{1}{4}(q - 2)q(q + 1) \end{pmatrix},$$

$$B_{3} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}q^{3} - q^{2} + \frac{1}{2}q \\ 0 & 0 & \frac{1}{4}q^{2}(q - 1) & \frac{1}{4}(q^{2} - 3q + 2)q \\ 0 & \frac{1}{2}q(q - 1) & \frac{1}{4}(q^{2} - 3q + 2)q & \frac{1}{4}(q^{2} - 3q + 2)q \\ 1 & \frac{1}{2}(q - 2)(q + 1) & \frac{1}{4}(q - 2)q(q + 1) & \frac{1}{4}q(q^{2} - 5q + 6) \end{pmatrix},$$

With $A'_i(A'_0, A'_1, A'_2, A'_3) = (A'_0, A'_1, A'_2, A'_3)L_i$, we have

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$$L_{1} = \begin{pmatrix} 0 & q-1 & 0 & 0 \\ 1 & q-2 & 0 & 0 \\ 0 & 0 & 0 & q-1 \\ 0 & 0 & 1 & q-2 \end{pmatrix},$$

$$L_{2} = \begin{pmatrix} 0 & 0 & q^{2}-1 & 0 \\ 0 & 0 & 0 & q^{2}-1 \\ 1 & 0 & q-2 & q(q-1) \\ 0 & 1 & q & -2+q^{2}-q \end{pmatrix},$$

$$L_{3} = \begin{pmatrix} 0 & 0 & 0 & q^{3}-q^{2}-q+1 \\ 0 & 0 & q^{2}-1 & q^{3}-2q^{2}-q+2 \\ 0 & q-1 & q(q-1) & q^{3}-2q^{2}-q+2 \\ 1 & q-2 & -2+q^{2}-q & 4+q^{3}-2q^{2}-q \end{pmatrix}$$

(2) We have the following computational result for the scheme $(G, \{R'_i\}_{i=0}^3)$ of Theorem 7 (i): Let A_i and A'_i denote the adjacency matrices of R'_i and D'_i respectively. With $A_i(A_0, A_1, A_2, A_3) = (A_0, A_1, A_2, A_3)B_i$, we have

$$B_{1} = \begin{pmatrix} 0 & q^{3} + q^{2} - q^{3} - 1 & 0 & 0 \\ 1 & -q^{3} + q^{2} + q^{4} - 2 & q^{3}(q^{2} - 1)/2 & (q^{2} - 2q + 1)q^{3}/2 \\ 0 & (q - 1)q^{2}(q + 1) & (q - 1)(q^{4} + q^{3} - q^{2} + q + 2)/2 & (q - 1)q(q^{3} - q^{2} - q + 1)/2 \\ 0 & (q - 1)q^{2}(q + 1) & q(q^{3} - q^{2} - q + 1)(q + 1)/2 & (q^{4} - 2 - 3q^{3} + 3q^{2} + q)(q + 1)/2 \end{pmatrix},$$

$$B_{2} = \begin{pmatrix} 0 & 0 & \frac{1}{2}q^{6} - \frac{1}{2}q^{4} + \frac{1}{2}q^{3} - \frac{1}{2}q & 0 \\ 0 & \frac{1}{2}q^{3}(q^{2} - 1) & \frac{1}{4}(q^{5} - 2q^{3} + 2q^{2} + q - 2)q & \frac{1}{4}(-1 + q^{4} + 2q - 2q^{3})q^{2} \\ 1 & \frac{1}{2}(q - 1)(q^{4} + q^{3} - q^{2} + q + 2) & \frac{1}{4}(q^{5} - 3q^{3} + 3q^{2} + q - 6)q & \frac{1}{4}q(q^{4} - 2 - q^{3} + 3q)(q - 1) \\ 0 & \frac{1}{2}q(q^{3} - q^{2} - q + 1)(q + 1) & \frac{1}{4}q(q^{4} - 2 - q^{3} + 3q)(q + 1) & \frac{1}{4}q(q^{4} - 2 - 3q^{3} + 2q^{2} + q)(q + 1) \end{pmatrix},$$

$$B_{3} = \begin{pmatrix} 0 & 0 & 0 & -q^{2} + \frac{1}{2}q^{3} - q^{5} + \frac{1}{2}q + \frac{1}{2}q^{4} + \frac{1}{2}q^{6} \\ 0 & \frac{1}{2}(q^{2} - 2q + 1)q^{3} & \frac{1}{4}(-1 + q^{4} + 2q - 2q^{3})q^{2} & \frac{1}{4}(q^{5} - 4q^{4} + 6q^{3} - 2q^{2} - 3q + 2)q \\ 0 & \frac{1}{2}(q - 1)q(q^{3} - q^{2} - q + 1) & \frac{1}{4}q(q^{4} - 2 - q^{3} + 3q)(q - 1) & \frac{1}{4}q(q^{4} - 2 - 3q^{3} + 2q^{2} + q)(q - 1) \\ 1 & \frac{1}{2}(q^{4} - 2 - 3q^{3} + 3q^{2} + q)(q + 1) & \frac{1}{4}q(q^{4} - 2 - 3q^{3} + 2q^{2} + q)(q - 1) \end{pmatrix}$$

With $A'_i(A'_0, A'_1, A'_2, A'_3) = (A'_0, A'_1, A'_2, A'_3)L_i$, we have

$$L_{1} = \begin{pmatrix} 0 & q^{4} - q^{3} + q - 1 & 0 & 0 \\ 1 & q^{2} - 2 & q(q^{2} - 1) & (q^{3} - 2q^{2} - q + 2)q \\ 0 & q(q - 1) & q^{2}(q - 1) & (q^{3} - q^{2} - q + 1)(q - 1) \\ 0 & (q - 2)q & q^{3} - q^{2} - q + 1 & q^{4} - 2q^{3} + 4q - 2 \end{pmatrix},$$

$$L_{2} = \begin{pmatrix} 0 & 0 & q^{5} + q^{2} - q^{3} - 1 & 0 \\ 0 & q(q^{2} - 1) & q^{2}(q^{2} - 1) & q^{5} - q^{4} - 2q^{3} + 2q^{2} + q - 1 \\ 1 & q^{2}(q - 1) & -q^{3} + q^{2} + q^{4} - 2 & q^{2}(q - 1)(q^{2} - 1) \\ 0 & q^{3} - q^{2} - q + 1 & q^{2}(q^{2} - 1) & q^{5} - q^{4} - 2q^{3} + 3q^{2} + q - 2 \end{pmatrix},$$

$$L_{3} = \begin{pmatrix} 0 & 0 & 0 & q^{6} - q^{5} + 2q^{3} - q^{2} - q^{4} - q + 1 \\ 0 & (q^{3} - 2q^{2} - q + 2)q & q^{5} - q^{4} - 2q^{3} + 2q^{2} + q - 1 & q^{6} - 2q^{5} - q^{4} + 6q^{3} - 2q^{2} - 4q + 2 \\ 0 & (q^{3} - q^{2} - q + 1)(q - 1) & q^{2}(q - 1)(q^{2} - 1) & (q - 1)(q^{5} - q^{4} - 2q^{3} + 3q^{2} + q - 2) \\ 1 & q^{4} - 2q^{3} + 4q - 2 & q^{5} - q^{4} - 2q^{3} + 3q^{2} + q - 2 & q^{6} - 2q^{5} - q^{4} + 6q^{3} - 4q^{2} - 6q + 4 \end{pmatrix} \right).$$

(3) We have the following computational result for the scheme $(H, \{R''_i\}_{i=0}^3)$ of Theorem 7 (ii): Let A_i and A'_i denote the adjacency matrices of R''_i and D''_i respectively. Since this scheme is self dual, we have Q = P (One can check directly that $P^2 = q^9 I$). With $A_i(A_0, A_1, A_2, A_3) = (A_0, A_1, A_2, A_3)B_i$, we have

$$B_{1} = \begin{pmatrix} 0 & q^{8} + q^{5} + q^{2} - q^{6} - q^{3} - 1 & 0 & 0 \\ 1 & q^{7} - 2q^{5} + 2q^{4} - q^{3} + q^{2} - 2 & \frac{1}{2}(q^{5} - 2q^{3} + 2q^{2} - 1)q^{3} & \frac{1}{2}(q^{5} - 2q^{4} + 4q^{2} - 4q + 1)q^{3} \\ 0 & (q^{5} - 2q^{3} + 2q^{2} - 1)q^{2} & \frac{1}{2}q^{8} - q^{6} + q^{5} + \frac{1}{2}q^{4} - \frac{3}{2}q^{3} + q^{2} + \frac{1}{2}q - 1 & \frac{1}{2}(q^{7} - 2q^{6} + 4q^{4} - 5q^{3} + q^{2} + 2q - 1)q \\ 0 & (q^{5} - 2q^{3} + 2q^{2} + 2q - 1)q^{2} & \frac{1}{2}(q^{7} - 2q^{5} + 2q^{4} + q^{3} - 3q^{2} + 1)q & \frac{1}{2}q^{8} - q^{7} + 2q^{5} - \frac{5}{2}q^{4} - \frac{3}{2}q^{3} + 2q^{2} - \frac{1}{2}q - 1 \end{pmatrix},$$

$$B_{2} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}q - \frac{1}{2}q^{4} + \frac{1}{2}q^{3} + \frac{1}{2}q^{6} - \frac{1}{2}q^{7} + \frac{1}{2}q^{9} & 0 \\ 0 & \frac{1}{2}(q^{5} - 2q^{3} + 2q^{2} - 1)q^{3} & \frac{1}{4}(q^{8} - 2q^{6} + 2q^{5} + q^{4} - 3q^{3} + 2q^{2} + q - 2)q & \frac{1}{4}(q^{7} - 2q^{6} + 4q^{4} - 5q^{3} + q^{2} + 2q - 1)q^{2} \\ 1 & \frac{1}{2}q^{8} - q^{6} + q^{5} + \frac{1}{2}q^{4} - \frac{3}{2}q^{3} + q^{2} + \frac{1}{2}q - 1 & \frac{1}{4}(q^{8} - 2q^{6} + 2q^{5} + q^{4} - 3q^{3} + 2q^{2} + q - 2)q & \frac{1}{4}(q^{7} - 2q^{6} + 4q^{4} - 5q^{3} + q^{2} + 2q - 1)q^{2} \\ 0 & \frac{1}{2}(q^{7} - 2q^{5} + 2q^{4} + q^{3} - 3q^{2} + 1)q & \frac{1}{4}(q^{8} - 2q^{6} + 2q^{5} + 2q^{4} - 7q^{3} + 5q^{2} + q - 2)q & \frac{1}{4}(q^{8} - 2q^{7} + 4q^{5} - 6q^{4} + 3q^{3} + 3q^{2} - 5q + 2)q \\ 0 & \frac{1}{2}(q^{7} - 2q^{5} + 2q^{4} + q^{3} - 3q^{2} + 1)q & \frac{1}{4}(q^{8} - 2q^{6} + 2q^{5} - 3q^{3} + 3q^{2} + q - 2)q & \frac{1}{4}(q^{8} - 2q^{7} + 4q^{5} - 6q^{4} + q^{3} + 5q^{2} - q - 2)q \end{pmatrix} \end{pmatrix},$$

$$B_{3} =$$

$$\begin{pmatrix} 0 & 0 & \frac{1}{2}q - q^2 + \frac{1}{2}q^4 + \frac{1}{2}q^3 - q^5 + \frac{1}{2}q^6 - q^8 + \frac{1}{2}q^7 + \frac{1}{2}q^9 \\ 0 & \frac{1}{2}(q^5 - 2q^4 + 4q^2 - 4q + 1)q^3 & \frac{1}{4}(q^7 - 2q^6 + 4q^4 - 5q^3 + q^2 + 2q - 1)q^2 & \frac{1}{4}(q^8 - 4q^7 + 6q^6 - 2q^5 - 7q^4 + 9q^3 - 2q^2 - 3q + 2)q \\ 0 & \frac{1}{2}(q^7 - 2q^6 + 4q^4 - 5q^3 + q^2 + 2q - 1)q & \frac{1}{4}(q^8 - 2q^7 + 4q^5 - 6q^4 + 3q^3 + 3q^2 - 5q + 2)q & \frac{1}{4}(q^8 - 4q^7 + 6q^6 - 2q^5 - 6q^4 + 9q^3 - 3q^2 - 3q + 2)q \\ 1 & \frac{1}{2}q^8 - q^7 + 2q^5 - \frac{5}{2}q^4 - \frac{3}{2}q^3 + 2q^2 - \frac{1}{2}q - 1 & \frac{1}{4}(q^8 - 2q^7 + 4q^5 - 6q^4 + q^3 + 5q^2 - q - 2)q & \frac{1}{4}(q^8 - 4q^7 + 6q^6 - 2q^5 - 6q^4 + 13q^3 + 3q^2 - 11q + 6)q \end{pmatrix}$$

In this case, we have $A_i = A'_i$ and $B_i = L_i$ for $0 \le i \le 3$.

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