# Correlation for permutations 

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#### Abstract

In this note we investigate correlation inequalities for 'up-sets' of permutations, in the spirit of the Harris-Kleitman inequality. We focus on two well-studied partial orders on $S_{n}$, giving rise to differing notions of up-sets. Our first result shows that, under the strong Bruhat order on $S_{n}$, up-sets are positively correlated (in the Harris-Kleitman sense). Thus, for example, for a (uniformly) random permutation $\pi$, the event that no point is displaced by more than a fixed distance $d$ and the event that $\pi$ is the product of at most $k$ adjacent transpositions are positively correlated. In contrast, under the weak Bruhat order we show that this completely fails: surprisingly, there are two up-sets each of measure $1 / 2$ whose intersection has arbitrarily small measure.

We also prove analogous correlation results for a class of nonuniform measures, which includes the Mallows measures. Some applications and open problems are discussed.


## 1 Introduction

Let $X=\{1,2, \ldots, n\}=[n]$. A family $\mathcal{F} \subset \mathcal{P}(X)=\{A: A \subset X\}$ is an up-set if given $F \in \mathcal{F}$ and $F \subset G \subset X$ then $G \in \mathcal{F}$. The well-known

[^0]and very useful Harris-Kleitman inequality [7, 9] guarantees that any two up-sets from $\mathcal{P}(X)$ are positively correlated. In other words, if $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$ are both up-sets then
$$
\frac{|\mathcal{A} \cap \mathcal{B}|}{2^{n}} \geq \frac{|\mathcal{A}|}{2^{n}} \times \frac{|\mathcal{B}|}{2^{n}} .
$$

The result has been very influential, and was extended several times to cover more general contexts [6, 8, 1]. However, for the most part, these results tend to focus on distributive lattices (such as $\mathcal{P}(X)$ ) and it is natural to wonder whether correlation persists outside of this setting.

In this note we aim to explore analogues of the Harris-Kleitman inequality for sets of permutations. There are two particularly natural notions for what it means for a family of permutations to be an up-set here, and the level of correlation that can be guaranteed in these settings turns out to differ greatly.

We write $S_{n}$ for the set of all permutations of [ $n$ ], which throughout the paper we regard as ordered $n$-tuples of distinct elements of $[n]$. That is, if $\mathbf{a} \in S_{n}$ then $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ where $\left\{a_{k}\right\}_{k \in[n]}=[n]$. Given $\mathbf{a} \in S_{n}$ and $i \in[n]$, we write $\operatorname{pos}(\mathbf{a}, i)$ for the position of $i$ in $\mathbf{a}$, i.e. $\operatorname{pos}(\mathbf{a}, i)=k$ if $a_{k}=i$. Given $1 \leq i<j \leq n$, the pair $\{i, j\}$ is said to be an inversion in a if $\operatorname{pos}(\mathbf{a}, i)>\operatorname{pos}(\mathbf{a}, j)$. We will write inv $(\mathbf{a})$ for the set of all inversions in $\mathbf{a}$. A pair $\{i, j\} \in \operatorname{inv}(\mathbf{a})$ is adjacent in a if $\operatorname{pos}(\mathbf{a}, i)=\operatorname{pos}(\mathbf{a}, j)+1$.

Definition. Given a family of permutations $\mathcal{A} \subset S_{n}$, we say that:
(i) $\mathcal{A}$ is a strong up-set if given $\mathbf{a} \in \mathcal{A}$, any permutation obtained from $\mathbf{a}$ by swapping the elements in a pair $\{i, j\} \in \operatorname{inv}(\mathbf{a})$ is also in $\mathcal{A}$.
(ii) $\mathcal{A}$ is a weak up-set if given $\mathbf{a} \in \mathcal{A}$, any permutation obtained from $\mathbf{a}$ by swapping the elements in an adjacent pair $\{i, j\} \in \operatorname{inv}(\mathbf{a})$ is also in $\mathcal{A}$.

We remark that both strong and weak up-sets have natural interpretations in the context of posets (see Chapter 2 of [4]). Given $\mathbf{a}, \mathbf{b} \in S_{n}$ write $\mathbf{a} \leq_{s} \mathbf{b}$ if $\mathbf{b}$ can be reached from $\mathbf{a}$ by repeatedly swapping inversions. We write $\mathbf{a} \leq_{w} \mathbf{b}$ if $\mathbf{b}$ can be reached from $\mathbf{a}$ by repeatedly swapping adjacent inversions. These relations give well-studied partial orders, known as the strong Bruhat order and the weak Bruhat order respectively. A (strong or weak) up-set is then simply a family which is closed upwards in the corresponding order ${ }^{11}$.

[^1]

The weak order for $n=3$


The strong order for $n=3$

It is clear that every strong up-set is also a weak up-set, but the opposite relation is not true. For $i, j \in[n]$ let $\mathcal{U}_{i j}=\left\{\mathbf{a} \in S_{n}: i\right.$ occurs before $j$ in $\left.\mathbf{a}\right\}$. If $i<j$, then $\mathcal{U}_{i j}$ is a weak up-set but not a strong up-set (except $\mathcal{U}_{1 n}$ ). For example, when $n=3$, the family $\mathcal{U}_{12}=\{123,132,312\}$ is a weak up-set but not a strong up-set (because 213 differs from 312 by swapping 2 and 3 into increasing order but $213 \notin \mathcal{U}_{12}$ ).

Our first result is that strong up-sets are positively correlated in the sense of Harris-Kleitman. That is, if $\mathcal{A}, \mathcal{B} \subset S_{n}$ are strong up-sets then

$$
\frac{|\mathcal{A} \cap \mathcal{B}|}{n!} \geq \frac{|\mathcal{A}|}{n!} \times \frac{|\mathcal{B}|}{n!}
$$

As we will also consider non-uniform measures, we phrase this in a more probabilistic way. We will say that a probability measure $\mu$ on $S_{n}$ is positively associated (for the strong order) if $\mu(\mathcal{A} \cap \mathcal{B}) \geq \mu(\mathcal{A}) \cdot \mu(\mathcal{B})$ for all strong upsets $\mathcal{A}$ and $\mathcal{B}$.

Theorem 1. The uniform measure $\mu(\mathcal{A})=\frac{|\mathcal{A}|}{n!}$ on $S_{n}$ is positively associated (for the strong order).

For weak up-sets the situation is more complicated. We saw that both $\mathcal{U}_{12}$ and $\mathcal{U}_{23}$ are weak up-sets and each has size $n!/ 2$. But $\mathcal{U}_{12} \cap \mathcal{U}_{23}$ is the family of permutations in which $1,2,3$ occur in ascending order. So, for $n \geq 3$ we have

$$
\frac{\left|\mathcal{U}_{12} \cap \mathcal{U}_{23}\right|}{n!}=\frac{1}{6}<\frac{1}{4}=\frac{\left|\mathcal{U}_{12}\right|}{n!} \times \frac{\left|\mathcal{U}_{23}\right|}{n!} .
$$

A plausible guess might be that every two up-sets $\mathcal{A}$ and $\mathcal{B}$ with size $n!/ 2$ achieve at least this level of correlation. Surprisingly, this turns out to be far from the truth; such an $\mathcal{A}$ and $\mathcal{B}$ can be almost disjoint.

Theorem 2. Let $0<\alpha, \beta<1$ be fixed. Then there are weak up-sets $\mathcal{A}, \mathcal{B} \subset$ $S_{n}$ with $|\mathcal{A}|=\lfloor\alpha n!\rfloor,|\mathcal{B}|=\lfloor\beta n!\rfloor$ and $|\mathcal{A} \cap \mathcal{B}|=\max \left(|\mathcal{A}|+|\mathcal{B}|-\left|S_{n}\right|, 0\right)+o(n!)$.

The correlation given in Theorem 2 is (essentially) minimal, since any two families $\mathcal{A}, \mathcal{B} \subset S_{n}$ satisfy $|\mathcal{A} \cap \mathcal{B}| \geq \max \left(|\mathcal{A}|+|\mathcal{B}|-\left|S_{n}\right|, 0\right)$.

Theorem 2 shows in quite a strong sense that the uniform measure on $S_{n}$ is not positively associated under the weak order. Our next result will prove positive association for a wider collection of measures under the strong order, giving a different generalisation of Theorem 1 .

Before describing these measures, we first give an alternative representation of elements of $S_{n}$ (essentially the Lehmer encoding of permutations see Chapter 11.4 of [10]). Given $\mathbf{a} \in S_{n}$ we can associate a vector $\mathbf{f}(\mathbf{a})=$ $\left(f_{1}, \ldots, f_{n}\right) \in G_{n}:=[1] \times[2] \times \cdots[n]$, with

$$
f_{j}:=|\{i \in[j]: \operatorname{pos}(\mathbf{a}, i) \leq \operatorname{pos}(\mathbf{a}, j)\}| .
$$

In other words, $f_{j}$ describes where element $j$ appears in the $n$-tuple a in relation to the elements from $[j]$. This gives a bijection between $S_{n}$ and $G_{n}$, and our positively associated measures on $S_{n}$ are built from this connection.

Definition. Let $X_{1}, \ldots, X_{n}$ be independent random variables, where each $X_{k}$ takes values in $[k]$. The independently generated measure $\mu$ defined by $\left\{X_{k}\right\}_{k \in[n]}$ is the following probability measure on $S_{n}$ : given $\mathbf{a} \in S_{n}$ we have

$$
\mu(\mathbf{a}):=\prod_{k \in[n]} \mathbb{P}\left(X_{k}=\mathbf{f}(\mathbf{a})_{k}\right) .
$$

We simply say that $\mu$ is independently generated if this holds for some such collection of $\left\{X_{k}\right\}_{k \in[n]}$.

Our second positive result applies to independently generated measures.
Theorem 3. Every independently generated probability measure on $S_{n}$ is positively associated.

We note that the uniform measure on $S_{n}$ is independently generated, taking $X_{k}$ to simply be uniform on $[k]$. Thus Theorem 3 implies Theorem 1 ,

We note that one special case of an independently generated measure is the Mallows measure [11]. Recalling the definition of $\operatorname{inv}(\mathbf{a})$ above, the Mallows measure with parameter $0<q \leq 1$ is defined by setting

$$
\mu(\mathbf{a}) \propto q^{\mid \operatorname{inv(} \mathbf{a}) \mid} .
$$

That is, $\mu(\mathbf{a})=\left(\sum_{\mathbf{a} \in S_{n}} q^{|\operatorname{inv}(\mathbf{a})|}\right)^{-1} \cdot q^{|\operatorname{inv}(\mathbf{a})|}$.
Our results in fact go beyond independently generated measures, and it turns out that here a key idea is a notion of up-set that sits 'between' the weak and strong up-sets above. This notion, which we call 'grid up-sets' (defined in Section 2), provides an environment that is suitable for FKG-like inequalities. This approach will allow us to strengthen Theorem 3 to apply to measures satisfying more general conditions.

Before closing the introduction, we note that while we have stated our results for up-sets, it is easy to obtain equivalent down-set versions of Theorems $1-3$ (for example, see Chapter 19 of [5]). These follow by noting that a set $A$ in a partial order $(P,<)$ is an up-set if and only if $A^{c}=P \backslash A$ is a downset. Indeed, if $\mu$ is a probability measure on $P$ and $\mu(A \cap B) \geq \mu(A) \cdot \mu(B)$ then we obtain the complementary inequality

$$
\begin{aligned}
\mu\left(A^{c} \cap B^{c}\right)=1-\mu(A)-\mu(B)+\mu(A \cap B) & \geq 1-\mu(A)-\mu(B)+\mu(A) \cdot \mu(B) \\
& =\mu\left(A^{c}\right) \cdot \mu\left(B^{c}\right) .
\end{aligned}
$$

The plan of the paper is as follows. In Section 2 we prove our positive association results. Here we give a self-contained proof of Theorem 3. We also introduce grid up-sets and use them to extend Theorem 3. In Section 3 we prove Theorem 2, constructing weak up-sets with bad correlation properties. Section 4 gives some applications of our main results, to families of permutations defined with bounded 'displacements', sequential domination properties, as well as to left-compressed set systems. Finally, in Section 5, we raise some questions and directions for further work.

## 2 Correlation for strong up-sets

In this section we will prove Theorem 3. As noted in the introduction, the uniform case is an immediate corollary (Theorem 1). The proof will use induction on $n$. To relate a family of permutations of $[n]$ with a family of permutations of some smaller ground set, we 'slice' according to position of element $n$. Given a family $\mathcal{A} \subset S_{n}$ and $k \in[n]$ let $\mathcal{A}_{k} \subset S_{n-1}$ denote those permutations obtained by deleting the appearance of element ' $n$ ' from $\mathbf{a} \in \mathcal{A}$ with $\operatorname{pos}(\mathbf{a}, n)=k$. That is:

$$
\mathcal{A}_{k}:=\left\{\left(a_{1}, a_{2}, \ldots, a_{n-1}\right) \in S_{n-1}:\left(a_{1}, \ldots, a_{k-1}, n, a_{k}, \ldots, a_{n-1}\right) \in \mathcal{A}\right\}
$$

In the next simple lemma we collect two properties of the slice operation which will be useful later.

Lemma 4. If $\mathcal{A} \subset S_{n}$ is a strong up-set and the slices $\mathcal{A}_{1}, \mathcal{A}_{2}, \ldots, \mathcal{A}_{n} \subset S_{n-1}$ are defined as above then:
(i) $\mathcal{A}_{k}$ is a strong up-set for all $k \in[n]$, and
(ii) $\mathcal{A}_{1} \subset \mathcal{A}_{2} \subset \mathcal{A}_{3} \cdots \subset \mathcal{A}_{n}$.

Proof. Part (i) is immediate. To see (ii), note that if $\mathbf{a} \in \mathcal{A}_{k}$ then we have $\left(a_{1}, \ldots, a_{k-1}, n, a_{k}, \ldots, a_{n-1}\right) \in \mathcal{A}$. Now, as $\mathcal{A}$ is a strong up-set and $n>a_{k}$, the pair $\left\{a_{k}, n\right\} \in \operatorname{inv}(\mathbf{a})$ and we find $\left(a_{1}, \ldots, a_{k}, n, a_{k+1}, \ldots, a_{n-1}\right) \in \mathcal{A}$, giving $\mathbf{a} \in \mathcal{A}_{k+1}$.

We will also need the following simple and standard arithmetic inequality, which will be used to relate the conditional probabilities of the slices in $S_{n-1}$ to probabilities in $S_{n}$. We provide a proof for completeness.
Lemma 5. Let $u_{1}, \ldots, u_{n}, v_{1}, \ldots, v_{n}, t_{1}, \ldots, t_{n} \in[0, \infty)$ with $u_{1} \leq \ldots \leq u_{n}$, $v_{1} \leq \ldots \leq v_{n}$ and $\sum_{k=1}^{n} t_{k} \leq 1$. Then

$$
\sum_{k=1}^{n} t_{k} u_{k} v_{k} \geq\left(\sum_{k=1}^{n} t_{k} u_{k}\right)\left(\sum_{k=1}^{n} t_{k} v_{k}\right)
$$

Proof. For convenience set $u_{0}=v_{0}=0$. Then, for all $k \in[n]$ set $x_{k}=$ $u_{k}-u_{k-1}$ and $y_{k}=v_{k}-v_{k-1}$. Note that the conditions on $u_{k}$ and $v_{k}$ give $x_{k}, y_{k} \geq 0$. Now,

$$
\sum_{k=1}^{n} t_{k} u_{k} v_{k}=\sum_{k=1}^{n} t_{k}\left(x_{1}+\cdots+x_{k}\right)\left(y_{1}+\cdots+y_{k}\right)=\sum_{i, j} r_{i, j} x_{i} y_{j}
$$

where

$$
r_{i, j}= \begin{cases}t_{i}+\cdots+t_{n} & \text { if } i \geq j \\ t_{j}+\cdots+t_{n} & \text { if } i \leq j\end{cases}
$$

Similarly,

$$
\left(\sum_{k=1}^{n} t_{k} u_{k}\right)\left(\sum_{\ell=1}^{n} t_{\ell} v_{\ell}\right)=\left(\sum_{k=1}^{n} t_{k}\left(\sum_{i=1}^{k} x_{i}\right)\right)\left(\sum_{\ell=1}^{n} t_{\ell}\left(\sum_{j=1}^{\ell} y_{j}\right)\right)=\sum_{i, j} s_{i, j} x_{i} y_{j},
$$

where $s_{i, j}=\left(t_{i}+\cdots+t_{n}\right)\left(t_{j}+\cdots+t_{n}\right)$. As $t_{k} \geq 0$ for all $k \in[n]$ and $\sum_{k=1}^{n} t_{k} \leq 1$, we see that $r_{i, j} \geq s_{i, j}$ for all $i, j$ and the result follows.

Proof of Theorem 3. We wish to show that if $\mu$ is an independently generated probability measure on $S_{n}$ and $\mathcal{A}, \mathcal{B} \subset S_{n}$ are strong up-sets in $S_{n}$ then $\mu(\mathcal{A} \cap \mathcal{B}) \geq \mu(\mathcal{A}) \cdot \mu(\mathcal{B})$. We will prove this by induction on $n$. The statement is trivial for $n=1$. Assuming that the statement holds for $n-1$, we will prove it for $n$.

To begin, note that as $\mu$ is independently generated on $S_{n}$, it is defined by independent random variables $\left\{X_{i}\right\}_{i \in[n]}$. Take $\nu$ to denote the independently generated measure on $S_{n-1}$ defined by the independent random variables $\left\{X_{i}\right\}_{i \in[n-1]}$. By definition of $\mu$, if $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right) \in S_{n}$ with $a_{k}=n$ then setting $\mathbf{a}_{[n-1]}:=\left(a_{1}, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n}\right) \in S_{n}$ we have

$$
\mu(\mathbf{a})=\nu\left(\mathbf{a}_{[n-1]}\right) \cdot \mathbb{P}\left(X_{n}=k\right)
$$

It follows that given any family $\mathcal{F} \subset S_{n}$ we have $\mu\left(\mathcal{F} \mid X_{n}=k\right)=\nu\left(\mathcal{F}_{k}\right)$. Note that the measure $\nu$ does not depend on $k$, which is important below.

With this in hand, suppose that $\mathcal{A}, \mathcal{B} \subset S_{n}$ are strong up-sets. Then

$$
\mu(\mathcal{A} \cap \mathcal{B})=\sum_{k \in[n]} \mathbb{P}\left(X_{n}=k\right) \mu\left(\mathcal{A} \cap \mathcal{B} \mid X_{n}=k\right)=\sum_{k \in[n]} \mathbb{P}\left(X_{n}=k\right) \nu\left((\mathcal{A} \cap \mathcal{B})_{k}\right)
$$

where the second equality follows by the previous paragraph. Clearly we have $(\mathcal{A} \cap \mathcal{B})_{k}=\mathcal{A}_{k} \cap \mathcal{B}_{k}$. Moreover, as both $\mathcal{A}_{k}$ and $\mathcal{B}_{k}$ are strong up-sets by Lemma 4 (i) and $\nu$ is independently generated, by induction we have $\nu\left(\mathcal{A}_{k} \cap \mathcal{B}_{k}\right) \geq \nu\left(\mathcal{A}_{k}\right) \cdot \nu\left(\mathcal{B}_{k}\right)$. Applying this above gives

$$
\mu(\mathcal{A} \cap \mathcal{B}) \geq \sum_{k \in[n]} \mathbb{P}\left(X_{n}=k\right) \cdot \nu\left(\mathcal{A}_{k}\right) \cdot \nu\left(\mathcal{B}_{k}\right)=\sum_{k \in[n]} t_{k} u_{k} v_{k}
$$

where $t_{k}=\mathbb{P}\left(X_{n}=k\right), u_{k}=\nu\left(\mathcal{A}_{k}\right)$ and $v_{k}=\nu\left(\mathcal{B}_{k}\right)$. Note now from Lemma 4 (ii) that we have $u_{1} \leq \ldots \leq u_{n}, v_{1} \leq \ldots \leq v_{n}$ and $\sum_{k \in[n]} t_{k}=1$. Thus the hypothesis of Lemma 5 applies, and this lemma gives

$$
\begin{aligned}
\mu(\mathcal{A} \cap \mathcal{B}) \geq \sum_{k \in[n]} t_{k} u_{k} v_{k} & \geq\left(\sum_{k \in[n]} t_{k} u_{k}\right)\left(\sum_{k \in[n]} t_{k} v_{k}\right) \\
& =\left(\sum_{k \in[n]} \mathbb{P}\left(X_{n}=k\right) \nu\left(\mathcal{A}_{k}\right)\right)\left(\sum_{k \in[n]} \mathbb{P}\left(X_{n}=k\right) \nu\left(\mathcal{B}_{k}\right)\right) \\
& =\mu(\mathcal{A}) \cdot \mu(\mathcal{B})
\end{aligned}
$$

This completes the proof of the theorem.

In contrast to this self-contained proof, our second proof will use the machinery of the FKG inequality in the following form.

Theorem 6 (FKG inequality [6]). Let $L$ be a finite distributive lattice and let $\mu$ be a probability measure on $L$ satisfying

$$
\mu(x \wedge y) \cdot \mu(x \vee y) \geq \mu(x) \cdot \mu(y)
$$

for all $x, y \in L$. Then any up-sets $\mathcal{A}, \mathcal{B} \subset L$ satisfy $\mu(\mathcal{A} \cap \mathcal{B}) \geq \mu(\mathcal{A}) \cdot \mu(\mathcal{B})$.
To make use of Theorem [6 recall that the permutations $S_{n}$ are in one-to-one correspondence with elements of the grid $G_{n}:=[1] \times[2] \times \cdots \times[n]$, where $\mathbf{a} \in S_{n}$ is indentified with $\mathbf{f}(\mathbf{a}) \in G_{n}$. Using this correspondence we will transfer the 'grid' partial order $\leq_{g}$ on $G_{n}$ to $S_{n}$, where $\mathbf{f} \leq_{g} \mathbf{g}$ for $\mathbf{f}, \mathbf{g} \in G_{n}$ if $\mathbf{f}_{i} \leq \mathbf{g}_{i}$ for all $i \in[n]$.

Definition. The grid order $\leq_{g}$ on $S_{n}$ is given by defining $\mathbf{a} \leq_{g} \mathbf{b}$ if $\mathbf{f}(\mathbf{a}) \leq_{g}$ $\mathbf{f}(\mathbf{b})$ when viewed as elements of $G_{n}$. A family $\mathcal{A} \subset S_{n}$ is a grid up-set if whenever $\mathbf{a} \in \mathcal{A}$ and $\mathbf{b} \in S_{n}$ with $\mathbf{a} \leq_{g} \mathbf{b}$ then $\mathbf{b} \in \mathcal{A}$.

We now use the FKG inequality to $G_{n}$ to give a second proof of Theorem 3. In fact this approach strengthens the result in two ways: it applies to grid up-sets rather than just strong up-sets, and it applies to measures satisfying a more general FKG-type condition.

Let $\mathbf{a}, \mathbf{b} \in S_{n}$. As $G_{n}$ is a distributive lattice we can define $\mathbf{a} \vee \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ in $S_{n}$ in natural way: let $\mathbf{a} \vee \mathbf{b}, \mathbf{a} \wedge \mathbf{b}$ be the unique elements of $S_{n}$ with:

$$
\mathbf{f}(\mathbf{a} \vee \mathbf{b})_{k}=\max \left\{\mathbf{f}(\mathbf{a})_{k}, \mathbf{f}(\mathbf{b})_{k}\right\} ; \quad \mathbf{f}(\mathbf{a} \wedge \mathbf{b})_{k}=\min \left\{\mathbf{f}(\mathbf{a})_{k}, \mathbf{f}(\mathbf{b})_{k}\right\} .
$$

Theorem 7. Suppose that $\mu$ is a probability measure on $S_{n}$ with

$$
\begin{equation*}
\mu(\mathbf{a} \vee \mathbf{b}) \cdot \mu(\mathbf{a} \wedge \mathbf{b}) \geq \mu(\mathbf{a}) \cdot \mu(\mathbf{b}) \tag{1}
\end{equation*}
$$

for all $\mathbf{a}, \mathbf{b} \in S_{n}$. Then any grid up-sets $\mathcal{A}, \mathcal{B} \subset S_{n}$ satisfy $\mu(\mathcal{A} \cap \mathcal{B}) \geq$ $\mu(\mathcal{A}) \cdot \mu(\mathcal{B})$.

Proof of Theorem 7. Transfer $\mu$ from $S_{n}$ to $G_{n}$, by setting $\mu(\mathbf{f}(\mathbf{a}))=\mu(\mathbf{a})$ for all $\mathbf{a} \in S_{n}$. As $\mathbf{f}: S_{n} \rightarrow G_{n}$ is a bijection this defines $\mu$ on $G_{n}$. By choice of the operations $\vee$ and $\wedge$ on $S_{n}$ above, (1) implies that

$$
\mu(\mathbf{f} \vee \mathbf{g}) \cdot \mu(\mathbf{f} \wedge \mathbf{g}) \geq \mu(\mathbf{f}) \cdot \mu(\mathbf{g})
$$

for all $\mathbf{f}, \mathbf{g} \in G_{n}$. The result now follows by applying Theorem 6 to $G_{n}$.

To complete our second proof of Theorem 3, by Theorem 7, it is enough to show that (i) every strong up-set is a grid up-set and (ii) that (1) holds for independently generated measures. This is content of the next two lemmas.

Lemma 8. If $\mathbf{a}, \mathbf{b} \in S_{n}$ with $\mathbf{a} \leq_{g} \mathbf{b}$ then $\mathbf{a} \leq_{s} \mathbf{b}$. Consequently, every strong up-set in $S_{n}$ is also a grid up-set.

Proof. Suppose that $\mathbf{a}, \mathbf{b} \in S_{n}$ where $(\mathbf{a}, \mathbf{b})$ is a covering relation in the grid order. That is, there is $i \in[n]$ with $\mathbf{f}(\mathbf{b})_{i}=\mathbf{f}(\mathbf{a})_{i}+1$ and $\mathbf{f}(\mathbf{a})_{j}=\mathbf{f}(\mathbf{b})_{j}$ for all $j \neq i$. It suffices to show that $\mathbf{a} \leq_{s} \mathbf{b}$ by transitivity, since every relation in $\leq_{g}$ can be expressed as a sequence of covering relations.

Let $\operatorname{pos}(\mathbf{a}, i)=k$ and take $\ell>k$ minimal so that $\operatorname{pos}(\mathbf{a}, j)=\ell$ for some $j<i$; such a choice of $\ell$ must exist since $(\mathbf{a}, \mathbf{b})$ is a covering relation with $\mathbf{f}(\mathbf{b})_{i}=\mathbf{f}(\mathbf{a})_{i}+1$. It is clear that $\mathbf{a}$ and $\mathbf{b}$ differ only in position $k$ and $\ell$, where $a_{k}=b_{\ell}=i$ and $a_{\ell}=b_{k}=j$. Thus $\{i, j\} \in \operatorname{inv}(\mathbf{a})$ and swapping these entries we obtain $\mathbf{b}$, i.e. $\mathbf{a} \leq_{s} \mathbf{b}$.

Lastly, if $\mathcal{A}$ is a strong up-set with $\mathbf{a} \in \mathcal{A}$ and $\mathbf{a} \leq_{g} \mathbf{b}$ then $\mathbf{a} \leq_{s} \mathbf{b}$, and so $\mathbf{b} \in \mathcal{A}$. Thus $\mathcal{A}$ is a grid up-set, as required.

Lemma 9. Inequality (1) holds for every independently generated probability measure $\mu$ on $S_{n}$.

Proof. Suppose that $\mu$ is an independently generated probability measure on $S_{n}$, defined by the independent random variables $\left\{X_{k}\right\}_{k \in[n]}$. Then for every $\mathbf{a} \in S_{n}$ we have

$$
\mu(\mathbf{a})=\prod_{k \in[n]} \mathbb{P}\left(X_{k}=\mathbf{f}(\mathbf{a})_{k}\right)
$$

Then given $\mathbf{a}, \mathbf{b} \in S_{n}$ and $\mathbf{a} \vee \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ as above, we have

$$
\begin{aligned}
\mu(\mathbf{a} \vee \mathbf{b}) \cdot \mu(\mathbf{a} \vee \mathbf{b})= & \prod_{k \in[n]}\left[\mathbb{P}\left(X_{k}=\max \left(\mathbf{f}(\mathbf{a})_{k}, \mathbf{f}(\mathbf{b})_{k}\right)\right)\right. \\
& \left.\quad \times \mathbb{P}\left(X_{k}=\min \left(\mathbf{f}(\mathbf{a})_{k}, \mathbf{f}(\mathbf{b})_{k}\right)\right)\right] \\
= & \prod_{k \in[n]}\left(\mathbb{P}\left(X_{k}=\mathbf{f}(\mathbf{a})_{k}\right) \cdot \mathbb{P}\left(X_{k}=\mathbf{f}(\mathbf{b})_{k}\right)\right)=\mu(\mathbf{a}) \cdot \mu(\mathbf{b}) .
\end{aligned}
$$

Thus (11) holds with equality for all $\mathbf{a}, \mathbf{b} \in S_{n}$, as required.

Above we defined the grid order on $S_{n}$ in such a way that it was isomorphic to the usual product ordering on $G_{n}$. Analysing the proof of Lemma 8 more carefully gives an alternative description of the grid order on $S_{n}$ in terms of certain switches. Given $1 \leq i<j \leq n$, recall that $\{i, j\}$ is an inversion in a if $\operatorname{pos}(\mathbf{a}, j)=k<\ell=\operatorname{pos}(\mathbf{a}, i)$. We will say that $\{i, j\}$ is a dominated inversion in a if additionally $a_{m} \geq i, j$ for all $m \in[k, \ell]$. Then $\mathbf{a} \leq_{g} \mathbf{b}$ if $\mathbf{b}$ can be reached from a by a sequence of operations, each consisting of swapping the elements from a dominated inversion.


The grid order for $n=3$

## 3 No correlation for weak up-sets

In this section we construct weak up-sets which are very far from being positively correlated. We will need the following simple concentration result.
Lemma 10. Let $0<\gamma, \delta, \varepsilon<1$. Let $U, V \subset[n]$ with $|U|=\gamma n$ and $|V|=$ $\delta n$. Select $\mathbf{a} \in S_{n}$ uniformly at random and consider the random variable $N(\mathbf{a}):=\left|\left\{i \in U: a_{i} \in V\right\}\right|$. Then $\mathbb{P}(N>(\gamma+\varepsilon)|V|) \rightarrow 0$ as $n \rightarrow \infty$.
Proof. For each $i \in[n]$ let $1_{i}: S_{n} \rightarrow\{0,1\}$ denote the Bernoulli random variable with $1_{i}(\mathbf{a})=1$ iff $a_{i} \in V$. Then $\mathbb{E}\left[1_{i}\right]=|V| / n$ for all $i \in[n]$. Noting that $N=\sum_{i \in U} 1_{i}$, linearity of expectation gives $\mathbb{E}[N]=\gamma|V|$.

To calculate the variance of $N$, note that $\mathbb{E}\left[1_{i} \cdot 1_{j}\right] \leq|V|^{2} / n^{2}$ for $i \neq j$. Since $N=\sum_{i \in U} 1_{i}$, this gives

$$
\mathbb{E}\left[N^{2}\right]=\sum_{i \in U} \mathbb{E}\left[1_{i}^{2}\right]+\sum_{i \neq j \in U} \mathbb{E}\left[1_{i} 1_{j}\right] \leq \gamma|V|+\gamma^{2}|V|^{2}
$$

and so $\operatorname{Var}(N)=\mathbb{E}\left[N^{2}\right]-(\mathbb{E}[N])^{2} \leq \gamma|V|$. Chebyshev's inequality then gives $\mathbb{P}(N>(\gamma+\varepsilon)|V|) \leq \mathbb{P}(|N-\mathbb{E}[N]| \geq \varepsilon|V|) \leq \gamma /\left(\varepsilon^{2}|V|\right) \rightarrow 0$ as $n \rightarrow \infty$.

We are now ready for the proof of Theorem 2.
Proof of Theorem 2. Given $0<\alpha, \beta, \varepsilon<1$, we require to find weak up-sets $\mathcal{A}, \mathcal{B} \subset S_{n}$ for large $n$, which satisfy $|\mathcal{A}| \geq \alpha n!,|\mathcal{B}| \geq \beta n!$ and $|\mathcal{A} \cap \mathcal{B}| \leq$ $(\max (\alpha+\beta-1,0)+5 \varepsilon) n$ !. Indeed, by deleting minimal elements from such $\mathcal{A}$ and $\mathcal{B}$ we obtain weak up-sets of size $\lfloor\alpha n!\rfloor$ and $\lfloor\beta n!\rfloor$ as in the theorem.

To begin, set $m=\left\lceil\left(\frac{\alpha}{\alpha+\beta}\right) n\right\rceil$ so that $\frac{m}{n-m}=\frac{\alpha}{\beta}+o(1)$. Consider the function $g: S_{n} \rightarrow[m]$ where $g(\mathbf{a})$ equals the number of elements from $[m]$ which do not appear after element $m$ in $\mathbf{a}$. That is,

$$
g(\mathbf{a}):=|\{i \in[m]: \operatorname{pos}(\mathbf{a}, i) \leq \operatorname{pos}(\mathbf{a}, m)\}| .
$$

Noting that $g$ is non-decreasing under switching inversions, we see that $\mathcal{A}:=$ $\left\{\mathbf{a} \in S_{n}: g(\mathbf{a}) \geq(1-\alpha) m\right\}$ is a weak up-set in $S_{n}$. Also noting that the families $L_{i}=\left\{\mathbf{a} \in S_{n}: g(\mathbf{a})=i\right\}$ for $i \in[m]$ partition $S_{n}$ into equal-sized sets, we obtain $|\mathcal{A}|=\sum_{i \in[(1-\alpha) m, m]}\left|L_{i}\right| \geq \alpha n!$.

Our second family $\mathcal{B}$ is defined similarly. Let $h: S_{n} \rightarrow[n-m+1]$, where $h(\mathbf{a})$ equals the number of elements from $[m, n]:=\{m, m+1, \ldots, n\}$ which do not appear before element $m$ in $\mathbf{a}$. That is,

$$
h(\mathbf{a}):=|\{i \in[m, n]: \operatorname{pos}(\mathbf{a}, i) \geq \operatorname{pos}(\mathbf{a}, m)\}| .
$$

Reasoning as above, we find $\mathcal{B}:=\left\{\mathbf{a} \in S_{n}: h(\mathbf{a}) \geq(1-\beta)(n-m+1)\right\}$ is a weak up-set and $|\mathcal{B}| \geq \beta n!$.

Having defined both families, it only remains to upper bound $|\mathcal{A} \cap \mathcal{B}|$. Here it is helpful to consider two further families.

- For $\mathbf{a} \in S_{n}$ let $N_{1}(\mathbf{a}):=\left|\left\{k \in U_{1}: a_{k} \in V_{1}\right\}\right|$, where $U_{1}=[(1-\alpha-\varepsilon) n]$ and $V_{1}=[m]$. Then $\mathcal{E}_{1}:=\left\{\mathbf{a} \in S_{n}: N_{1}(\mathbf{a}) \geq(1-\alpha)\left|V_{1}\right|\right\}$.
- For $\mathbf{a} \in S_{n}$ let $N_{2}(\mathbf{a}):=\left|\left\{k \in U_{2}: a_{k} \in V_{2}\right\}\right|$, where $U_{2}=[(\beta+\varepsilon) n, n]$ and $V_{2}=[m, n]$. Then $\mathcal{E}_{2}:=\left\{\mathbf{a} \in S_{n}: N_{2}(\mathbf{a}) \geq(1-\beta)\left|V_{2}\right|\right\}$.

The functions $N_{1}$ and $N_{2}$ are defined as in Lemma 10, and so we have $\left|\mathcal{E}_{1}\right|,\left|\mathcal{E}_{2}\right| \leq \varepsilon n$ !, provided $n \geq n_{0}(\alpha, \beta, \varepsilon)$.

We claim that every $\mathbf{a} \in \mathcal{C}:=(\mathcal{A} \cap \mathcal{B}) \backslash\left(\mathcal{E}_{1} \cup \mathcal{E}_{2}\right)$ satisfies

$$
\begin{equation*}
\operatorname{pos}(\mathbf{a}, m) \in I:=[(1-\alpha-\varepsilon) n,(\beta+\varepsilon) n] . \tag{2}
\end{equation*}
$$

Note that this will complete the proof of the theorem, since it gives
$|\mathcal{A} \cap \mathcal{B}| \leq|\mathcal{C}|+\left|\mathcal{E}_{1}\right|+\left|\mathcal{E}_{2}\right| \leq\left(\frac{|I|+1}{n}\right) n!+2 \varepsilon n!\leq(\max (\alpha+\beta-1,0)+5 \varepsilon) n!$.
To prove the claim, take $\mathbf{a} \in \mathcal{A} \cap \mathcal{B}$. Note that if $\operatorname{pos}(\mathbf{a}, m)<(1-\alpha-\varepsilon) n$ then $\mathbf{a} \in \mathcal{E}_{1}$ since

$$
\begin{aligned}
N_{1}(\mathbf{a})=\left|\left\{k \in U_{1}: a_{k} \in V_{1}\right\}\right| & =|\{i \in[m]: \operatorname{pos}(\mathbf{a}, i) \leq(1-\alpha-\varepsilon) n\}| \\
& \geq g(\mathbf{a}) \geq(1-\alpha) m=(1-\alpha)\left|V_{1}\right|
\end{aligned}
$$

The first equality is by definition of $N_{1}$, the second equality holds by double counting, the first inequality follows from by definition of $g$ and the fact that $\operatorname{pos}(\mathbf{a}, m) \leq(1-\alpha-\epsilon) n$, and the final inequality holds as $\mathbf{a} \in \mathcal{A}$.

Similarly, if $\operatorname{pos}(\mathbf{a}, m)>(\beta+\varepsilon) n$ then $\mathbf{a} \in \mathcal{E}_{2}$, since

$$
\begin{aligned}
N_{2}(\mathbf{a})=\left|\left\{k \in U_{2}: a_{k} \in V_{2}\right\}\right| & =|\{i \in[m, n]: \operatorname{pos}(\mathbf{a}, i) \geq(\beta+\varepsilon) n\}| \\
& \geq h(\mathbf{a}) \geq(1-\beta)(n-m+1)=(1-\beta)\left|V_{2}\right|
\end{aligned}
$$

Again, the first two equalities hold by definition of $N_{2}$ and by double counting respectively. The first inequality follows from the definition of $h$ and the fact that $\operatorname{pos}(\mathbf{a}, m)>(\beta+\varepsilon) n$, and the final inequality holds as $\mathbf{a} \in \mathcal{B}$.

We have shown that if $\mathbf{a} \in(\mathcal{A} \cap \mathcal{B}) \backslash\left(\mathcal{E}_{1} \cup \mathcal{E}_{2}\right)=\mathcal{C}$ then a satisfies (2) which, as described above, completes the proof.

## 4 Examples and an application

Several natural families of permutations enjoy the property of being strong up-sets. In the first subsection we present a number of examples of these. Together these provide a wide variety of families for which positive correlation results can be deduced from Theorem 1 and Theorem 3. For instance, we shall see that for a random permutation a $\in S_{n}$ (chosen uniformly or following an independently generated measure), the event that no element is displaced by more than a fixed distance $d$ by a and the event that a contains at most $k$ inversions are positively correlated. Likewise, each of these events is positively correlated with the event that at least $u$ elements from $\{1, \ldots, v\}$ occur among the first $w$ positions in a.

In the second subsection we will give an application of Theorem 1 to the correlation of left-compressed set families.

### 4.1 Examples of strong up-sets

## Layers

For each $k \in\left[0,\binom{n}{2}\right]$ let $\mathcal{L}_{k}:=\left\{\mathbf{a} \in S_{n}:|\operatorname{inv}(\mathbf{a})|=k\right\}$. Then it is easily seen that the family $\mathcal{L}_{\geq k}:=\cup_{i \geq k} \mathcal{L}_{i}$ is a strong up-set. In words, this is the set of all permutations which can be written as a product of at most $\binom{n}{2}-k$ adjacent transpositions.

## Band-like permutations

Our next example is based on considering how much each element is moved by a permutation. Given a permutation $\mathbf{a} \in S_{n}$ and an element $i \in[n]$, the displacement of $i$ in a is given by $\operatorname{disp}(\mathbf{a}, i):=|i-\operatorname{pos}(\mathbf{a}, i)|$. We will say a is a $t$-band permutation if $\operatorname{disp}(\mathbf{a}, i) \leq t$ for all $1 \leq i \leq n$.

Lemma 11. The $t$-band permutations in $S_{n}$ form a strong up-set.
Proof. Suppose that $\mathbf{a} \in S_{n}$ is a $t$-band permutation and that $\{i, j\} \in \operatorname{inv}(\mathbf{a})$. Let $\mathbf{b}$ be the permutation obtained from a by swapping $i$ and $j$. It is clear that $\operatorname{disp}(\mathbf{a}, k)=\operatorname{disp}(\mathbf{b}, k)$ for all $k \notin\{i, j\}$. A simple case check also gives
(a) $\operatorname{disp}(\mathbf{b}, i)+\operatorname{disp}(\mathbf{b}, j) \leq \operatorname{disp}(\mathbf{a}, i)+\operatorname{disp}(\mathbf{a}, j)$, and
(b) $|\operatorname{disp}(\mathbf{b}, i)-\operatorname{disp}(\mathbf{b}, j)| \leq|\operatorname{disp}(\mathbf{a}, i)-\operatorname{disp}(\mathbf{a}, j)|$.

As $\mathbf{a}$ is a $t$-band permutation we have $\operatorname{disp}(\mathbf{a}, i), \operatorname{disp}(\mathbf{a}, j) \leq t$ and so it follows that $\operatorname{disp}(\mathbf{b}, i), \operatorname{disp}(\mathbf{b}, j) \leq t$, i.e. $\mathbf{b}$ is also a $t$-band permutation.

In fact this argument shows rather more. Given $\mathbf{a} \in S_{n}$, the displacement list $\mathbf{d}(\mathbf{a})$ is the vector given by:

$$
\mathbf{d}(\mathbf{a}):=(\operatorname{disp}(\mathbf{a}, 1), \ldots, \operatorname{disp}(\mathbf{a}, n))
$$

Now, given a set of vectors $\mathcal{D} \subset\{0,1, \ldots, n-1\}^{n}$, we can form the family of permutations $\mathcal{A}(\mathcal{D}):=\left\{\mathbf{a} \in S_{n}: \mathbf{d}(\mathbf{a}) \in \mathcal{D}\right\} \subset S_{n}$. That is, those permutations in $S_{n}$ whose displacement lists lie in $\mathcal{D}$.

Definition. A set of permutations $\mathcal{A}$ is said to be band-like if $\mathcal{A}=\mathcal{A}(\mathcal{D})$ for some set $\mathcal{D} \subset\{0,1, \ldots, n-1\}^{n}$ which is closed under:

- reordering the entries,
- decreasing any entry,
- replacing two entries of an element of $\mathcal{D}$ with new entries so that neither the sum or difference of these entries increases.

The argument of Lemma 11 shows that:
Lemma 12. Any band-like set of permutations in $S_{n}$ is a strong up-set.
In addition to $t$-band permutations, examples of band-like sets include $\left\{\mathbf{a} \in S_{n}: \sum_{i=1}^{n} \operatorname{disp}(\mathbf{a}, i) \leq t\right\}$ and $\left\{\mathbf{a} \in S_{n}: \sum_{i=1}^{n} \operatorname{disp}(\mathbf{a}, i)^{2} \leq t\right\}$.

## Sequentially dominating permutations

Our final example arises from assigning weight and thresholds as follows. Given a sequence of real weights $\mathbf{w}=\left(w_{1}, \ldots, w_{n}\right)$ with $w_{1} \geq w_{2} \geq \cdots \geq w_{n}$ and thresholds $\mathbf{t}=\left(t_{1}, \ldots, t_{n}\right)$ we consider the family

$$
\mathcal{D}(\mathbf{w}, \mathbf{t}):=\left\{\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right) \in S_{n}: \sum_{i=1}^{m} w_{a_{i}} \geq t_{m} \text { for all } m\right\}
$$

Since the weights are decreasing, such families are closed under swapping inversions and so form strong up-sets.

Some common families arise in this way, including the families of permutations which satisfy 'at least $a$ elements from $\{1, \ldots, b\}$ occur among the first $c$ positions'. Indeed, such families can be written as $\mathcal{D}(\mathbf{w}, \mathbf{t})$, where

$$
\mathbf{w}=(\underbrace{1, \ldots, 1}_{b}, \underbrace{0, \ldots, 0}_{n-b}),
$$

with $t_{i}=a$ if $i=c$, and $t_{i}=0$ otherwise.
Many specific examples follow from these general families. For instance,
Corollary 13. Let a be a random permutation chosen under an independently generated probability measure on $S_{n}$. Then, for any $k, l, m, u, v, w \in \mathbb{N}$, any two of the following events are positively correlated:

- There are at most $k$ inversions in a,
- No element is displaced by more than l by a,
- The sum over all elements of the displacements in $\mathbf{a}$ is at most $m$,
- The first $w$ positions of $\mathbf{a}$ contain at least $u$ of the elements $\{1, \ldots, v\}$.

Amusingly, the families of permutations constructed in the proof of Theorem 2 (our non-correlation result for weak up-sets) can be described using weights in a superficially similar way to a sequentially dominated family. Given a non-increasing sequence of weights and any thresholds, we may define the set of all permutations satisfying that the sum of all entries up to and including element $m$ is at least $t_{m}$ for all $m$. More precisely,

$$
\mathcal{D}^{\prime}(\mathbf{w}, \mathbf{t}):=\left\{\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right) \in S_{n}: \sum_{i=1}^{m} w_{a_{i}} \geq t_{a_{m}} \text { for all } m\right\} .
$$

In general this is not an up-set in the strong or weak sense. However, if we take weights $u_{1}=u_{2}=\ldots u_{k}=1, u_{k+1}=\cdots=u_{n}=0$ with threshold $s_{k}=k / 2$ and weights $v_{1}=v_{2}=\ldots v_{k}=0, v_{k+1}=\cdots=v_{n}=-1$ with threshold $t_{k}=-k / 2$ then the two families $\mathcal{D}^{\prime}(\mathbf{u}, \mathbf{s})$ and $\mathcal{D}^{\prime}(\mathbf{v}, \mathbf{t})$ are precisely those constructed in the proof of Theorem 2.

### 4.2 Maximal chains and left-compressed up-sets

A family of sets $\mathcal{A} \subset \mathcal{P}(X)$ is left-compressed if for any $1 \leq i<j \leq n$, whenever $A \in \mathcal{A}$ with $i \notin A, j \in A$ we also have $(A \backslash\{j\}) \cup\{i\} \in \mathcal{A}$. See [5] for background and a number of useful applications of compressions. It is not hard to show that if $\mathcal{A}$ and $\mathcal{B}$ are left-compressed $r$-uniform families (that is, each consists of $r$-element subsets of $[n]$ ) then they are positively correlated in the sense that

$$
\frac{|\mathcal{A} \cap \mathcal{B}|}{\binom{n}{r}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}} \times \frac{|\mathcal{B}|}{\binom{n}{r}}
$$

However, in general left-compressed families may not be positively correlated; indeed, they may simply be disjoint if the families have different sizes. Below we use Theorem 1 to give a natural measure of the similarity of non-uniform families from which positive correlation for left-compressed families follows.

A maximal chain in $\mathcal{P}(X)$ is a nested sequence of sets $C_{0} \subset C_{1} \subset \cdots \subset C_{n}$ with $C_{i} \subset X$ and $\left|C_{i}\right|=i$. A permutation a of $X$ can be thought of as a
maximal chain in $\mathcal{P}(X)$ by identifying a with the family of sets forming initial segments from $\mathbf{a}$; that is setting $C_{i}:=\left\{a_{1}, \ldots, a_{i}\right\}$ for all $i \in[0, n]$.

If $\mathcal{A}$ is a family of sets, we write $c(\mathcal{A})$ for the number of maximal chains which contain an element of $\mathcal{A}$. Note that if $\mathcal{A}$ is $r$-uniform then the probability that a uniformly random maximal chain meets $\mathcal{A}$ is proportional to $|\mathcal{A}|$ and so in this case $c(\mathcal{A}) / n!=|\mathcal{A}| /\binom{n}{r}$. If $\mathcal{A}$ and $\mathcal{B}$ are families of sets then we write $c(\mathcal{A}, \mathcal{B})$ for the number of maximal chains that meet both $\mathcal{A}$ and $\mathcal{B}$. We will use $c(\mathcal{A})$ as our measure of the size of $\mathcal{A}$ and $c(\mathcal{A}, \mathcal{B})$ as our measure of the intersection (or similarity) of $\mathcal{A}$ and $\mathcal{B}$. With this notion, the following Theorem can be interpreted as saying that left-compressed families are positively correlated.
Theorem 14. If $\mathcal{A}$ and $\mathcal{B}$ are left-compressed families from $\mathcal{P}(X)$ then

$$
\frac{c(\mathcal{A}, \mathcal{B})}{n!} \geq \frac{c(\mathcal{A})}{n!} \times \frac{c(\mathcal{B})}{n!} .
$$

Proof. Let $C(\mathcal{A})$ denote the set of all permutations of $X$ which correspond to chains meeting $\mathcal{A}$ and $C(\mathcal{B})$ be the set of all permutations of $X$ which correspond to chains meeting $\mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are left-compressed $C(\mathcal{A})$ and $C(\mathcal{B})$ are strong up-sets in $S_{n}$. Applying Theorem 1 gives the result.

We remark that while the functional $c(\mathcal{A}) / n$ ! is thought of as a measure $\mathcal{A}$, it is not a probability measure on $\mathcal{P}(X)$ since additivity fails (e.g. consider the partition $\mathcal{P}(X)=\cup_{i}\binom{X}{i}$.

A number of further variations on this result are possible (e.g. if $\mathcal{A}$ is left-compressed and $\mathcal{B}$ is right-compressed then $\mathcal{A}$ and $\mathcal{B}$ are negatively correlated). For example, given a family $\mathcal{F} \subset \mathcal{P}(X)$ and a maximal chain $\mathcal{C}$, let $N_{\mathcal{F}}(\mathcal{C}):=|\mathcal{C} \cap \mathcal{F}|$. Identifying permutations a $\in S_{n}$ with maximal chains as above, we obtain the following.
Theorem 15. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$ be left-compressed families and suppose that $\mathcal{C}$ is a maximal chain from $\mathcal{P}(X)$ chosen uniformly at random. Then for any $k, l$ we have

$$
\mathbb{P}\left(N_{\mathcal{A}}(\mathcal{C}) \geq k, N_{\mathcal{B}}(\mathcal{C}) \geq l\right) \geq \mathbb{P}\left(N_{\mathcal{A}}(\mathcal{C}) \geq k\right) \cdot \mathbb{P}\left(N_{\mathcal{B}}(\mathcal{C}) \geq l\right)
$$

## 5 Open questions

One general question is to determine which other measures on $S_{n}$ satisfy positive association. A particularly appealing class of measures to consider
here are those given by a 1 -dimensional spatial model. Spatial models of this kind are much studied in statistical physics. See [3, 2] for examples of such results.

Let $x(1), x(2), \ldots, x(n) \in \mathbb{R}$ with $x(1) \leq x(2) \leq \cdots \leq x(n)$. We will regard these as $n$ particles placed in increasing order on the real line. A permutation $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right) \in S_{n}$ gives rise to a permutation of these particles. In this model, any point $x(i)$ is displaced by $|x(i)-x(\operatorname{pos}(\mathbf{a}, i))|$. The total displacement is $\sum_{i}|x(i)-x(\operatorname{pos}(\mathbf{a}, i))|$. We define the associated measure on $S_{n}$ by

$$
\mu(\mathbf{a}) \propto q^{\sum_{i}|x(i)-x(\operatorname{pos}(\mathbf{a}, i))|} .
$$

More generally, given a non-decreasing function $V: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$, define

$$
\mu(\mathbf{a}) \propto q^{\sum_{i} V(x(i)-x(\operatorname{pos}(\mathbf{a}, i)))} .
$$

These definitions are special cases of the well-studied Boltzmann measures in which points are picked in $\mathbb{R}^{d}$ and more general functions in the exponent of $q$ are allowed.

We suspect that all measures defined in this way have positive association. However we do not have a proof of this, even in special cases. The following three cases all seem interesting.

Question 1 (Equally spaced points). Is the measure $\mu$ defined by

$$
\mu(\mathbf{a}) \propto q^{\sum_{i}|i-\operatorname{pos}(\mathbf{a}, i)|}
$$

positively associated?
This corresponds to taking $x(i)=i$ and $V(u)=|u|$.
Question 2 (Middle gap). Let $m(\mathbf{a})=\left|\left\{k: 1 \leq k \leq n / 2, n / 2<a_{k} \leq n\right\}\right|$ be the number of elements which are moved 'across the middle gap' by $\mathbf{a}$. Is the measure $\mu$ defined by

$$
\mu(\mathbf{a}) \propto q^{m(\mathbf{a})}
$$

positively associated?
This corresponds to taking $x(i)=0$ for $i \in\left[\frac{n}{2}\right]$ and $x(i)=1$ if $i \in\left[\frac{n}{2}+1, n\right]$ and $V(u)=1$ if $u<0$ and $V(u)=0$ otherwise.

Question 3 (Fixed points). Let $f(\mathbf{a})=\left|\left\{k: a_{k}=k\right\}\right|$ be the number of fixed points of a. Is the measure $\mu$ defined by

$$
\mu(\mathbf{a}) \propto q^{n-f(\mathbf{a})}
$$

positively associated?
This corresponds to taking any distinct $\{x(i)\}_{i \in[n]}$ and setting $V(0)=0$ and $V(u)=1$ otherwise.

Lastly, the correlation behaviour seen in the strong and weak orders are extreme, with the first displaying Harris-Kleitman type correlation (Theorem 1) and the second displaying worst possible correlation (Theorem22). It seems interesting to understand how correlation behaviour emerges between these extremes.

Definition. Given $t \in[n]$, a family of permutations $\mathcal{A} \subset S_{n}$ is a $t$-up-set if given $\mathbf{a} \in \mathcal{A}$, any permutation obtained from a by swapping the elements in a pair $\{i, j\} \in \operatorname{inv}(\mathbf{a})$ with $|\operatorname{pos}(\mathbf{a}, i)-\operatorname{pos}(\mathbf{a}, j)| \leq t$ is also in $\mathcal{A}$.

Note that if $t=1$ then a $t$-up-set is simply a weak up-set. On the other hand, for $t=n$ then a $t$-up-set is a strong up-set. Thus we can think of $t$ -up-sets as interpolating between the weak and strong notions as we increase $t \in[n]$. It seems natural to investigate the correlation behaviour of $t$-up-sets.

Question 4 (Correlation for $t$-up-sets). Given $\alpha>0$, does there exist $\beta>0$ such that the following holds: given $n \in \mathbb{N}$ and $t=\lceil\alpha n\rceil$, any two $t$-up-sets $\mathcal{A}, \mathcal{B} \subset S_{n}$ with $|\mathcal{A}|,|\mathcal{B}| \geq \alpha n$ ! satisfy $|\mathcal{A} \cap \mathcal{B}| \geq \beta n$ !.

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[^1]:    ${ }^{1}$ Note that this is in agreement with the usual notion of an up-set in $\mathcal{P}(X)$, starting from the poset $(\mathcal{P}(X), \subseteq)$.

