Chromatic numbers of Kneser-type graphs

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Abstract

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that there are two different types of independent sets¹ in these graphs. The first example of an independent set in G(n, r, s) is a star: the family of sets containing a fixed (s+1)-set. It has cardinality $\binom{n-s-1}{r-s-1}$ which is asymptotically $\frac{n^{r-s-1}}{(r-s-1)!} = \Theta(n^{r-s-1})$ if r, s are fixed. It was proved by Frankl and Füredi [FF] that if r > 2s + 1 and n is sufficiently large then the star is the maximal independent set in G(n, r, s). Recently the result has been significantly extended in [KL] to the regime where s is fixed and C < r < n/C for some absolute constant C.

In the case $r \leq 2s + 1$ the maximal independent sets can not be classified is any reasonable way. All known constructions come from design theory (the connection will be indicated in Section 2) and explicit constructions are known in very special cases only. In particular, Rödl [Rö], using a probabilistic method, showed that $\alpha(G(n,r,s)) \ge (1+o(1))n^s \frac{(2r-2s-1)!}{r!(r-s-1)!} = \Theta(n^s)$ (see [BKK] for more details). The matching

¹Recall that the set of vertices of a graph is independent if any two of its vertices are not adjacent. The independence number $\alpha(G)$ is the size of the largest independent set of G. For any graph G one has $\chi(G) \ge |V|/\alpha(G)$.

upper bound is known if (r - s) is a power of a prime [FW] and it is a major open problem to obtain the same upper bound without the assumption that (r - s) is a power of a prime.

The intermediate case r = 2s + 1 is of particular interest. For this choice of parameters both star and design-type constructions give lower bounds of the same order of magnitude $\Theta(n^s)$, so one may wonder which construction gives a better estimate. Miraculously, both constructions result in exactly the same bound $\alpha(G(n, 2s + 1, s)) \ge (1 + o(1))\frac{n^s}{s!}$ (although, the *o*-term is slightly better in the design-type construction).

It is not difficult to see that the chromatic number of G(n, r, s) has the order of magnitude $\Theta(\frac{n^r}{\alpha(G(n, r, s))})$ which explains the aforementioned dichotomy. The problem now is to determine the constant C (depending on r, s) such that $\chi(G(n, r, s)) \sim Cn^{\min\{s+1, r-s\}}$.

For r > 2s + 1 all known upper bounds on $\chi(G(n, r, s))$ are obtained using Turán numbers (see [BKK], [S]). The author is unaware of any improvements of the trivial inequality $\chi(G(n, r, s)) \ge {n \choose r} / \alpha(G(n, r, s)) \sim n^{s+1} \frac{(r-s-1)!}{r!}$ for r > 2s + 1, except for the case s = 0 in which the chromatic number is known exactly and equals to $\chi(G(n, r, 0)) = n - 2r + 2$, $(n \ge 2r)$ (this is a celebrated result of Lovász [L]).

In this paper we will consider the region $r \leq 2s+1$. The simplest upper bound on the chromatic number of G(n, r, s) is this case is the maximal degree bound: $\chi(G(n, r, s)) \leq \Delta(G(n, r, s)) + 1 \sim n^{r-s} \frac{r!}{s!((r-s)!)^2}$ which already gives the correct order of magnitude. Some non-trivial estimates were obtained by the author in [Z], for instance, $\chi(G(n, r, s)) \leq (1 + o(1))n^{r-s}$ which improves the maximal degree bound if (r-s) is less than \sqrt{s} . The first result of the present paper is the following sharp result.

Theorem 1.1. Let r > s. Then $\chi(G(n, r, s)) \leq (1 + o(1))n^{r-s} \frac{(r-s-1)!}{(2r-2s-1)!}$ as $n \to \infty$.

Note that if $r \leq 2s + 1$ and (r - s) is a power of a prime then the bound in Theorem 1.1 coincides with known lower bounds. In fact, Theorem 1.1 is a simple corollary of recent results of Keevash [K2]. Results of [K2] require n to be extremely large compared to r, s whereas in most of the applications of graphs G(n, r, s) one needs to consider r, s growing with n. Moreover, in applications to combinatorial geometry one typically requires r, s to grow linearly with n.

The main aim of this paper is to present a different approach to the problem. Namely, we develop a new elementary approach and solve the special case (r, s) = (4, 2), which is the first unsolved case in the region $r \leq 2s + 1$. The best known upper bound on the chromatic number of G(n, 4, 2) is $\frac{n^2}{2} + 100n$ [BKK]. Note that if we consider the family of vertices of G(n, 4, 2) which contain element $\{1\}$, the induced subgraph will be isomorphic to G(n - 1, 3, 1). This means that any proper coloring of G(n, 4, 2) will automatically lead to a proper coloring of G(n, 3, 1), so it is important to understand how to color G(n - 1, 3, 1) first. In Section 3.1 we provide a simple proof of the inequality $\chi(G(n, 3, 1)) \leq \frac{(n-1)(n-2)}{6}$ for $n = 2^t$ being a power of 2.

Theorem 1.2. $\chi(G(n, 4, 2)) \leq (1 + o(1))\frac{n^2}{6}$.

Of course, Theorem 1.2 is a particular case of Theorem 1.1 but techniques developed in the proof of this result may be of independent interest.

In addition, in order to prove Theorem 1.2 we need to estimate the *list chromatic number* of G(n, r, s). Recall that the list chromatic number $\chi_{list}(G)$ of a graph G is the minimal number k such that for any arrangement of sets L(v), $v \in V(G)$, each L(v) of size k, there are colors $c(v) \in L(v)$ such that each edge is not monochromatic. In the end of Section 3.6 we prove the following.

Lemma 1.3. Fix r, s and let $n \to \infty$, then $\chi_{list}(G(n, r, s)) = O(n^{s+1} \log n)$.

It would be interesting to obtain more estimates on $\chi_{list}(G(n, r, s))$ in various asymptotic regimes as well.

In Section 2 we prove Theorem 1.1 and in Section 3.2 we prove Theorem 1.2, Section 4 contains some final remarks.

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2 Proof of Theorem 1.1

Recall from [K2] that a family \mathcal{F} of *r*-element subsets of an *n*-set *X* forms an (n, r, s)-design if every *s*-element subset of [n] belongs to exactly one element of \mathcal{F} . A complete resolution of $\binom{[n]}{r}$ is a partition of $\binom{[n]}{r}$ into (n, r, r - 1)-designs, each of which is partitioned into (n, r, r - 2)-designs, and so on, down to (n, r, 1)-designs.

Keevash [K2] proved that a complete resolution of $\binom{[n]}{r}$ always exists provided that $n \equiv r \pmod{\gcd[r]}$ and n is sufficiently large.

In particular, given n as above for any p > q there is an (n, r, p)-design which can be decomposed into (n, r, q)-designs.

We relate decompositions of designs and colorings of G(n, r, s) by the following simple claim.

Claim 2.1. Suppose that there is an (n, 2r-s-1, r)-design which can be decomposed into (n, 2r-s-1, s)-designs. Then $\chi(G(n, r, s)) \leq N := \frac{\binom{n}{r}}{\binom{2r-s-1}{r}} / \frac{\binom{n}{s}}{\binom{2r-s-1}{s}}$.

Proof. If possible, take a decomposition of an (n, 2r - s - 1, r)-design into (n, 2r - s - 1, s)-designs: $\mathcal{D} = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_N$ where N is as in the statement of the claim. As \mathcal{D} is an (n, 2r - s - 1, r)-design, this decomposition induces a decomposition of $\binom{[n]}{r}$ into N classes: an r-set A belongs to a class *i* if there is a set $X \in \mathcal{D}_i$ such that $A \subset X$. If two r-sets A, B belong to the same class *i* then there are $X, Y \in \mathcal{D}_i$ such that $A \subset X, B \subset Y$. Because \mathcal{D}_i is an (n, 2r - s - 1, s)-design we have either X = Y and $|A \cap B| \ge s + 1$ or $|X \cap Y| \le s - 1$ and $|A \cap B| \le s - 1$. In both cases A and B are not connected by an edge and we have constructed a proper coloring of G(n, r, s) using N colors.

For any sufficiently large n we take n' such that $n+r! \ge n' \ge n$ and $n' \equiv r \pmod{gcd[r]}$. By Keevash's theorem and the above claim we obtain

$$\chi(G(n,r,s)) \leqslant \chi(G(n',r,s)) \leqslant (1+o(1))n'^{r-s} \frac{r!(r-s-1)!}{r!(2r-s-1)!} / \frac{s!(2r-2s-1)!}{s!(2r-s-1)!} = (1+o(1))n^{r-s} \frac{(r-s-1)!}{(2r-2s-1)!}$$

3 Proof of Theorem 1.2

3.1 Sketch of the proof

In the proof of Theorem 1.1 we colored graph G(n, r, s) using a decomposition of a certain design into other designs. In the case of G(n, 4, 2) our strategy will be the same but the main difficulty is to construct the required designs without use of heavy machinery. To color G(n, 4, 2) we need to build a decomposition of an (n, 5, 4)-design into (n, 5, 2)-designs, but we will be able to provide only an approximate version of this decomposition and this will suffice for our purposes. We call a family $\mathcal{F} \subset {[n] \choose r}$ an approximate (n, r, s)-design if each s-element set is contained in at most one set from \mathcal{F} and if the number of s-element subsets which are not contained in any set of \mathcal{F} is $o({n \choose s})$.

Our proof is largely inspired by the proof of the following simple result about G(n, 3, 1):

Theorem 3.1 ([BKR]). Let $n = 2^t$. Then $\chi(G(n, 3, 1)) \leq \frac{(n-1)(n-2)}{6}$.

Note that this bound is tight (see [BKK]).

Proof. Identify [n] with $V = \mathbb{F}_2^t$. Let us say that two triples of vectors are equivalent if one of them can be obtained from another by a translation. It is easy to see that there are exactly $\binom{|V|}{3}/|V| = \frac{(n-1)(n-2)}{6}$ equivalence classes.

Now we prove that each class form an independent set, i.e. any two sets from the same class intersect in an even number of elements. Take a pair of equivalent triples $\{a, b, c\}, \{a, d, e\}$, by definition we have $\{a, b, c\} + v = \{a, d, e\}$ for some $v \in V \setminus 0$. Note that $a + b \neq a + d, a + e$, so a + b = d + e because $a + b = (a + v) + (b + v) \in \{a + d, a + e, d + e\}$. By the same reasoning a + c = d + e and, therefore, b = c. A contradiction.²

Now we describe ideas of the proof.

At first we note that we may assume that $n = p^2 - 1$, where p is a prime. So we can identify [n] with the set $\mathbb{F}_p^2 \setminus 0$. Denote $V = \mathbb{F}_p^2$, $\mathfrak{G} = GL_2(\mathbb{F}_p)$ and consider the family $\mathcal{A} = \{\{v_1, \ldots, v_5\} : v_i \in V \setminus 0, v_1 + \ldots + v_5 = 0, v_1, \ldots, v_5 \text{ are pairwise non-collinear}\}$. It is easy to see that \mathcal{A} is an approximate (n, 5, 4)-design and $|\mathcal{A}| \sim \frac{n^4}{120}$.

First, we show how to color in a small number of colors all 4-element sets which are contained in sets from \mathcal{A} . In order to color them properly it is enough to color sets from \mathcal{A} in such a way that two sets of the same color intersect in at most 1 element. To do this, we divide \mathcal{A} into orbits under the action of \mathfrak{G} : $\mathcal{A} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_l$. We note that $l \sim \frac{n^2}{120}$ because $|\mathcal{A}| \sim \frac{n^4}{120}$ and $|\mathfrak{G}| \sim n^2$. The idea is to color each orbit separately from the others. The main observation is that inside of each orbit the induced subgraph has a very special structure: to see this, let $A_1, A_2 \in \mathcal{A}_i$ be an adjacent pair of vertices from the same orbit, that is $|A_1 \cap A_2| \ge 2$; since \mathcal{A}_i is an orbit, there is $g \in \mathfrak{G}$ such that $A_2 = gA_1$, so the map g maps a pair of elements of A_1 to another pair of elements of A_1 . Since any linear map on a vector space is completely determined by its action on a basis elements, this in particular implies that the degree of any vertex in \mathcal{A}_i is always at most $20 \cdot 19$. Let E_i be the set of $g \in \mathfrak{G}$ such that $|gA_1 \cap A_1| \ge 2$. We conclude that the graph induced on \mathcal{A}_i has the structure of a Cayley graph on \mathfrak{G} with the set of generators E_i .

For most of the orbits \mathcal{A}_i we are able to gain enough control on the local structure of this Cayley graph using algebraic tools. This allows us to construct a proper coloring of a large neighborhood of any vertex in the orbit \mathcal{A}_i . But there may be some "global" obstructions to extend these "local" colorings to the whole orbit. To overcome this, we construct a reasonably small set $\mathcal{A}^{wall} \subset \mathcal{A}$ such that for almost all orbits \mathcal{A}_i the set $\mathcal{A}_i \setminus \mathcal{A}^{wall}$ splits into tiny components for which the local coloring can be applied. It remains to color the set \mathcal{A}^{wall} and all remaining orbits which have not yet been colored. This can be done by estimating the maximal degrees of the corresponding induced subgraphs and using other crude bounds. In particular, the following simple observation is used.

Claim 3.2. Let H be a graph. Suppose that the vertex set of H is covered by a system of subsets: $V(H) = \bigcup_{i=1}^{m} A_i$. Suppose that for any $v \in V(H)$ the number of edges between v and the set $\bigcup_{A_i \neq v} A_i$ is at most d. Suppose that for any $i \chi_{list}(H|_{A_i}) \leq l$ holds. Then $\chi(G) \leq l + d$.

3.2 Beginning of the proof

By Prime Number Theorem, for any natural *n* there is a prime *p* such that $p^2 - 1 \ge n$ and $p^2 - 1 \sim n$. Because G(n, 4, 2) can be embedded into $G(p^2 - 1, 4, 2)$ we may assume that $n = p^2 - 1$. Consequently, we may identify [n] with $\mathbb{F}_p^2 \setminus 0$. Denote $V = \mathbb{F}_p^2$ and let $\mathfrak{G} = GL_2(\mathbb{F}_p)$ be the general linear group of *V*.

²The presented proof differs from that given in [BKR]. I found this proof on the cite of Moscow Mathematical Olympiad (https://olympiads.mccme.ru/mmo/2012/75mmo.pdf, page 44, in Russian). The original proof is by induction on the power of 2, and it generalizes to the bound $\chi(G(n,3,1)) \leq \frac{n(n-1)}{6} + cn$ for arbitrary n [BKR]. Consequently, in [BKK] the same idea was applied to G(n,4,2) and the bound $\chi(G(n,4,2)) \leq \frac{n^2}{2} + 100n$ was obtained. The last inequality is the best previously known upper bound for the chromatic number of G(n,4,2).

First, we introduce some notation. Let $\mathcal{A} = \{\{v_1, \ldots, v_5\}: v_i \in V \setminus 0, v_1 + \ldots + v_5 = 0, v_i, v_j \text{ are not collinear}\}$ and, similarly, $\mathcal{A}_{ord} = \{(v_1, \ldots, v_5) : v_i \in V \setminus 0, v_1 + \ldots + v_5 = 0, v_i, v_j \text{ are not collinear}\}$ (note that $\mathcal{A} \subset {V \choose 5}$ but $\mathcal{A}_{ord} \subset V^5$). There is a natural projection $\pi : \mathcal{A}_{ord} \to \mathcal{A}$.

Throughout the proof we use indexations like d_{ij} or \prod_{ij} omitting the range of i, j. Unless otherwise specified, it means that the range is $1 \leq i, j \leq 5$ and $i \neq j$. For any family \mathcal{F} and a set S we denote by $\mathcal{F}(S)$ the subfamily of sets from \mathcal{F} containing S.

For linearly independent vectors $a, b \in V$, denote by $g_{a,b}$ the linear map which maps the standard basis e_1, e_2 to a, b. Note that $g_{a,b} \in \mathfrak{G}$ and the matrix of this operator is just (a, b). Denote by $g_{ij} : \mathcal{A}_{ord} \to \mathfrak{G}$ a function which maps a sequence $A = (a_1, \ldots, a_5) \in \mathcal{A}_{ord}$ to the operator g_{a_i, a_j} , i.e. $g_{ij}(A) = g_{a_i, a_j}$.

A dependence ω of length t is a sequence (d_{ij}) (here $1 \leq i, j \leq 5, i \neq j$) of integers such that $\sum_{i\neq j} d_{ij} = 0$ and $\sum_{i< j} |d_{ij} + d_{ji}| = 2t$. We think of each dependence ω as of a map $\mathcal{A}_{ord} \to \mathbb{F}_p$ defined as follows:

$$\omega(A) = \prod_{i \neq j} \det(g_{ij}(A))^{d_{ij}},\tag{1}$$

A dependence ω is called trivial if $\omega(A) = 1$ for any $A \in \mathcal{A}_{ord}$. Otherwise ω is called nontrivial. Given two dependencies $\omega = (d_{ij})$ and $\omega' = (d'_{ij})$ one can define the product: $\omega \omega' := (d_{ij} + d'_{ij})$. In view of (1), it will be sometimes convenient to denote a dependence $\omega = (d_{ij})$ as $\omega = \prod \det(g_{ij})^{d_{ij}}$.

Fix $t = n^{0.01}$, denote by $\mathcal{A}_{ord}^{short}$ the set of all sequences $A \in \mathcal{A}_{ord}$ such that there is a *nontrivial* dependence ω of length at most t and $\omega(A) = 1$. Denote $\mathcal{A}^{short} = \pi(\mathcal{A}_{ord}^{short})$ and let $\mathcal{A}^{long} := \mathcal{A} \setminus \mathcal{A}^{short}$ and $\mathcal{A}_{ord}^{long} := \mathcal{A}_{ord} \setminus \mathcal{A}_{ord}^{short}$.

Now we define sets \mathcal{A}_{ord}^{wall} and \mathcal{A}^{wall} . Let *E* be the set of functions $\mathcal{A}_{ord} \to \mathfrak{G}$ of the form $g_{ij}g_{kl}^{-1}(A) :=$ $g_{ij}(A)g_{kl}^{-1}(A)$ where $i \neq j, k \neq l$ and $(i,j) \neq (k,l)$. The motivation of this definition is that two element $hA, h'A \in \mathfrak{G}A$ of the orbit of A intersect in at least two elements if and only if $h^{-1}h' = e(A)$ for some $e \in E$.

Each element $g_{ij}g_{kl}^{-1}$ of E determines a dependence $\omega = \det g_{ij}g_{kl}^{-1}$ that is ω is a sequence (d_{rs}) with $d_{ij} = 1, d_{kl} = -1$ and all other coordinates are set to be zero. For a sequence of elements $g_1, \ldots, g_m \in E$ we denote by $g_1g_2\ldots g_m: \mathcal{A}_{ord} \to \mathfrak{G}$ a function which maps a sequence A to the operator $g_1(A)g_2(A)\ldots g_m(A)$, also we denote by $det(g_1g_2...g_m)$ the product of dependencies $det(g_i)$.

Abusing the notation, for an arbitrary family \mathcal{F} and elements α, β denote $\mathcal{F}(\alpha, \beta) := \mathcal{F}(\{\alpha, \beta\})$. In Section 3.4 we will prove the following lemma.

Lemma 3.3. There are sets $\mathcal{A}_{ord}^{wall} \subset \mathcal{A}_{ord}$ and $\mathcal{A}^{wall} = \pi(\mathcal{A}_{ord}^{wall})$ such that: 1. Choose any $A \in \mathcal{A}_{ord}^{long}$ and $g_1, \ldots, g_m \in E$. Suppose that the dependence $\omega = \det(g_1g_2\ldots g_m)$ is of length at least t/3 and $\omega(A) = 1$. Then for some *i* we have $g_1g_2 \dots g_i(A)A \in \mathcal{A}_{ord}^{wall}$.

2. For any pair of vectors α, β we have $|\mathcal{A}^{wall}(\alpha, \beta)| = o(n^2)$.

Let us briefly explain the meaning of these definitions and the role of dependencies in the proof. For the sake of simplicity, here we ignore the difference between ordered and unordered families. Let $\mathcal{A}_1 \subset \mathcal{A}^{long}$ be an orbit of the action of \mathfrak{G} and let A be a representative of \mathcal{A}_1 . Let us connect two different sets from \mathcal{A}_1 if they intersect in at least two elements. By definition of E, two sets h_1A and h_2A are connected by an edge if there is $g \in E$ such that $h_2^{-1}h_1 = g(A)$. Thus, for each cycle $\mathcal{C} = \{h_1A, \ldots, h_mA\} \subset \mathcal{A}_1$ of the resulting graph we can construct a dependence $\omega = \det(g_1 \dots g_m)$ where $g_i \in E$ are such that $g_i(A) = h_{i-1}^{-1}h_i$. Note that $\omega(A) = 1$ and vertices of \mathcal{C} can be represented as follows:

$$\mathcal{C} = \{h_1 A, h_1 g_1(A) A, h_1 g_1 g_2(A) A, \dots, h_1 g_1 \dots g_{m-1}(A) A\}$$

We call a cycle \mathcal{C} nontrivial if the corresponding dependence is nontrivial. Now the first conclusion of Lemma 3.3 implies that any nontrivial cycle C in a "long" orbit A_1 must intersect A^{wall} (to see this, note the identity $h_1g_1(A)A = g_1(h_1A)h_1A$. Thus, the graph induced on the set $\mathcal{A}_1 \setminus \mathcal{A}^{wall}$ contains only "trivial" cycles which will allow us to color this set optimally.

Finally, we form a set $\mathcal{A}_{good} = \mathcal{A} \setminus (\mathcal{A}^{short} \cup \mathcal{A}^{wall})$ and construct a graph G on \mathcal{A}_{good} in which two sets are adjacent if they intersect in at least two places.

Lemma 3.4. $\chi(G) \leq (1+o(1))\frac{n^2}{6}$.

Given this lemma we can color in $(1 + o(1))\frac{n^2}{6}$ colors vertices of G(n, 4, 2) which are contained in some set from \mathcal{A}_{good} (just like in the proof of Theorem 1.1). Denote the set of all remaining vertices by U, it remains to prove that this set can be colored in a small number of colors.

Lemma 3.5. $\chi(G(n, 4, 2)|_U) = o(n^2).$

Clearly, the combination of these lemmas implies Theorem 1.2.

The rest of the proof is organized as follows. In Section 3.3 we prove auxiliary results about trivial and nontrivial dependencies. In Section 3.4 we construct the set \mathcal{A}^{wall} . In Sections 3.5 and 3.6 we prove Lemmas 3.4 and 3.5 respectively.

3.3 Dependencies

We begin with a simple observation.

Claim 3.6. Take $A \in \mathcal{A}_{ord}$, $h \in \mathfrak{G}$ and arbitrary dependence ω . Then $\omega(hA) = \omega(A)$.

Proof. Note that for any $g \in \mathfrak{G}$ $g_{ha,hb} = hg_{a,b}$ and that $g_{ij}(hA) = g_{ha_i,ha_j} = hg_{a_i,a_j} = hg_{ij}(A)$. So,

$$\omega(hA) = \prod_{i \neq j} \det(g_{ij}(hA))^{d_{ij}} = \prod_{i \neq j} \det(hg_{ij}(A))^{d_{ij}} = \det(h)^{\sum d_{ij}} \prod_{i \neq j} \det(g_{ij}(A))^{d_{ij}} = \omega(A).$$

Denote by $\mathcal{A}_{ord}(\alpha,\beta)$ the set of sequences $A \in \mathcal{A}_{ord}$ such that $A = (\alpha,\beta,x_1,x_2,x_3)$ for some $x_i \in V$. The next lemma states that the set of sufficiently degenerate sequences $A \in \mathcal{A}_{ord}$ is rather sparse.

Lemma 3.7. $|\mathcal{A}_{ord}^{short}(\alpha,\beta)| = O(n^{1.7})$ for any linearly independent $\alpha,\beta \in V \setminus 0$.

Proof. Note that there are at most $2^{30}t^{10}$ dependencies of length at most t which determine different functions $\mathcal{A} \to \mathbb{F}_p$. Indeed, take a dependence $\omega = (d_{ij})$ of length $\leq t$, then from (1) we have:

$$\omega = \prod_{ij} \det(g_{ij}^{d_{ij}}) = (-1)^s \prod_{i < j} \det(g_{ij})^{d_{ij} + d_{ji}}$$

because det $g_{ij} = -\det g_{ji}$. So the dependence ω can be recovered as a function from the values of $d_{ij} + d_{ji}$ and $(-1)^s$. By definition, $|d_{ij} + d_{ji}| \leq 2t$ so there are at most $2 \cdot (4t)^{10} < 2^{30}t^{10}$ choices of ω which are different as functions.

For each nontrivial ω of length at most t we bound the number of sequences $A \in \mathcal{A}_{ord}(\alpha, \beta)$ satisfying $\omega(A) = 1$. Suppose that $\omega(A) = 1$ for all $A \in \mathcal{A}_{ord}(\alpha, \beta)$. Consider a sequence $A = (A_1, \ldots, A_5) \in \mathcal{A}_{ord}$, then there exists $h \in \mathfrak{G}$ such that $hA_1 = \alpha, hA_2 = \beta$ so $hA \in \mathcal{A}_{ord}(\alpha, \beta)$. By Claim 3.6 and our assumption, $\omega(A) = \omega(hA) = 1$, so ω is trivial. A contradiction.

Now we note that ω determines a rational function $\tilde{\omega} : \mathbb{F}_p^4 \to \mathbb{F}_p$: let $\tilde{\omega}(x, y) = \omega(\alpha, \beta, x, y, -\alpha - \beta - x - y)$. Each determinant is a degree 2 polynomial, therefore, $\tilde{\omega}(x, y) = \frac{P(x,y)}{Q(x,y)}$ where P and Q have degrees at most 4t. The number of $A \in \mathcal{A}_{ord}(\alpha, \beta)$ for which $\omega(A) = 1$ is less than the number of solutions of the equation R(x, y) = P(x, y) - Q(x, y) = 0. From the previous paragraph we derive that R is a nontrivial polynomial of degree at most 4t. By Sparse Zeros Lemma ([BF], p. 86) R has at most $4tp^3$ roots.

Altogether, we have $|\mathcal{A}_{ord}^{short}(\alpha,\beta)| \leq (2^{30}t^{10})(4tp^3) \leq 2^{34}n^{1.61} = O(n^{1.7}).$

Now we prove that short trivial dependencies are indeed "trivial".

Lemma 3.8. Let $\omega = (d_{ij})$ be a trivial dependence of length at most $t = n^{0.01}$. Then $d_{ij} + d_{ji} = 0$ for any $i \neq j$ and the sum $D = \sum_{i < j} d_{ij}$ is even.

Proof. As we have mentioned before, we have

$$\omega(A) = \prod_{i \neq j} \det(g_{a_i, a_j})^{d_{ij}} = (-1)^D \prod_{i < j} \det(g_{a_i, a_j})^{d_{ij} + d_{ji}}.$$

Analogously to the previous lemma, ω may be written as a fraction $\frac{P(x_1,\ldots,x_4)}{Q(x_1,\ldots,x_4)}$ of polynomials in 8 variables of degrees at most 4t (each x_i represents a vector of two variables (x_i^1, x_i^2)). As before, we consider the polynomial R = P - Q. Since the dependence ω is trivial, for any vectors $x_1, x_2, x_3, x_4 \in V$ such that $(x_1, x_2, x_3, x_4, -x_1 - x_2 - x_3 - x_4) \in \mathcal{A}_{ord}$ it follows that $R(x_1, x_2, x_3, x_4) = 0$. Condition $(x_1, x_2, x_3, x_4, -x_1 - x_2 - x_3 - x_4) \in \mathcal{A}_{ord}$ means that these five vectors are in general position. An easy calculation yields that the number of such tuples is at least $p^8 - 10p^7 > 4tp^7$ so by Sparse Zeros Lemma R must vanish, i.e. $1 \equiv \frac{P}{Q} \equiv (-1)^D \prod_{i < j} \det(x_i, x_j)^{d_{ij} + d_{ji}}$. But determinants $\det(x_i, x_j)$ are pairwise coprime for i < j so each multiple must be equal to 1 (indeed, $\det(x_i, x_j)$ is an irreducible polynomial of degree 2 in variables $x_i^1, x_i^2, x_j^1, x_j^2$; two different polynomials $\det(x_i, x_j)$ can not be proportional). The lemma follows.

3.4 Construction of \mathcal{A}^{wall}

In this section we prove the following crucial lemma:

Lemma 3.9. There are sets $\mathcal{A}_{ord}^{wall} \subset \mathcal{A}_{ord}$ and $\mathcal{A}^{wall} = \pi(\mathcal{A}_{ord}^{wall})$ such that:

1. Choose any $A \in \mathcal{A}_{ord}^{long}$ and $g_1, \ldots, g_m \in E$. Suppose that the dependence $\omega = \det(g_1g_2\ldots g_m)$ is of length at least t/3 and $\omega(A) = 1$. Then for some i we have $g_1g_2\ldots g_i(A)A \in \mathcal{A}_{ord}^{wall}$.

2. For any pair of vectors α, β we have $|\mathcal{A}^{wall}(\alpha, \beta)| = o(n^2)$.

The idea behind the proof is very simple: in each orbit, we sample a random set of relatively small "boxes" which will almost surely cover most of the orbit. By our assumption that the orbit is in \mathcal{A}^{long} we conclude that a "nontrivial" cycle does not fit in a box, so it must intersect its boundary. We put \mathcal{A}^{wall} to be the union of the boundaries of the sampled boxes.

Proof. Consider the orbit decomposition of \mathcal{A}_{ord}^{long} under the action of \mathfrak{G} : $\mathcal{A}_{ord}^{long} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_l$. Note that by Lemma 3.7 $|\mathcal{A}_{ord}^{long}| \sim |\mathcal{A}_{ord}| \sim n^4$, also we know that $|\mathfrak{G}| \sim n^2$, consequently, $l \sim \frac{n^4}{|\mathfrak{G}|} \sim n^2$. Choose a representative $A_j \in \mathcal{A}_j$ for each j.

We start by constructing \mathcal{A}_{ord}^{wall} in each orbit separately:

Claim 3.10. For any j = 1, ..., l there is a set $\mathcal{A}_j^{wall} \subset \mathcal{A}_j$ such that $|\mathcal{A}_j^{wall}| = O(n^2 t^{-0.5})$ and the following holds for any $A \in \mathcal{A}_j$ and $g_1, ..., g_m \in E$. Suppose that the dependence $\omega = \det(g_1g_2...g_m)$ is of length at least t/3 and $\omega(A) = 1$. Then for some i we have $g_1g_2...g_i(A)A \in \mathcal{A}_j^{wall}$.

Proof. Take a representative $A \in \mathcal{A}_j$, consider two sets $T = \{\det g_{ij}(A) \mid i \neq j\}$ and $\tilde{T} = \{\det g_{ij}(A) \mid i < j\}$ both lying in \mathbb{F}_p . Note that |T| = 20 and $|\tilde{T}| = 10$ because $A \in \mathcal{A}_{ord}^{long}$. For a positive integer λ let us define a box B_{λ} and the boundary of the box ∂B_{λ} as follows:

$$B_{\lambda} = \left\{ \pm \prod_{a \in \tilde{T}} a^{\lambda_a} \mid \lambda_a \in [-\lambda, \lambda], \sum_{a \in \tilde{T}} \lambda_a = 0 \right\},$$
$$\partial B_{\lambda} = \left\{ \pm \prod_{a \in \tilde{T}} a^{\lambda_a} \mid \lambda_a \in [-\lambda, \lambda], \sum_{a \in \tilde{T}} \lambda_a = 0, \exists b : \lambda_b = \pm \lambda \right\}.$$

Here are some properties of these objects:

Claim 3.11. If $\lambda \leq t/30$ then: 1. $|B_{\lambda}| \geq (2\lambda)^{10}$ and $|\partial B_{\lambda}| \leq 2^{16} \cdot \lambda^9$. 2. All products of the form $\pm \prod_{a \in \tilde{T}} a^{\lambda_a}$ where $\lambda_a \in [-\lambda, \lambda]$, $\sum_{a \in \tilde{T}} \lambda_a = 0$ are distinct.

Proof. Note that the bound on $|\partial B_{\lambda}|$ is obvious and the bound on $|B_{\lambda}|$ is an immediate consequence of the Part 2.

Suppose that two different products of the given form do coincide. Bringing everything on the left hand side we obtain an equality $\pm \prod_{a \in \tilde{T}} a^{\lambda_a} = 1$ where $\lambda_a \in [-2\lambda, 2\lambda]$ and $\sum \lambda_a = 0$. Expressing this in terms of g_{ij} we obtain an equation $\pm \prod_{i>j} \det(g_{ij})^{\lambda_{ij}}(A) = 1$ where λ_{ij} obey the same conditions. Construct a dependence ω in the following way: let $\omega = (\lambda_{ij})$ if the sign before the product is positive and let $\omega = (\lambda_{ij}) \cdot \det(g_{12}g_{21}^{-1})$ if the sign is negative.

We see that $\omega(A) = 1$ and ω has length at most $10 \cdot 2\lambda + 2 < t$. Since $A \in \mathcal{A}_{ord}^{long}$ we deduce that ω is trivial, and so each $\lambda_{ij} = 0$, a contradiction because initially we chose two different products (note that we considered only λ_{ij} with i < j).

Now we fix $\lambda = t/1000$ and $q = \frac{p}{\lambda^{9.5}}$. Choose independently at random q nonzero residues $\rho_1, \ldots, \rho_q \in \mathbb{F}_p^{\times} = \mathbb{F}_p \setminus 0$ and consider random sets

$$C = \bigcup_{i=i}^{q} \rho_i \cdot B_{\lambda}, \quad \partial C = \bigcup_{i=i}^{q} \rho_i \cdot \partial B_{\lambda}, \quad R = \mathbb{F}_p^{\times} \setminus C.$$

Let us bound their cardinalities. Every residue does not belong to $\rho_i B_\lambda$ with probability $1 - \frac{|B_\lambda|}{n-1}$ so

$$\mathbb{E}|R| \le p \left(1 - \frac{|B_{\lambda}|}{p-1}\right)^{q}$$

Next, $|\partial C| \leq q |\partial B_{\lambda}| = O(q\lambda^9) = O(\frac{p}{\sqrt{\lambda}})$. Therefore, there is a choice of ρ_i -s so that $|\partial C \cup R| = O(p\lambda^{-0.5})$. Let \mathcal{A}_j^{wall} be the set of all $hA \in \mathcal{A}_j$ such that det $h \in \partial C \cup R$. We claim that this is the right choice of \mathcal{A}_j^{wall} .

First, it is straightforward that $|\mathcal{A}_{j}^{wall}| = O(n^{2}t^{-0.5})$. Next, take an arbitrary $hA \in \mathcal{A}_{j}$ and elements $g_{1}, \ldots, g_{m} \in E$ such that $\omega = \det(g_{1} \ldots g_{m})$ is of length at least t/3 and $1 = \omega(hA) = \omega(A)$. Define dependencies $\omega_{i} = \det(g_{1} \ldots g_{i})$ and note that lengths of ω_{i} and ω_{i+1} differ by at most 2. So there is $k \in [1, m]$ such that the length of ω_{k} lies in the interval $(10\lambda, t/60]$. Thus, by Claim 3.11 $\omega_{k}(A) \in B_{t/30}$ and it has a unique product representation, so $\omega_{k}(A) \notin B_{2\lambda}$ because otherwise it would have length at most $10 \times 2\lambda/2$ (the length is the half of the sum of degrees). We conclude that residues det h and $\omega_{k}(A) \det h = \det h$ and $\omega_{i}(A) \det h$ lie in different boxes (unless they do not lie in any box ρB_{λ} at all, in which case we are done). Suppose that $\det h, \omega_{i+1}(A) \det h \in \rho_{f} B_{\lambda}$ but $\omega_{i}(A) \det h \notin \rho_{f} B_{\lambda}$. It follows that $\omega_{i+1}(A) \det h \in \rho_{f} \partial B_{\lambda}$ and so $g_{1} \ldots g_{i+1}(hA)hA \in \mathcal{A}_{j}^{wall}$ but this is what we needed to prove. The claim is proved.

Now we glue \mathcal{A}_{ord}^{wall} from \mathcal{A}_{j}^{wall} . In view of Claim 3.10, we only need to ensure that all sets $\mathcal{A}_{ord}^{wall}(\alpha,\beta)$ are small. Let us choose uniformly at random operators $\gamma_j \in \mathfrak{G}$ for $j = 1, \ldots, l$ and consider sets $\mathcal{B}_j = \gamma_j \mathcal{A}_j^{wall}$ and $\mathcal{B} = \bigcup \mathcal{B}_j$. We claim that with high probability we can take $\mathcal{A}_{ord}^{wall} := \mathcal{B}$, so we only need to prove that $|(\pi \mathcal{B})(\alpha,\beta)|$ is small for any pair of linearly independent vectors α,β . Define $\xi_{\alpha,\beta,j} = |(\pi \mathcal{B}_j)(\alpha,\beta)|$ and $\xi_{\alpha,\beta} = |(\pi \mathcal{B})(\alpha,\beta)|$. Clearly, $\xi_{\alpha,\beta} \leq \sum_j \xi_{\alpha,\beta,j}$ and for any $j \ \xi_{\alpha,\beta,j} \leq 20$ because there is no two elements

 $h_1A, h_2A \in \mathcal{A}_j$ which contain α, β on the same places. Let $Q = \frac{n^2}{\sqrt{\log n}}$. As variables $\xi_{\alpha,\beta,j}$ are independent and probability that $\xi_{\alpha,\beta,j} > 0$ is at most $P = \frac{|\mathcal{A}_j|}{p} = O(\frac{1}{\sqrt{t}})$ we conclude that

$$\mathbb{P}(\xi_{\alpha,\beta} > Q) < \binom{l}{Q/20} P^{Q/20} < 2^{l}O(t^{-0.5})^{Q/20} < e^{c_{1}n^{2} - c_{2}Q\log n} = O(e^{-cn^{2}\sqrt{\log n}}) = O(n^{-10}),$$

(the condition $\xi_{\alpha,\beta} > Q$ implies that there are at least Q/20 nonzero ξ -s). Thus, with high probability $\xi_{\alpha,\beta} \leq Q$ for all α, β . The lemma is proved.

3.5 Proof of Lemma 3.4

Let us consider the decomposition of \mathcal{A}^{long} into the orbits $\mathcal{A}_1, \ldots, \mathcal{A}_l$ under the action of \mathfrak{G} (note that this is not the same decomposition as in the proof of Lemma 3.9 because now we work inside \mathcal{A} instead of \mathcal{A}_{ord}). We have $|\mathcal{A}^{long}| \sim \frac{n^4}{120}$ and $\mathfrak{G} \sim n^2$ so $l \sim \frac{n^2}{120}$. Thus, to prove Lemma 3.4 we only need to color each set $\mathcal{A}_j \setminus \mathcal{A}^{wall}$ in 20 colors.

Lemma 3.12. For any $j = 1, \ldots, l$ we have $\chi(G|_{\mathcal{A}_j \setminus \mathcal{A}^{wall}}) \leq 20$.

Proof. Let us fix \mathcal{A}_j and its representative $A \in \mathcal{A}_j$ and $A_{ord} \in \pi^{-1}A$. Two vertices gA and hA are adjacent in G if and only if $|gA \cap hA| \ge 2$ or equivalently $|g^{-1}hA \cap A| \ge 2$, that is, some two elements of A are mapped by $g^{-1}h$ into other two elements of A. By definition, this means that $g^{-1}h = e(A_{ord})$ for some $e \in E$ (see Section 3.2), that is $h = ge(A_{ord})$. The induced subgraph $G|_{\mathcal{A}_j \setminus \mathcal{A}^{wall}}$ splits into connected components $\mathcal{C}_1, \ldots, \mathcal{C}_m$. Take a representative h_iA from the component \mathcal{C}_i .

Every vertex $hA \in \mathcal{C}_i$ has a representation

$$hA = h_i g_1 f_1^{-1} g_2 f_2^{-1} \dots g_q f_q^{-1} A, \tag{2}$$

where $g_s, f_s \in S(A) := \{g_{ij}(A_{ord})\}$ and for all s the vertex $h_i g_1 f_1^{-1} \dots g_s f_s^{-1} A$ lies in \mathcal{C}_i . Consider an arbitrary bijection $\psi : S(A) \to \mathbb{Z}_{20}$ such that $\psi(g_{ij}(A)) = \psi(g_{ji}(A)) + 10 \pmod{20}$. Now we define a coloring c of $\mathcal{A}_j \setminus \mathcal{A}^{wall}$ as follows:

$$c(h) = \sum_{s=1}^{q} \psi(g_s) - \psi(f_s) \pmod{20}$$

Let us suppose for a moment that this definition does not depend on the choice of the representation (2) of hA. Then we can write any two adjacent vertices hA, h'A in the form:

$$hA = h_i g_1 f_1^{-1} g_2 f_2^{-1} \dots g_q f_q^{-1} A$$

$$h'A = h_i g_1 f_1^{-1} g_2 f_2^{-1} \dots g_q f_q^{-1} g'(f')^{-1} A,$$

and so $c(h') - c(h) = \psi(g') - \psi(f') \neq 0 \pmod{20}$ that is the coloring c is proper.

So it remains to check correctness of the definition of the coloring c. Take a vertex $hA \in C_i$ and two its representations of the from (2). Bringing everything to the left hand side we obtain

$$h_i x_1 y_1^{-1} x_2 y_2^{-1} \dots x_u y_u^{-1} A = h_i A,$$

for some $x_s, y_s \in S(A)$ and we need to prove that $\sum_{s=1}^u \psi(x_s) - \psi(y_s) = 0 \pmod{20}$. Note that for any s the vertex $h_i x_1 y_1^{-1} \dots x_s y_s^{-1} A$ has to lie inside C_i . We can write each multiple $x_s y_s^{-1}$ as $e_s(A_{ord})$ for some $e_s \in E$. Consider the dependence $\omega = \det(e_1 \dots e_u) =: (d_{ij})$. By definition, $\omega(A_{ord}) = \prod \det(x_s y_s^{-1}) = 1$ and $\sum_{s=1}^u \psi(x_s) - \psi(y_s) = \sum_{i \neq j} d_{ij} \psi(g_{ij}) \pmod{20}$.

Suppose that the length of ω is at most t. Then ω is trivial because $A_{ord} \in \mathcal{A}_{ord}^{long}$. Then Lemma 3.8 applies and we obtain:

$$\sum_{i \neq j} d_{ij}\psi(g_{ij}) = \sum_{i < j} d_{ij}\psi(g_{ij}) + d_{ji}\psi(g_{ji}) = \sum_{i < j} \psi(g_{ji})(d_{ji} + d_{ij}) + 10 \cdot d_{ij} = 10 \sum_{i < j} d_{ij} = 0 \pmod{20}$$

and we are done.

Now suppose that the length of ω is at least t > t/3. Then Lemma 3.9 applied to the sequence e_1, \ldots, e_u and $h_i A_{ord}$ yields that there exists s such that $h_i e_1 \ldots e_s(A_{ord}) A_{ord} \in \mathcal{A}_{ord}^{wall}$. But this means that $h_i x_1 y_1^{-1} \dots x_s y_s^{-1} A \notin \mathcal{C}_i$ because $\mathcal{C}_i \cap \mathcal{A}^{wall} = \emptyset$. We arrived at a contradiction, Lemma 3.12 is proved.

Proof of Lemma 3.5 3.6

We should color all 4-element subsets which are not subsets of any element of \mathcal{A}_{good} . Take an arbitrary $X = \{x_1, x_2, x_3, x_4\} \subset V \setminus 0$ and denote $A = X \cup \{-x_1 - x_2 - x_3 - x_4\}$. Let U_1 be the set of all 4-element sets X such that $A \notin A$, that is a pair of elements of A is collinear (this includes the cases then the sum of x_i -s equals 0). Finally, let U_2 be the set of all 4-element sets X such that $A \in \mathcal{A}^{short} \cup \mathcal{A}^{wall}$. Clearly, $U = U_1 \cup U_2$ and we need to show that $\chi(G(n, 4, 2)|_{U_i}) = o(n^2)$ for i = 1, 2. Note that in this section we are working with 4-element subsets of $V \setminus 0$ and, in particular, $U_1, U_2 \subset {\binom{V \setminus 0}{4}}$.

Let us begin with U_2 , we will deduce the desired bound using the inequality $\chi(G(n,4,2)|_{U_2}) \leq$ $\Delta(G(n,4,2)|_{U_2})+1$. Take a vertex $X \in U_2$, by Lemmas 3.7 and 3.9 there are $o(n^2)$ sets from $\mathcal{A}^{wall} \cup \mathcal{A}^{short}$ intersecting X in at least two places. Thus, the maximal degree of considered induced subgraph is $o(n^2)$.

Now we focus on U_1 .

Recall that the list chromatic number $\chi_{list}(H)$ of a graph H is the smallest number k such that the following holds. For each assignment of sets $L(v), v \in V(H)$ of cardinality at least k there is a proper coloring c of H such that $c(v) \in L(v)$ for any $v \in V(H)$. We need the following general result.

Claim 3.13. Let H be a graph. Suppose that the vertex set of H is covered by a system of subsets: $V(H) = \bigcup_{i=1}^{m} A_i$. Suppose that for any $v \in V(H)$ the number of edges between v and the set $\bigcup_{A_i \neq v} A_i$ is at most d. Suppose that for any $i \chi_{list}(H|_{A_i}) \leq l$ holds. Then $\chi(G) \leq l + d$.

Proof. Suppose that we have already colored $B = A_1 \cup \ldots \cup A_i$ in at most l + d colors. Let us show that the set $C = A_{i+1} \setminus B$ can also be colored in at most l + d colors. For a vertex $v \in C$ let L(v) be the list of colors in which v can be colored without contradicting the coloring of B. By assumption, v has at most d neighbours in B, therefore, $|L(v)| \ge l$. Since $C \subset A_{i+1}$, we can pick a color from each L(v) so that the resulting coloring of $C \cup B = A_1 \cup \ldots \cup A_{i+1}$ is proper. The claim now follows by induction.

To apply Claim 3.13 we construct a covering system of U_1 as follows. Let U^* be the set of quadruples $\{x_1,\ldots,x_4\}$ for which $-x_1 - x_2 - x_3 - x_4$ either equals 0 or is proportional to x_i for some i. For a pair of collinear nonzero vectors α, β let $U(\alpha, \beta)$ be the set of all $X \in U_1$ containing α and β . It follows that $U_1 = U^* \cup \bigcup_{\alpha \sim \beta} U(\alpha, \beta)$ (here $\alpha \sim \beta$ means that α and β are collinear). Indeed, if $X = \{x_1, \ldots, x_4\} \in U_1$ then either $x_i \sim x_j$ for some $i \neq j$, which means that $X \in U(x_i, x_j)$, either $x_i \sim -x_1 - x_2 - x_3 - x_4$ for some i, which means that $X \in U^*$. Let d_X be the number of edges between X and the subset $\bigcup P$ of U_1 , where the union is taken over $P \in \{U^*, U_{\alpha,\beta}\}$ such that $X \notin P$.

Claim 3.14. $d_X = O(n^{3/2})$ for any $X \in U_1$.

Proof. We count the number of neighbors Y of X in $\bigcup P$. The intersection $Z = X \cap Y$ can be fixed in 6 ways. There are at most 4np sets $Y \in U^*$ containing Z. If $Y \in U(\alpha, \beta)$ for $\{\alpha, \beta\} \not\subset X$ then we can choose two last elements of Y in at most np ways. So, altogether, there are at most $O(n^{3/2})$ neighbors of X.

Using a similar argument one can prove that $\Delta(G(n, 4, 2)|_{U^*}) = O(n)$ and, by the trivial inequality $\chi_{list}(H) \leq \Delta(H) + 1$, we obtain $\chi_{list}(G(n, 4, 2)|_{U^*}) = O(n)$. Now we bound the list chromatic number of the graph induced on $U(\alpha, \beta)$. Clearly, this subgraph is isomorphic to a subgraph of G(n, 2, 0). So it remains to bound the list chromatic number of the latter graph. We will prove a slightly more general result.

Lemma 3.15. Fix r, s and let $n \to \infty$, then $\chi_{list}(G(n, r, s)) = O(n^{s+1} \log n)$.

Proof. The assertion follows immediately from the standard fact that $\chi_{list}(G) \leq \chi(G) \log |G|$ for any graph G.

Thus, we checked assumptions of Claim 3.13 with $d, l = O(n^{3/2})$. Therefore, $\chi(G(n, 4, 2)|_{U_1}) = o(n^2)$.

4 Remarks

In this Section we very briefly discuss the limitations of the presented techniques.

Unfortunately, the presented approach depends heavily on the particular choice of the parameters (r, s) = (4, 2) and it is not completely clear how one can extend it to larger values of r, s or to other families of graphs.

In particular, if one tries to generalize the method to an arbitrary graph G(n, r, s) with $r \leq 2s + 1$ one faces the following problems:

1. What should replace the set \mathcal{A} , that is an approximate design which is invariant under an action of a large group (more precisely, a group of large transitivity, for instance, $GL_t(\mathbb{F}_p)$)? It is only straightforward to come up with such a family in the case then s = r - 2: identify [n] with $\mathbb{F}_p^s \setminus 0$ and consider a family of sets which elements sum up to 0. The action of the general linear group $GL_s(\mathbb{F}_p)$ will preserve this family.

2. During the proof, it was essential to work with determinants of linear operators instead of operators themselves. For instance, it allowed us to use polynomial methods in Section 3.3, the commutativity of \mathbb{F}_p^{\times} allowed us to construct the "wall" in Section 3.4: indeed, the key observation was that the size of the boundary of a box is much smaller than the size of the box itself. It is completely false if one tries to apply this idea to $GL_2(\mathbb{F}_p)$ directly. But for $s \ge 3$ the determinant is not strong enough to capture all the edges of the graph: the set E of linear operators in \mathbb{F}_p^s which map a fixed set A to a set which intersects A in at least s elements always contains operators of determinant 1, unless s = 2.

3. Splitting the graph into "structured" (4-element subsets of elements of \mathcal{A}^{good} in our case) and "degenerate" (the set U in our case) parts will also become harder. Consider, for instance, the graph G = G(n, r, r-2). We would like to prove that $\chi(G(n, r, r-2)) \sim \frac{n^2}{6}$. But note that G(n, r, r-2) contains a lot of copies of the graph G(n - r + 3, 3, 1) which has almost the same chromatic number and has only $O(n^3)$ vertices. This means that the "degenerate" part of the graph should be very carefully defined so that it will not accidentally contain a copy of G(n - r + 3, 3, 1) inside. Note that the case (r, s) = (4, 2) the set U contains many subgraphs isomorphic to G(n, 2, 0) but, luckily for us, the chromatic number of the latter graph grows linearly.

The most natural candidates for a future generalization of the approach are the graphs G(n, 5, 2) and G(n, 5, 3) both of which represent some of the new difficulties mentioned above. Also one may consider a simpler sequence of graphs, namely graphs $G(n, r, \ge s)$ which have the same sets of vertices as G(n, r, s) but two vertices of $G(n, r, \ge s)$ are connected if their intersection contains at least s elements. This simplification eliminates the need of approximate designs in the coloring.

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