

## AD-NILPOTENT IDEALS AND THE SHI ARRANGEMENT

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ABSTRACT. We extend the Shi bijection from the Borel subalgebra case to parabolic subalgebras. In the process, the  $I$ -deleted Shi arrangement  $\mathbf{Shi}(I)$  naturally emerges. This arrangement interpolates between the Coxeter arrangement  $\mathbf{Cox}$  and the Shi arrangement  $\mathbf{Shi}$ , and breaks the symmetry of  $\mathbf{Shi}$  in a certain symmetrical way. Among other things, we determine the characteristic polynomial  $\chi(\mathbf{Shi}(I), t)$  of  $\mathbf{Shi}(I)$  explicitly for  $A_{n-1}$  and  $C_n$ . More generally, let  $\mathbf{Shi}(G)$  be an arbitrary arrangement between  $\mathbf{Cox}$  and  $\mathbf{Shi}$ . Armstrong and Rhoades recently gave a formula for  $\chi(\mathbf{Shi}(G), t)$  for  $A_{n-1}$ . Inspired by their result, we obtain formulae for  $\chi(\mathbf{Shi}(G), t)$  for  $B_n$ ,  $C_n$  and  $D_n$ .

## 1. INTRODUCTION

Let  $\mathfrak{g}$  be a finite-dimensional complex simple Lie algebra of rank  $l$ . Fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . Then we have the root system  $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ . Let  $V$  be the real vector space spanned by  $\Delta$ . We denote by  $\langle \cdot, \cdot \rangle$  the canonical inner product on  $V$  which is induced from  $\mathfrak{h}$  of the Killing form of  $\mathfrak{g}$ . For convenience, we will equip  $V$  with an inner product  $(\cdot, \cdot)$  which is a suitable scalar multiple of the canonical one. For any root  $\alpha \in \Delta$ , let  $\mathfrak{g}_\alpha$  be the root space relative to  $\alpha$ . Let  $\Pi = \{\alpha_1, \dots, \alpha_l\} \subseteq \Delta^+$  be a fixed choice of simple and positive root systems of  $\Delta$ , respectively. Let  $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ . Then  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$  is a Borel subalgebra of  $\mathfrak{g}$ .

The abelian ideals of a Borel subalgebra were studied by Kostant [10, 11] in connection with the representation theory of semisimple Lie groups. In particular, D. Peterson's following theorem was detailed in [11]: there are  $2^l$  abelian ideals of  $\mathfrak{b}$ , regardless of the type of  $\mathfrak{g}$ . Peterson's approach was to give a bijection between the abelian ideals of  $\mathfrak{b}$  and a certain set of elements in the affine Weyl group  $\widehat{W}$  of  $\mathfrak{g}$ . This surprising result led Cellini and Papi to find similarities for *ad-nilpotent* ideals of  $\mathfrak{b}$ , i.e., the ideals of  $\mathfrak{b}$  which are contained in  $\mathfrak{n}$ . For example, they showed how to associate to any ad-nilpotent ideal  $\mathfrak{i}$  of  $\mathfrak{b}$  a uniquely determined element  $w_i \in \widehat{W}$  in [5]. In [6], they gave further a bijection between the set of ad-nilpotent ideals of  $\mathfrak{b}$  and the set of  $W$ -orbits in  $\check{Q}/(h+1)\check{Q}$ , where  $\check{Q}$  is the coroot lattice and  $h$  is the Coxeter number.

On the other hand, we note that a bijection between the set of all ad-nilpotent ideals of  $\mathfrak{b}$  and the dominant regions of the now-called Shi arrangement had been given by Shi in [17]. Thus, Theorems 3.2 and 3.6 there count the number of all ad-nilpotent ideals of  $\mathfrak{b}$ . To state his result, let us recall some notations concerning hyperplane arrangements.

A *hyperplane arrangement* is a finite collection of affine hyperplanes in an Euclidean space. For example, the *Coxeter arrangement* associated with  $\Delta^+$  is the arrangement in  $V$  defined

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by

$$(1) \quad \mathbf{Cox} := \{H_{\alpha,0} \mid \alpha \in \Delta^+\}.$$

Here for  $\alpha \in \Delta^+$  and  $k \in \mathbb{Z}$ , we define a hyperplane

$$(2) \quad H_{\alpha,k} := \{v \in V \mid (v, \alpha) = k\}.$$

If  $\mathcal{A}$  is a hyperplane arrangement in  $V$ , the connected components of  $V - \bigcup_{H \in \mathcal{A}} H$  are called *regions*. For example, there are  $|W|$  regions of  $\mathbf{Cox}$ , where  $W$  is the Weyl group associated to  $\Delta(\mathfrak{g}, \mathfrak{h})$ . For later use, let us single out the *dominant region* of  $\mathbf{Cox}$  as follows:

$$(3) \quad V_{\infty} := \{v \in V \mid (v, \alpha) > 0, \forall \alpha \in \Delta^+\}.$$

By the idea of Postnikov and Stanley in [12], a *deformation* of the Coxeter arrangement is an affine arrangement each of whose hyperplanes is parallel to some one in  $\mathbf{Cox}$ . The *Shi arrangement*  $\mathbf{Shi}$  associated to  $\Delta^+$  can be viewed as such an example:

$$(4) \quad \mathbf{Shi} := \mathbf{Cox} \cup \{H_{\alpha,1} \mid \alpha \in \Delta^+\}.$$

This arrangement was defined by Shi in the study of the Kazhdan-Lusztig cellular structure of the affine Weyl group of type  $A$ , see Chapter 7 of [15]. A region of  $\mathbf{Shi}$  is called *dominant* if it is contained in  $V_{\infty}$ . For any  $\mathfrak{h}$ -stable subset  $\mathfrak{u}$  of  $\mathfrak{g}$ , let  $\Phi_{\mathfrak{u}} \subset \Delta$  be the subset defined so that

$$\mathfrak{u} = \mathfrak{u} \cap \mathfrak{h} + \sum_{\alpha \in \Phi_{\mathfrak{u}}} \mathfrak{g}_{\alpha}.$$

Now let us cite the Shi bijection from Theorem 1.4 of [17] as follows:

**Theorem 1.1. (Shi)** *There exists a natural bijective map from the set of all the ad-nilpotent ideals of  $\mathfrak{b}$  to the set of all the dominant regions of the hyperplane arrangement  $\mathbf{Shi}$ . The map sends  $\mathfrak{i}$  to  $\{v \in V_{\infty} \mid (v, \beta) > 1, \forall \beta \in \Phi_{\mathfrak{i}}; (v, \beta) < 1, \forall \beta \in \Delta^+ \setminus \Phi_{\mathfrak{i}}\}$ .*

The first purpose of this paper is to generalize the Shi bijection from the Borel subalgebra case to parabolic subalgebras, see Theorem 2.1. In the process, the  $I$ -deleted Shi arrangement  $\mathbf{Shi}(I)$  naturally emerges, where  $I$  is an arbitrary subset of  $\Pi$ . This arrangement interpolates between  $\mathbf{Cox}$  and  $\mathbf{Shi}$ , see (12). More generally, let us consider

$$(5) \quad \mathbf{Shi}(G) = \mathbf{Cox} \cup \{H_{\alpha,1} \mid \alpha \in G\},$$

where  $G$  is any subset of  $\Delta^+$ .

Recall that the fundamental combinatorial object associated with a hyperplane arrangement  $\mathcal{A}$  in  $V$  is its *intersection poset*  $L(\mathcal{A})$ , which is defined as the set of nonempty intersections of hyperplanes from  $\mathcal{A}$ , partially ordered by the *reverse* inclusion of subspaces. As an invariant distilled from  $\mathcal{L}(\mathcal{A})$ , the *characteristic polynomial*  $\chi(\mathcal{A}, t) \in \mathbb{Z}[t]$  of  $\mathcal{A}$  is defined by

$$(6) \quad \chi(\mathcal{A}, t) = \sum_{x \in L(\mathcal{A})} \mu(V, x) t^{\dim(x)},$$

where  $\mu : L(\mathcal{A}) \times L(\mathcal{A}) \rightarrow \mathbb{Z}$  is the Möbius function of the poset  $L(\mathcal{A})$ , see (3.15) and section 3.11 of [18].

When  $\mathfrak{g} = A_{n-1}$ , by the finite field method of Crapo and Rota [7], Armstrong and Rhoades recently gave a formula for  $\chi(\mathbf{Shi}(G), t)$  in [1]. See Theorem 4.3. Analyzing its proof, one sees that there are two key features: for a large prime  $p$ , let  $f(S)$  be the number of vectors

in  $\mathbb{F}_p^l - \text{Cox}_p$  satisfying  $(H_{\alpha,1})_p$  for all  $\alpha \in S$ . Then the first feature is that  $\chi(\text{Shi}(G), p)$  can be expressed as an alternating sum

$$(7) \quad \sum_{S \subseteq G} (-1)^{|S|} f(S),$$

and the summation is reduced to certain subsets of  $G$ . The second one is that  $f(S)$  is shown to be dependent only on  $|S|$ .

We find a uniform way to express the first feature. Indeed, as recorded in Lemma 3.7, regardless of the type of  $\mathfrak{g}$ , one can always express  $\chi(\text{Shi}(G), p)$  as (7), and it suffices to take the summation over the *quasi-antichains* of  $\Delta^+$  (cf. Definition 3.6) contained in  $G$ . A detailed study of quasi-antichains will be given in Section 3. In particular, we will show that they are in bijection with the elements of  $\mathcal{L}(\text{Cox})$ . When  $\mathfrak{g} = A_{n-1}, B_n, C_n$  or  $D_n$ , we put

$$(8) \quad \text{Stir}(G, k) := \#\left\{\text{quasi-antichains of } \Delta^+ \text{ contained in } G \text{ with size } n - k\right\}.$$

In particular if  $\mathfrak{g} = A_{n-1}$ , as we shall see in (16), the definition (8) agrees with the original terminology  $\text{Stir}(G, k)$  in [1], and  $\text{Stir}(\Delta^+, n - k)$  is nothing but the number of partitions of  $[n] := \{1, 2, \dots, n\}$  into  $k$  blocks. The latter is usually denoted by  $S(n, k)$ , and termed as the *Stirling number of the second kind*. Recall from (1.93) and (1.94d) of [18] that we have the recurrence

$$(9) \quad S(n, k) = kS(n - 1, k) + S(n - 1, k - 1),$$

and the combinatorial identity

$$(10) \quad \sum_{k=0}^n S(n, k)(t)_k = t^n.$$

Here  $(t)_k = t(t - 1) \cdots (t - k + 1)$  is the falling factorial. It is interpreted as 1 whenever  $k \leq 0$ , and will be used throughout this paper. We will give some analogs of (9) and (10) in Lemma 5.2.

By explicit calculations, we find that the second feature still holds when  $\mathfrak{g}$  is  $B_n$  or  $C_n$ . This leads us to

**Theorem 1.2.** (Theorem 5.5 and Remark 5.6) *Let  $\mathfrak{g}$  be  $B_n$  or  $C_n$ ,  $n \geq 2$ . For any subset  $G \subseteq \Delta^+$ , the characteristic polynomial of  $\text{Shi}(G)$  is given by*

$$\chi(\text{Shi}(G), t) = \sum_{k=0}^n (-1)^k \text{Stir}(G, n - k) 2^{n-k} \left(\frac{t-1}{2} - k\right)_{n-k}.$$

When  $\mathfrak{g} = D_n$ ,  $f(S)$  no longer depends only on  $|S|$ . However, by a more careful analysis, we still obtain a formula for  $\chi(\text{Shi}(G), p)$ , see Theorem 6.4. Our explicit calculations also show that when  $\mathfrak{g}$  is classical and  $p$  is large enough,  $f(S)$  is nonzero if and only if  $S$  is a quasi-antichain of  $\Delta^+$ , see Remark 3.8. Thus the reduction of the alternating sum (7) to quasi-antichains turns out to be precise.

Now let us specialize the general story to the  $I$ -deleted Shi arrangement  $\text{Shi}(I)$ . Going from  $\text{Shi}$  to  $\text{Shi}(I)$ , the symmetry is broken kind of symmetrically. Thus we may expect  $\text{Shi}(I)$  to behave better than an arbitrary  $\text{Shi}(G)$ . Indeed, when  $\mathfrak{g} = A_{n-1}$ , we find that: the polynomial  $\chi(\text{Shi}(I), t)$  factors into nonnegative integers and it depends only on  $|I|$ ;

moreover, the cone of  $\mathbf{Shi}(I)$  is free in the sense of Terao [19], see Theorem 4.6. These results are based on the works of Athanasiadis [2, 3], Armstrong and Rhoades [1]. When  $\mathfrak{g} = C_n$ , we find that:  $\chi(\mathbf{Shi}(I), t)$  always factors into nonnegative integers; moreover, if  $I$  contains  $2e_n$ ,  $\chi(\mathbf{Shi}(I), t)$  depends only on  $|I|$ , and the same conclusion holds if  $I$  does not contain  $2e_n$ , see Theorem 5.1. These results are obtained via Theorem 1.2 and Lemma 5.2.

The paper is organized as follows: we generalize the Shi bijection in Section 2. We collect some preliminaries on the characteristic polynomial of a hyperplane arrangement in Section 3. In particular, the finite field method is described there. Moreover, we introduce the quasi-antichains and give a detailed study of them. Section 4 is devoted to the study of  $\mathbf{Shi}(I)$  for  $A_{n-1}$ , while Section 5 handles the  $C_n$  case. Finally, a formula for  $\chi(\mathbf{Shi}(G), t)$  is deduced for  $D_n$  in Section 6.

## 2. A GENERALIZATION OF THE SHI BIJECTION

This section is devoted to generalizing the Shi bijection (see Theorem 1.1) from the Borel subalgebra case to parabolic subalgebras. Let us begin with some preliminaries. We endow  $\Delta^+$  with the usual partial ordering. Namely,  $\alpha \leq \beta$  if  $\beta - \alpha = \sum_{\gamma \in \Delta^+} c_\gamma \gamma$ , where the  $c_\gamma$  are some non-negative real numbers. Any subset  $P$  of  $\Delta^+$  inherits a partial ordering from  $(\Delta^+, \leq)$ . Let us denote the corresponding poset by  $(P, \leq)$  or simply by  $P$ . Recall that a *dual order ideal* of  $P$  is a subset  $J$  of  $P$  such that if  $t \in J$  and  $t \leq s$  for  $s \in P$ , then  $s \in J$ . Recall also that an *antichain* of  $P$  is a subset of  $P$  consisting of pairwise non-comparable elements. Note that there is a canonical bijection between the dual order ideals of  $P$  and the antichains of  $P$ : given a dual order ideal, we send it to the set of its minimal elements; the inverse map sends the antichain  $\{a_1, \dots, a_k\}$  to the dual order ideal which is the union of the principal dual order ideals  $V_{a_1}, \dots, V_{a_k}$ , where  $V_a = \{b \in P \mid a \leq b\}$ .

Fix a subset  $I \subseteq \Pi$ . Let  $\Delta_I$  be the sub root system of  $\Delta$  spanned by  $I$ , and put  $\Delta_I^+ = \Delta_I \cap \Delta^+$ . Let

$$\mathfrak{p}_I = \mathfrak{h} + \sum_{\alpha \in \Delta_I \cup \Delta^+} \mathfrak{g}_\alpha$$

be the standard parabolic subalgebra of  $\mathfrak{g}$  corresponding to  $I$ . Recall that an ideal  $\mathfrak{i}$  of  $\mathfrak{p}_I$  is called *ad-nilpotent* if for all  $x \in \mathfrak{i}$ ,  $\text{ad}_{\mathfrak{p}_I} x$  is nilpotent. Let

$$(11) \quad C_I = \{\beta \in \Delta^+ \setminus \Delta_I \mid \forall \alpha \in \Delta_I^+, \beta - \alpha \notin \Delta\}.$$

We define the *I-deleted Shi arrangement* as

$$(12) \quad \mathbf{Shi}(I) := \text{Cox} \cup \{H_{\alpha,1} \mid \alpha \in C_I\}.$$

Since  $\mathbf{Shi}(\Pi)$  is  $\text{Cox}$  and  $\mathbf{Shi}(\emptyset)$  is  $\mathbf{Shi}$ , we see that  $\mathbf{Shi}(I)$  interpolates between the Coxeter arrangement and the Shi arrangement. Again a region of  $\mathbf{Shi}(I)$  is called dominant if it is contained in  $V_\infty$ . For any ad-nilpotent ideal  $\mathfrak{i}$  of  $\mathfrak{p}_I$ , we put  $\Psi_{\mathfrak{i}} = \cup_{\alpha} \{\beta \in C_I \mid \alpha \leq \beta\}$ , where  $\alpha$  runs over all the minimal elements of  $(\Phi_{\mathfrak{i}}, \leq)$ . Then we have

**Theorem 2.1.** *There exists a natural bijective map from the set of all the ad-nilpotent ideals of  $\mathfrak{p}_I$  to the set of all the dominant regions of the hyperplane arrangement  $\mathbf{Shi}(I)$ . This map sends  $\mathfrak{i}$  to  $\{v \in V_\infty \mid (v, \beta) > 1, \forall \beta \in \Psi_{\mathfrak{i}}; (v, \beta) < 1, \forall \beta \in C_I \setminus \Psi_{\mathfrak{i}}\}$ .*

We note that the number of the ad-nilpotent ideals for  $\mathfrak{p}_I$  was enumerated by Righi in Theorem 5.12 of [13] for  $\mathfrak{g}$  classical, and in [14] for  $\mathfrak{g}$  exceptional using GAP4. As a consequence, we also know the number of dominant regions of  $\text{Shi}(I)$ . Theorem 2.1 will be proved by collecting the bijections in the following subsections.

2.1. Recall that a subset  $I$  of  $\Pi$  is fixed. Let us put

$$\mathcal{F}_I := \{\Phi \subseteq \Delta^+ \setminus \Delta_I : \text{if } \alpha \in \Phi, \beta \in \Delta^+ \cup \Delta_I \text{ and } \alpha + \beta \in \Delta^+, \text{ then } \alpha + \beta \in \Phi\}.$$

As noted in section 3 of [13], we have a bijection

$$(13) \quad \{\text{ad-nilpotent ideals of } \mathfrak{p}_I\} \rightarrow \mathcal{F}_I; \mathfrak{i} \mapsto \Phi_{\mathfrak{i}}.$$

2.2. For any  $\Phi \in \mathcal{F}_I$ , let  $A(\Phi)$  be the set of all the minimal elements of  $(\Phi, \leq)$ . We note that  $A(\Phi)$  is contained in  $C_I$ . Indeed, let  $\beta$  be any minimal element of  $(\Phi, \leq)$ , and take any  $\alpha \in \Delta_I^+$ , it suffices to show that  $\beta - \alpha$  is not a root. Assuming the contrary gives  $\beta + (-\alpha) \in \Delta^+ \setminus \Delta_I$ . Since  $\beta \in \Phi$ ,  $-\alpha \in \Delta^+ \cup \Delta_I$  and  $\Phi \in \mathcal{F}_I$ , we conclude that  $\beta - \alpha \in \Phi$ , which contradicts to the minimality of  $\beta$ . Thus  $A(\Phi)$  is an antichain of  $(C_I, \leq)$  and we have a well-defined map

$$(14) \quad \mathcal{F}_I \rightarrow \{\text{antichains of } C_I\}; \Phi \mapsto A(\Phi).$$

Actually, the above map is bijective. To show this, it suffices to prove that for any antichain  $A$  of  $C_I$ , the set

$$\Phi(A) = \bigcup_{\beta \in A} \{\alpha \in \Delta^+ : \alpha \geq \beta\}$$

belongs to  $\mathcal{F}_I$ , which is precisely the content of Proposition 1.4 of [4] in view of the following

**Lemma 2.2.** *We have  $C_I = \{\beta \in \Delta^+ \setminus \Delta_I \mid \forall \alpha \in I, \beta - \alpha \notin \Delta\}$ .*

*Proof.* Fix any  $\beta \in \Delta^+ \setminus \Delta_I$  such that  $\beta - \alpha$  is not a root for any  $\alpha \in I$ , it suffices to show that  $\beta - \gamma$  is not a root for any  $\gamma \in \Delta_I^+$ . Let us prove this by induction on the height of  $\gamma$ . There is nothing to prove when  $\text{ht}(\gamma) = 1$ . Suppose that we have proved it for all  $\gamma' \in \Delta_I^+$  with  $\text{ht}(\gamma') < r$ . Now take  $\gamma \in \Delta_I^+$  be such that  $\text{ht}(\gamma) = r$ . Choose  $\alpha \in I$  such that  $(\gamma, \alpha) > 0$ . Then  $\gamma - \alpha \in \Delta_I^+$ . By assumption,  $\beta - \alpha$  is not a root, thus  $(\beta, \alpha) \leq 0$  and  $(\beta - \gamma, \alpha) < 0$ . Thus  $\beta - \gamma$  is not a root since otherwise  $\beta - (\gamma - \alpha)$  would be a root, contradicting to the induction hypothesis since  $\text{ht}(\gamma - \alpha) = r - 1$ .  $\square$

2.3. We take the canonical bijection from  $\{\text{antichains of } C_I\}$  to  $\{\text{dual order ideals of } C_I\}$ .

2.4. Given any dual order ideal  $\Phi$  of  $C_I$ , define

$$R_\Phi = \{v \in V_\infty \mid (v, \beta) > 1, \forall \beta \in \Phi; (v, \beta) < 1, \forall \beta \in C_I \setminus \Phi\}.$$

Then the map

$$(15) \quad \{\text{dual order ideals of } C_I\} \rightarrow \{\text{dominant regions of } \text{Shi}(I)\}; \Phi \mapsto R_\Phi.$$

is well-defined in view of the following

**Lemma 2.3.** *The set  $R_\Phi$  is non-empty. Hence it is a dominant region of  $\text{Shi}(I)$ .*

*Proof.* Let  $\tilde{\Phi}$  be the unique dual order ideal of  $\Delta^+$  containing  $\Phi$ , that is,

$$\tilde{\Phi} = \bigcup_{\beta \in \Phi} \{\alpha \in \Delta^+ : \alpha \geq \beta\}.$$

We claim that  $C_I \setminus \Phi \subseteq \Delta^+ \setminus \tilde{\Phi}$ . Indeed, if there exists  $\alpha \in C_I \setminus \Phi$  such that  $\alpha \in \tilde{\Phi}$ , then  $\alpha = \beta + \gamma$  for some  $\beta \in \Phi$  and  $\gamma \in \Delta^+$ . Since  $\alpha \in C_I$  and  $\Phi$  is a dual order ideal of  $C_I$ , this would imply that  $\alpha \in \Phi$ , which is absurd. Thus the claim follows, and we have  $R_{\tilde{\Phi}} \subseteq R_{\Phi}$ , where  $R_{\tilde{\Phi}} = \{v \in V_{\infty} \mid (v, \beta) > 1, \forall \beta \in \tilde{\Phi}; (v, \beta) < 1, \forall \beta \in \Delta^+ \setminus \tilde{\Phi}\}$ . Note that  $R_{\tilde{\Phi}}$  is non-empty by Theorem 1.1. Thus,  $R_{\Phi}$  is non-empty as well. Then it is immediate that  $R_{\Phi}$  is a dominant region of  $\text{Shi}(I)$ .  $\square$

Given any dominant region  $R$  of  $\text{Shi}(I)$ , we say that  $R$  is *above* the hyperplane  $H_{\beta,1}$ ,  $\beta \in \Delta^+$ , if it contains  $R$  and the origin of  $V$  in different half spaces. Now define

$$\tau(R) = \{\beta \in C_I \mid R \text{ is above the hyperplane } H_{\beta,1}\}.$$

We note that  $\tau(R)$  is a dual order ideal of  $C_I$ . Indeed, take any  $\gamma \in C_I$  such that  $\beta \leq \gamma$  for some  $\beta \in \tau(R)$ , then since  $R$  is contained in  $V_{\infty}$ , we have

$$(v, \gamma) = (v, \beta) + (v, \gamma - \beta) \geq (v, \beta) > 1, \forall v \in R.$$

Therefore,  $\gamma \in \tau(R)$  as desired. Now it is obvious that the map (15) is bijective with the inverse given by  $\tau$ .

### 3. CHARACTERISTIC POLYNOMIAL, THE FINITE FIELD METHOD AND QUASI-ANTICHAINS

Let  $\mathcal{A}$  be a collection of hyperplanes in the Euclidean space  $\mathbb{R}^n$ . In this section, we will collect some preliminaries concerning the determination of  $\chi(\mathcal{A}, t)$ . In particular, the finite field method due to Crapo and Rota [7] will be described. Moreover, we will introduce quasi-antichains, and give a detailed study of them.

**3.1. Characteristic polynomial and Poincaré polynomial.** The *Poincaré polynomial* of  $\mathcal{A}$  is defined by

$$P(\mathcal{A}, t) = \sum_{x \in L(\mathcal{A})} \mu(V, x) (-t)^{n - \dim(x)}.$$

Comparing it with (6), one sees easily that

$$P(\mathcal{A}, t) = (-t)^n \chi(\mathcal{A}, -\frac{1}{t}) \text{ and } \chi(\mathcal{A}, t) = t^n P(\mathcal{A}, -\frac{1}{t}).$$

**Example 3.1.** Consider  $G_2$ . Let  $\Pi = \{\alpha_1, \alpha_2\}$ , where  $\alpha_1$  is the short simple root and  $\alpha_2$  is the long simple root. Then one can easily compute from definition that

- If  $I = \{\alpha_1\}$ , then  $\chi(\text{Shi}(I), t) = (t - 3)(t - 5)$ ;
- If  $I = \{\alpha_2\}$ , then  $\chi(\text{Shi}(I), t) = (t - 4)(t - 5)$ .

**Remark 3.2.** Since the negative of the coefficient of  $t$  in  $\chi(\text{Shi}(I), t)$  is  $|C_I| + |\Delta^+|$ , in general, we shall not expect  $\chi(\text{Shi}(I), t)$  to depend only on the cardinality of  $I$ .

Suppose that the normals to the hyperplanes of  $\mathcal{A}$  span a subspace of  $V \subseteq \mathbb{R}^n$  of dimension  $r(\mathcal{A})$ . We call  $r(\mathcal{A})$  the *rank* of  $\mathcal{A}$ . We say a region of  $\mathcal{A}$  is *relatively bounded* if its intersection with  $V$  is bounded. When  $V = \mathbb{R}^n$ , a region of  $\mathcal{A}$  is relatively bounded if and only if it is bounded. A good reason to study the characteristic polynomial for hyperplane arrangements is given by the following classic theorem of Zaslavsky.

**Theorem 3.3.** (Section 2 of [20]) *For any hyperplane in  $\mathbb{R}^n$ , we have*

- *The number of regions of  $\mathcal{A}$  is  $(-1)^n \chi(\mathcal{A}, -1)$ ;*
- *The number of relatively bounded regions of  $\mathcal{A}$  is  $(-1)^{r(\mathcal{A})} \chi(\mathcal{A}, 1)$ .*

Based on Shi's determination of the number of regions of **Shi** [15, 16], Headley obtained the characteristic polynomial of **Shi**.

**Theorem 3.4.** (Theorem 2.4 of [8]) *Let  $\mathfrak{g}$  be a finite-dimensional complex simple Lie algebra with rank  $l$  and Coxeter number  $h$ . Let  $V$  be the real vector space spanned by  $\Delta(\mathfrak{g}, \mathfrak{h})$ . Then for the Shi arrangement in  $V$ , we have*

$$P(\text{Shi}, t) = (1 + ht)^l.$$

Recall that  $h$  is the Coxeter number, which equals to  $n + 1, 2n, 2n$  and  $2(n - 1)$  for  $A_n, B_n, C_n$  and  $D_n$  respectively, see Section 3.18 of [9]. Let us denote the multiset of the roots of  $\chi(\mathcal{A}, t)$  by  $\exp(\mathcal{A})$ . For example,  $\exp(\text{Shi}) = \{h^l\}$  by the above theorem. Here and in what follows we write  $\{a_1^{m_1}, a_2^{m_2}, \dots, a_r^{m_r}\}$  for a multiset, where each  $m_i$  stands for the multiplicity of  $a_i$ . The Factorization Theorem of Terao [19] states that the characteristic polynomial  $\chi(\mathcal{A}, t)$  factors over the nonnegative integers for any free arrangement  $\mathcal{A}$ .

**3.2. The finite field method.** Athanasiadis [2] offered a different approach to Theorem 3.4 which did not rely on Shi's result. There the main tool was the *finite field method* of Crapo and Rota [7] which turned out to be very useful. Let us describe this method. Suppose that the defining equations for the hyperplanes in  $\mathcal{A}$  have coefficients in  $\mathbb{Z}$ . Let  $p \in \mathbb{Z}$  be a prime number and consider a hyperplane  $H \in \mathcal{A}$  with defining equation  $a_1 x_1 + \dots + a_n x_n = b$ , where  $a_i, b \in \mathbb{Z}$ . Then we define the following subset  $H_p$  of the finite vector space  $\mathbb{F}_p^n$  by reducing the coefficients of  $H$  modulo  $p$ :

$$H_p := \{(x_1, \dots, x_n) \in \mathbb{F}_p^n \mid a_1 x_1 + \dots + a_n x_n = b\}.$$

We note that when  $p$  is large enough, each  $H_p$  is a hyperplane in  $\mathbb{F}_p^n$ , and we call  $\mathcal{A}_p := \{H_p \mid H \in \mathcal{A}\}$  the *reduced hyperplane arrangement* of  $\mathcal{A}$ .

**Theorem 3.5.** ([7]) *Let  $p \in \mathbb{Z}$  be a large prime, and let  $\mathcal{A}$  be a collection of hyperplanes in  $\mathbb{R}^n$  whose hyperplanes have defining equations with coefficients in  $\mathbb{Z}$ . Then the characteristic polynomial of  $\mathcal{A}$  satisfies*

$$\chi(\mathcal{A}, p) = \# \left( \mathbb{F}_p^n - \bigcup_{H \in \mathcal{A}} H_p \right).$$

*That is,  $\chi(\mathcal{A}, p)$  counts the number of points in the complement of the reduced arrangement  $\mathcal{A}_p$  in the finite vector space  $\mathbb{F}_p^n$ .*

A proof of the above theorem may also be found on pages 199–200 of [2].

**3.3. Expressing  $\chi(\text{Shi}(G), p)$  as an alternating sum over the quasi-antichains.** When  $\mathfrak{g}$  is a finite-dimensional simple Lie algebra over  $\mathbb{C}$  with rank  $l$ , the arrangements **Shi** and **Cox**, thus any arrangement between them, can be realized as arrangements whose hyperplanes have defining equations in  $\mathbb{Z}$ . Let  $G$  be any subset of  $\Delta^+$ . Recall the arrangement **Shi**( $G$ ) defined in (5). We shall prepare a lemma for the computation of its characteristic polynomial.

**Definition 3.6.** We call a subset  $S \subseteq \Delta^+$  a *quasi-antichain* if for any two distinct elements  $\alpha, \beta$  in  $S$ , the difference  $\alpha - \beta$  is not a nonzero integer multiple of any root. We call the cardinality of a quasi-antichain its *size*.

Of course, any antichain of  $(\Delta^+, \leq)$  is automatically a quasi-antichain of  $\Delta^+$ . The notion of quasi-antichain will be illustrated vividly in the following sections. It is motivated by the following

**Lemma 3.7.** *Let  $p$  be a large prime. We have*

$$\chi(\text{Shi}(G), p) = \sum_{\substack{S \subseteq G \text{ is a quasi} \\ \text{-antichain of } \Delta^+}} (-1)^{|S|} f(S),$$

where  $f(S)$  is the number of vectors in  $\mathbb{F}_p^l - \text{Cox}_p$  satisfying  $(H_{\alpha,1})_p$  for all  $\alpha \in S$ .

*Proof.* By Theorem 3.5, it suffices to count the number of vectors in  $\mathbb{F}_p^l - \text{Shi}(G)_p$ . By the principle of inclusion-exclusion (see for example Chapter 2 of [18]), we have

$$\chi(\text{Shi}(G), p) = \sum_{S \subseteq G} (-1)^{|S|} f(S).$$

The lemma follows from the observation that  $f(S) = 0$  if  $S \subseteq G$  is not a quasi-antichain of  $\Delta^+$ . Indeed, in such a case, we can find two distinct roots  $\alpha$  and  $\beta$  in  $S$  such that  $\alpha - \beta = k\gamma$ , where  $k$  is a positive integer and  $\gamma \in \Delta^+$ . Now if there exists  $v$  in  $\mathbb{F}_p^l - \text{Cox}_p$  satisfying the equations  $(H_{\alpha,1})_p$  and  $(H_{\beta,1})_p$ , it would satisfy  $(H_{k\gamma,0})_p$  as well. Since  $p$  is large enough, it would be a solution of  $(H_{\gamma,0})_p$ , contradicting to the assumption that  $v \notin \text{Cox}_p$ .  $\square$

**Remark 3.8.** As we shall see in Corollaries 4.4, 5.4 and 6.3, when  $\mathfrak{g}$  is classical and  $p$  is large enough,  $f(S)$  is nonzero if and only if  $S$  is a quasi-antichain of  $\Delta^+$ .

**Lemma 3.9.** *Any quasi-antichain  $S = \{\beta_1, \dots, \beta_k\} \subseteq \Delta^+$  is linearly independent. In particular,  $0 \leq k \leq l$ .*

*Proof.* Suppose that  $k > 0$ . Observe that  $(\beta_i, \beta_j) \leq 0$  for any  $i \neq j$ . Indeed, otherwise  $\beta_i - \beta_j$  would be a root, contradicting to the assumption that  $S$  is a quasi-antichain. Now the same argument of page 9 of [9] verifies the linear independence of  $S$ .  $\square$

### 3.4. Quasi-antichains and elements of $\mathcal{L}(\text{Cox})$ .

**Definition 3.10.** We say that a subset  $S \subseteq \Delta^+$  is *sub-closed* if  $\alpha - \beta = k\gamma$ , where  $\alpha, \beta \in S$ ,  $k \in \mathbb{N}^+$  and  $\gamma \in \Delta^+$ , implies that  $\gamma \in S$ . Here “sub” stands for subtraction.



It is obvious that any quasi-antichain of  $\Delta^+$  is sub-closed. For any subset  $S \subseteq \Delta^+$ , we put  $H_S := \bigcap_{\alpha \in S} H_{\alpha,0}$ , and make the convention that  $H_\emptyset = V$ . Moreover, we denote the intersection of all the sub-closed subsets of  $\Delta^+$  containing  $S$  by  $(S)$ . One sees easily that  $(S)$  is the smallest sub-closed subset of  $\Delta^+$  containing  $S$ .

**Lemma 3.11.** *For any subset  $S \subseteq \Delta^+$ , we have  $H_{(S)} = H_S$ .*

*Proof.* The desired conclusion follows from the observation that if  $\alpha - \beta = k\gamma$ , where  $\alpha, \beta, \gamma \in \Delta^+$ , and  $k \in \mathbb{N}^+$ , then

$$H_{\alpha,0} \cap H_{\beta,0} \cap H_{\gamma,0} = H_{\alpha,0} \cap H_{\beta,0}.$$

□

**Lemma 3.12.** *For any sub-closed subset  $S \subseteq \Delta^+$ , there exists a unique quasi-antichain  $S' \subseteq S$  such that  $H_S = H_{S'}$ .*

*Proof.* Existence: let us do induction on  $|S|$ . When  $|S| = 0$  or  $1$ , there is nothing to prove. Suppose that  $S'$  exists when  $|S| \leq k$ . Now let us consider the case that  $|S| = k + 1$  and  $S$  is not a quasi-antichain. By Definition 3.6,

$$A := \{\alpha \in S \mid \exists \beta \in S \text{ s.t. } \alpha - \beta = k\gamma \text{ for some } \gamma \in \Delta^+ \text{ and } k \in \mathbb{N}^+\}$$

is non-empty. Pick up a maximal element  $\alpha_0$  of  $(A, \leq)$ . Then there exists  $\beta_0 \in S$  such that  $\alpha_0 - \beta_0 = k_0\gamma_0$ , for some  $\gamma_0 \in \Delta^+$  and  $k_0 \in \mathbb{N}^+$ . Note that  $\gamma_0 \in S$  since  $S$  is sub-closed. We claim that  $S_1 := S \setminus \{\alpha_0\}$  is still sub-closed. Indeed, for any  $\alpha, \beta \in S_1$  such that  $\alpha - \beta = k\gamma$ , where  $\gamma \in \Delta^+$  and  $k \in \mathbb{N}^+$ , we have  $\gamma \in S$  since  $S$  is sub-closed. Notice that  $\gamma \neq \alpha_0$ . Otherwise, we would have  $\alpha \in A$ ,  $\alpha_0 \leq \alpha$  and  $\alpha \neq \alpha_0$ , contradicting to our choice of  $\alpha_0$ . Thus  $\gamma \in S_1$  and the claim holds. Since  $|S_1| = k$ , by induction hypothesis, there exists a quasi-antichain  $S' \subseteq S_1$  such that  $H_{S'} = H_{S_1}$ . Since

$$H_{\alpha_0,0} \cap H_{\beta_0,0} \cap H_{\gamma_0,0} = H_{\beta_0,0} \cap H_{\gamma_0,0},$$

we conclude that  $H_S = H_{S_1} = H_{S'}$ .

Uniqueness: reviewing the previous paragraph, we see that every element of  $S \setminus S'$  can be expressed as a linear combination with coefficients in  $\mathbb{N}^+$  of two or more elements of  $S'$ . Moreover,  $S'$  is linearly independent by Lemma 3.9. Thus  $S'$  can be *characterized* as the set of all roots  $\alpha \in S$  such that  $\alpha$  is not expressible as a linear combination with coefficients in  $\mathbb{N}^+$  of two or more elements of  $S$ . The latter set is uniquely determined by  $S$ . Thus  $S'$  must be unique as well. □

**Lemma 3.13.** *If  $S_1, S_2 \subseteq \Delta^+$  are two quasi-antichains such that  $H_{S_1} = H_{S_2}$ , then  $S_1 = S_2$ .*

*Proof.* On one hand,  $S_1$  and  $S_2$  are two quasi-antichains contained in  $(S_1 \cup S_2)$ . On the other hand, by Lemma 3.11,

$$H_{(S_1 \cup S_2)} = H_{S_1 \cup S_2} = H_{S_1} \cap H_{S_2} = H_{S_1} = H_{S_2}.$$

Thus  $S_1 = S_2$  by Lemma 3.12. □

**Theorem 3.14.** *The quasi-antichains of  $\Delta^+$  are in bijection with the elements of  $\mathcal{L}(\text{Cox})$ , regardless of the type of  $\mathfrak{g}$ .*

*Proof.* This is a combination of Lemmas 3.11, 3.12 and 3.13. □

4. CHARACTERISTIC POLYNOMIAL OF  $\text{Shi}(I)$ : TYPE  $A$ 

Let  $\mathfrak{g} = A_{n-1}$ ,  $n \geq 2$ . We choose  $\Delta^+ = \{e_i - e_j | 1 \leq i < j \leq n\}$ . We will refer to  $e_i - e_j$  simply by  $ij$ . They span the real vector space  $V = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0\}$ . The corresponding set of simple roots is  $\Pi = \{12, 23, \dots, (n-1)n\}$ . Recall from (2) that the arrangements Cox and Shi etc are defined within  $V$ . This section is mainly devoted to investigating the characteristic polynomial and the freeness of  $\text{Shi}(I)$ .

**4.1. Set partitions and quasi-antichains.** Put  $[n] := \{1, 2, \dots, n\}$ . We say that  $\pi = \{B_1, B_2, \dots, B_k\}$ , where each  $B_i$  is a nonempty subset of  $[n]$ , is a *partition of  $[n]$  into  $k$  blocks* if we have the disjoint union

$$[n] = B_1 \sqcup B_2 \sqcup \dots \sqcup B_k.$$

The *arc diagram* of  $\pi$  is drawn as follows: place the numbers  $1, 2, \dots, n$  on a line and draw an arc between each pair  $i < j$  such that

- $i$  and  $j$  are in the same block of  $\pi$ ; and
- there is no  $i < l < j$  such that  $i, l, j$  are in the same block of  $\pi$ .

A partition  $\pi$  of  $[n]$  is called *nonnesting* if it does not contain arcs  $ij$  and  $kl$  such that  $i < k < l < j$ —that is, no arc of  $\pi$  “nests” inside another. Figure 1 displays the arc diagrams for the partitions  $\{\{1, 3\}, \{2\}, \{4, 5\}\}$  and  $\{\{1, 5\}, \{2\}, \{3, 4\}\}$  of  $[5]$ . The first partition is nonnesting, while the second one is nesting since the arc 34 lies inside the arc 15.

Let  $\Gamma_n$  be the staircase Young diagram. Following Shi [17], let us fill the positive roots of  $A_{n-1}$  into the boxes of  $\Gamma_{n-1}$  as follows: put the root  $ij$  into the  $(n+1-j, i)$ -th box  $B_{n+1-j, i}$ . Figure 2 is an example with  $n = 5$ .

We will *identify*  $\Delta^+$  with  $\Gamma_{n-1}$  in this way, and transfer the partial ordering  $\leq$  on  $\Delta^+$  to a partial ordering on  $\Gamma_{n-1}$  accordingly. We note that for any two positive roots  $\alpha$  and  $\beta$ ,  $\alpha \leq \beta$  if and only if the box of  $\beta$  is to the north, or the west, or the northwest of the box of  $\alpha$ . This describes the poset structure of  $\Gamma_{n-1}$ . By a bit of abuse of notation, we will refer to this poset simply by  $\Gamma_{n-1}$ . We can characterize the quasi-antichains of  $\Gamma_{n-1}$  as follows: a subset  $S$  of  $\Gamma_{n-1}$  is a quasi-antichain if and only if in each row and column of  $\Lambda_n$ , there is at most one box of  $S$ . Now let us state an easy observation which should be well-known.

**Lemma 4.1.** *There exists a bijective map from the set of all the quasi-antichains of  $\Gamma_{n-1}$  to the set of all partitions of  $[n]$ . Moreover, this map sends the quasi-antichains of  $\Gamma_{n-1}$  with size  $n - k$  to the partitions of  $[n]$  with  $k$  blocks.*

*Proof.* Given any quasi-antichain  $A$ , let us draw an arc between the indices  $i$  and  $j$  for each element  $ij \in A$ . Then we end up with the arc graph of a partition  $\pi(A)$ . We map  $A$  to  $\pi(A)$ , which is easily seen to be bijective. Since a partition has  $k$  blocks if and only if its arc diagram has  $n - k$  arcs, the second statement follows directly.  $\square$

**Remark 4.2.** One sees easily that the map in the above lemma sends the set of all the antichains of  $\Gamma_{n-1}$  to the set of all the nonnesting partitions of  $[n]$ .

FIGURE 1. Two partitions of  $[5]$ 

15	25	35	45
14	24	34	
13	23		
12			

FIGURE 2. The Young diagram  $\Gamma_4$  for  $A_4$ 

**4.2. The characteristic polynomial and the freeness of  $\text{Shi}(I)$ .** For any subset  $G \subseteq \Gamma_{n-1}$ , recall that

$$\text{Shi}(G) = \{x_i - x_j = 0, 1 \leq i < j \leq n\} \cup \{x_i - x_j = 1, ij \in G\}.$$

Recall also that in [1] a partition  $\pi$  of  $[n]$  is called a  $G$ -partition if all its arcs are contained in  $G$ , and the  $G$ -Stirling number  $\text{Stir}(G, k)$  is defined as the number of  $G$ -partitions with  $k$  blocks. Thus  $\text{Stir}(\Gamma_{n-1}, k) = S(n, k)$ . Moreover, in view of Lemma 4.1, we have

$$(16) \quad \text{Stir}(G, k) = \#\left\{\text{quasi-antichains of } \Gamma_{n-1} \text{ contained in } G \text{ with size } n - k\right\}.$$

This reinterpretation of  $\text{Stir}(G, k)$  motivates our uniform definition (8). Now let us state a result of Armstrong and Rhoades in [1], which is deduced by the finite field method described in Theorem 3.5.

**Theorem 4.3.** (Theorem 3.2 of [1]) *For any subset  $G \subseteq \Gamma_{n-1}$ , the characteristic polynomial of  $\text{Shi}(G)$  is given by*

$$\chi(\text{Shi}(G), t) = \sum_{k=0}^{n-1} (-1)^k \text{Stir}(G, n - k) (t - k - 1)_{n-1-k}.$$

**Corollary 4.4.** *Let  $p$  be a large prime. For any subset  $S \subseteq \Gamma_{n-1}$  with cardinality  $0 \leq k \leq n - 1$ , the following are equivalent:*

- (i)  $S$  is a quasi-antichain of  $\Gamma_{n-1}$ ;
- (ii) in each row and column of  $\Gamma_{n-1}$ , there is at most one box of  $S$ ;

- (iii)  $f(S)$  is nonzero;
- (iv)  $f(S) = (p - k - 1)!/(p - n)!$ .

*Proof.* It follows from §4.1 and the proof of Theorem 4.3. □

**Example 4.5.** When  $G$  is the empty set, by Theorem 4.3, we have

$$\chi(\mathbf{Cox}, p) = (p - 1)_{n-1}.$$

When  $G = \Gamma_{n-1}$ , by Theorem 4.3 and  $(a)_n = (-1)^n(n - a - 1)_n$ , we have

$$\begin{aligned} \chi(\mathbf{Shi}, p) &= \sum_{k=0}^{n-1} (-1)^k S(n, n-k)(p-k-1)_{n-k-1} \\ &= \sum_{k=1}^n (-1)^{n-1} S(n, k)(n-p-1)_{k-1} \\ &= \frac{(-1)^{n-1}}{n-p} \sum_{k=0}^n S(n, k)(n-p)_k \\ &= (p-n)^{n-1}, \end{aligned}$$

where the penultimate equality holds since  $S(n, 0) = 0$  for  $n > 0$ , and the final equality uses (10).

**Theorem 4.6.** Let  $\mathfrak{g}$  be  $A_{n-1}$  and let  $I$  be any subset of  $\Pi$  with cardinality  $r \geq 1$ . Then  $\chi(\mathbf{Shi}(I), t)$  is independent of the  $r$  elements that  $I$  contains, and

$$(17) \quad \exp(\mathbf{Shi}(I)) = \{(n-r)^{n-r}, (n-r+1)^1, (n-r+2)^1, \dots, (n-1)^1\}.$$

In particular, the number of regions is  $(n-r+1)^{n-r-1}n!/(n-r)!$  and the number of bounded regions is  $(n-r-1)^{n-r-1}(n-2)!/(n-r-2)!$ . Moreover, the cone of  $\mathbf{Shi}(I)$  is free.

*Proof.* Fix any subset  $I$  of  $\Pi$  with cardinality  $r \geq 1$ . As noticed by Righi in §5.1 of [13], the poset  $(C_I, \leq)$  is isomorphic to  $\Gamma_{n-1-r}$ . See Figure 3 for two examples. Moreover, by Corollary 4.4(ii),  $\mathbf{Stir}(C_I, n-k)$  equals to the number of quasi-antichains of  $\Gamma_{n-1-r}$  with size  $k$ . By Lemma 4.1, the latter is  $S(n-r, n-r-k)$ , which is nonzero only if  $0 \leq k \leq n-r$ . Therefore by Theorem 4.3, we have

$$\chi(\mathbf{Shi}(I), t) = \sum_{k=0}^{n-r} (-1)^k S(n-r, n-r-k)(t-k-1)_{n-k-1}.$$

This tells us that  $\chi(\mathbf{Shi}(I), t)$  depends only on  $r$ . To deduce its explicit expression, the first way is quoting the formula (10), which gives

$$\sum_{k=0}^{n-r} S(n-r, k)(t)_k = t^{n-r}.$$

Then after some elementary calculations similar to those presented in Example 4.5, one arrives at (17). Alternatively, since  $\chi(\mathbf{Shi}(I), t)$  is independent of the  $r$  elements that  $I$  contains, we can focus on the special case that  $I_0 = \{12, \dots, r(r+1)\}$ . Then  $C_{I_0} = \{ij \mid r+1 \leq i < j \leq n\}$ , and (17) follows from Theorem 2.2 of [3].

For the last statement, we use Theorem 4.1 of [3]. It suffices to rule out the following two possibilities:

- (a) there exists  $1 \leq i < j < k \leq n$  such that  $ij, jk \in C_I$  but  $ik \notin C_I$ ;
- (b) there exists four distinct numbers  $i_1 < j_1, i_2 < j_2 \in [n]$  such that  $i_1j_1, i_2j_2$  gives *all* the edges between the vertices  $\{i_1, j_1, i_2, j_2\}$  in  $C_I$ .

Suppose that (a) happens. Then by Lemma 2.2, the (not necessarily distinct) simple roots  $i(i+1), (j-1)j, j(j+1), (k-1)k$  are not contained in  $I$ . Then  $ik \notin C_I$  means that there is a simple root  $\alpha \in I$  such that  $ik - \alpha$  is a root. Thus,  $\alpha$  has to be  $i(i+1)$  or  $(k-1)k$ , which is absurd. Now suppose that (b) happens. Without loss of generality, we assume that  $i_1 < i_2$ . Then there are three cases: (1)  $i_1 < j_1 < i_2 < j_2$ ; (2)  $i_1 < i_2 < j_1 < j_2$ ; (3)  $i_1 < i_2 < j_2 < j_1$ . Suppose that (3) happens. Then by Lemma 2.2, the (not necessarily distinct) simple roots  $i_1(i_1+1), (j_1-1)j_1, i_2(i_2+1), (j_2-1)j_2$  are not contained in  $I$ . Thus we would have  $i_1j_2 \in C_I$ , contradiction. (1) and (2) can be ruled out similarly.  $\square$

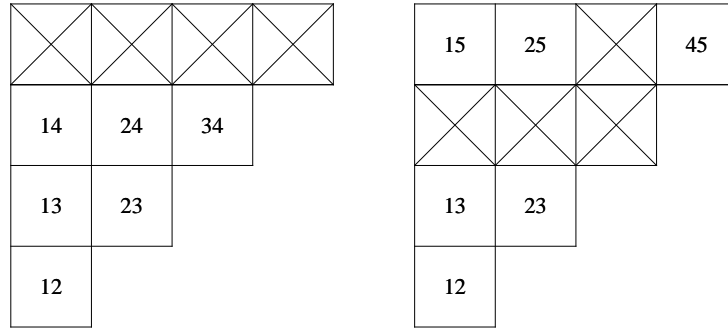


FIGURE 3.  $\mathfrak{g} = A_4$ ,  $I = \{45\}$ ,  $I' = \{34\}$

### 5. CHARACTERISTIC POLYNOMIAL OF $\text{SHI}(I)$ : TYPE $C$

Let  $\mathfrak{g}$  be of type  $C_n$ ,  $n \geq 2$ . We choose  $\Delta^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\} \cup \{2e_i \mid 1 \leq i \leq n\}$ . Then  $\Pi = \{e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, 2e_n\}$ . They span the real vector space  $\mathbb{R}^n$ . This section is devoted to proving the following

**Theorem 5.1.** *Let  $\mathfrak{g}$  be  $C_n$  and let  $I$  be any subset of  $\Pi$  with cardinality  $1 \leq r \leq n-1$ . Then when  $I$  contains  $2e_n$ ,  $\chi(\text{Shi}(I), t)$  is independent of the other  $r-1$  simple roots that  $I$  contains, and we have*

$$(18) \quad \exp(\text{Shi}(I)) = \{(2n-2r+1)^{n-r+1}, (2n-2r+3)^1, (2n-2r+5)^1, \dots, (2n-1)^1\}.$$

*In particular, the number of regions is  $(2n-2r+2)^{n-r}(2n)!!/(2n-2r)!!$  and the number of bounded regions is  $(2n-2r)^{n-r}(2n-2)!!/(2n-2r-2)!!$ . Similarly, when  $I$  does not contain  $2e_n$ ,  $\chi(\text{Shi}(I), t)$  is independent of the  $r$  simple roots that  $I$  contains, and we have*

$$(19) \quad \exp(\text{Shi}(I)) = \{(2n-2r)^{n-r}, (2n-2r+1)^1, (2n-2r+3)^1, \dots, (2n-1)^1\}.$$

*In particular, the number of regions is  $(2n-2r+1)^{n-r}(2n)!!/(2n-2r)!!$  and the number of bounded regions is  $(2n-2r-1)^{n-r}(2n-2)!!/(2n-2r-2)!!$ .*

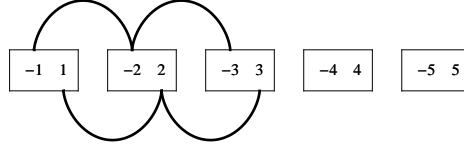
**5.1. Quasi-antichains: characterization and representation.** Following Shi [17], let us fill the positive roots of  $C_n$  into the staircase Young diagram  $\Lambda_n$  as follows: for  $1 \leq i \leq n$ , put  $2e_i$  into the  $(i, i)$ -th box  $B_{i,i}$ ; for  $1 \leq i < j \leq n$ , put  $e_i + e_j$  (resp.  $e_i - e_j$ ) into the  $(i, j)$ -th box  $B_{i,j}$  (resp. the  $(i, 2n+1-j)$ -th box  $B_{i,2n+1-j}$ ). The  $C_4$  case is illustrated in Figure 4.

$2e_1$	$e_1 + e_2$	$e_1 + e_3$	$e_1 + e_4$	$e_1 - e_4$	$e_1 - e_3$	$e_1 - e_2$
	$2e_2$	$e_2 + e_3$	$e_2 + e_4$	$e_2 - e_4$	$e_2 - e_3$	
		$2e_3$	$e_3 + e_4$	$e_3 - e_4$		
			$2e_4$			

FIGURE 4. The staircase Young diagram  $\Lambda_4$  for  $C_4$

We will *identify*  $\Delta^+$  with  $\Lambda_n$  in this way, and transfer the partial ordering  $\leq$  on  $\Delta^+$  to a partial ordering on  $\Lambda_n$  accordingly. We note that for any two positive roots  $\alpha$  and  $\beta$ ,  $\alpha \leq \beta$  if and only if the box of  $\beta$  is to the north, or the west, or the northwest of the box of  $\alpha$ . This describes the poset structure of  $\Lambda_n$ . By a bit of abuse of notation, we will refer to this poset simply by  $\Lambda_n$ . For later reference, we denote by  $T_{n-1}$  the subdiagram obtained from  $\Lambda_n$  by deleting the  $n$ -th column. The diagram  $T_{n-1}$  inherits a partial ordering from  $\Lambda_n$ . Let  $L_1 = \{2e_i \mid 1 \leq i \leq n\}$ , and define  $L_j$ ,  $2 \leq j \leq n-1$ , to be the collection of roots on the  $j$ -th row and the  $j$ -th column of  $\Lambda_n$ . We can characterize the quasi-antichains of  $\Lambda_n$  as follows: a subset  $S$  of  $\Lambda_n$  is a quasi-antichain if and only if in each row and column of  $\Lambda_n$ , and in each  $L_j$ ,  $1 \leq j \leq n-1$ , there is at most one box of  $S$ .

Put  $\pm[n] = \{\pm 1, \dots, \pm n\}$ . As in [2], we call the elements of  $\pm[n]$  the *signed integers* from 1 to  $n$ . Let  $S \subseteq \Delta^+$  be a quasi-antichain with cardinality  $n-k$ . Let us represent  $S$  in the following vivid way: put  $n$  boxes on a line and fill  $-i, i$  in the  $i$ -th box for  $1 \leq i \leq n$ . If  $2e_i \in S$  (there is at most one such  $i$ ), delete the  $i$ -th box. If  $e_i - e_j \in S$  (resp.  $e_i + e_j \in S$ ), for some  $1 \leq i < j \leq n$ , draw an arc between  $i$  and  $j$  (resp.  $i$  and  $-j$ ) from the below, and draw an arc between  $-i$  and  $-j$  (resp.  $-i$  and  $j$ ) from the above. Carrying out this process for all the  $n-k$  roots in  $S$ , we call the resulting graph the *signed partition associated to  $S$*  and denote it by  $\pi^\pm(S)$ . For example, when  $n = 5$  and  $S = \{e_1 - e_2, e_2 - e_3\}$ , the corresponding signed partition  $\pi^\pm(S)$  is given in Figure 5.

FIGURE 5. A signed partition for  $C_5$ 

We read  $\pi^\pm(S)$  from the above in the following way: when there is an arc between two signed integers, single them out and join them with an arrow pointing to the right; when there is no arc on a box, single out the positive number in it. For example, reading Figure 5 from the above gives

$$-1 \rightarrow -2 \rightarrow -3, 4, 5.$$

Similarly, we read  $\pi^\pm(S)$  from the below in the following way: when there is an arc between two signed integers, single them out, swap them, and then join them with an arrow pointing to the right; when there is no arc on a box, single out the negative number in it. For example, reading Figure 5 from the below gives

$$3 \rightarrow 2 \rightarrow 1, -4, -5.$$

We call  $3 \rightarrow 2 \rightarrow 1$  the *negative* of  $-1 \rightarrow -2 \rightarrow -3$ , call  $-4$  the negative of 4 etc. Then up to a choice of sign, there are 3 *essentially different* ordered parts for the signed partitions in Figure 5. One sees easily that similar things hold in general. Namely, no matter  $S$  contains a root of the form  $2e_i$  or not, reading  $\pi^\pm(S)$  from the above and the below always gives  $2k$  ordered parts; moreover, the negative of each part occurs exactly once, and up to a sign, there are  $k$  essentially different ordered parts of  $\pi^\pm(S)$ . As we shall see in the next subsection,  $\pi^\pm(S)$  is introduced to facilitate the counting of  $f(S)$  in the way of Athanasiadis [2].

Let  $S(\Lambda_n, n - k)$  (resp.  $S(T_n, n - k)$ ) be the number of quasi-antichains of  $\Lambda_n$  (resp.  $T_n$ ) with size  $k$ , which is nonzero only if  $0 \leq k \leq n$ . The following lemma gives analogs of (9) and (10).

**Lemma 5.2.** (i) For  $\Lambda_n$ , we have the recurrence

$$(20) \quad S(\Lambda_n, k) = S(\Lambda_{n-1}, k - 1) + (2k + 1)S(\Lambda_{n-1}, k)$$

and the identity

$$(21) \quad \sum_{k=0}^n S(\Lambda_n, k) 2^k (t)_k = (2t + 1)^n.$$

(ii) For  $T_n$ , we have the recurrence

$$(22) \quad S(T_n, k) = S(T_{n-1}, k - 1) + (2k + 2)S(T_{n-1}, k)$$

and the identity

$$(23) \quad \sum_{k=0}^n S(T_n, k) 2^k (t-1)_k = (2t)^n.$$

*Proof.* We only provide the proof for the  $\Lambda_n$  case. The  $T_n$  case is entirely similar. Since the subdiagram obtained from  $\Lambda_n$  by deleting the first row is isomorphic to  $\Lambda_{n-1}$  as posets, we denote it by  $\Lambda_{n-1}$ . A quasi-antichain  $S$  of  $\Lambda_n$  with size  $k$  can be formed in two ways: no element of  $S$  is chosen in the first row of  $\Lambda_n$ ; one element of  $S$  is chosen in the first row of  $\Lambda_n$ . The first number is  $S(\Lambda_{n-1}, n-1-k)$ . To figure out the second number, we observe that for any box  $B_{i,j}$  of  $\Lambda_n$ , where  $i \geq 2$ , thanks to the existence of  $L_1, \dots, L_{n-1}$ , there are *always* two elements in the first row of  $\Lambda_n$  which can not be chosen to form a quasi-antichain together with  $B_{i,j}$ . Suppose that we have an arbitrary quasi-antichain  $S'$  with size  $k-1$  of  $\Lambda_{n-1}$  at hand, and we want to add one element  $B_{1,j}$  from the first row such that  $S' \cup \{B_{1,j}\}$  is a quasi-antichain of  $\Lambda_n$ . Now for each element in  $S'$ , by the previous observation, there are two candidates in the first row that should be avoided. Moreover, there is no overlapping among these candidates since  $S'$  is a quasi-antichain. Thus there are exactly  $2n+1-2k$  allowable choices for  $B_{1,j}$ . Therefore, the second number is  $(2n+1-2k)S(\Lambda_{n-1}, n-k)$ , and we have

$$(24) \quad S(\Lambda_n, n-k) = S(\Lambda_{n-1}, n-1-k) + (2n-2k+1)S(\Lambda_{n-1}, n-k),$$

(20) is a reformulation of (24). Finally, (21) follows from (20) and an induction argument.  $\square$

**5.2. The formula for  $f(S)$ .** Let  $S$  be a quasi-antichain of  $\Delta^+$  with size  $n-k$ . By definition,  $f(S)$  equals to the number of  $n$ -tuples  $(x_1, \dots, x_n) \in \mathbb{F}_p^n$  satisfying  $(H_{\alpha,1})_p$ , for all  $\alpha \in S$ , as well as  $x_i \neq 0$ , for  $1 \leq i \leq n$ ;  $x_i \pm x_j \neq 0$ , for  $1 \leq i < j \leq n$ . Let us adopt the way of Athanasiadis [2] to count  $f(S)$ . Namely, we think of each  $n$ -tuple in  $\mathbb{F}_p^n$  as a map from  $\pm[n]$  to  $\mathbb{F}_p$ , sending  $i$  to the class  $x_i \in \mathbb{F}_p$ , and  $-i$  to the class  $-x_i$ . We think of the elements of  $\mathbb{F}_p$  as boxes arranged and labeled cyclically with the classes mod  $p$ . The top box is labeled with the zero class, the clockwise next box is labeled with the class 1 mod  $p$  etc. Then the  $n$ -tuples in  $\mathbb{F}_p^n$  become *placements* of the signed integers from 1 to  $n$  in the  $p$  boxes, and  $f(S)$  counts the number of those satisfying  $(H_{\alpha,1})_p$ ,  $\forall \alpha \in S$ , as well as the following

- (a) when a signed integer is placed in the class  $a$ , its negative is placed in the class  $-a$ ;
- (b) the zero class is always empty; and
- (c) distinct signed integers are placed in distinct classes.

**Lemma 5.3.** *Let  $S$  be a quasi-antichain of  $\Delta^+$  with size  $n-k$ . Let  $p$  be a large prime, then*

$$f(S) = \frac{(p-2n+2k-1)!!}{(p-2n-1)!!}.$$

*Proof.* For convenience, we call the classes from 1 to  $(p-1)/2$  (resp., from  $(p+1)/2$  to  $p-1$ ), both included, the *right half* (resp. *left half*) of the circle. By the definition of  $\pi^\pm(S)$ ,  $f(S)$  equals to the number of ways of placing the  $2k$  ordered parts of  $\pi^\pm(S)$  on the circle such that (a-c) are met. Namely, we should place them on the circle such that

- each ordered part is consecutive and clockwise; and there is no overlapping;



- each ordered part is either entirely on the right half or entirely on the left half; if it is on the right half, then its negative is on the left half according to (a).

Thus we can focus on what happens only on the right half, and it boils down to put the  $k$  essentially different ordered parts there, with each part a choice of sign.

Note that there are two types of  $S$ . The first type is that  $S$  contains  $2e_i$  for some  $i \in [n]$ . Then  $-i$  is sent to the  $(p-1)/2$  class, and  $i$  to the  $(p+1)/2$  class. Since there are  $(p-1)/2 - n$  empty boxes on the right half circle in total, the allowable ways to place the  $k$  essentially different parts there is  $f(S) = 2^k \prod_{j=1}^k ((p-1)/2 - n + j)$ , which equals to  $(p-2n+2k-1)!!/(p-2n-1)!!$ , as desired.

Now suppose that  $S$  does not contain any  $2e_i$ . Then there are two cases:

- (i) the class  $(p-1)/2$  is empty. Therefore, to place the  $k$  essentially different parts in the right half circle, we should always avoid the class  $(p-1)/2$ . Since besides the class  $(p-1)/2$ , there are  $(p-1)/2 - n - 1$  empty boxes there in total, the allowable ways are easily seen to be  $2^k \prod_{j=1}^k ((p-1)/2 - n - 1 + j)$ .
- (ii) the class  $(p-1)/2$  is filled. Then there must be an ordered part of  $\pi^\pm(S)$  which is placed entirely on the right half circle such that it ends with the class  $(p-1)/2$ . We have  $2k$  different ways to choose this ordered part. Then we have to place the remaining  $k-1$  essentially different ordered parts in the remaining classes of the right half. Since there is a choice of sign for each of them, and there are  $(p-1)/2 - n$  empty boxes on the right half in total, the allowable ways are  $(2k)2^{k-1} \prod_{j=1}^{k-1} ((p-1)/2 - n + j)$ .

Summing the numbers in (i) and (ii) gives the *same* value of  $f(S)$  as in the first type. This finishes the proof.  $\square$

**Corollary 5.4.** *Let  $p$  be a large prime. For any subset  $S \subseteq \Lambda_n$  with cardinality  $0 \leq k \leq n$ , the following are equivalent:*

- (i)  $S$  is a quasi-antichain of  $\Lambda_n$ ;
- (ii) in each row and column of  $\Lambda_n$ , and in each  $L_j$ ,  $1 \leq j \leq n-1$ , there is at most one box of  $S$ ;
- (iii)  $f(S)$  is nonzero;
- (iv)  $f(S) = (p-2k-1)!!/(p-2n-1)!!$ .

*Proof.* It follows from §5.1 and Lemma 5.3.  $\square$

**Theorem 5.5.** *Let  $\mathfrak{g}$  be  $C_n$ ,  $n \geq 2$ . Then for any subset  $G \subseteq \Lambda_n$ , the characteristic polynomial of  $\text{Shi}(G)$  is given by*

$$\chi(\text{Shi}(G), t) = \sum_{k=0}^n (-1)^k \text{Stir}(G, n-k) 2^{n-k} \left(\frac{t-1}{2} - k\right)_{n-k}.$$

*Proof.* This follows from Lemma 3.7 and Lemma 5.3.  $\square$

**Remark 5.6.** Since the positive roots for  $B_n$  are  $\{e_i - e_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}$ , one sees immediately that Lemma 5.3, thus the theorem above, holds for  $B_n$ .

**Example 5.7.** When  $G$  is the empty set, by Theorem 5.5, we have

$$\chi(\text{Cox}, p) = 2^n \left( \frac{p-1}{2} \right)_n.$$

When  $G = \Lambda_n$ , by Theorem 5.5,

$$\chi(\text{Shi}, p) = \sum_{k=0}^n (-1)^k S(\Lambda_n, n-k) 2^{n-k} \left( \frac{p-1}{2} - k \right)_{n-k}.$$

Then using (21) and a sequence of steps analogous to those in Example 4.5, we have

$$\chi(\text{Shi}, p) = (p - 2n)^n.$$

**5.3. Proof of Theorem 5.1 when  $I$  contains  $2e_n$ .** Fix any subset  $I \subseteq \Pi$  with cardinality  $1 \leq r \leq n-1$  which contains  $2e_n$ . As noticed by Righi in §5.2 of [13], the poset  $(C_I, \leq)$  is isomorphic to  $T_{n-r}$ . See Figure 6 for two examples. Moreover, by Corollary 5.4(ii),  $\text{Stir}(C_I, n-k)$  equals to the number of quasi-antichains of  $T_{n-r}$  with size  $k$ . Note that the latter is  $S(T_{n-r}, n-r-k)$ , which is nonzero only if  $0 \leq k \leq n-r$ . Now by Theorem 5.5,

$$\chi(\text{Shi}(I), t) = \sum_{k=0}^{n-r} (-1)^k S(T_{n-r}, n-r-k) (t - 2k - 1)_{n-k}.$$

This tells us that  $\chi(\text{Shi}(I), t)$  is independent of the other  $r-1$  simple roots that  $I$  contains. To arrive at (18), after some elementary calculations similar to those in Example 5.7, it boils down to show

$$(25) \quad \sum_{k=0}^{n-r} S(T_{n-r}, k) 2^k (t-1)_k = (2t)^{n-r},$$

which is an easy consequence of Lemma 5.2(ii).

**5.4. Proof of Theorem 5.1 when  $I$  does not contain  $2e_n$ .** Fix any subset  $I \subseteq \Pi$  with cardinality  $1 \leq r \leq n-1$  which does not contain  $2e_n$ . This case is similar to the previous one. The only difference is that now the poset  $(C_I, \leq)$  is isomorphic to  $\Lambda_{n-r}$ . Then replacing  $T_{n-r}$  in the previous case by  $\Lambda_{n-r}$  and quoting Lemma 5.2(i) instead finish the proof.  $\square$

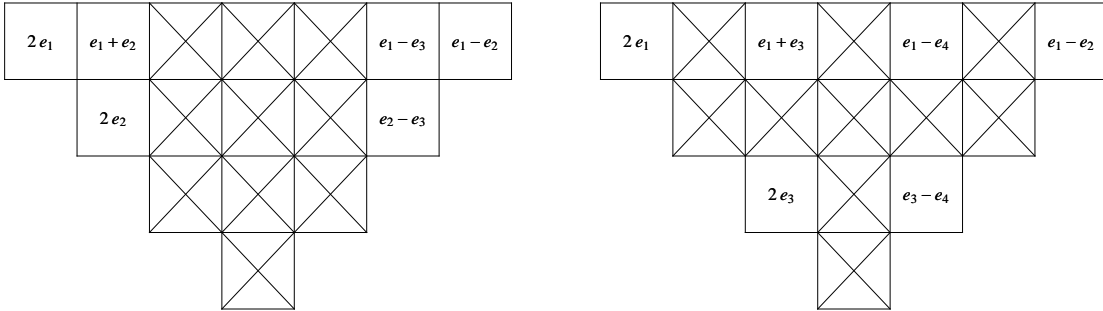


FIGURE 6.  $\mathfrak{g} = C_4$ ,  $I = \{e_3 - e_4, 2e_4\}$ ,  $I' = \{e_2 - e_3, 2e_4\}$

6. CHARACTERISTIC POLYNOMIAL OF  $\mathbf{SHI}(G)$ : TYPE  $D$ 

Let  $\mathfrak{g} = D_n$ ,  $n \geq 4$ . We choose  $\Delta^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq n\}$ , then  $\Pi = \{e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}$ . They span the real vector space  $\mathbb{R}^n$ . In this section, we will give a formula for the characteristic polynomial of  $\mathbf{SHI}(G)$ , where  $G$  is an arbitrary subset of  $\Delta^+$ . As suggested by Lemma 3.7, this can be achieved by finding a formula for  $f(S)$ , where  $S$  is an arbitrary quasi-antichain of  $\Delta^+$  contained in  $G$  with size  $n - k$ .

**Example 6.1.** Let us consider  $D_4$ . Then

- $f(S) = (p - 4)(p - 5)$  when  $S = \{e_1 + e_3, e_2 - e_3\}$ ;
- $f(S) = (p - 3)(p - 5)$  when  $S = \{e_1 + e_2, e_3 + e_4\}$ .

Thus  $f(S)$  no longer depends only on  $|S|$  for  $D_n$ . However, by a more careful analysis, we can still obtain a formula for  $f(S)$ . Indeed, similar to the  $C_n$  case, we associate to  $S$  a signed partition  $\pi^\pm(S)$ . Reading  $\pi^\pm(S)$  from the above and the below gives  $2k$  ordered parts, where the negative of each part occurs exactly once. Thus up to a sign, there are  $k$  essentially different ordered parts. Let  $n_1(S)$  be one half of the number of ordered parts with length 1 in  $\pi^\pm(S)$ . It is necessarily a nonnegative integer. By definition,  $f(S)$  equals to the number of  $n$ -tuples  $(x_1, \dots, x_n) \in \mathbb{F}_p^n$  satisfying  $(H_{\alpha,1})_p$ , for all  $\alpha \in S$ , as well as  $x_i \pm x_j \neq 0$ , for  $1 \leq i < j \leq n$ . Again we adopt the way of Athanasiadis to count [2]. Then the  $n$ -tuples in  $\mathbb{F}_p^n$  become *placements* of the signed integers from 1 to  $n$  in the  $p$  boxes, and  $f(S)$  counts the number of those satisfying  $(H_{\alpha,1})_p$ ,  $\forall \alpha \in S$ , as well as the following

- (a) when a signed integer is placed in the class  $a$ , its negative is placed in the class  $-a$ ;
- (b) there is at most one signed integer placed in each nonzero class;
- (c) there exists at most one  $i \in [n]$  such that both  $i$  and  $-i$  are placed in the zero class.

**Lemma 6.2.** Let  $S$  be a quasi-antichain of  $\Delta^+$  with size  $n - k$ . Let  $p$  be a large prime, then

$$f(S) = \frac{(p - 2n + 2k + 1)!!}{(p - 2n + 1)!!} - n_1(S) \frac{(p - 2n + 2k - 1)!!}{(p - 2n + 1)!!},$$

where  $n_1(S)$  is one half of the number of ordered parts with length 1 in  $\pi^\pm(S)$ .

*Proof.* For convenience, we call the zero class combined with the right (resp. left) half the *extended* right (resp. left) half. By the definition of  $\pi^\pm(S)$ ,  $f(S)$  equals to the number of placements of the  $2k$  ordered parts of  $\pi^\pm(S)$  on the circle such that (a-c) are met. Namely, we should place them on the circle such that

- each ordered part is consecutive and clockwise; and except for in the zero class, there is no overlapping;
- each ordered part is either entirely on the extended right half or entirely on the extended left half; if it is placed on the extended right half, then its negative should be placed on the extended left half according to (a).

Thus we can focus on what happens only on the extended right half, and it boils down to put the  $k$  essentially different ordered parts there, with each part a choice of sign. To deduce an explicit expression for  $f(S)$ , we note that there are two cases:

- (i) the zero class is empty. Then the number of allowable placements is already counted by Lemma 5.3. Namely, it is  $(p - 2n + 2k - 1)!! / (p - 2n - 1)!!$ .

- (ii) the zero class is filled. In such a case, if an ordered part with length 1 is placed in the zero class, then its negative must be placed there as well; while for an ordered part with length greater than 1, we can place it (or its negative) entirely on the extended right half starting with the zero class. Thus, there are  $2k - n_1(S)$  ways to use an ordered part to fill the zero class. Then it boils down to place the remaining  $k - 1$  essentially different ordered parts on the right half. Since there are  $(p - 1)/2 - n + 1$  empty boxes there in total, similar to the second type of Lemma 5.3, one can count that the latter number is  $(p - 2n + 2k - 1)!!/(p - 2n + 1)!!$ . Thus the total number is  $(2k - n_1(S))(p - 2n + 2k - 1)!!/(p - 2n + 1)!!$

Summing up the numbers in (i) and (ii) gives the desired expression for  $f(S)$ .  $\square$

**Corollary 6.3.** *Let  $\mathfrak{g}$  be  $D_n$ ,  $n \geq 4$ . Let  $p$  be a large prime. Let  $S \subseteq \Delta^+$  be any subset with cardinality  $0 \leq k \leq n$ . Then  $S$  is a quasi-antichain of  $\Delta^+$  if and only if  $f(S) \neq 0$ .*

Let  $G$  be any subset of  $\Delta^+$ . We put

$$(26) \quad \text{Stir}_1(G, k) = \sum_S n_1(S),$$

where  $S$  runs over all the quasi-antichains of  $\Delta^+$  contained in  $G$  with size  $n - k$ .

**Theorem 6.4.** *Let  $\mathfrak{g}$  be  $D_n$ ,  $n \geq 4$ . Then for any subset  $G \subseteq \Delta^+$ , the characteristic polynomial of  $\text{Shi}(G)$  is given by*

$$\chi(\text{Shi}(G), t) = \sum_{k=0}^n (-1)^k \left\{ \text{Stir}(G, n-k)(t-2k+1) - \text{Stir}_1(G, n-k) \right\} 2^{n-1-k} \left( \frac{t-1}{2} - k \right)_{n-1-k}.$$

*Proof.* This follows from Lemma 3.7 and Lemma 6.2.  $\square$

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