# Dyck tilings, increasing trees, descents, and inversions 

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#### Abstract

Cover-inclusive Dyck tilings are tilings of skew Young diagrams with ribbon tiles shaped like Dyck paths, in which tiles are no larger than the tiles they cover. These tilings arise in the study of certain statistical physics models and also Kazhdan-Lusztig polynomials. We give two bijections between cover-inclusive Dyck tilings and linear extensions of tree posets. The first bijection maps the statistic (area + tiles) $/ 2$ to inversions of the linear extension, and the second bijection maps the "discrepancy" between the upper and lower boundary of the tiling to descents of the linear extension.


## 1 Introduction

Kenyon and Wilson KW11a] and Shigechi and Zinn-Justin [SZJ12] independently introduced the notion of "cover-inclusive Dyck tilings" (defined below). The probabilities of certain events pertaining to the double-dimer model and spanning trees are given by formulas that involve counting these Dyck tilings KW11a, KW11b. Dyck tilings are also relevant to the study of Kazhdan-Lusztig polynomials [SZJ12]. More recently, Dyck tilings have arisen in connection with fully packed loop systems [FN12, Remark 2.10] and other contexts [Fay13].

We give two new bijections between Dyck tilings and perfect matchings, which when restricted to Dyck tilings with a certain "lower path," become bijections to linear extensions of a tree poset. The first bijection is compatible with the number of inversions of a linear extension, and gives a bijective proof of a formula that was conjectured by Kenyon and Wilson [KW11a, Conjecture 1] and proved non-bijectively by Kim [Kim12]. The second bijection is compatible with descents of the linear extension, and leads to a new enumeration formula. We also conjecture a third enumeration formula.

### 1.1 Background

Dyck paths of order $n$ are often defined as staircase lattice paths on an $n \times n$ square grid, from the lower-left corner to the upper-right corner, which do not go below the diagonal.

[^0]

( () ( () ) ) ( ) ( ( ) ) ( ) ( ( ) ) ) ( )

$1 \begin{array}{llllllllll}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 101112131415161718192021222324\end{array}$

Figure 1: A Dyck path $\lambda$ of order $n=12$ is shown in bold on the left. Above it is its associated (rotated) Young diagram. Below the Dyck path is its corresponding Dyck word, its balanced parentheses expression, and coordinates for the step positions. The horizontal dotted lines depict the chords between the matching up and down steps. In the middle is the planted plane tree corresponding to the Dyck path, where each node other than the root corresponds to a chord. On the right is the set of chords of the Dyck path, where each chord is represented as the pair of coordinates of the matching up and down steps. (See Sta99, Exercise 6.19].) A linear extension of the chord poset of $\lambda$ can be represented by placing the numbers $1, \ldots, n$ on the chords of the diagram on the left (so that if one chord is nested within another, it gets a higher number), or equivalently by placing the numbers $0, \ldots, n$ on the vertices of the planted plane tree (so that the numbers increase when going up the tree), or equivalently by ordering the set of chords represented as pairs (so that if one pair is nested within another, it occurs later). See also Figure 4.

Each such Dyck path has associated with it a Young diagram formed from the boxes above and to the left of it. If we rotate the lattice by $45^{\circ}$ and dilate it by a factor of $\sqrt{2}$, then each step of the Dyck path is either $(+1,+1$ ) (an up or "U" step) or $(+1,-1)$ (a down or "D" step). (See Figure 1.) This rotated form will be more convenient for us to work with. A Dyck path's sequence of $U$ and $D$ steps, when concatenated, forms a word which is called a Dyck word. If the U steps are written as "(" and the down steps are written as ")", then the Dyck word is a balanced parentheses expression.

We define a chord of a Dyck path $\lambda$ to be the horizontal segment between an up step and the matching down step, as shown in Figure 1. The chords of a Dyck path $\lambda$ naturally form a chord poset $P_{\lambda}$, where nesting is the order relation, i.e., one chord is above another chord in the partial order if its horizontal span lies within the span of the other chord, or equivalently, if the ( and ) corresponding to the first chord are nested within the (and ) corresponding to the second chord. If we adjoin a bottom-most element to the chord poset, we call the result the tree poset (see Figure 1), since its Hasse diagram is a planted plane tree, i.e., a tree embedded in the upper half-plane with a single distinguished vertex (the root) on the boundary of the half-plane, where two such embedded trees are considered equivalent if their embeddings are isotopic. Combinatorially, a planted plane tree is a tree with a distinguished root vertex, such that the children of any vertex are ordered. There is a


$$
3,7,10,12,6,1,4,9,8,5,11,2
$$

(d) Preorder word, list of labels of $L$ encountered in left-to-right depth first search, $L \circ L_{0}^{-1}$.

$$
6,12,1,7,10,5,2,9,8,3,11,4
$$

(e) Element of $\mathscr{L}\left(P_{\lambda}\right)$, inverse preorder word, standardization of $\ell_{1}, \ldots, \ell_{n}, L_{0} \circ L^{-1}$.

Figure 2: Natural labelings of a planted plane tree and their associated permutations.
natural bijection between Dyck paths of order $n$ and planted plane trees with $n+1$ vertices (see [Sta99, Exercise 6.19], or Figure 1).

A natural labeling of a poset $P$ with $n$ elements is an order-preserving bijection $L$ : $P \rightarrow[n]$, where $[n]$ denotes $\{1,2, \ldots, n\}$. For the tree poset associated with $\lambda$, it is more convenient to take a natural labeling of the chord poset $P_{\lambda}$, and then label the root by 0 , as shown in Figure 2. A planted plane tree with a natural labeling is called an increasing planted plane tree. As is well known, there are $(2 n-1)!!$ increasing planted plane trees on $n+1$ vertices (see BFS92, Corollary 1(iv)]). Given a labeled tree $L$, if we delete all vertices with labels larger than $k \leq n$, the result is a labeled tree $L^{(k)}$ on $k+1$ vertices (including the root labeled 0 ). Given $L^{(k-1)}$, there are $2 k-1$ possible positions where the vertex with label $k$ may be attached to the labeled tree $L^{(k-1)}$. (Each time a new vertex gets added to the tree, the subsequent vertex has one less possible attachment location but three new ones: just before, just after, and on top of the new vertex.) Thus, to each labeled tree $\left(P_{\lambda}, L\right)$ we can derive a sequence of attachment sites $p_{1}, \ldots, p_{n}$, where $0 \leq p_{i}<2 i-1$. Any such sequence determines a poset $P_{\lambda}$ together with a natural labeling $L$ of $P_{\lambda}$, and the map from sequences to pairs $\left(P_{\lambda}, L\right)$ is a bijection. In terms of the endpoints of the chords, this sequence of insertion locations is given by

$$
p_{i}=\#\left\{j<i: \ell_{j}<\ell_{i}\right\}+\#\left\{j<i: r_{j}<\ell_{i}\right\}
$$

where $\ell_{i}$ and $r_{i}$ denote the left and right endpoints of the chord labeled $i$, as in Figure 2 c .
Let $L_{0}$ be the natural labeling of $P_{\lambda}$ which orders the chords by their left endpoints (Figure 2 a). The preorder word of a natural labeling $L$ of $P_{\lambda}$ is $L \circ L_{0}^{-1}$, which is the permutation on $[n]$ obtained by reading the labels of $L$ (excluding 0 ) in a left-to-right depthfirst order (Figure 2 d). The inverse of the preorder word, $\sigma=L_{0} \circ L^{-1}$, turns out to be a more natural object. It can also be obtained as the "standardization" of the sequence of left-endpoints $\ell_{1}, \ldots, \ell_{n}$, i.e., $\sigma$ is the permutation on $[n]$ for which $\sigma_{i}<\sigma_{j}$ iff $\ell_{i}<\ell_{j}$ (Figure 2 e ). If $\omega$ is a natural labeling of the poset $P$, Stanley Sta72] defines

$$
\mathscr{L}(P, \omega)=\left\{\omega \circ L^{-1}: L \text { is a natural labeling of } P\right\}
$$

and these are sometimes called the linear extensions of the labeled poset $(P, \omega)$. We will abbreviate $\mathscr{L}\left(P_{\lambda}\right)=\mathscr{L}\left(P_{\lambda}, L_{0}\right)$.

It is also well known that there are $(2 n-1)!$ ! perfect matchings on the numbers $1, \ldots, 2 n$. Given the sequence $p_{1}, \ldots, p_{n}$, one natural way to associate a perfect matching match $\left(p_{1}, \ldots, p_{n}\right)$ with it is to take $\operatorname{match}\left(p_{1}, \ldots, p_{n-1}\right)$, increment all the numbers that are bigger then $p_{n}$, and then adjoin pair $\left(p_{n}+1,2 n\right)$. We define the min-word of the matching to be the list of the smaller item of each pair, sorted in order of the larger item in each pair, see Figure 4 .

Given two Dyck paths $\lambda$ and $\mu$ of order $n$, if the path $\mu$ is at least as high as the path $\lambda$ at each horizontal position, then we write $\mu \subset \lambda$ (since the Young diagram associated with $\mu$ is a subset of $\lambda$ 's Young diagram), and we write $\lambda / \mu$ for the skew Young diagram which consists of the boxes between $\lambda$ and $\mu$.

Dyck tiles, also called "Dyck strips" in SZJ12, are ribbon tiles (connected skew shapes that do not contain a $2 \times 2$ rectangle) in which the leftmost and rightmost boxes are at the same height, and no box within the tile is below these endpoints. (If each vertex in a Dyck path is replaced with a box, and the boxes are glued together, then the result is a Dyck tile, which explains the terminology.) A tiling of a skew Young diagram by Dyck tiles is a Dyck tiling. We say that one Dyck tile covers another Dyck tile if the first tile has at least one box whose center lies straight above the center of a box in the second tile. A Dyck tiling is called cover-inclusive if for each pair of its tiles, when the first tile covers the second tile, then the horizontal extent of the first tile is included as a subset within the horizontal extent of the second tile. We denote by $\mathcal{D}(\lambda, \mu)$ the set of all cover-inclusive Dyck tilings of shape $\lambda / \mu$, and let

$$
\mathcal{D}(\lambda, *)=\bigcup_{\mu} \mathcal{D}(\lambda / \mu) .
$$

Figure 3 shows all the cover-inclusive Dyck tilings of a certain skew shape.

### 1.2 Connections between Dyck tilings and increasing trees

It was observed empirically that there is a close connection between Dyck tilings and linear extensions of planted plane trees. More specifically, for a Dyck path $\lambda$ of order $n$, the total number of cover-inclusive Dyck tilings of skew shape $\lambda / \mu$ for some $\mu$ was conjectured [KW11a, Conjecture 1] and subsequently proven [Kim12] to be

$$
\begin{equation*}
|\mathcal{D}(\lambda, *)|=\frac{n!}{\prod_{\text {chords } c \text { of } \lambda}|c|}, \tag{1}
\end{equation*}
$$

where $|c|$ is the length of the chord $c$ as measured in terms of number of up steps in $\lambda$ between and including its ends. This formula has the form of the tree hook-length formula of Knuth [Knu73, pg. 70] for the number of linear extensions of the tree poset $P_{\lambda}$. These formulas call out for a bijection between cover-inclusive Dyck tilings and linear extensions. Here we give two such bijections. These bijections were in part inspired by a bijection due to Aval, Boussicault, and Dasse-Hartaut ABDH13] (see also [ABN11]), to which one of our bijections specializes in the case where $P_{\lambda}$ is the antichain. Our bijections actually provide refined enumeration formulas, which relate statistics (defined below) on the Dyck tilings to well-studied statistics on the permutations.





Figure 3: All the cover-inclusive Dyck tilings of a particular skew shape. (This figure first appeared in KW11a.) The generating function for the tilings of this skew shape by number of tiles is $t^{7}+2 t^{9}+4 t^{11}+5 t^{13}+5 t^{15}+4 t^{17}+2 t^{19}+t^{21}$, which, as discussed in [SZJ12], is closely related to a Kazhdan-Lusztig polynomial.

For two Dyck paths $\lambda$ and $\mu$, let $\operatorname{dis}(\lambda, \mu)$ be the "discrepancy" between $\lambda$ and $\mu$, i.e., the number of locations where $\lambda$ has a down step while $\mu$ has an up step (which also equals half the number of locations where $\lambda$ and $\mu$ step in opposite directions, i.e., half the Hamming distance between the Dyck words of $\lambda$ and $\mu$ ). For a skew shape $\lambda / \mu$ we define $\operatorname{dis}(\lambda / \mu)=\operatorname{dis}(\lambda, \mu)$.

For a Dyck tiling $T$ of $\operatorname{shape} \operatorname{sh}(T)=\lambda / \mu$, we $\operatorname{define} \operatorname{dis}(T)=\operatorname{dis}(\lambda, \mu)$, and $\operatorname{tiles}(T)$ to be the number of tiles in $T$, and $\operatorname{area}(T)$ to be the number of boxes of the skew shape $\lambda / \mu$. We define

$$
\operatorname{art}(T)=(\underline{\operatorname{area}}(T)+\underline{\operatorname{tiles}}(T)) / 2 .
$$

The art statistic is always integer-valued since each tile has odd area, and appears to be more natural than the area statistic.

Recall that the inversion statistic of a permutation $\sigma$ on $[n]$ is defined by

$$
\operatorname{inv}(\sigma)=\#\{(i, j): 1 \leq i<j \leq n, \sigma(i)>\sigma(j)\}
$$

and the descent statistic is defined by

$$
\operatorname{des}(\sigma)=\#\left\{i<n: \sigma_{i}>\sigma_{i+1}\right\} .
$$

In §2 we give our two bijections, which we call DTS and DTR, which stand for "Dyck tiling strip" and "Dyck tiling ribbon" respectively, for reasons that will become apparent. The functions DTS and DTR are bijections from the sequences $p_{1}, \ldots, p_{n}$ to cover-inclusive Dyck tilings of order $n$ (without restrictions on the lower path $\lambda$ or upper path $\mu$ ). As discussed above, these sequences $p_{1}, \ldots, p_{n}$ are also in bijection with increasing planted plane trees, so we can write $\operatorname{DTS}(\lambda, \sigma)$ and $\operatorname{DTR}(\lambda, \sigma)$ for the maps which take the labeled


Figure 4: Objects associated with the Dyck path $\lambda=$ UDUUDUDD. Row 1 shows natural labelings of the chord poset $P_{\lambda}$. Row 2 shows essentially the same thing - natural labelings of the planted plane tree associated with $\lambda$. Row 3 shows the labels of the planted plane tree listed in depth-first search order, where children are searched in left-to-right order. This is equivalent to listing the chord labels in order of the left endpoint. Row 4 shows the inverse of the permutation from the sixth row, with marks at the descents. These are the permutations $\sigma$ of $\mathscr{L}\left(P_{\lambda}\right)$. Row 5 shows the left and right endpoints of the chords of $\lambda$, listed in the order of the natural labeling. Row 6 shows the growth sequences $p_{1}, \ldots, p_{n}$ which correspond to the increasing planted plane trees in row 2 . Row 7 shows perfect matchings on $1, \ldots, 2 n$ that correspond to the sequences $p_{1}, \ldots, p_{n}$ in row 6 . Row 8 shows these same perfect matchings, represented as a $2 \times n$ array of numbers, where the columns are sorted, and the bottom row is sorted, together with markings at the descents in the top row. The top row of each $2 \times n$ array is the min-word $w$. Rows 9 and 10 show the cover-inclusive Dyck tilings $\operatorname{DTS}(\lambda, \sigma)$ and $\operatorname{DTR}(\lambda, \sigma)$ whose lower path is $\lambda$. The last three rows give statistics on these objects. Row 11 gives $\operatorname{des}(w)=\operatorname{des}(\sigma)=\operatorname{des}(\ell)=\operatorname{dis}(\mathrm{DTR})$. Row 12 gives $\operatorname{inv}(\sigma)=\operatorname{inv}(\ell)=\operatorname{art}($ DTS $)$. Row 13 gives $\operatorname{inv}(w)=$ tiles $(D T S)=$ nestings(matching).
tree defined by $\lambda$ and $\sigma$ to the sequence $p_{1}, \ldots, p_{n}$ and then to the Dyck tiling. These three bijections are compatible with one another in the sense that the lower paths of $\operatorname{DTS}(\lambda, \sigma)$ and $\operatorname{DTR}(\lambda, \sigma)$ are both $\lambda$. Thus, for a given Dyck path $\lambda$ of order $n$, the maps $\operatorname{DTS}(\lambda, \cdot)$ and $\operatorname{DTR}(\lambda, \cdot)$ are bijections from $\mathscr{L}\left(P_{\lambda}\right)$ to $\mathcal{D}(\lambda, *)$, see Figure 4. Furthermore, if $\sigma \in \mathscr{L}\left(P_{\lambda}\right)$, then

$$
\operatorname{art}(\operatorname{DTS}(\lambda, \sigma))=\operatorname{inv}(\sigma)
$$

and

$$
\operatorname{dis}(\operatorname{DTR}(\lambda, \sigma))=\operatorname{des}(\sigma)
$$

It is natural to ask if there is a bijection from linear extensions in $\mathscr{L}\left(P_{\lambda}\right)$ to cover-inclusive Dyck tilings with lower path $\lambda$ that simultaneously maps inv to art and des to dis, but as can be seen from Figure 4, no such bijection exists.

Björner and Wachs [BW89] gave the following $q$-analog of Knuth's tree hook-length formula:

$$
\sum_{\sigma \in \mathscr{L}\left(P_{\lambda}\right)} q^{\operatorname{inv}(\sigma)}=\frac{[n]_{q}!}{\prod_{\text {vertices } v \in P_{\lambda}}[\mid \text { subtree rooted at } v \mid]_{q}},
$$

where $[n]_{q}=1+q+\cdots+q^{n-1}$ and $[n]_{q}!=[1]_{q} \cdots[n]_{q}$. Using this formula together with the DTS bijection, we get a bijective proof of the following theorem, originally proven in Kim12] using inductive computation:
Theorem 1.1. [KW11a, Conjecture 1][Kim12] Given a Dyck path $\lambda$ of order n, we have

$$
\begin{equation*}
\sum_{\text {Dyck tilings } T \in \mathcal{D}(\lambda, *)} q^{\operatorname{art}(T)}=\frac{[n]_{q}!}{\prod_{\text {chords } \operatorname{cof} \lambda}[|c|]_{q}} . \tag{2}
\end{equation*}
$$

In turn, the DTR bijection implies

## Theorem 1.2.

$$
\begin{equation*}
\sum_{\text {Dyck tilings } T \in \mathcal{D}(\lambda, *)} z^{\operatorname{dis}(T)}=\sum_{\sigma \in \mathscr{L}\left(P_{\lambda}\right)} z^{\operatorname{des}(\sigma)} . \tag{3}
\end{equation*}
$$

It is evident from the form of (2) that if $\lambda_{1}$ and $\lambda_{2}$ are two Dyck paths whose corresponding trees are isomorphic when we ignore their embeddings in the plane, then

$$
\sum_{T \in \mathcal{D}\left(\lambda_{1}, *\right)} q^{\operatorname{art}(T)}=\sum_{T \in \mathcal{D}\left(\lambda_{2}, *\right)} q^{\operatorname{art}(T)} .
$$

The corresponding fact is not obvious for the dis statistic, but Stanley [Sta72, Thm. 9.1 and Prop. 14.1] proved that for any naturally labeled poset $(P, \omega)$,

$$
\sum_{\sigma \in \mathscr{L}(P, \omega)} z^{\operatorname{des}(\sigma)}
$$

is independent of the labeling $\omega$, so then it follows from (3) that

$$
\sum_{T \in \mathcal{D}\left(\lambda_{1}, *\right)} z^{\mathrm{dis}(T)}=\sum_{T \in \mathcal{D}\left(\lambda_{2}, *\right)} z^{\mathrm{dis}(T)} .
$$

These sums can be computed recursively using [Sta72, 12.6(ii) and 12.2].
Experimentally the tiles statistic also appears to behave nicely in this fashion:

Conjecture 1.3. If $\lambda_{1}$ and $\lambda_{2}$ are two Dyck paths whose corresponding trees are isomorphic when we ignore their embeddings in the plane, then

$$
\sum_{T \in \mathcal{D}\left(\lambda_{1}, *\right)} t^{\mathrm{tiles}(T)}=\sum_{T \in \mathcal{D}\left(\lambda_{2}, *\right)} t^{\mathrm{tiles}(T)}
$$

We have confirmed this equation by direct computation for all Dyck paths $\lambda_{1}$ and $\lambda_{2}$ of order 8 or less.

In contrast, the area statistic does not have this property, nor do any of the joint statistics between art, dis, and tiles.

The Dyck tilings $\operatorname{DTS}\left(p_{1}, \ldots, p_{n}\right)$ and $\operatorname{DTR}\left(p_{1}, \ldots, p_{n}\right)$ can also be understood in terms of the perfect matching match $\left(p_{1}, \ldots, p_{n}\right)$, as we discuss in §2. We shall see, for example, that

$$
\operatorname{dis}\left(\operatorname{DTR}\left(p_{1}, \ldots, p_{n}\right)\right)=\operatorname{des}\left(\min -\operatorname{word}\left(\operatorname{match}\left(p_{1}, \ldots, p_{n}\right)\right)\right)
$$

and

$$
\operatorname{tiles}\left(\operatorname{DTS}\left(p_{1}, \ldots, p_{n}\right)\right)=\operatorname{inv}\left(\min -\operatorname{word}\left(\operatorname{match}\left(p_{1}, \ldots, p_{n}\right)\right)\right) .
$$

An interesting special case for our bijections is when the lower path $\lambda$ is the "zig-zag" path

$$
\operatorname{zigzag}_{n}=\text { UDUDUD } \ldots \mathrm{UD}=(\mathrm{UD})^{n}
$$

so that $P_{\lambda}$ is the antichain, and $\mathscr{L}\left(P_{\lambda}\right)$ is the set of all permutations. In this case, the bijection DTR restricts to the bijection between permutations and Dyck tableaux in ABDH13, as discussed in §4.

We show in §6 that when $\lambda=\operatorname{zigzag}_{n}$,

$$
\left.\operatorname{art}\left(\operatorname{DTR}^{\operatorname{zigzag}_{n}}, \sigma\right)\right)=\operatorname{mad}(\sigma)
$$

where mad is a Mahonian statistic on permutations defined in [CSZ97] (which we review later). The composition of the two Dyck tiling bijections DTS ${ }^{-1}\left(\operatorname{zigzag}_{n}, \cdot\right) \circ$ DTR $\left(\operatorname{zigzag}_{n}, \cdot\right)$ gives a bijection mapping mad to inv which is different than the one given in [CSZ97].

If we further restrict to the case where both $\lambda=\operatorname{zigzag}_{n}$ and all the Dyck tiles have size 1, then the Dyck tiling is determined by its upper path $\mu$. In §5 we show that in this case, both DTS $\left(\operatorname{zigzag}_{n}, \cdot\right)$ and DTR $_{\left(\operatorname{zigzag}_{n}, \cdot\right) \text { restrict to (classical) bijections from permutations }}$ avoiding the pattern 231 to Dyck paths.

## 2 Dyck tiling bijections

### 2.1 Bijections with increasing trees

Recall that $\lambda / \mu$ denotes the skew shape between the lower Dyck path $\lambda$ and upper Dyck path $\mu$. We choose coordinates so that when $\lambda$ and $\mu$ are of order $n$, they start at $(-n, 0)$, each step is either $(+1,+1)$ or $(+1,-1)$, and the ending location is $(+n, 0)$. We coordinatize a Dyck tile by the Dyck path formed from the lower corners of the boxes contained within the Dyck tile. Column $s \in \mathbb{Z}$ refers to the set of points $(s, y)$.

Given a Dyck path $\rho$ and a column $s$, we define the spread of $\rho$ at $s$ to be the Dyck path $\rho^{\prime}$ whose points are

$$
\begin{aligned}
& \{(x, y)+(-1,0):(x, y) \in \rho \text { and } x \leq s\} \cup \\
& \{(x, y)+(0,1) \quad:(x, y) \in \rho \text { and } x=s\} \cup \\
& \{(x, y)+(+1,0):(x, y) \in \rho \text { and } x \geq s\} .
\end{aligned}
$$

Notice that the spread of $\rho$ at $s$ makes sense whether or not Dyck path $\rho$ overlaps column $s$.
Given a Dyck path $\rho^{\prime}$ and a column $s$, we define the contraction of $\rho^{\prime}$ at $s$ to be the Dyck path $\rho$ whose spread at $s$ is $\rho^{\prime}$. Not all Dyck paths $\rho^{\prime}$ will have a contraction at $s$. If there is a contraction at $s$, then it is unique.

We define the spread and contraction of a Dyck tile at column $s$ by taking the spread or contraction of the Dyck path that coordinatizes the Dyck tile. We define the spread and contraction of a Dyck tiling at column $s$ by taking the spread or contraction of the upper and lower bounding Dyck paths as well as of all the Dyck tiles within the tiling.

Given a Dyck tiling $T$ of $\lambda / \mu$, a column $s$ is eligible if:

1. The upper boundary $\mu$ contains an up step that ends in column $s$.
2. The intersection of $\mu$ with column $s$ is not the top corner of a Dyck tile of $T$ containing just one box.

There is always at least one eligible column of a Dyck tiling $T$, since the leftmost step of $\mu$ ends at an eligible column. We define the special column of a Dyck tiling to be its rightmost eligible column.

We now define two growth processes on Dyck tilings, one of which is used in the DTS bijection, the other in the DTR bijection. These processes are very similar - they both involve spreading the Dyck tiling at a certain column $s$ and adding boxes to the right of the new column. In the strip-grow process, we add a "broken strip" of one-box tiles from the growth site to the right boundary of the tile, so that a portion of $\mu$ is pushed up and left at a $45^{\circ}$ angle; see Figures 5 and 6. In the ribbon-grow process, we add a "ribbon" of one-box tiles from the growth site right-wards to the special column, so that a portion of $\mu$ is pushed up; see Figures 5 and 6 .

Formally, given a Dyck tiling $T$ of order $n$, and a column $s$ such that $-n \leq s \leq n$, we define strip-grow $(T, s)$ to be the Dyck tiling formed by first spreading $T$ at $s$ to get $T^{\prime \prime}$, and then adding a NE "broken strip" of one-box tiles to each up (NE) step of the upper boundary of $T$ from the growth site until the right boundary of $T^{\prime \prime}$ to obtain $T^{\prime}=\operatorname{strip-grow}(T, s)$.

Similarly, we define ribbon-grow $(T, s)$ to be the Dyck tiling formed by first spreading $T$ at $s$ to get $T^{\prime \prime}$, and then if $T^{\prime \prime \prime}$ 's special column $Q$ is to the right of $s$, adding a ribbon of one-box tiles on top of $T^{\prime \prime}$ at columns that are strictly between columns $s$ and $Q$ to obtain $T^{\prime}=\operatorname{ribbon-grow}(T, s)$. Notice that the upper boundary of tiling $T^{\prime \prime}$ has a down step starting at column $s$, and an up step ending at column $Q$, so adding this ribbon of one-box tiles to $T^{\prime \prime}$ results in a valid Dyck tiling $T^{\prime}$.

If $T$ is cover-inclusive, then both strip-grow $(T, s)$ and ribbon-grow $(T, s)$ are also coverinclusive. The maps strip-grow and ribbon-grow are the grow maps.


Figure 5: The growth process for the DTS bijection (left column) and the DTR bijection (right column). The spread and contract steps are the same for both bijections, as is the definition of the special column. In the DTS bijection, a "broken strip" of one-box tiles is always added to each up step to the right of the growth site, which has the effect of pushing up-and-left the upper boundary (as indicated by the arrows). In the DTR bijection, when the growth is at a column to the left of the special column, a "ribbon" of one-box tiles is added from the growth site to the special column, which has the effect of pushing up the upper boundary of the tiling (as indicated by the arrows).






$$
\begin{array}{ccccc}
12 & 9 & & 11 \\
7 & 10 & 4 & 8 & 5 \\
7 & & 1 & 1 & \\
3 & 6 & 1 & 2
\end{array}
$$

Figure 6: Example showing the bijections from increasing planted plane trees (linear extensions of a chord poset) to Dyck tilings, with the DTS bijection on the left, and the DTR bijection on the right. The increasing planted plane tree is built up one vertex at a time in numerical order of the vertex labels, while the Dyck tilings are built up, starting from the empty Dyck tiling, by a sequence of the growth steps illustrated in Figure 5, with newly added tiles shown in gray. At each stage $k$, the linear extension of the chord poset $P_{\lambda^{(k)}}$ is shown together with the $k^{\text {th }}$ Dyck tiling, whose lower boundary is $\lambda^{(k)}$. The preorder word of the final linear extension is $3,7,10,12,6,1,4,9,8,5,11,2$, whose inverse is $6,12,1,7,10,5,2,9,8,3,11,4$, which has 6 descents and 34 inversions. The discrepancy between the upper and lower paths of the final tiling on the right is 6 , and (area + tiles) $/ 2$ of the final tiling on the left is 34 .

Column $s$ is eligible for $T^{\prime}$. Because we added a strip or a ribbon to the spread of $T$ to get $T^{\prime}$, there are no eligible columns of $T^{\prime}$ to the right of $s$, so $s$ is $T^{\prime \prime}$ s special column in both growth processes.

Given a cover-inclusive Dyck tiling $T^{\prime}$ of order $n+1$, we now define strip-shrink $\left(T^{\prime}\right)$ and ribbon-shrink $\left(T^{\prime}\right)$, which we will show to be the inverses of the corresponding grow maps. First, we identify $T^{\prime}$ 's special column $s$. For the strip-shrink map, we then remove all the one-box tiles to the right of $s$ that are part of an up step in $T^{\prime \prime}$ s upper boundary. This operation gives a new cover-inclusive Dyck tiling $T^{\prime \prime}$ since the speciality of $s$ ensures that to the right of $s$ all up steps of $\mu$ are part of one-box tiles, so removing them will keep the upper boundary a Dyck path. For the ribbon-shrink map, we find the column $r$ for which the tiling $T^{\prime}$ has one-box Dyck tiles on top of columns $s+1, s+2, \ldots, r-1$ but not on top of column $r$. (If there are no such one-box Dyck tiles, then $r=s+1$.) We then remove these one-box Dyck tiles to obtain a new cover-inclusive Dyck tiling $T^{\prime \prime}$.

For both the strip-shrink and ribbon-shrink maps, since $s$ was special, the upper boundary $\mu^{\prime}$ of $T^{\prime}$ makes an up step from $s-1$ to $s$. Because $s+1$ was not eligible in $T^{\prime}$, either $\mu^{\prime}$ makes a down step from column $s$ to $s+1$, or it makes an up step but there is a one-box Dyck tile on the top border, which is then removed in $T^{\prime \prime}$. In $T^{\prime \prime}$, the top tile in column $s$ has a peak at $s$. Because $T^{\prime}$ is cover-inclusive, $T^{\prime \prime}$ is also cover-inclusive, and so any tile of $T^{\prime \prime}$ which intersects columns $s-1, s$, or $s+1$ in fact intersects all three columns, and has a peak at $s$. Therefore $T^{\prime \prime}$ can be contracted at column $s$ to obtain a new cover-inclusive Dyck tiling $T$. The maps from $T^{\prime}$ to $(T, s)$ are the shrink maps.

Lemma 2.1. The strip/ribbon-grow maps and the strip/ribbon-shrink maps are inverses and define a bijection from pairs $(T, s)$, where $T$ is a cover-inclusive Dyck tiling of order $n$ and $s$ is an integer between $-n$ and $n$ inclusive, to cover-inclusive Dyck tilings $T^{\prime}$ of order $n+1$.

Proof. Suppose that we apply one of the grow maps to $(T, s)$ to get $T^{\prime}$ and then apply the corresponding shrink map to $T^{\prime}$. We already saw that $s$ is the special column of $T^{\prime}$, which is then recovered by the shrink map.

Consider the strip-grow followed by the strip-shrink map. The addition of a one-box tile strip to the right of $s$ ensures that $s$ is the special column of $T^{\prime}$ and thus shrinking $T^{\prime}$ results in removing that same strip of boxes.

Consider the ribbon-grow followed by the ribbon-shrink map. Let $q$ denote $T$ 's special column. If $q \leq s$ then no new one-box Dyck tiles are added by the ribbon-grow map, column $s+1$ of $T^{\prime}$ does not contain a one-box Dyck tile, and no one-box Dyck tiles are removed by the shrink map. If $q>s$, then $T^{\prime \prime \prime}$ 's special column is $q+1$, and $q-s$ one-box Dyck tiles are added by the grow map, one in each of columns $s+1, \ldots, q$. Because $q+1$ is $T^{\prime \prime}$ 's special column, there is no one-box Dyck tile of either $T^{\prime \prime}$ or $T^{\prime}$ in column $q+1$. Thus ribbon-shrink will remove precisely the one-box tiles that ribbon-grow added to columns $s+1, \ldots, q$.

In either the strip or ribbon cases, the shrink map removes precisely those one-box Dyck tiles that the grow map added. The contraction of the shrink map undoes the spreading of the grow map, so shrinking $T^{\prime}$ results in $(T, s)$.

Next, suppose that we apply the shrink map to $T^{\prime}$ to get $(T, s)$ and then apply the grow map to $(T, s)$. Column $s$ was $T^{\prime \prime}$ 's special column.

Consider the strip-shrink followed by the strip-grow map. Since $s$ was $T^{\prime}$ 's special column, all up steps of $T^{\prime \prime}$ s upper boundary to the right of $s$ are the top boundary of a one-box tile,
which strip-shrink then removes. Each of these up-steps still exist in $T^{\prime \prime}$ (translated by $(+1,-1))$ and in $T^{\prime}($ translated by $(0,-1))$. The strip-grow map then adds back the one-box tiles at these locations.

Consider the ribbon-shrink followed by the ribbon-grow map. Let $Q$ denote the first column to the right of $s$ for which $T^{\prime}$ does not have a one-box Dyck tile (such a column exists). If $Q>s+1$, then upon removing the topmost one-box Dyck tiles in columns $s+1, \ldots, Q-1$ to get cover-inclusive Dyck tiling $T^{\prime \prime}$, column $Q$ is an eligible column of $T^{\prime \prime}$, and the rightmost such column (since $s<Q$ was special for $T^{\prime}$ ), so $Q$ is $T^{\prime \prime \prime}$ 's special column. Upon shrinking, $Q-1>s$ is $T$ 's special column, so the ribbon-grow map adds back in the one-box Dyck tiles to positions $s+1, \ldots, Q-1$. Otherwise, $s=Q-1$ is $T^{\prime \prime \prime}$ 's special column, ribbon-shrink does not remove any one-box tiles, the special column of $T$ is $\leq s$, and the ribbon-grow map does not add any one-box tiles.

In either the strip or ribbon cases, the spreading of the grow map undoes the contraction of the shrink map, and the grow map then adds one-box Dyck tiles precisely in the positions where the shrink map removed them, so growing $(T, s)$ results in $T^{\prime}$.

The bijections DTS and DTR are given by repeated application of the strip-grow and ribbon-grow maps respectively. More precisely, we first do a minor change of coordinates,

$$
p_{i}=(i-1)+s_{i},
$$

so that $0 \leq p_{i} \leq 2(i-1)$, and then define

$$
\operatorname{DTS}\left(p_{1}, \ldots, p_{n}\right)= \begin{cases}\varnothing & n=0 \\ \operatorname{strip-grow}\left(\operatorname{DTS}\left(p_{1}, \ldots, p_{n-1}\right), p_{n}-(n-1)\right) & n>0\end{cases}
$$

and

$$
\operatorname{DTR}\left(p_{1}, \ldots, p_{n}\right)= \begin{cases}\varnothing & n=0 \\ \text { ribbon-grow }\left(\operatorname{DTR}\left(p_{1}, \ldots, p_{n-1}\right), p_{n}-(n-1)\right) & n>0\end{cases}
$$

Theorem 2.2. The maps DTS and DTR are bijections from integer sequences $p_{1}, \ldots, p_{n}$ with $0 \leq p_{i} \leq 2(i-1)$ to cover-inclusive Dyck tilings of order $n$.

Proof. Immediate from Lemma 2.1.

### 2.2 Comparison of statistics

Next we compare the bijections DTS and DTR to the bijection from integer sequences $p_{1}, \ldots, p_{n}$ to increasing planted plane trees. The strip-grow and ribbon-grow maps introduce a new chord in the lower boundary of the Dyck tiling; the existing chords may be stretched, but their relative order is unchanged. (By induction, the lower boundaries of $\operatorname{DTS}\left(p_{1}, \ldots, p_{n}\right)$ and $\operatorname{DTR}\left(p_{1}, \ldots, p_{n}\right)$ are the same.) If we keep track of the order in which the chords are introduced, the result is a natural labeling $L$ of the chord poset $P_{\lambda}$ of the lower boundary $\lambda$, which together comprise an increasing planted plane tree, as shown in Figure 6. In fact, this increasing planted plane tree is the one given by the standard bijection from sequences
$p_{1}, \ldots, p_{n}$ to increasing planted plane trees. We can represent an increasing planted plane tree by a Dyck path $\lambda$ and a permutation $\sigma \in \mathscr{L}\left(P_{\lambda}\right)$, as shown in Figure 2. Given such a pair $(\lambda, \sigma)$, we define

$$
\operatorname{DTS}(\lambda, \sigma)=\operatorname{DTS}\left(\text { sequence } p_{1}, \ldots, p_{n} \text { which yields labeled tree defined by }(\lambda, \sigma)\right)
$$

and
$\operatorname{DTR}(\lambda, \sigma)=\operatorname{DTR}\left(\right.$ sequence $p_{1}, \ldots, p_{n}$ which yields labeled tree defined by $\left.(\lambda, \sigma)\right)$.
We therefore obtain the following theorem:
Theorem 2.3. For each Dyck path $\lambda$, the maps $\operatorname{DTS}(\lambda, \cdot)$ and $\operatorname{DTR}(\lambda, \cdot)$ are bijections from linear extensions in $\mathscr{L}\left(P_{\lambda}\right)$ to cover-inclusive Dyck tilings with lower path $\lambda$.

Next we compare statistics of the Dyck tiling $T$ to statistics of the permutation $\sigma$. To do this, it is convenient to view an increasing planted plane tree as having its edges labeled rather than its vertices, and when the tree is represented by a Dyck path $\lambda$, for the labels to reside on the up steps of $\lambda$. The preorder word $\left(\sigma^{-1}\right)$ is the listing of these labels in order. Both grow maps will insert a new chord into $\lambda$ at a position $s$. Let $n$ be the order of Dyck path $\lambda$, and let $\lambda^{\prime}$ and $\sigma^{\prime}$ be the Dyck path and linear extension associated with strip-grow $(T, s)$ or ribbon-grow $(T, s)$. The word $\left(\sigma^{\prime}\right)^{-1}$ is just $\sigma^{-1}$ with $n+1$ inserted at the location which is the number of up steps of $\lambda$ to the left of $s$.

Theorem 2.4. For each Dyck path $\lambda$ and linear extension $\sigma \in \mathscr{L}\left(P_{\lambda}\right)$,

$$
\operatorname{art}(\operatorname{DTS}(\lambda, \sigma))=\operatorname{inv}(\sigma)
$$

Proof. By the above discussion,

$$
\operatorname{inv}\left(\left(\sigma^{\prime}\right)^{-1}\right)-\operatorname{inv}\left(\sigma^{-1}\right)=\# \text { up steps of } \lambda \text { to the right of } s .
$$

Notice that with $T^{\prime}=\operatorname{strip}-\operatorname{grow}(T, s)$, we have $\operatorname{tiles}\left(T^{\prime}\right)-\operatorname{tiles}(T)=\left(n-s-\mu_{s}\right) / 2$, which is the number of up steps of $\mu$ to the right of column $s$, and that area $\left(T^{\prime}\right)-\operatorname{area}(T)=$ $\mu_{s}-\lambda_{s}+\left(n-s-\mu_{s}\right) / 2$. (Here $\rho_{s}$ denotes the height of Dyck path $\rho$ in column $s$.) In other words, $\operatorname{art}\left(T^{\prime}\right)-\operatorname{art}(T)=\left(n-s-\lambda_{s}\right) / 2$, which we can write as

$$
\operatorname{art}(\operatorname{strip-grow}(T, s))-\operatorname{art}(T)=\# \text { up steps of } \lambda \text { to the right of } s .
$$

Upon combining these equations and using induction, and using the fact that $\operatorname{inv}\left(\sigma^{-1}\right)=$ $\operatorname{inv}(\sigma)$, the theorem follows.

Theorem 2.5. For each Dyck path $\lambda$ and linear extension $\sigma \in \mathscr{L}\left(P_{\lambda}\right)$,

$$
\operatorname{dis}(\operatorname{DTR}(\lambda, \sigma))=\operatorname{des}(\sigma)
$$

Proof. Let $T^{\prime}=\operatorname{ribbon-grow}(T, s)$. By the above discussion, we see that the $n+1$ occurs before the $n$ in $\sigma^{\prime}$ iff $T^{\prime \prime}$ s special column $s$ is smaller than $T$ 's special column. Thus

$$
\operatorname{des}\left(\sigma^{\prime}\right)-\operatorname{des}(\sigma)= \begin{cases}1 & \text { ribbon-grow map added a ribbon } \\ 0 & \text { otherwise }\end{cases}
$$

Notice that spreading $T$ at $s$ does not change the discrepancy between the lower and upper paths. If the ribbon-grow map added a ribbon to the spread of $T$ at $s$, then $T^{\prime}$ has one extra location, the place between $s$ and $s+1$, where the upper path $\mu^{\prime}$ goes up while the lower path $\lambda^{\prime}$ goes down. Thus $\operatorname{dis}\left(T^{\prime}\right)-\operatorname{dis}(T)=\operatorname{des}\left(\sigma^{\prime}\right)-\operatorname{des}(\sigma)$. Induction completes the proof.

Proof of Theorem 1.1. Immediate from Theorem 2.3 and Theorem 2.4 and the theorem of Björner and Wachs [BW89] on the $q$-hook-length formula for the $q$-distribution of the inversion statistic of linear extensions.

Proof of Theorem 1.2. Immediate from Theorem 2.3 and Theorem 2.5.

## 3 Histoires d'Hermite

Next we compare the growth of the perfect matching match $\left(p_{1}, \ldots, p_{n}\right)$ with the Dyck tilings $\operatorname{DTS}\left(p_{1}, \ldots, p_{n}\right)$ and $\operatorname{DTR}\left(p_{1}, \ldots, p_{n}\right)$. The shape of a perfect matching of $\{1, \ldots, 2 n\}$ is the Dyck path of order $n$ which has an up-step at the location of the smaller item in each pair, and a down-step at the location of the larger items. For each pair $(a, b)$ of the matching, we can record the number of other pairs $(c, d)$ which nest it, i.e., for which $c<a<b<d$. If we record these nesting numbers on the down-steps of the Dyck path, the resulting labeled Dyck path is called an histoire d'Hermite, and the perfect matching can be recovered from it. The down-steps can also be labeled according to crossings, i.e., the number of other pairs $(c, d)$ for which $a<c<b<d$, but for our purposes it is more convenient to work with nestings, and to record the nesting numbers on the up steps of the Dyck path.

Kim [Kim12] and Konvalinka showed how to transform a cover-inclusive Dyck tiling into an histoire d'Hermite, which we illustrate in Figure 7 without defining it formally. Each number on the up step counts the number of tiles of the tiling which are encountered by the exploration process of gray paths (shown in Figure 7), and each tile is encountered exactly once.

Theorem 3.1. The histoire d'Hermite arising from exploring $\operatorname{DTS}\left(p_{1}, \ldots, p_{n}\right)$ from the left is the same as the histoire d'Hermite arising from match $\left(p_{1}, \ldots, p_{n}\right)$ resulting from recording the nesting numbers on the up steps.

Proof. The theorem is trivially true when $n=0$. Suppose that it is true for $n$, and $T$ is a Dyck tiling of order $n$. The strip-grow map modifies the upper boundary $\mu$ of $T$ by inserting an up step at the special column, and appending a down step at the end. When we update the perfect matching, a new smaller element is inserted at the location specified by $p_{n+1}$, which is matched with $2(n+1)$, so by induction, the upper boundary of the Dyck tiling is the shape of the perfect matching. The strip-grow map adds one-box tiles at each of the


Figure 7: The growth of the DTS Dyck tiling from Figure 6 (on left) together with its corresponding perfect matching (on right). The histoire d'Hermite is shown with the perfect matching, where the numbers count nestings, and with the Dyck tiling, where the numbers count tiles.
up steps to the right of the new special column, so the numbers in the associated histoire d'Hermite after the special column are incremented. In the perfect matching, after the new pair is added, the nesting numbers of each pair whose smaller element is to the right of the insertion point are incremented. This completes the induction.

From this we see that the number of tiles in $\operatorname{DTS}\left(p_{1}, \ldots, p_{n}\right)$ equals the nesting number of $\operatorname{match}\left(p_{1}, \ldots, p_{n}\right)$, which in turn is the number of inversions in min-word $\left(\operatorname{match}\left(p_{1}, \ldots, p_{n}\right)\right)$.

Theorem 3.2. If $(\lambda, \sigma)$ is the labeled tree arising from $\left(p_{1}, \ldots, p_{n}\right)$, then

$$
\operatorname{DES}(\sigma)=\operatorname{DES}\left(\min -\operatorname{word}\left(\operatorname{match}\left(p_{1}, \ldots, p_{n}\right)\right)\right)
$$

In particular, $\operatorname{dis}\left(\operatorname{DTR}\left(p_{1}, \ldots, p_{n}\right)\right)=\operatorname{des}\left(\min -\operatorname{word}\left(\operatorname{match}\left(p_{1}, \ldots, p_{n}\right)\right)\right)$.
Proof. Recalling the construction of the perfect matching, we see that $i$ is a descent of the min-word when $p_{i+1} \leq p_{i}$. In the construction of the increasing planted plane tree, this is precisely when the node labeled $i+1$ occurs to the left of the node labeled $i$ in the left-to-right depth-first search order, which occurs precisely when $\sigma_{i+1}<\sigma_{i}$.

## 4 Dyck tableaux

In this section we explain how the DTR bijection from Dyck tilings to perfect matchings is related to the work by Aval, Boussicault, and Dasse-Hartaut ABDH13] on what they call Dyck tableaux. We make use of a bijection from Dyck tilings of a skew shape $\lambda / \mu$ to bounded weakly increasing labelings of the planted plane tree associated with $\lambda$, which is illustrated in Figure 8. This bijection appears in [KW11a, Prop. 1.11] in the case $\lambda=$ zigzag $_{n}$, and in [SZJ12, Sect. 4] for general skew shapes. For the reader's convenience, we review the bijection in Proposition 4.1.


Figure 8: On the left is a cover-inclusive Dyck tiling of a certain skew shape $\lambda / \mu$, in which the chords of $\lambda$ have been labeled according to the number Dyck tiles above the chord which cover both endpoints of the chord. Given $\lambda / \mu$ and these labels, the cover-inclusive Dyck tiling can be recovered. These chord labels define a weakly increasing labeling of the tree poset associated with $\lambda$, as shown in the middle. On the right is shown the maximum values of these labels for any cover-inclusive Dyck tiling of $\mathcal{D}(\lambda, \mu)$.

Proposition 4.1 ([SZJ12]). Let $\lambda / \mu$ be a skew shape. For each chord $c$ of the lower Dyck path $\lambda$, let $h_{c}$ denote the minimal thickness of the portion of the skew shape $\lambda / \mu$ between the endpoints of the chord, i.e., $h_{c}$ is the maximal number of Dyck tiles that can fit in the shape $\lambda / \mu$ and cover chord c. There is a bijection between the cover-inclusive Dyck tilings $T$ of $\lambda / \mu$ and the weakly increasing assignments of nonnegative integers to the poset of chords $P_{\lambda}$, such that the number $g_{c}$ assigned to chord c satisfies $0 \leq g_{c} \leq h_{c}$. This bijection satisfies $\sum_{c} g_{c}=(\operatorname{area}(T)-\operatorname{tiles}(T)) / 2$.

Proof. First observe that in any cover-inclusive Dyck tiling of skew shape $\lambda / \mu$, every tile is shaped like the portion of the lower boundary $\lambda$ directly beneath it. We can assign to chord $c$ a number $g_{c}$, where $0 \leq g_{c} \leq h_{c}$, which encodes the number of tiles directly above $c$ in which the boxes in columns $\ell$ and $r$ are in the same tile, as shown in Figure 8. Since we are tiling with Dyck tiles, if a chord $c^{\prime}$ is above chord $c$, then $g_{c^{\prime}} \geq g_{c}$, so this labeling of the chord poset $P_{\lambda}$ is weakly increasing.

The map from weakly increasing labelings of the chords in $P_{\lambda}$ to Dyck tilings is as follows. Inductively we add Dyck tiles on top of chords, starting from the lowest chords. If $c_{1}$ covers $c_{2}$ in the poset, then we add $g_{c_{1}}-g_{c_{2}} \geq 0$ new Dyck tiles whose endpoints are exactly above the endpoints of $c_{1}$. By construction, this ensures cover-inclusiveness since smaller tiles are added on top of larger ones. By definition of $h_{c}$, all added tiles will fit in $\lambda / \mu$.

It is straightforward to check that these maps are inverses.
The tree structure leads to a recursive algorithm for enumerating the number of bounded weakly increasing labelings of trees by the statistic $\sum_{c} g_{c}$, which Lascoux and Schützenberger showed to be equivalent to computing Kazhdan-Lusztig polynomials for pairs of Grassmannian permutations [S81].

Using the above bijection, we see that cover-inclusive Dyck tilings are a natural generalization of the Dyck tableaux recently introduced by Aval, Boussicault, and Dasse-Hartaut in ABDH13. They defined a Dyck tableau of order $n$ to be a skew shape between the zig-zag path zigzag ${ }_{n+1}$ and upper Dyck path $\mu$ containing exactly $n$ dots, such that each column of the skew shape going through the valleys of zigzag $_{n+1}$ contain exactly one dot, as shown in Figure 9.

Proposition 4.2. There is a bijection between the cover-inclusive Dyck tilings whose lower path is the zig-zag path $\operatorname{zigzag}_{n}=(U D)^{n}$ of $n$ up-down steps and Dyck tableaux of order $n$.


Figure 9: The bijection between cover-inclusive Dyck tilings with $\lambda=\operatorname{zigzag}_{n}$ and Dyck tableaux. The dot heights encode the chord labels.

Proof. Let the dot-height of a dot in zigzag $_{n+1} / \mu$ be the number of boxes in the column of the dot, which are in zigzag $_{n+1} / \mu$ and below the dot. These dot heights are naturally associated with the chords of zigzag $_{n}$, and are independent of one another. Now $P_{\text {zigzag }_{n}}$ is just the antichain on $n$ points, so the weakly increasing condition is vacuously true, and the maximum dot heights are precisely the maximum number of Dyck tiles that can fit within $\lambda / \mu$ and cover the chord associated with the dot's column (see Figure 9). Thus, Dyck tilings with lower path zigzag ${ }_{n}$ and Dyck tableaux of order $n$ are different representations of the same object.

With this interpretation of Dyck tableaux as cover-inclusive Dyck tilings in $\mathcal{D}\left(\right.$ zigzag $\left._{n}, *\right)$, the bijection $\operatorname{DTR}\left(\operatorname{zigzag}_{n}, \cdot\right)$ is equivalent to the bijection given in ABDH13.

## 5 Dyck tilings and 231-avoiding permutations

We consider Dyck tilings whose lower path $\lambda$ is the zigzag path, i.e., $\lambda=\operatorname{zigzag}_{n}=(\mathrm{UD})^{n}$. The poset $P_{\lambda}$ is just an antichain of $n$ points, so that its linear extensions are exactly the permutations on $n$ letters, i.e., $\mathscr{L}\left(P_{\text {zigzag }_{n}}\right)=S_{n}$, the symmetric group on $n$ letters. We consider 231-avoiding permutations in the usual sense - namely, permutations $\sigma$, such that there are no indices $i<j<k$ for which $\sigma_{k}<\sigma_{i}<\sigma_{j}$.
 231-avoiding permutations in $S_{n}$ and Dyck tilings whose lower path is zigzag $_{n}$ and which contain only one-box tiles.

Proof. The map DTS places a tile above a position iff there is an element to the left of the position that is larger than an element to the right of the position, i.e., if it is surrounded by a " $2,1$. ." Thus, the image of DTS has no long tiles iff there are no 231 's in the permutation. The map DTR places a tile above a position iff there is an element to the left of the position which is one larger than an element to the right of the position. But this occurs precisely when there is an element to the left of the position that is larger than an element to the right of the position. Thus the image of DTR also has no long tiles iff there are no 231's in the permutation.

Reflecting $\mu$ about the $y$-axis and reversing the permutation $\sigma$, i.e., $\sigma_{n}, \ldots, \sigma_{1}$, gives a bijection between Dyck paths and 132-avoiding permutations. For the case DTR, this bijection has been given by Knuth in [Knu75, Problem 2.2.1-4].

## 6 Dyck tilings and the mad statistic

In this section we relate Dyck tilings to the permutation statistic mad, which was defined by Clarke, Steingrímsson, and Zeng [CSZ97], and whose definition we now review. For a word $w=w_{1} \cdots w_{n}$ of order $n$, the descent set of $w$ is denoted by

$$
\operatorname{DES}(w)=\left\{i<n: w_{i}>w_{i+1}\right\}
$$

The statistic mad of a permutation $\sigma$ is then defined by

$$
\begin{aligned}
\operatorname{desdif}(\sigma) & =\sum_{i \in \operatorname{DES}(\sigma)}\left(\sigma_{i}-\sigma_{i+1}\right), \\
\operatorname{res}(\sigma) & =\sum_{i \in \operatorname{DES}(\sigma)} \#\left\{k<i: \sigma_{i}>\sigma_{k}>\sigma_{i+1}\right\}, \\
\operatorname{mad}(\sigma) & =\operatorname{desdif}(\sigma)+\operatorname{res}(\sigma) .
\end{aligned}
$$

This can also be written as

$$
\operatorname{mad}(\sigma)=\sum_{i \in \operatorname{DES}(\sigma)}\left[1+\#\left\{k>i+1: \sigma_{i}>\sigma_{k}>\sigma_{i+1}\right\}+2 \times \#\left\{k<i: \sigma_{i}>\sigma_{k}>\sigma_{i+1}\right\}\right]
$$

Clarke et al. gave a bijective proof that mad is a Mahonian statistic CSZ97, i.e., that it is equidistributed with inv.

In this section we prove
Theorem 6.1. For each permutation $\sigma$ of order $n$,

$$
\operatorname{art}\left(\operatorname{DTR}^{\left.\left(\operatorname{zigzag}_{n}, \sigma\right)\right)=\operatorname{mad}(\sigma) . . . ~}\right.
$$

When we combine Theorem 6.1 with Theorem 2.4, we obtain an involution

$$
\operatorname{DTS}\left(\operatorname{zigzag}_{n}, \cdot\right)^{-1} \circ \operatorname{DTR}^{\left(\operatorname{zigzag}_{n}, \cdot\right)}
$$

on permutations of order $n$ which goes by way of Dyck tilings and which maps mad to inv. This involution of course shows that mad is equidistributed with inv. We do not see any connection between this involution and the one given by Clarke et al.

Lemma 6.2. Suppose $\lambda$ is a Dyck tiling and $\sigma \in \mathscr{L}\left(P_{\lambda}\right)$. In the increasing planted plane tree associated with $\lambda$ and $\sigma$, let $\ell_{i}$ and $r_{i}$ be the left and right endpoints of the chord of $P_{\lambda}$ labeled $i$, as shown in Figure 2. Letting $T=\operatorname{DTR}(\lambda, \sigma)$, we have

$$
\begin{align*}
& \operatorname{area}(T)=\sum_{i \in \operatorname{DES}(\ell)}\left(\ell_{i}-r_{i+1}\right)  \tag{4}\\
& \operatorname{tiles}(T)=\sum_{i \in \operatorname{DES}(\ell)}\left(\ell_{i}-r_{i+1}-2 \times \#\left\{j>i+1: r_{i+1}<\ell_{j}<\ell_{i}\right\}\right), \tag{5}
\end{align*}
$$

where $\ell$ denotes the word $\ell_{1} \cdots \ell_{n}$.
Proof. We let $\ell_{i}^{(k)}$ and $r_{i}^{(k)}$ denote the left and right endpoints of the chord labeled $i$ after the $k^{\text {th }}$ ribbon-grow map. Recall that $p_{1}, \ldots, p_{n}$ gives the sequence of growth locations, and that $s_{i}=p_{i}-(i-1)$. We have $\ell_{i}^{(i)}=p_{i}+1$ and $r_{i}^{(i)}=p_{i}+2$. Suppose $i>1$. If $s_{i} \geq s_{i-1}$, then the $i^{\text {th }}$ ribbon-grow map adds no new one-box tiles, $\ell_{i}^{(i)}>\ell_{i-1}^{(i)}$, and so $\ell_{i}^{(n)}>\ell_{i-1}^{(n)}$, so $i-1 \notin \operatorname{DES}\left(\ell^{(n)}\right)$. If on the other hand $s_{i}<s_{i-1}$, then the number of one-box tiles that the $i^{\text {th }}$ ribbon-grow map adds is $s_{i-1}-s_{i}=p_{i-1}-p_{i}-1=\ell_{i-1}^{(i)}-r_{i}^{(i)}$, and $i-1 \in \operatorname{DES}\left(\ell^{(n)}\right)$. Later on in the growth process, new chords may get added between $r_{i}$ and $\ell_{i-1}$, which of course
increases the difference between them. For $j>i$, this happens iff $r_{i}^{(j)}<\ell_{j}^{(j)}<r_{j}^{(j)}<\ell_{i-1}^{(j)}$, which happens iff $r_{i}^{(j)}<\ell_{j}^{(j)}<\ell_{i-1}^{(j)}$, which happens iff $r_{i}^{(n)}<\ell_{j}^{(n)}<\ell_{i-1}^{(n)}$, and when this happens the distance between $r_{i}$ and $\ell_{i-1}$ increases by 2 (i.e., $r_{i}^{(j)}-\ell_{i-1}^{(j)}=r_{i}^{(j-1)}-\ell_{i-1}^{(j-1)}+2$ ) and otherwise increases by 0 . This establishes the tiles formula (5). Notice that any such chord $j$ also increases by 2 the area of one of the tiles produced by chord $i$. This establishes the area formula (4).

For a word $w$, the inversion set of $w$ is denoted by

$$
\operatorname{INV}(w)=\left\{(i, j): i<j \text { and } w_{j}<w_{i}\right\}
$$

Lemma 6.3. For a Dyck path $\lambda$ and a natural labeling $L$ of $P_{\lambda}$, let $\ell_{i}$ and $r_{i}$ denote the left and right endpoints of the chord labeled $i$. Then $\operatorname{INV}(\ell) \subset \operatorname{INV}(r)$. In particular, if $\sigma$ and $\tau$ are the standardizations of $\ell$ and $r$ respectively, then $\operatorname{INV}(\sigma) \subset \operatorname{INV}(\tau)$.

Proof. Suppose $i<j$. Let $x$ and $y$ denote the chords of $P_{\lambda}$ which are labeled $i$ and $j$ by $L$. Since $L$ is natural, either $x<y$ in $P_{\lambda}$, or $x$ and $y$ are incomparable in $P_{\lambda}$.

If $x<y$ in $P_{\lambda}$, then $\ell_{i}<\ell_{j}<r_{j}<r_{i}$. Thus $(i, j)$ is not an inversion of $\ell$, but it is an inversion of $r$.

If $x$ and $y$ are incomparable in $P_{\lambda}$, then either $\ell_{i}<r_{i}<\ell_{j}<r_{j}$, or else $\ell_{j}<r_{j}<\ell_{i}<r_{i}$. In this case, $(i, j)$ is either an inversion in both $\ell$ and $r$, or an inversion in neither of them.

If $\sigma$ and $\tau$ are two permutations on $[n]$ with $\operatorname{INV}(\sigma) \subset \operatorname{INV}(\tau)$, then we define

$$
\operatorname{desdif}(\sigma, \tau)=\sum_{i \in \operatorname{DES}(\sigma)}\left(\sigma_{i}-\sigma_{i+1}+\tau_{i}-\tau_{i+1}\right)
$$

For a word $w=w_{1} \cdots w_{n}$ and a descent $i \in \operatorname{DES}(w)$, we define the set $\operatorname{REM}_{i}(w)$ of right embraced numbers of $w$ with respect to descent $i$ by

$$
\operatorname{REM}_{i}(w)=\left\{k>i: w_{i}>w_{k}>w_{i+1}\right\}
$$

Lemma 6.4. Let $\lambda$ be a Dyck path, $\sigma \in \mathscr{L}\left(P_{\lambda}\right)$, and $T=\operatorname{DTR}(\lambda, \sigma)$. Let $\ell_{i}$ and $r_{i}$ denote the left and right endpoints of the chord labeled $i$ in the increasing planted plane tree associated with $\lambda, \sigma$. Recall that $\sigma$ is the standardization of $\ell$, and let $\tau$ denote the standardization of $r$. Then

$$
\begin{align*}
& \operatorname{area}(T)=\operatorname{desdif}(\sigma, \tau)-\operatorname{des}(\sigma)-\sum_{i \in \operatorname{DES}(\sigma)}\left|\operatorname{REM}_{i}(\sigma) \Delta \operatorname{REM}_{i}(\tau)\right|  \tag{6}\\
& \operatorname{tiles}(T)=\operatorname{desdif}(\sigma, \tau)-\operatorname{des}(\sigma)-\sum_{i \in \operatorname{DES}(\sigma)}\left(\left|\operatorname{REM}_{i}(\sigma)\right|+\left|\operatorname{REM}_{i}(\tau)\right|\right) \tag{7}
\end{align*}
$$

where $A \Delta B=(A \cup B) \backslash(A \cap B)$.
Proof. Suppose $i \in \operatorname{DES}(\ell)$, i.e., $\ell_{i+1}<\ell_{i}$. Then in fact $\ell_{i+1}<r_{i+1}<\ell_{i}<r_{i}$. To characterize $\ell_{i}-r_{i+1}$, consider any other chord, say with label $j(j \neq i$ and $j \neq i+1)$, which has at least one step between $r_{i+1}$ and $\ell_{i}$. There are three cases as follows:

Case 1: Both $\ell_{j}$ and $r_{j}$ are between $r_{i+1}$ and $\ell_{i}$, i.e., $\ell_{i+1}<r_{i+1}<\ell_{j}<r_{j}<\ell_{i}<r_{i}$. Because the chords are noncrossing, this case happens iff both $\ell_{i+1}<\ell_{j}<\ell_{i}$ and $r_{i+1}<r_{j}<r_{i}$.

Case 2: Only $\ell_{j}$ is between $r_{i+1}$ and $\ell_{i}$, which happens iff $\ell_{i+1}<r_{i+1}<\ell_{j}<\ell_{i}<r_{i}<r_{j}$. It is easy to see that this case occurs iff $j<i$ and $\ell_{i+1}<\ell_{j}<\ell_{i}$ and $r_{i+1}<r_{i}<r_{j}$.

Case 3: Only $r_{j}$ is between $r_{i+1}$ and $\ell_{i}$. This case is similar to case 2, and occurs iff $j<i$ and $\ell_{j}<\ell_{i+1}<\ell_{i}$ and $r_{i+1}<r_{j}<r_{i}$.

By considering chords $j$ that fall into one of these three cases, and using the fact that $\sigma$ is the standardization of $\ell$ and $\tau$ is the standardization of $r$, we obtain

$$
\begin{align*}
& \ell_{i}-r_{i+1}=1+2 \times \#\left\{j: \sigma_{i+1}<\sigma_{j}<\sigma_{i} \text { and } \tau_{i+1}<\tau_{j}<\tau_{i}\right\} \\
&+\#\left\{j<i: \sigma_{i+1}<\sigma_{j}<\sigma_{i} \text { and } \tau_{i+1}<\tau_{i}<\tau_{j}\right\} \\
&+\#\left\{j<i: \sigma_{j}<\sigma_{i+1}<\sigma_{i} \text { and } \tau_{i+1}<\tau_{j}<\tau_{i}\right\} \\
&=1+\left(\sigma_{i}-\sigma_{i+1}-1\right)+\left(\tau_{i}-\tau_{i+1}-1\right) \\
&-\#\left\{j>i: \sigma_{i+1}<\sigma_{j}<\sigma_{i} \text { and } \tau_{i+1}<\tau_{i}<\tau_{j}\right\} \\
&-\#\left\{j>i: \sigma_{j}<\sigma_{i+1}<\sigma_{i} \text { and } \tau_{i+1}<\tau_{j}<\tau_{i}\right\} \\
&=\sigma_{i}-\sigma_{i+1}+\tau_{i}-\tau_{i+1}-1-\left|\operatorname{REM}_{i}(\sigma) \Delta \operatorname{REM}_{i}(\tau)\right| . \tag{8}
\end{align*}
$$

Upon summing over descents of $\sigma$ and using (4), we obtain (6).
Recall equation (5). For $j>i+1$, we have $r_{i+1}<\ell_{j}<\ell_{i}$ if and only if $r_{i+1}<r_{j}<\ell_{i}$, which in turn occurs if and only if both $\ell_{i+1}<\ell_{j}<\ell_{i}$ and $r_{i+1}<r_{j}<r_{i}$. Thus

$$
\#\left\{j>i+1: r_{i+1}<\ell_{j}<\ell_{i}\right\}=\left|\operatorname{REM}_{i}(\sigma) \cap \operatorname{REM}_{i}(\tau)\right|
$$

and combining this with (5) and (8) yields (77).
Proof of Theorem 6.1. We use Lemma 6.4, and observe that when $\lambda=\operatorname{zigzag}_{n}$ we have $\sigma=\tau$, so when we evaluate $\operatorname{art}(T)=(\operatorname{area}(T)+\operatorname{tiles}(T)) / 2$, we obtain

$$
\operatorname{art}\left(\operatorname{DTR}\left(\operatorname{zigzag}_{n}, \sigma\right)\right)=\operatorname{desdif}(\sigma, \sigma)-\operatorname{des}(\sigma)-\sum_{i \in \operatorname{DES}(\sigma)}\left|\operatorname{REM}_{i}(\sigma)\right|
$$

Now $\operatorname{desdif}(\sigma, \sigma)=2 \times \operatorname{desdif}(\sigma)$, and it is easy to see that

$$
\operatorname{res}(\sigma)=\operatorname{desdif}(\sigma)-\operatorname{des}(\sigma)-\sum_{i \in \operatorname{DES}(\sigma)} \#\left\{k>i: \sigma_{i+1}<\sigma_{k}<\sigma_{i}\right\}
$$

so we obtain

$$
\operatorname{art}\left(\operatorname{DTR}\left(\operatorname{zigzag}_{n}, \sigma\right)\right)=\operatorname{desdif}(\sigma)+\operatorname{res}(\sigma)=\operatorname{mad}(\sigma)
$$

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