# ARC-TRANSITIVE CAYLEY GRAPHS ON NONABELIAN SIMPLE GROUPS WITH PRIME VALENCY 

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#### Abstract

In 2011, Fang et al. in (J. Combin. Theory A 118 (2011) 1039-1051) posed the following problem: Classify non-normal locally primitive Cayley graphs of finite simple groups of valency $d$, where either $d \leq 20$ or $d$ is a prime number. The only case for which the complete solution of this problem is known is of $d=3$. Except this, a lot of efforts have been made to attack this problem by considering the following problem: Characterize finite nonabelian simple groups which admit non-normal locally primitive Cayley graphs of certain valency $d \geq 4$. Even for this problem, it was only solved for the cases when either $d \leq 5$ or $d=7$ and the vertex stabilizer is solvable. In this paper, we make crucial progress towards the above problems by completely solving the second problem for the case when $d \geq 11$ is a prime and the vertex stabilizer is solvable.


Keywords. Cayley graph, simple group, arc-transitive graph.

## 1. Introduction

Throughout this paper, graphs are assumed to be finite undirected graphs without loops and multiple edges, and groups are assumed to be finite. Let $G$ be a permutation group on a set $\Omega$, and let $\alpha \in \Omega$. Denote by $G_{\alpha}$ the stabiliser of $\alpha$ in $G$, that is, the subgroup of $G$ fixing the point $\alpha$. The group $G$ is semiregular if $G_{\alpha}=1$ for every $\alpha \in \Omega$, and regular if $G$ is transitive and semiregular.

For a graph $\Gamma$, denote by $V(\Gamma), E(\Gamma)$ and $\operatorname{Aut}(\Gamma)$ its vertex set, edge set and full automorphism group, respectively. For a vertex $v \in V(\Gamma)$, let $\Gamma(v)$ be the neighbourhood of $v$ in $\Gamma$. An $s$-arc in $\Gamma$ is an ordered $(s+1)$-tuple $\left(v_{0}, v_{1}, \ldots, v_{s}\right)$ of vertices of $\Gamma$ such that $v_{i-1}$ is adjacent to $v_{i}$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i<s$. A graph $\Gamma$, with $G \leq \operatorname{Aut}(\Gamma)$, is said to be $(G, s)$-arc-transitive or $G$-regular if $G$ is transitive on the $s$-arc set of $\Gamma$ or $G$ is regular on the vertex set $V(\Gamma)$ of $\Gamma$, respectively. For short, a 1 -arc means an arc, and $(G, 1)$-arc-transitive means $G$-arc-transitive. If a graph $\Gamma$ is $G$-regular, then $\Gamma$ is also called a Cayley graph of $G$, and the Cayley graph is normal if $G$ is normal in $\operatorname{Aut}(\Gamma)$. A graph $\Gamma$ is said to be $s$-arc-transitive if it is $(\operatorname{Aut}(\Gamma), s)$ -arc-transitive. In particular, 0-arc-transitive is vertex-transitive, and 1-arc-transitive is arc-transitive or symmetric.

A fair amount of work have been done on symmetric Cayley graphs on non-abeian simple groups in the literature. One of the remarkable achievements in this research field is the complete classification of cubic non-normal symmetric Cayley graphs of nonabelian simple groups, and it turns out that up to isomorphism, there are only two cubic non-normal symmetric Cayley graphs of non-abelian simple groups which are both cubic

[^0]5 -arc-transitive Cayley graphs on $\mathrm{A}_{47}$ (see [14, 26, 25]). Recall that a graph $\Gamma$ is called locally primitive if for any $v \in V(\Gamma)$, the stabilizer $\operatorname{Aut}(\Gamma)_{v}$ of $v$ in $\operatorname{Aut}(\Gamma)$ is primitive on $\Gamma(v)$. In view of the fact that every cubic symmetric graph is locally primitive, a natural question arises: What can we say about locally primitive non-normal symmetric Cayley graphs of non-abelian simple groups?

On locally primitive graphs, Weiss [23] conjectured that there is a function $f$ defined on the positive integers such that, whenever $\Gamma$ is a $G$-vertex-transitive locally primitive graph of valency $d$ with $G \leq \operatorname{Aut}(\Gamma)$ then, for any vertex $v \in V(\Gamma),\left|G_{v}\right| \leq f(d)$. By Conder et al. [1], Weiss conjecture is true for vertex-transitive locally primitive $d$-valent graphs if $d \leq 20$ or $d$ is a prime number, and by Spiga [21]. Weiss conjecture is also true if the restriction $G^{\Gamma(v)}$ of $G$ on $\Gamma(v)$ contains an abelian regular subgroup, that is, of affine type. In 2007, Fang et al. [8, Theorem 1.1] shown that for any valency $d$ for which the Weiss conjecture holds, all but finitely many locally primitive Cayley graphs of valency $d$ on the finite nonabelian simple groups are normal, and based on this, the following problem was proposed:

Problem 1.1. [8, Problem 1.2] Classify non-normal locally primitive Cayley graphs of finite simple groups of valency $d$, where either $d \leq 20$ or $d$ is a prime number.

As mentioned above, this problem has been completely solved by Li et al. for the case when $d=3$. For the case when $d \geq 4$, however, it is quite difficult to give a complete solution of Problem 1.1. Because of this, researchers have focused on the following slightly easier problem.

Problem 1.2. Characterize finite nonabelian simple groups which admit non-normal locally primitive Cayley graphs of certain valency $d \geq 4$.

Clearly, a tetravalent graph is locally primitive if and only if the graph is 2-arctransitive. In 2004, Fang et al [7] proved that except 22 groups given in [7, Table 1], every tetravalent 2-arc-transitive Cayley graph $\Gamma$ of a non-abelian simple group $G$ is normal, and based on this, in 2018, Du and Feng [5] proved that there are exactly 7 non-abelian simple groups which admit at least one non-normal 2-arc-transitive Cayley graph, thus giving a complete solution of Problem 1.2 for the case when $d=4$.

There are also some partial solutions of Problem 1.2 for the case when $d$ is a prime number. It is easy to see that a graph with prime valency is locally primitive if and only if it is symmetric. Fang et al in [8] constructed an infinite family of $p$-valent non-normal symmetric Cayley graphs of the alternating groups for all prime $p \geq 5$, and using a result in [9] on the automorphism groups of Cayley graphs of non-abelian simple groups, they also gave all possible candidates of finite nonabelian simple groups which might have a pentavalent non-normal symmetric Cayley graph. This was recently improved by Du et al [6] by proving that there are only 13 finite nonabelian simple groups which admit a pentavalent non-normal symmetric Cayley graph.

More recently, Pan et al [17] considered Problem 1.2 for the case when $d=7$, and they proved that for a 7 -valent Cayley graph $\Gamma$ of a non-abelian simple group $G$ with solvable vertex stabilizer, either $\Gamma$ is normal, or $\operatorname{Aut}(\Gamma)$ has a normal arc-transitive nonabelian simple subgroup $T$ such that $G<T$ and $(G, T)=\left(\mathrm{A}_{6}, \mathrm{~A}_{7}\right),\left(\mathrm{A}_{20}, \mathrm{~A}_{21}\right),\left(\mathrm{A}_{62}, \mathrm{~A}_{63}\right)$ or $\left(\mathrm{A}_{83}, \mathrm{~A}_{84}\right)$, and for each of these 4 pairs $(G, T)$, there do exist a 7 -valent $G$-regular $T$-arc-transitive graph.

In this paper, we shall prove the following theorem which generalizes the result in [17] to all prime valent cases, and hence gives a solution of Problem 1.2 for the case when $d$ is a prime and the vertex-stabilizer is solvable.

Theorem 1.3. Let $G$ be a non-abelian simple group and $\Gamma$ a connected arc-transitive Cayley graph of $G$ with prime valency $p \geq 11$. If $\operatorname{Aut}(\Gamma)_{v}$ is solvable for $v \in V(\Gamma)$, then either $G \unlhd \operatorname{Aut}(\Gamma)$, or $\operatorname{Aut}(\Gamma)$ has a normal subgroup $T$ with $G<T$ such that $\Gamma$ is $T$-arc-transitive and $(G, T, p)$ is one of the following four triples:

$$
\left(\mathrm{A}_{5}, \operatorname{PSL}(2,11), 11\right),\left(\mathrm{A}_{5}, \operatorname{PSL}(2,29), 29\right),\left(\mathrm{M}_{22}, \mathrm{M}_{23}, 23\right),\left(\mathrm{A}_{n-1}, \mathrm{~A}_{n}, p\right)
$$

where $n=p k \ell$ with $k \mid \ell$ and $\ell \mid(p-1)$, and $k$ and $\ell$ have the same parity.
Conversely, we show that all the first three triples as well as the fourth triple in case of $n=p$ can happen.

Theorem 1.4. Use the same notation as Theorem 1.3. If $(G, T, p)$ is one of the following triples:

$$
\left(\mathrm{A}_{5}, \operatorname{PSL}(2,11), 11\right),\left(\mathrm{A}_{5}, \operatorname{PSL}(2,29), 29\right),\left(\mathrm{M}_{22}, \mathrm{M}_{23}, 23\right),\left(\mathrm{A}_{p-1}, \mathrm{~A}_{p}, p\right)
$$

then there exists a p-valent symmetric Cayley graph $\Gamma$ of $G$ such that $\operatorname{Aut}(\Gamma)_{v}$ is solvable for some $v \in V(\Gamma)$.

Let $p$ be a prime and $\ell, k$ integers with $k \mid \ell$ and $\ell \mid(p-1)$ such that $k$ and $\ell$ have the same parity. The triple $(p, \ell, k)$ is called conceivable if there exists an arc-transitive Cayley graph of the alternating group $\mathrm{A}_{p k \ell-1}$ with valency $p$ and its automorphism group has solvable vertex stabilizer. We have been unable to determine all the conceivable triples $(p, \ell, k)$, and we would like to leave it as an open problem for future research.

Problem 1.5. Determine conceivable triples $(p, \ell, k)$.
By Theorem 1.4, $(p, 1,1)$ is conceivable for each prime $p \geq 5$, and by $[6],(5,4,2)$ is conceivable, but not $(5,2,2)$. For the case $p=7$, it was shown in [17] that $(7,1,1)$, $(7,3,1),(7,3,3)$ and $(7,6,2)$ are the only conceivable triples.

The paper is organized as follows. In Section 2 we introduce some preliminary results on nonabelian simple groups and arc-transitive graphs with prime valency. Then we prove Theorem 1.3 in Section 3 and Theorem 1.4 in Section 4.

## 2. Preliminary

In this section, we introduce some preliminary results that will be used latter.
For a positive integer $n$, we use $\mathbb{Z}_{n}$ to denote the cyclic group of order $n$. For a group $G$ and a subgroup $H$ of $G$, denote by $N_{G}(H)$ and $C_{G}(H)$ the normalizer and the centralizer of $H$ in $G$ respectively. Given two groups $N$ and $H$, denote by $N \times H$ the direct product of $N$ and $H$, by $N . H$ an extension of $N$ by $H$, and if such an extension is split, then we write $N: H$ instead of $N . H$.

The following proposition is an exercise in Dixon and Mortimer's textbook [4, p.49].
Proposition 2.1. Let $n$ be a positive integer and $p$ a prime. Let $p^{\nu(n)}$ be the largest power of $p$ which divides $n!$. Then $\nu(n)=\sum_{i=1}\left\lfloor\frac{n}{p^{2}}\right\rfloor<\frac{n}{p-1}$.

The next proposition is called the Frattini argument on transitive permutation group, and we refer to $[4, ~ p .9]$.

Proposition 2.2. Let $G$ be a transitive permutation group on $\Omega, H$ a subgroup of $G$ and $v \in \Omega$. Then $H$ is transitive if and only if $G=H G_{v}$.

We denote by $\operatorname{Aut}(G)$ the automorphism group of a group $G$, and by $\operatorname{Inn}(G)$ the inner automorphism group of $G$ consisting of these automorphisms of $G$ induced by all element of $G$ by conjugation on $G$. Then $\operatorname{Inn}(G)$ is normal in $\operatorname{Aut}(G)$, and the quotient group $\operatorname{Aut}(G) / \operatorname{Inn}(G)$ is called the outer automorphism of $G$, denoted by Out $(G)$. The following proposition is a direct consequence of the classification of finite simple groups (see [13, Table 5.1.A-C] for example).
Proposition 2.3. Let $T$ be a finite non-abelian simple group. Then $\operatorname{Out}(T)$ is solvable.
Let $G$ and $E$ be two groups. We call an extension $E$ of $G$ by $N$ a central extension of $G$ if $E$ has a central subgroup $N$ such that $E / N \cong G$, and if further $E$ is perfect, that is, the derived group $E^{\prime}$ equals to $E$, we call $E$ a covering group of $G$. A covering group $E$ of $G$ is called a double cover if $|E|=2|G|$. Schur [20] proved that for every non-abelian simple group $G$ there is a unique maximal covering group $M$ such that every covering group of $G$ is a factor group of $M$ (see [12, Kapitel V, S23]). This group $M$ is called the full covering group of $G$, and the center of $M$ is the Schur multiplier of $G$, denoted by Mult $(G)$. By Kleidman and Liebeck [13, Theorem 5.1.4] and Du et al [6, Proposition 2.6], we have the following proposition.

Proposition 2.4. $\operatorname{Mult}\left(\mathrm{A}_{n}\right)=\mathbb{Z}_{2}$ with $n \geq 8$. For $n \geq 5, \mathrm{~A}_{n}$ has a unique double cover $2 . \mathrm{A}_{n}$, and for $n \geq 7$, all subgroups of index $n$ of $2 . \mathrm{A}_{n}$ are isomorphic to $2 . \mathrm{A}_{n-1}$.

By Kleidman and Liebeck [13, Proposition 5.3.7], we have the following proposition.
Proposition 2.5. Let $r$ be a prime power and $f$ a positive integer. If $\mathrm{A}_{n} \leq \mathrm{GL}(f, r)$ with $n \geq 9$, then $f \geq n-2$.

Let $\Gamma$ be a connected graph and $G$ a group of automorphisms of $\Gamma$. For $v \in V(\Gamma)$, denote by $G_{v}^{\Gamma(v)}$ the induced permutation group of the natural action of $G_{v}$ on the neighbourhood $\Gamma(v)$. Let $G_{v}^{*}$ be the subgroup of $G_{v}$ fixing every vertex in $\Gamma(v)$. Then $G_{v}^{*}$ is the kernel of the natural action of $G_{v}$ on $\Gamma(v)$, and hence $G_{v} / G_{v}^{*} \cong G_{v}^{\Gamma(v)}$. By the connectivity of $\Gamma$, there exists a path $v=v_{0}, v_{1}, v_{2}, \cdots, v_{m}$ such that $G_{v_{0} v_{1} \cdots v_{m}}^{*}:=$ $G_{v_{0}}^{*} \cap G_{v_{1}}^{*} \cap \cdots \cap G_{v_{m}}^{*}=1$. Clearly,

$$
1=G_{v_{0} v_{1} \cdots v_{m}}^{*} \unlhd G_{v_{0} v_{1} \cdots v_{m-1}}^{*} \unlhd \cdots \unlhd G_{v_{0} v_{1}}^{*} \unlhd G_{v_{0}}^{*}=G_{v}^{*} \unlhd G_{v}
$$

and for $0 \leq i<m$, we have $G_{v_{0} v_{1} \cdots v_{i}}^{*} / G_{v_{0} v_{1} \cdots v_{i+1}}^{*} \cong\left(G_{v_{0} v_{1} \cdots v_{i}}^{*}\right)^{\Gamma\left(v_{i+1}\right)}$. Then we can easily obtain the following proposition, and this was known from a series of lectures given by Cai Heng Li in Peking University in 2013.

Proposition 2.6. Let $\Gamma$ be a connected graph and let $G$ be a vertex-transitive group of automorphisms of $\Gamma$. Then $G_{v}$ is nonsolvable if and only if $G_{v}^{\Gamma(v)}$ is nonsolvable.

For self-containing, we give a short proof of the following proposition, which is mainly owed to an anonymous referee (also see [11] for another proof).

Proposition 2.7. Let $\Gamma$ be a connected $G$-arc-transitive graph of prime valency $p \geq 5$, and let $(u, v)$ be an arc of $\Gamma$. Assume that $G_{v}$ is solvable. Then $G_{u v}^{*}=1$ and $G_{v} \cong$ $\mathbb{Z}_{k} \times\left(\mathbb{Z}_{p}: \mathbb{Z}_{\ell}\right)$ with $k|\ell|(p-1)$, where $\mathbb{Z}_{p}: \mathbb{Z}_{\ell} \leq \operatorname{AGL}(1, p)$.

Proof. It follows from [23] that $G_{u v}^{*}=1$. Let $P$ be a Sylow $p$-subgroup of $G_{v}$. Note that $G_{v}^{\Gamma(v)}$ is a transitive solvable group of prime degree. By the Burnside Theorem (also see [4, Theorem 3.5B]), $G_{v} / G_{v}^{*} \cong G_{v}^{\Gamma(v)} \cong \mathbb{Z}_{p}: \mathbb{Z}_{\ell} \leq \operatorname{AGL}(1, p)$ with $\ell \mid(p-1)$ and $G_{u v} / G_{v}^{*} \cong \mathbb{Z}_{\ell}$. In particular, $P G_{v}^{*} / G_{v}^{*} \unlhd G_{v} / G_{v}^{*}$, and so $P G_{v}^{*} \unlhd G_{v}$. Since $G_{u}^{*}=$ $G_{u}^{*} / G_{u v}^{*}=G_{u}^{*} /\left(G_{u}^{*} \cap G_{v}^{*}\right) \cong G_{u}^{*} G_{v}^{*} / G_{v}^{*} \leq G_{u v} / G_{v}^{*} \cong \mathbb{Z}_{\ell}$, we have $G_{v}^{*} \cong \mathbb{Z}_{k}$ with $k \mid \ell$, and then $\left|G_{v}\right|=p k \ell$ with $k|\ell|(p-1)$. Since $G_{u v}=G_{u v} / G_{u v}^{*}=G_{u v} /\left(G_{u}^{*} \cap G_{v}^{*}\right) \lesssim$ $G_{u v} / G_{u}^{*} \times G_{u v} / G_{v}^{*} \cong \mathbb{Z}_{\ell} \times \mathbb{Z}_{\ell}, G_{u v}$ is abelian of exponent $\ell$. Let $G_{u v} / G_{v}^{*}=\left\langle a G_{v}^{*}\right\rangle \cong \mathbb{Z}_{\ell}$. Then $\langle a\rangle \cong \mathbb{Z}_{\ell}$ and $\langle a\rangle \cap G_{v}^{*}=1$. It follows that $G_{u v}=\langle a\rangle \times G_{v}^{*}$.

Since $\left|G_{v}^{*}\right|=\ell \mid(p-1), P G_{v}^{*}$ has a unique Sylow $p$-subgroup $P$ and hence $P G_{v}^{*}=$ $P \times G_{v}^{*}$. Then $P$ is characteristic in $P G_{v}^{*}$, and since $P G_{v}^{*} \unlhd G_{v}$, we have $P \unlhd G_{v}$. It follows that $G_{v}=P: G_{u v}=P:\left(\langle a\rangle \times G_{v}^{*}\right)=G_{v}^{*} \times(P:\langle a\rangle) \cong \mathbb{Z}_{k} \times\left(\mathbb{Z}_{p}: \mathbb{Z}_{\ell}\right)$.

Taking normal quotient graphs is a useful method for studying arc-transitive graphs. Let $\Gamma$ be an $X$-vertex-transitive graph, where $X \leq \operatorname{Aut}(\Gamma)$ has an intransitive normal subgroup $N$. The normal quotient graph $\Gamma_{N}$ of $\Gamma$ induced by $N$ is defined to be a graph with vertex set $\left\{\alpha^{N} \mid \alpha \in V(\Gamma)\right\}$, the set of all $N$-orbits on $V(\Gamma)$, such that two vertices $B, C \in\left\{\alpha^{N} \mid \alpha \in V(\Gamma)\right\}$ are adjacent if and only if some vertex in $B$ is adjacent in $\Gamma$ to some vertex in $C$. If $\Gamma$ and $\Gamma_{N}$ have the same valency, then $\Gamma$ is called a normal cover of $\Gamma_{N}$. The following proposition is a special case of [15, Lemma 2.5], which slightly improves a remarkable result of Praeger [18, Theorem 4.1].

Proposition 2.8. Let $\Gamma$ be a connected $X$-arc-transitive graph of prime valency, with $X \leq \operatorname{Aut}(\Gamma)$, and let $N \unlhd X$ have at least three orbits on $V(\Gamma)$. Then the following statements hold.
(1) $N$ is semi-regular on $V(\Gamma), X / N \leq \operatorname{Aut}\left(\Gamma_{N}\right), \Gamma_{N}$ is a connected $X / N$-arctransitive graph, and $\Gamma$ is a normal cover of $\Gamma_{N}$.
(2) $X_{v} \cong(X / N)_{\Delta}$ for any $v \in V(\Gamma)$ and $\Delta \in V\left(\Gamma_{N}\right)$.

## 3. Proof of Theorem 1.3

Throughout this section we make the following assumption.
Assumption: $\Gamma$ is a symmetric graph of prime valency $p \geq 11$ with $v \in V(\Gamma)$, Aut $(\Gamma)_{v}$ is solvable, and $G \leq \operatorname{Aut}(\Gamma)$ is a non-abelian simple group and transitive on $V(\Gamma)$.

The proof of the following lemma is straightforward, but will be used frequently latter.
Lemma 3.1. Let $X=H: K$ be a transitive permutation group on $\Omega$. Let $w \in \Omega$. If $H$ is transitive, then $K$ is isomorphic to $X_{w} / H_{w}$.
Proof. Since $H$ is transitive, $X=H X_{w}$ by Proposition 2.2. So $K \cong X / H=H X_{w} / H \cong$ $X_{w} /\left(X_{w} \cap H\right)=X_{w} / H_{w}$.

The product of all minimal normal subgroups of a group $X$ is called the socle of $X$, denoted by $\operatorname{soc}(X)$, and the largest normal solvable subgroup of $X$ is called the radical of $X$, denoted by $\operatorname{rad}(X)$.

Lemma 3.2. Let $G, \Gamma, p$ and $v$ be as given in Assumption. Let $\Gamma$ be $X$-arc-transitive with $G \leq X \leq \operatorname{Aut}(\Gamma)$, and let $\operatorname{rad}(X)=1$. Then either $\operatorname{soc}(X)=G$, or $\Gamma$ is $\operatorname{soc}(X)$ -arc-transitive with $G<\operatorname{soc}(X)$ and one of the following holds:
(1) $(G, \operatorname{soc}(X))=\left(\mathrm{A}_{n-1}, \mathrm{~A}_{n}\right)$ with $n \geq 6$, and $(\operatorname{soc}(X))_{v}$ is transitive on $\{1,2, \cdots, n\}$.
(2) $(G, \operatorname{soc}(X))=\left(\mathrm{M}_{22}, \mathrm{M}_{23}\right)$, and $(\operatorname{soc}(X))_{v}=\mathbb{Z}_{23}$.
(3) $(G, \operatorname{soc}(X))=\left(\mathrm{A}_{5}, \operatorname{PSL}(2,11)\right)$, and $(\operatorname{soc}(X))_{v}=\mathbb{Z}_{11}$.
(4) $(G, \operatorname{soc}(X))=\left(\mathrm{A}_{5}, \operatorname{PSL}(2,29)\right)$, and $(\operatorname{soc}(X))_{v}=\mathbb{Z}_{29}: \mathbb{Z}_{7}$.

In particular, $\Gamma$ is a Cayley graph of $G$ for cases (2)-(4).
Proof. Let $N$ be a minimal normal subgroup of $X$. Since $\operatorname{rad}(X)=1$, we have $N=$ $T_{1} \times \cdots \times T_{d} \cong T^{d}$ for a non-abelian simple group $T$. Write $K=N G$.

Assume that $G \unlhd X$. If $N \cap G=1$, applying Lemma 3.1 with $K=G: N$ we have that $N \cong(K)_{v} / G_{v}$ is solvable, a contradiction. Therefore, $N \cap G \neq 1$, forcing $G \leq N$, and since $G$ is normal, the minimality of $N$ implies $N=G$. By the arbitrariness of $N$, we have $\operatorname{soc}(X)=G$.

In what follows we assume that $G \nexists X$. If $\Gamma$ is bipartite, then the transitivity of $G$ on $V(\Gamma)$ implies that $G$ has a normal subgroup of index 2 , contradicting the simplicity of $G$. Thus, $\Gamma$ is not bipartite. Therefore $N$ has either one or at least three orbits on $V(\Gamma)$. We claim that the latter cannot occur.

We argue by contradiction and we suppose that $N$ has at least three orbits on $V(\Gamma)$. By Proposition 2.8, $N$ is semiregular on $V(\Gamma)$, and so $|N|=|T|^{d}$ is a divisor of $|V(\Gamma)|$. In particular, $|N|||G|$. Since $N$ has at least three orbits, $| G|\geq 3| N \mid$ and hence $N \cap G=1$.

Consider the conjugate action of $G$ on $N$, and since $G$ is simple, the action is trivial or faithful. If it is trivial then $K=N \times G$, and by Lemma 3.1, $N \cong K_{v} / G_{v}$ is solvable, a contradiction. It follows that the conjugate action of $G$ on $N$ is faithful, and hence we may assume $G \leq \operatorname{Aut}(N)$.

Note that $\operatorname{Aut}(N) \cong \operatorname{Aut}(T)^{d}: \mathrm{S}_{d}$. Set $M=\operatorname{Aut}(T)^{d}$ and $M_{1}=\operatorname{Inn}(N) \cong T^{d}$. Then $\left|M_{1}\right|=|N|, M_{1} \unlhd M, M \unlhd \operatorname{Aut}(N)$ and $M_{1} \unlhd \operatorname{Aut}(N)$. Clearly, $G \cap M_{1}=1$ as $|G| \geq 3|N|=$ $3\left|M_{1}\right|$. If $G \cap M \neq 1$ then $G \leq M$ and hence $G \cong G /\left(G \cap M_{1}\right) \cong G M_{1} / M_{1} \leq M / M_{1} \cong$ $\operatorname{Out}(T)^{d}$, which is impossible because $\operatorname{Out}(T)$ is solvable by Propostion 2.3. This means that $G \cap M=1$, and therefore, $G \cong G /(G \cap M) \cong G M / M \leq \operatorname{Aut}(N) / M \cong \mathrm{~S}_{d}$. Recall that $|N|=|T|^{d}$ and $|N|||G|$. Then for any prime $p$ with $p||T|$, we have $p^{d} \mid d!$, and by Proposition 2.1, $d<\frac{d}{p-1}$, a contradiction.

We have just shown that $N$ has one orbit, that is, $N$ is transitive on $V(\Gamma)$. If $N \cap G=1$, Lemma 3.1 implies that $G \cong K_{v} / N_{v}$ is solvable, a contradiction. Therefore, $G \leq N$, and by the arbitrariness of $N, X$ has only one minimal normal subgroup, that is, $\operatorname{soc}(X)=N$.

Since $G$ is not normal in $X$, we have $G<N$, and hence $N_{v} \neq 1$ as $\Gamma$ is $G$-vertextransitive. Clearly, we may chose $v$ such that $N_{v}^{\Gamma(v)} \neq 1$. Since $\Gamma$ has prime valency and $N_{v}^{\Gamma(v)} \unlhd X_{v}^{\Gamma(v)}, N_{v}^{\Gamma(v)}$ is transitive on $\Gamma(v)$, that is, $\Gamma$ is $N$-arc-transitive.

Recall that $N=T_{1} \times T_{2} \times \cdots \times T_{d} \cong T^{d}$. Suppose $d \geq 2$. If $T_{1}$ is transitive, then by Lemma 3.1, $T_{2} \times \cdots \times T_{d} \cong N_{v} /\left(T_{1}\right)_{v}$ is solvable, a contradiction. Thus, $T_{1}$ has at least three orbits, and hence $|G| \geq 3\left|T_{1}\right|$. In particular, $G \cap T_{1}=1$. By the simplicity of $G$, the conjugate action of $G$ on $T_{1}$ is trivial or faithful. If it is trivial then $G T_{1}=G \times T_{1}$, and by Lemma 3.1, $T_{1} \cong\left(G T_{1}\right)_{v} / G_{v}$ is solvable, a contradiction. Thus, the conjugate action of $G$ on $T_{1}$ is faithful and hence we may assume $G \leq \operatorname{Aut}\left(T_{1}\right)$. Since $|G| \geq 3\left|T_{1}\right|=3\left|\operatorname{lnn}\left(T_{1}\right)\right|$, we have $G \cap \operatorname{Inn}\left(T_{1}\right)=1$ and hence $G=G /\left(G \cap \operatorname{lnn}\left(T_{1}\right)\right) \cong$ $G \operatorname{lnn}\left(T_{1}\right) / \operatorname{lnn}\left(T_{1}\right) \leq \operatorname{Aut}\left(T_{1}\right) / \operatorname{Inn}\left(T_{1}\right)=\operatorname{Out}\left(T_{1}\right)$, which is impossible because $\operatorname{Out}\left(T_{1}\right)$ is solvable. Thus, $\operatorname{soc}(X)=N=T$ is a non-abelian simple group.

By the Frattini argument, $T=G T_{v}$. Then the triple ( $T, G, T_{v}$ ) can be read out from [16], where $T_{v}$ is a group given in Proposition 2.7. Note that $p \geq 11$.

By [16, Proposition 4.2], $T$ cannot be any exceptional group of Lie type.
Assume that $T=\mathrm{A}_{n}$. By [16, Proposition 4.3], one of the following occurs:
(a) $G=\mathrm{A}_{n-1}, T=\mathrm{A}_{n}$ with $n \geq 6$ and $T_{v}$ is transitive on $\{1,2, \cdots, n\}$, or
(b) $G=\mathrm{A}_{n-2}, T=\mathrm{A}_{n}$ with $n=q^{f}$ for some prime $q$, and $T_{v} \leq \mathrm{A} \Gamma \mathrm{L}\left(1, q^{f}\right)$ is 2 -homogeneous on $\{1,2, \cdots, n\}$.

If (b) occurs, then $T_{v}$ is primitive on $\left\{1,2,3, \cdots, q^{f}\right\}$ because it is 2-homogeneous. By Proposition 2.7, $T_{v}$ has a normal subgroup $\mathbb{Z}_{p}$, and by the primitivity of $T_{v}, \mathbb{Z}_{p}$ is transitive and so regular on $\left\{1,2,3, \cdots, q^{f}\right\}$. It follows $q^{f}=p$ and $T_{v} \leq \operatorname{AGL}(1, p)=$ $\mathbb{Z}_{p}: \mathbb{Z}_{p-1}$. Moreover, since $\left|T_{v}\right|=\frac{|T|\left|G_{v}\right|}{|G|} \geq \frac{|T|}{|G|}=p(p-1)$, we have that $T_{v}=\operatorname{AGL}(1, p)=$ $\mathbb{Z}_{p}: \mathbb{Z}_{p-1}$. Thus, $A_{p}$ contains a cyclic subgroup $\mathbb{Z}_{p-1}$, which is impossible because $\mathbb{Z}_{p-1}$ contains odd permutations on $\{1,2,3, \cdots, p\}$. It follows that $T=\mathrm{A}_{n}, G=\mathrm{A}_{n-1}$ and $T_{v}$ is transitive on the $n$ points, which is the case (1) of the lemma.

Assume that $T$ is a sporadic simple group. By [16, Proposition 4.4], $G=\mathrm{M}_{22}, T=$ $\mathrm{M}_{23}$, and $T_{v}=\mathbb{Z}_{23}$ or $\mathbb{Z}_{23}: \mathbb{Z}_{11}$. Suppose on the contrary that $T_{v}=\mathbb{Z}_{23}: \mathbb{Z}_{11}$. We may let $T_{u v}=\mathbb{Z}_{11}$ for $u \in \Gamma(v)$. Since $\Gamma$ is $T$-arc-transitive, there is an element $g \in T$ interchanging $u$ and $v$, and hence $T_{u v}^{g}=T_{u^{g} v^{g}}=T_{u v}$, that is, $g \in N_{T}\left(T_{u v}\right)$. A computation with MAGMA [2] shows that there is only one conjugate class of $\mathbb{Z}_{11}$ in $\mathrm{M}_{23}$, and the normalizer of $\mathbb{Z}_{11}$ in $\mathbb{M}_{23}$ is $\mathbb{Z}_{11}: \mathbb{Z}_{5}$. Thus, $g \in \mathbb{Z}_{11}: \mathbb{Z}_{5}$ has odd order, which is impossible because $g$ interchanges $u$ and $v$. It follows that $T_{v}=\mathbb{Z}_{23}$, which is the case (2) of the lemma.

Assume that $T$ is a classical simple group of Lie type. Note that $T=G T_{v}, G$ is non-abelian simple and $T_{v}$ is solvable. Let $H$ is a maximal subgroup subject to that $T_{v} \leq H$ and $H$ is solvable. Then $T=G H$, and $(T, G, H)$ is listed in [16, Table 1.1 and Table 1.2]. Clearly, $|T: G|\left|\left|T_{v}\right|\right||H|$. For an integer $m$ and a prime $r$, we use $m_{r}$ to denote the largest $r$-power dividing $m$.

By Proposition $2.7, T_{v}=\mathbb{Z}_{k} \times\left(\mathbb{Z}_{p}: \mathbb{Z}_{\ell}\right)$ with $k|\ell| p-1$, where $\mathbb{Z}_{p}: \mathbb{Z}_{\ell} \leq \operatorname{AGL}(1, p)$. Let $P$ and $Q$ be the maximal normal $r$-subgroup of $T_{v}$ and $H$ respectively. Then $Q \cap T_{v} \leq P$, and since $T_{v} /\left(T_{v} \cap Q\right) \cong Q T_{v} / Q \leq H / Q$, we have $\left|T_{v}\right|_{r} \leq\left|T_{v} \cap Q\right| \cdot|H / Q|_{r} \leq|P||H / Q|_{r}$. Clearly, $\left|T_{v}\right|_{p}=p$ and hence $|T: G|_{p} \leq p$.

Suppose that $r \neq p$ and $r\left|\left|T_{v}\right|\right.$. If $P$ is not contained in $\mathbb{Z}_{k}$, then $1 \neq P \mathbb{Z}_{k} / \mathbb{Z}_{k} \unlhd$ $T_{v} / \mathbb{Z}_{k} \cong \mathbb{Z}_{p}: \mathbb{Z}_{\ell}$, which is impossible because $\mathbb{Z}_{p}$ is the unique minimal normal subgroup of $\mathbb{Z}_{p}: \mathbb{Z}_{\ell}$. Therefore $P \leq \mathbb{Z}_{k}$. It follows from $k \mid \ell$ that $|P|^{2} \leq\left|T_{v}\right|_{r}$, and from $\left|T_{v}\right|_{r} \leq$ $|P||H / Q|_{r}$ that $|P| \leq|H / Q|_{r}$. Thus, $|T: G|_{r} \leq\left|T_{v}\right|_{r} \leq\left(|H / Q|_{r}\right)^{2}$.

Since $G$ is a non-abelian simple group, we may exclude Row 1 of [16, Table 1.1] and Rows $7-10,17$ and 21 of [16, Table 1.2 ], and since $p \geq 11$ and $p||H|$, we may exclude Rows $6,11-13,16-20,22$ and $24-27$ of [16, Table 1.2]. The remaining cases are Rows 2-9 of [16, Table 1.1], and Rows 1-5, 14, 15, 23 and 28 of [16, Table 1.2].

In what follows we write $q=r^{f}$ for some prime $r$ and positive integer $f$.
For Row 2 of [16, Table 1.1], $T=\operatorname{PSL}(4, q), G=\operatorname{PSp}(4, q)$, and $H=q^{3}: \frac{q^{3}-1}{(4, q-1)} .3$. By [13, Table 5.1A], $q^{2}| | T: G \mid$. Thus $r \neq p$. Note that $|H / Q|_{r}=1$ or 3 . Since $r^{2 f}=q^{2} \leq\left|T_{v}\right|_{r} \leq|H / Q|_{r}^{2}$, we have $r=3$ and $f=1$, that is, $q=3$. This is impossible because a computation with Magma shows $T=\operatorname{PSL}(4,3)$ has no factorization $T=G H$.

For Row 3 of [16, Table 1.1], $T=\operatorname{PSp}(2 m, q), G=\Omega^{-}(2 m, q)$, and $H=q^{m(m+1) / 2}$ : $\left(q^{m}-1\right) . m$ with $m \geq 2$ and $q$ even. Then $r=2$. By [13, Table 5.1A], $q^{m}| | T: G \mid$, implying $r \neq p$. Furthermore, $r^{f m}=q^{m} \leq\left|T_{v}\right|_{r} \leq|H / Q|_{r}^{2}=m_{r}^{2}$. It follows $r^{m_{2}} \leq$ $r^{f m} \leq m_{2}^{2}$, and this holds if and only if $m_{2}=2$ or 4 . If $m_{2}=2$, then $2^{f m} \leq m_{2}^{2}=4$ implies $f=1, m=2$, which is impossible because $\operatorname{PSp}(4,2) \cong \mathrm{S}_{6}$ is not a simple group. If $m_{2}=4$, then $2^{f m} \leq m_{2}^{2}=16$ implies that $m=4$ and $f=1$. In this case, $|H|=2^{12} \cdot 15$, contradicting that $p||H|$ with $p \geq 11$.

For Rows 4 and 5 of [16, Table 1.1], $T=\operatorname{PSp}(4, q), G=\operatorname{PSp}\left(2, q^{2}\right)$, and $H=q^{3}$ : $\frac{q^{2}-1}{(2, q-1)}$.2. By [13, Table 5.1A], $q^{2}| | T: G \mid$, and so $r \neq p$. Note that $|H / Q|_{r}=1$ or 2 . Since $r^{2 f}=q^{2} \leq\left|T_{v}\right|_{r} \leq|H / Q|_{r}^{2}$, we have that $r=2$ and $f=1$. This is impossible because $T=\operatorname{PSp}(4,2) \cong \mathrm{S}_{6}$ is not simple.

For Row 6 of [16, Table 1.1], $T=\operatorname{PSU}(2 m, q), G=\operatorname{SU}(2 m-1, q)$, and $H=q^{m^{2}}$ : $\frac{q^{2 m}-1}{q+1(2 m, q+1)} . m$ with $m \geq 2$. By [13, Table 5.1A], $q^{2 m-1}=r^{(2 m-1) f}| | T: G \mid$ and $r \neq$ $p$. Thus $r^{(2 m-1) f}=q^{2 m-1} \leq\left|T_{v}\right|_{r} \leq H /\left.Q\right|_{r} ^{2}=m_{r}^{2}$, implying $r^{2 m_{r}-1} \leq m_{r}^{2}$, which is impossible.

For Row 7 of [16, Table 1.1], $T=\mathrm{P} \Omega(2 m+1, q), G=\Omega^{-}(2 m, q)$, and $H=$ $\left(q^{m(m-1) / 2} \cdot q^{m}\right): \frac{q^{m}-1}{2} . m$ with $m \geq 3$ and $q$ odd. Then $r, m_{r} \geq 3$. By [13, Table 5.1A], $q^{m}=2^{f m}| | T: G \mid$ and hence $r \neq p$. Then $r^{f m}=q^{m} \leq\left|T_{v}\right|_{r} \leq|H / Q|_{r}^{2}=m_{r}^{2}$, and so $r^{m_{r}} \leq m_{r}^{2}$, which is impossible.

For Row 8 of [16, Table 1.1], $T=\mathrm{P} \Omega^{+}(2 m, q), G=\Omega(2 m-1, q)$, and $H=q^{m(m-1) / 2}$ : $\frac{q^{m}-1}{\left(4, q^{m}-1\right)} . m$ with $m \geq 5$. By [13, Table 5.1A], $q^{m-1}=r^{f(m-1)}| | T: G \mid$ and $r \neq p$. Then $r^{f(m-1)}=q^{m-1} \leq\left|T_{v}\right|_{r} \leq|H / Q|_{r}^{2}=m_{r}^{2}$. Note that the inequality $2^{x}>x^{2}$ always holds for $x \geq 5$. Thus $m_{r} \leq 4$. Since $r^{f(m-1)} \leq m_{r}^{2}$ and $m \geq 5$, we have that $r=2, m_{r}=4$ and $m=5$, which is impossible because $m_{r}=5_{2}=1$.

For Row 9 of [16, Table 1.1], $T=\mathrm{P} \Omega^{+}(8, q), G=\Omega(7, q)$, and $H=q^{6}: \frac{q^{4}-1}{\left(4, q^{4}-1\right)} \cdot 4$. By [13, Table 5.1A], $q^{3}=r^{3 f}| | T: G \mid$, and $r \neq p$. Then $r^{3 f}=q^{3} \leq\left|T_{v}\right|_{r} \leq|H / Q|_{r}^{2}=\left(4_{r}\right)^{2}$, implying $r=2$ and $f=1$. In this case, $|H|=2^{8} \cdot 15$, contradicting $p||H|$ with $p \geq 11$.

For Row 14 of [16, Table 1.2$], T=\operatorname{PSp}(4,11), H=11_{+}^{1+2}: 10 . \mathrm{A}_{4}$, and $G=$ $\operatorname{PSL}\left(2,11^{2}\right)$. By [13, Table 5.1 A$], 11^{2}| | T: G| |\left|T_{v}\right|$ and hence $p \neq 11$, which is impossible because $p$ is the largest prime divisor of $\left|T_{v}\right|$. Similarly, we may exclude Row 15 of [16, Table 1.2 ], because $T=\operatorname{PSp}(4,23), H=23_{+}^{1+2}: 22 . \mathrm{S}_{4}, G=\operatorname{PSL}\left(2,23^{2}\right)$, and $23^{2}| | T: G| |\left|T_{v}\right|$ by [13, Table 5.1A].

For Row 23 of [16, Table 1.2 ], $T=\Omega(7,3), H=3^{3+3}: 13: 3$ and $G=\operatorname{Sp}(6,2)$. Then $p=13$, and since $\left|T_{v}\right|=p k \ell$ with $k|\ell|(p-1)$, we have $3^{5} \nmid\left|T_{v}\right|$. However, $|T: G|=|\Omega(7,3)| /|\operatorname{Sp}(6,2)|=13 \cdot 3^{5}$ implies $3^{5}| | T_{v} \mid$, a contradiction. Similarly, we may exclude Row 28 of [16, Table 1.2 ] because $T=\mathrm{P} \Omega^{+}(8,3), H=3^{6}:\left(3^{3}: 13: 3\right)$ or $3^{3+6}: 13: 3, G=\Omega^{+}(8,2)$ and $|T: G|=13 \cdot 3^{7}$.

For Rows 1-5 of [16, Table 1.2 ], by Magma we obtain the following:
(a) $\left(G, T, T_{v}\right)=\left(\mathrm{A}_{5}, \operatorname{PSL}(2,11), \mathbb{Z}_{11}\right)$,
(b) $\left(G, T, T_{v}\right)=\left(\mathrm{A}_{5}, \operatorname{PSL}(2,11), \mathbb{Z}_{11}: \mathbb{Z}_{5}\right)$,
(c) $\left(G, T, T_{v}\right)=\left(\mathrm{A}_{5}, \operatorname{PSL}(2,19), \mathbb{Z}_{19}: \mathbb{Z}_{9}\right)$,
(d) $\left(G, T, T_{v}\right)=\left(\mathrm{A}_{5}, \operatorname{PSL}(2,29), \mathbb{Z}_{29}: \mathbb{Z}_{7}\right)$,
(e) $\left(G, T, T_{v}\right)=\left(\mathrm{A}_{5}, \operatorname{PSL}(2,29), \mathbb{Z}_{29}: \mathbb{Z}_{14}\right)$,
(f) $\left(G, T, T_{v}\right)=\left(\mathrm{A}_{5}, \operatorname{PSL}(2,59), \mathbb{Z}_{59}: \mathbb{Z}_{29}\right)$.

For case (b), $|V(\Gamma)|=\left|T: T_{v}\right|=12$ and hence $\Gamma$ is a complete graph of order 12, contradicting that $\operatorname{Aut}(\Gamma)_{v}$ is solvable. Similarly, cases (c),(e) and (f) cannot occur because $\Gamma$ is a complete graph of order 20,30 or 60 , respectively. Thus, we have (a) or (d), which is the case (3) or (4) of the lemma.

For cases (2)-(4), it is easy to see that $G_{v}=G \cap T_{v}=1$. Since $G$ is transitive, it is regular, that is, $\Gamma$ is Cayley graph of $G$.

Lemma 3.3. Let $G, \Gamma, p$ and $v$ be as given in Assumption and further assume that $G$ is regular on $V(\Gamma)$. Then $\operatorname{rad}(\operatorname{Aut}(\Gamma))$ has at least three orbits on $V(\Gamma)$, and if $\operatorname{rad}(\operatorname{Aut}(\Gamma)) G \unlhd \operatorname{Aut}(\Gamma)$ then $\operatorname{rad}(\operatorname{Aut}(\Gamma)) G=\operatorname{rad}(\operatorname{Aut}(\Gamma)) \times G$.

Proof. Set $A=\operatorname{Aut}(\Gamma), R=\operatorname{rad}(A)$ and $B=R G$. If $R$ is transitive on $V(\Gamma)$, then Lemma 3.1 implies that $G \cong B_{v} / R_{v}$ is solvable, a contradiction. Since $G$ is transitive, $\Gamma$ is not bipartite and hence $R$ has at least three orbits. Assume that $B \unlhd A$. To finish the proof, it suffices to show $B=R \times G$. This is clearly true for $R=1$.

Assume $R \neq 1$. Then $R \cap G=1$. Since $G$ is regular, $B_{v} \neq 1$, and since $B \unlhd A$, $\Gamma$ is $B$-arc-transitive. By Proposition 2.7, $B_{v}$ has a normal sylow $p$-subgroup $\mathbb{Z}_{p}$, and $\left|B_{v}\right|=p m$ with $(p, m)=1$. Note that $R G=B=G B_{v}$. Again by the regularity of $G$, we have $\left|B_{v}\right|=|R|=p m$. Let $R_{p}$ be a Sylow $p$-subgroup of $R$. We claim $R_{p} \unlhd B$.

Suppose to the contrary that $R_{p} \nexists B$. Since $R \unlhd B$ is solvable, by the Jordan-Holder Theorem, $B$ has a normal series: $1 \unlhd R_{1} \unlhd R_{2} \unlhd \cdots \unlhd R \unlhd B$ such that $R_{1} \unlhd B, R_{2} \unlhd B$, $R_{2} / R_{1} \cong \mathbb{Z}_{p}$ and $R_{1} \neq 1$. Since $(p, m)=1$, we have $p \nmid\left|R_{1}\right|$. Note that $R_{2} / R_{1} \unlhd B / R_{1}$ and $G R_{1} / R_{1} \cong G /\left(G \cap R_{1}\right)=G$. Since $R_{2} / R_{1} \cong \mathbb{Z}_{p}$, the conjugate action of $G R_{1} / R_{1}$ on $R_{2} / R_{1}$ must be trivial by the simplicity of $G$. It follows that $G R_{2} / R_{1}=R_{2} / R_{1} \times G R_{1} / R_{1}$, and hence, $G R_{1} / R_{1} \unlhd G R_{2} / R_{1}$, forcing $G R_{1} \unlhd G R_{2}$. Since $p\left|\left|R_{2}\right|, G R_{2}\right.$ is arc-transitive on $\Gamma$, and hence $G R_{1}$ is also arc-transitive because $\left|\left(G R_{1}\right)_{v}\right|=\left|R_{1}\right| \neq 1$. It follows $p\left|\left|R_{1}\right|\right.$, a contradiction. Thus, $R_{p} \unlhd B$, as claimed.

Let $C=C_{B}\left(R_{p}\right)$. Since $R_{p} \unlhd B$ and $R_{p} \cong \mathbb{Z}_{p}$, the conjugate action of $G$ on $R_{p}$ is trivial and so $R_{p} G=R_{p} \times G$. It follows that $G \leq C$ and $C=C \cap B=C \cap(R G)=(C \cap R) G$. Clearly, $R_{p} \leq C \cap R$ and hence $R_{p}$ is a Sylow $p$-subgroup of $C \cap R$. This implies that $C \cap R=R_{p} \times L$ where $L$ is a $p^{\prime}$-subgroup of $C \cap R$, and in particular, $L$ is characteristic in $C \cap R$ and so normal in $B$. Thus, $C=\left(R_{p} \times L\right) G=R_{p} \times L G$, and therefore, $L G \unlhd C$. Note that $C$ is arc-transitive because $G \leq C$ and $R_{p} \leq C$. If $L \neq 1$ then $(L G)_{v} \neq 1$ and then $L G \unlhd C$ implies that $L G$ is arc-transitive. This means that $p \|(L G)_{v} \mid$, and since $L G=G(L G)_{v}$, we have $\left|(L G)_{v}\right|=|L|$ and $p||L|$, which is impossible. It follows that $L=1$ and $C=R_{p} \times G$. Furthermore, $G \unlhd B$ and so $B=R \times G$.

Proof of Theorem 1.3: Let $G, \Gamma, p$ and $v$ as given in Assumption and further let $G$ be regular on $V(\Gamma)$. Write $A=\operatorname{Aut}(\Gamma), R=\operatorname{rad}(A)$ and $B=R G$. Then $R \cap G=1$ and $B / R \cong G$. By the Frattini argument, $B=G R=G B_{v}$, and so $|R|=\left|B_{v}\right|$.

Assume $R=1$. By Lemma 3.2, either $\operatorname{soc}(A)=G$, or $\Gamma$ is $\operatorname{soc}(A)$-arc-transitive and $G<\operatorname{soc}(A)$ with $(G, \operatorname{soc}(A))=\left(\mathrm{A}_{n-1}, \mathrm{~A}_{n}\right),\left(\mathrm{M}_{22}, \mathrm{M}_{23}\right),\left(\mathrm{A}_{5}, \operatorname{PSL}(2,11)\right)$ or $\left(\mathrm{A}_{5}, \operatorname{PSL}(2,29)\right)$.

Assume $R \neq 1$. By Lemma 3.3, $R$ has at least three orbits, and by Proposition 2.8, the quotient graph $\Gamma_{R}$ has valency $p$ with $A / R$-arc-transitive and $B / R$-vertex-transitive. Moreover, $(A / R)_{\Delta} \cong A_{v}$ is solvable for any $\Delta \in V\left(\Gamma_{R}\right)$. Write $I / R=\operatorname{soc}(A / R)$. Since
$B / R \cong G$, Lemma 3.2 implies that either $B / R=I / R \unlhd A / R$, or $\Gamma_{R}$ is $I / R$-arctransitive with $B / R<I / R$ and $(B / R, I / R)=\left(\mathrm{A}_{n-1}, \mathrm{~A}_{n}\right)$ with $(I / R)_{\Delta}$ being transitive on $\{1,2, \cdots, n\}$, or $\left(B / R, I / R,(I / R)_{\Delta}\right)=\left(\mathrm{M}_{22}, \mathrm{M}_{23}, \mathbb{Z}_{23}\right),\left(\mathrm{A}_{5}, \operatorname{PSL}(2,11), \mathbb{Z}_{11}\right)$ or $\left(\mathrm{A}_{5}, \operatorname{PSL}(2,29), \mathbb{Z}_{29}: \mathbb{Z}_{7}\right)$.

Case 1: $B / R=I / R \unlhd A / R$.
In this case, $B=G R \unlhd A$, and by Lemma 3.3, $B=G \times R$. It follows that $G$ is characteristic in $B$, and hence $G \unlhd A$.

Case 2: $\Gamma_{R}$ is $I / R$-arc-transitive with $B / R<I / R$ and $(B / R, I / R)=\left(\mathrm{A}_{n-1}, \mathrm{~A}_{n}\right)$ with $(I / R)_{\Delta}$ being transitive on $\{1,2, \cdots, n\}$, or $\left(B / R, I / R,(I / R)_{\Delta}\right)=\left(\mathrm{M}_{22}, \mathrm{M}_{23}, \mathbb{Z}_{23}\right)$, $\left(\mathrm{A}_{5}, \operatorname{PSL}(2,11), \mathbb{Z}_{11}\right)$ or $\left(\mathrm{A}_{5}, \operatorname{PSL}(2,29), \mathbb{Z}_{29}: \mathbb{Z}_{7}\right)$.

Let $\left(B / R, I / R,(I / R)_{\Delta}\right)=\left(\mathrm{M}_{22}, \mathrm{M}_{23}, \mathbb{Z}_{23}\right),\left(\mathrm{A}_{5}, \operatorname{PSL}(2,11), \mathbb{Z}_{11}\right)$ or $\left(\mathrm{A}_{5}, \operatorname{PSL}(2,29), \mathbb{Z}_{29}\right.$ : $\mathbb{Z}_{7}$ ). By Lemma 3.2, $\Gamma$ is a Cayley graph on $G B / R \cong G$. Since $\Gamma$ is a Cayley graph on $G$, we have that $|V(\Gamma)|=\left|V\left(\Gamma_{R}\right)\right|$, which contradicts the assumption $R \neq 1$. Thus $(B / R, I / R)=\left(\mathrm{A}_{n-1}, \mathrm{~A}_{n}\right)$ with $(I / R)_{\Delta}$ being transitive on $\{1,2, \cdots, n\}$.

First we claim $B=R \times G$. Suppose to the contrary that $B \neq R \times G$. Since $R$ is solvable, there exists a series of normal subgroups of $B: R_{0}=1<R_{1}<\cdots<R_{s}=B$ such that $R_{i} \triangleleft B$ and $R_{i+1} / R_{i}$ is an elementary abelian group for each $0 \leq i \leq s-1$. Since $R G \neq R \times G$, there exists $0 \leq j \leq s-1$ such that $G R_{i}=G \times R_{i}$ for any $0 \leq i \leq j$, but $G R_{j+1} \neq G \times R_{j+1}$.

Write $R_{j+1} / R_{j}=\mathbb{Z}_{r}^{f}$ for some prime $r$ and positive integer $f$. Note that $G \cap R_{i}=1$ for $0 \leq i \leq s$ and so $R_{i+1} G / R_{i} \cong G$ for $0 \leq i \leq s-1$. In particular, the conjugate action of $R_{j+1} G / R_{j}$ on $R_{j+1} / R_{j}$ is trivial or faithful. If it is trivial, then $R_{j+1} G / R_{j}=$ $\left(R_{j+1} / R_{j}\right)\left(R_{j} G / R_{j}\right)=R_{j+1} / R_{j} \times R_{j} G / R_{j}$, implying $R_{j} G \triangleleft R_{j+1} G$, and since $G R_{j}=$ $G \times R_{j}$, we have $G \unlhd R_{j+1} G$ and $G R_{j+1}=G \times R_{j+1}$, a contradiction. It follows that the conjugate action of $R_{j+1} G / R_{j}$ on $R_{j+1} / R_{j}$ is faithful, and we may assume $G \leq \mathrm{GL}(f, r)$.

Recall that $\left|B_{v}\right|=|R|$ and $R_{j+1} / R_{j}=\mathbb{Z}_{r}^{f}$. Then $r^{f}| | B_{v} \mid$, and since $\Gamma_{R}$ is $I / R$ -arc-transitive, $\Gamma$ is $I$-arc-transitive and Proposition 2.8 implies $I_{v} \cong(I / R)_{\Delta}$. Since $B / R<I / R$, we have $\left|B_{v}\right|\left|\left|I_{v}\right|\right.$ and so $\left.r^{f}\right|\left|(I / R)_{\Delta}\right|$. If $r=p$ then Proposition 2.7 implies $r^{2} \nmid\left|(I / R)_{\Delta}\right|$ and hence $G \leq \mathrm{GL}(1, p)$, a contradiction. It follows $r \neq p$, and again by Proposition 2.7, $r^{f} \mid(p-1)^{2}$.

Now $B / R=\mathrm{A}_{n-1} \leq \mathrm{GL}(f, r)$. By assumption, $p \geq 11$. Since $(I / R)_{\Delta}$ contains a normal subgroup $\mathbb{Z}_{p}$, we have $p \mid n$ and so $n-1 \geq 11-1=10$. By Proposition 2.5, $f \geq(n-1)-2 \geq p-3$ and so $(p-1)^{2} \geq r^{f} \geq 2^{p-3}$. This is impossible because the function $f(x)=2^{x-3}-(x-1)^{2}>0$ always holds for $x \geq 11$. This completes the proof of the claim, and hence $B=R \times G$.

Set $C=C_{I}(R)$. Then $G \leq C, C \unlhd I$ and $C \cap R \leq Z(C)$. Recall that $I / R=$ $\mathrm{A}_{n}$ or $\mathrm{M}_{23}$. Since $G \cong(R \times G) / R \leq C R / R \unlhd I / R$, we have $I=C R$, and since $Z(C) /(C \cap R) \unlhd C / C \cap R \cong C R / R=I / R$, we have $C \cap R=Z(C)$ and $C / Z(C) \cong I / R$. Furthermore, $C^{\prime} /\left(C^{\prime} \cap Z(C)\right) \cong C^{\prime} Z(C) / Z(C)=(C / Z(C))^{\prime}=C / Z(C) \cong I / R$, and so $Z\left(C^{\prime}\right)=C^{\prime} \cap Z(C), C=C^{\prime} Z(C)$ and $C^{\prime} / Z\left(C^{\prime}\right) \cong I / R$. It follows $C^{\prime}=\left(C^{\prime} Z(C)\right)^{\prime}=C^{\prime \prime}$, and hence $C^{\prime}$ is a covering group of $I / R$.

Suppose $Z\left(C^{\prime}\right) \neq 1$. Then Proposition 2.4 implies that $Z\left(C^{\prime}\right)=\mathbb{Z}_{2}$ and $C^{\prime} \cong 2 . \mathrm{A}_{n}$. Since $G \leq C$ and $C / C^{\prime}$ is abelian, we have $G \leq C^{\prime}$. So $G \times Z\left(C^{\prime}\right) \cong \mathrm{A}_{n-1} \times \mathbb{Z}_{2}$ is a subgroup of $C^{\prime} \cong 2$. $\mathrm{A}_{n}$, which is impossible by Proposition 2.4.

Thus, $Z\left(C^{\prime}\right)=1$. It follows $C^{\prime} \cong I / R$. Since $G<C$ and $C / C^{\prime}$ is abelian, we have $G<C^{\prime} \unlhd I$, and since $|I|=|I / R||R|=\left|C^{\prime}\right||R|$ and $C^{\prime} \cap R=1$, we have $I=C^{\prime} \times R$. Since $C^{\prime}$ is a nonabelian simple group, $C^{\prime}$ is characteristic in $I$, and hence $C^{\prime} \unlhd A$ because $I \unlhd A$. Since $G$ is regular on $\Gamma$ and $G<C^{\prime} \unlhd I, C^{\prime}$ has non-trivial stabilizer, and hence $\Gamma$ is $C^{\prime}$-arc-transitive on $\Gamma$. Note that $C^{\prime} \cong I / R=\mathrm{A}_{n}$.

Summing up, we have proved that either $G \unlhd A$, or $A$ has a normal arc-transitive subgroup $T$ such that $G<T$ and $(G, T)=\left(\mathrm{A}_{5}, \operatorname{PSL}(2,11)\right),\left(\mathrm{A}_{5}, \operatorname{PSL}(2,29)\right),\left(\mathrm{M}_{22}, \mathrm{M}_{23}\right)$ or $\left(\mathrm{A}_{n-1}, \mathrm{~A}_{n}\right)$ (for $R=1, T=\operatorname{soc}(A)$, and for $\left.R \neq 1, T=C^{\prime}\right)$. Let $(G, T)=\left(\mathrm{A}_{n-1}, \mathrm{~A}_{n}\right)$. Since $G$ is regular, $\left|T_{v}\right|=n$, and by Proposition $2.7, n=p k \ell$ with $k|\ell|(p-1)$. To finish the proof, we are left to show that $k$ and $\ell$ have the same parity.

Suppose to the contrary that $k$ and $\ell$ has different parity. Then $k$ is odd and $\ell$ is even as $k \mid \ell$. Since $(G, T)=\left(\mathrm{A}_{n-1}, \mathrm{~A}_{n}\right)$, we have $|T: G|=n$ and $T$ can be viewed as the alternating permutation group by the well-known right multiplication action of $T$ on the set $[T: G]$ of all right cosets of $G$ in $T$, still denoted by $\mathrm{A}_{n}$. By the regularity of $G$ on $\Gamma, T=G T_{v}$ and $G \cap T_{v}=1$, which implies that $T_{v} \leq \mathrm{A}_{n}$ is a regular permutation group on $[T: G]$. By Proposition $2.7, T_{v}=\mathbb{Z}_{k} \times\left(\mathbb{Z}_{p}: \mathbb{Z}_{\ell}\right)$, and so $T_{v}$ has a cyclic group $\mathbb{Z}_{\ell}$ with odd index $\left|T_{v}: \mathbb{Z}_{\ell}\right|=p k$. Let $\mathbb{Z}_{\ell}=\langle a\rangle$. Since $T_{v}$ is regular, $a$ is a product of $p k \ell$-cycles on $[T: G]$ in its distinct cycle decomposition, so an odd permutation as $\ell$ is even and $k p$ is odd, which is impossible because $T_{v} \leq \mathrm{A}_{n}$. This completes the proof.

## 4. Proof of Theorem 1.4

The goal of this section is to prove Theorem 1.4. To do that, we first describe a widely known construction for vertex-transitive and symmetric graphs, part of which is attributed to Sabidussi [19].

Let $G$ be a group, $H$ a subgroup of $G$, and $D$ a union of some double cosets of $H$ in $G$ such that $H \nsubseteq D$ and $D^{-1}=D$. Then the coset graph $\Gamma=\operatorname{Cos}(G, H, D)$ is defined as the graph with vertex-set $[G: H]$, the set of all right cosets of $H$ in $G$, and edge-set $E(\Gamma)=\{\{H g, H x g\}: g \in G, x \in D\}$. This graph is regular with valency $|D| /|H|$, and is connected if and only if $G=\langle D, H\rangle$, that is, if and only if $G$ is generated by $D$ and $H$. The group $G$ acts vertex-transitively on $\Gamma$ by right multiplication. More precisely, for $g \in G$, the permutation $\hat{g}_{H}: H x \mapsto H x g, x \in G$, on $[G: H]$ is an automorphism of $\operatorname{Aut}(\Gamma)$, and $\hat{G}_{H}:=\left\{\hat{g}_{H} \mid g \in G\right\}$ is a transitive subgroup of Aut $(\Gamma)$. The map $g \mapsto \hat{g}_{H}$, $g \in G$, is a homomorphism from $G$ to $S_{[G: H]}$, the well-known coset action of $G$ on $H$, and the kernel of this coset action is $H_{G}=\bigcap_{g \in G} H^{g}$, the largest normal subgroup of $G$ contained in $H$. It follows that $G / H_{G} \cong \hat{G}_{H}$. Furthermore, $\Gamma$ is $\hat{G}_{H}$-arc-transitive if and only if $D$ consists of just one double coset $H a H$. If $H_{G}=1$, we say that $H$ is core-free in $G$, and in this case, $G \cong \hat{G}_{H}$.

If $H=1$, denote $\operatorname{Cos}(G, H, D)$ and $\hat{G}_{H}$ by $\operatorname{Cay}(G, D)$ and $\hat{G}$, respectively. In this case, $\hat{G}$ is the right regular representation of $G$, and it is regular on the vertex set of $\operatorname{Cay}(G, D)$. By definition, $\operatorname{Cay}(G, D)$ is Cayley graph of $\hat{G}$, and for short, Cay $(G, D)$ is also called a Cayley graph of $G$ with respect to $D$.

Conversely, suppose $\Gamma$ is any graph on which the group $G$ acts faithfully and vertextransitively. Then it is easy to show that $\Gamma$ is isomorphic to the coset graph $\operatorname{Cos}(G, H, D)$, where $H=G_{v}$ is the stabiliser in $G$ of the vertex $v \in V(\Gamma)$, and $D$ is a union of double cosets of $H$, consisting of all elements of $G$ taking $v$ to one of its neighbours. Then
$H \nsubseteq D$ and $D^{-1}=D$. Moreover, if $G$ is arc-transitive on $\Gamma$ and $g$ is an element of $G$ that swaps $v$ with one of its neighbours, then $g^{2} \in H$ and $D=H g H$, and the valency of $\Gamma$ is $|D| /|H|=\left|H: H \cap H^{g}\right|$. Also $a$ can be chosen as a 2-element in $G$. In particular, if $L \leq G$ is regular on vertex set of $\Gamma$, then $\Gamma$ is also isomorphic to Cay $(L, S)$, where $S$ consists of all elements of $L$ taking $v$ to one of its neighbours with $S^{-1}=S$, and by the regularity, we have $S=D \cap L$. Thus, we have the following proposition.

Proposition 4.1. Let $\Gamma$ be a $G$-vertex-transitive graph and $L$ be a regular subgroup of $G$. Then $\Gamma \cong \operatorname{Cos}(G, H, D) \cong \operatorname{Cay}(L, S)$ with $S=L \cap D$, where $H=G_{v}$ for $v \in V(\Gamma)$, $D$ is a union of double cosets of $H$, consisting of all elements of $G$ taking $v$ to one of its neighbours, and $S$ consists of all elements of $L$ taking $v$ to one of its neighbours. Moreover, $\Gamma$ be $G$-arc-transitive if and only if $G$ has a 2-element $g$ such that $D=H g H$, and in this case, $\Gamma$ has valency $\left|H: H \cap H^{g}\right|$.

Let $\Gamma=\operatorname{Cos}(G, H, D)$ be a coset graph. We set $\operatorname{Aut}(G, H, D)=\left\{\alpha \in \operatorname{Aut}(G) \mid H^{\alpha}=\right.$ $\left.H, D^{\alpha}=D\right\}$. For any $\alpha \in \operatorname{Aut}(G, H, D)$, the permutation $\alpha_{H}: H x \mapsto H x^{\alpha}, x \in G$, on $[G: H]$ is an automorphism of $\Gamma$, and the map $\alpha \mapsto \alpha_{H}$ is a natural action of $\operatorname{Aut}(G, H, D)$ on $V(\Gamma)$. It follows that $\operatorname{Aut}(G, H, D) / K \cong \operatorname{Aut}(G, H, D)_{H}$, where $\operatorname{Aut}(G, H, D)_{H}=\left\{\alpha_{H} \mid \alpha \in \operatorname{Aut}(G, H, D)\right\}$ and $K$ is the kernel of the action. Furthermore, $\operatorname{Aut}(G, H, D)_{H} \leq \operatorname{Aut}(\Gamma)$. For $h \in H$, let $\tilde{h}$ be the inner automorphism of $G$ induced by $h$, that is, $\tilde{h}: g \mapsto h^{-1} g h, g \in G$. Then $\tilde{H}:=\{\tilde{h} \mid h \in H\} \leq \operatorname{Aut}(G, H, D)$ and hence $\tilde{H}_{H}:=\left\{\tilde{h}_{H} \mid h \in H\right\}$ is a subgroup of $\operatorname{Aut}(G, H, D)_{H}$.

The following proposition was proved by Wang, Feng and Zhou [22, Lemma 2.10], which is important for computing automorphism groups of coset graphs.

Proposition 4.2. Let $G$ be a finite group, $H$ a core-free subgroup of $G$ and $D$ a union of several double-cosets $H g H$ such that $H \nsubseteq D$ and $D=D^{-1}$. Let $\Gamma=\operatorname{Cos}(G, H, D)$ and $A=\operatorname{Aut}(\Gamma)$. Then $\hat{G}_{H} \cong G$, $\operatorname{Aut}(G, H, D)_{H} \cong \operatorname{Aut}(G, H, D), \tilde{H}_{H} \cong \tilde{H}$, and $\mathbf{N}_{A}\left(\hat{G}_{H}\right)=\hat{G}_{H} \operatorname{Aut}(G, H, D)_{H}$ with $\hat{G}_{H} \cap \operatorname{Aut}(G, H, D)_{H}=\tilde{H}_{H}$.

Now we are ready to prove Theorem 1.4 and this follows from Lemmas 4.3-4.6.
Let $x, y, t$ be permutations in $\mathrm{S}_{11}$ as following:

$$
\begin{aligned}
x & =(1,11,8,3,6,9,4,10,2,7,5) \\
y & =(2,10,6)(3,11,4)(7,8,9) \\
t & =(2,5)(3,9)(6,11)(8,10)
\end{aligned}
$$

Let $T=\langle x, t\rangle, H=\langle x\rangle, G=\langle y, t\rangle$. Define

$$
\Gamma=\operatorname{Cos}(T, H, H t H)
$$

Then a computation with Magma [2] shows that $T \cong \operatorname{PSL}(2,11), H \cong \mathbb{Z}_{11},\left|H \cap H^{t}\right|=$ 1 , and $G \cong \mathrm{~A}_{5}$. By Proposition 4.1, $\Gamma$ has valency 11 and $T$ acts arc-transitively on $\Gamma$. Since $11 X|G|, G$ acts semiregularly on $V(\Gamma)$, and since $|G|=|V(\Gamma)|, G$ is regular on $V(\Gamma)$. It follows that $\Gamma$ is a non-normal Cayley group of $\mathrm{A}_{5}$ with $\operatorname{PSL}(2,11)$-arctransitive. A direct computation with Magma shows that $\operatorname{Aut}(\Gamma) \cong \operatorname{PGL}(2,11)$ and this implies the following lemma.

Lemma 4.3. There exists an 11-valent symmetric Cayley graph $\Gamma$ of $A_{5}$ such that $\operatorname{Aut}(\Gamma) \cong \mathrm{PGL}(2,11)$. In particular, $\operatorname{Aut}(\Gamma)_{v}$ is solvable for $v \in V(\Gamma)$.

Let $x, y, t, z$ be permutations in $\mathrm{S}_{30}$ as following:

$$
\begin{aligned}
x= & (1,21,10,9,22,28,13,15,30,6,19,18,7,27,23,4,25,17,20,2,12,29,16,26,8,11, \\
& 3,24,5) \\
y= & (1,24,9)(2,6,5)(3,27,21)(4,12,20)(7,25,26)(8,10,13)(11,14,16)(15,30,23)(17, \\
& 28,29)(18,22,19) \\
t= & (1,3)(2,10)(4,11)(5,19)(6,24)(7,16)(8,17)(9,28)(12,27)(13,20)(14,22)(15,26) \\
& (18,30)(21,23) \\
z= & (2,18,23,10,29,9,17)(3,7,19,20,4,24,30)(5,22,27,13,28,6,16)(8,12,15,21,11, \\
& 25,26)
\end{aligned}
$$

Let $T=\langle x, t\rangle, H=\langle x, z\rangle, G=\langle y, t\rangle$. Define

$$
\Gamma=\operatorname{Cos}(T, H, H t H)
$$

Then a computation with Magma [2] shows that $T \cong \operatorname{PSL}(2,29), H \cong \mathbb{Z}_{29}: \mathbb{Z}_{7}$, $\left|H \cap H^{t}\right|=7$, and $G \cong A_{5}$. By Proposition 4.1, $\Gamma$ has valency 29 and $T$ acts arctransitively on $\Gamma$. Since $29 \nmid|G|, G$ acts semiregularly on $V(\Gamma)$, and since $|G|=|V(\Gamma)|$, $G$ is regular on $V(\Gamma)$. It follows that $\Gamma$ is a non-normal Cayley group of $\mathrm{A}_{5}$ with $\operatorname{PSL}(2,29)$-arc-transitive. A direct computation with MAGMA shows that Aut $(\Gamma) \cong$ $\operatorname{PGL}(2,29)$ and this implies the following lemma.

Lemma 4.4. There exists a 29-valent symmetric Cayley graph $\Gamma$ of $A_{5}$ such that $\operatorname{Aut}(\Gamma) \cong \mathrm{PGL}(2,29)$. In particular, $\operatorname{Aut}(\Gamma)_{v}$ is solvable for $v \in V(\Gamma)$.

Let $x, y, t$ be permutations in $\mathrm{S}_{23}$ as following:

$$
\begin{aligned}
x & =(1,4,6,7,2,19,3,11,9,20,13,23,16,8,21,5,14,22,18,15,17,10,12) \\
y & =(1,14,6,5,9,2,10,3,15,13,11)(4,22,16,19,17,8,21,7,12,18,23) \\
t & =(1,17)(3,9)(5,18)(6,13)(7,12)(10,19)(14,22)(21,23)
\end{aligned}
$$

Let $T=\langle x, t\rangle, H=\langle x\rangle, G=\langle y, t\rangle$. Define

$$
\Gamma=\operatorname{Cos}(T, H, H t H)
$$

Lemma 4.5. The above graph $\Gamma$ is 23-valent symmetric Cayley graph of $M_{22}$ and $\operatorname{Aut}(\Gamma)=\left(\hat{\mathrm{M}}_{23}\right)_{H} \cong \mathrm{M}_{23}$. In particular, Aut $(\Gamma)_{v}$ is solvable for $v \in V(\Gamma)$.
Proof. A computation with Magma [2] shows that $T \cong \mathrm{M}_{23}, H \cong \mathbb{Z}_{23},\left|H \cap H^{t}\right|=1$, and $G \cong \mathrm{M}_{22}$. By Proposition 4.1, $\Gamma$ has valency 23 and $T$ acts arc-transitively on $\Gamma$. Since $23 \chi|G|, G$ acts semiregularly on $V(\Gamma)$, and since $|G|=|V(\Gamma)|, G$ is regular on $V(\Gamma)$. It follows that $\Gamma$ is a non-normal Cayley group of $\mathrm{M}_{22}$ with $\mathrm{M}_{23}$-arc-transitive. However, we cannot compute $\operatorname{Aut}(\Gamma)$ with Magma because $|V(\Gamma)|$ is too large. By Proposition 4.1, we may let $\Gamma=\operatorname{Cay}(G, S)$ with $S=G \cap H t H$. Write $A=\operatorname{Aut}(\Gamma)$. By Magma, $S=\left\{s_{i} \mid 1 \leq i \leq 23\right\}$, where

$$
\begin{aligned}
& s_{1}=(1,14,6,5,9,2,10,3,15,13,11)(4,22,16,19,17,8,21,7,12,18,23), \\
& s_{2}=(1,11,13,15,3,10,2,9,5,6,14)(4,23,18,12,7,21,8,17,19,16,22), \\
& s_{3}=(1,15,5,2,12,18,16,14,21,13,7)(3,6,4,22,8,19,10,17,9,23,11), \\
& s_{4}=(1,7,13,21,14,16,18,12,2,5,15)(3,11,23,9,17,10,19,8,22,4,6), \\
& s_{5}=(1,9,14)(2,19,5,4,22,12)(3,21,6)(7,23,15,11,8,18)(10,13)(16,17), \\
& s_{6}=(1,14,9)(2,12,22,4,5,19)(3,6,21)(7,18,8,11,15,23)(10,13)(16,17),
\end{aligned}
$$

$$
\begin{aligned}
& s_{7}=(1,4,3)(2,6)(5,8,7,10,14,21)(9,12,17,22,16,13)(11,19,23)(15,18), \\
& s_{8}=(1,3,4)(2,6)(5,21,14,10,7,8)(9,13,16,22,17,12)(11,23,19)(15,18), \\
& s_{9}=(1,12)(2,19,3)(4,6,18,5,8,10)(7,11,23,16,14,22)(9,13)(15,17,21), \\
& s_{10}=(1,12)(2,3,19)(4,10,8,5,18,6)(7,22,14,16,23,11)(9,13)(15,21,17), \\
& s_{11}=(1,7,3,16,12)(2,11,23,22,14)(4,15,5,18,10)(6,9,13,8,17), \\
& s_{12}=(1,12,16,3,7)(2,14,22,23,11)(4,10,18,5,15)(6,17,8,13,9), \\
& s_{13}=(3,16,23,12,6)(4,11,22,18,10)(5,17,7,19,9)(8,14,15,21,13), \\
& s_{14}=(3,6,12,23,16)(4,10,18,22,11)(5,9,19,7,17)(8,13,21,15,14), \\
& s_{15}=(1,15,12,6,19)(2,11,13,14,7)(3,16,21,22,4)(5,10,17,9,23), \\
& s_{16}=(1,19,6,12,15)(2,7,14,13,11)(3,4,22,21,16)(5,23,9,17,10) \\
& s_{17}=(1,7)(3,8)(4,6)(9,19)(11,23)(12,15)(13,18)(14,21), \\
& s_{18}=(2,6)(3,10)(4,22)(8,16)(11,13)(12,18)(14,15)(21,23), \\
& s_{19}=(1,11)(2,16)(4,19)(6,12)(8,14)(9,13)(15,18)(17,22), \\
& s_{20}=(1,17)(3,9)(5,18)(6,13)(7,12)(10,19)(14,22)(21,23), \\
& s_{21}=(1,15)(5,16)(6,18)(7,19)(8,21)(9,23)(11,12)(17,22), \\
& s_{22}=(1,17)(2,9)(5,11)(6,19)(7,13)(8,23)(10,12)(14,15), \\
& s_{23}=(1,5)(2,4)(3,11)(8,13)(9,19)(10,15)(14,16)(18,23) .
\end{aligned}
$$

Let 1 be the identity in $G$. Then $1 \in V(\Gamma)$. Suppose to the contrary that $A_{1}$ is nonsolvable. By Proposition 2.6, the restriction $A_{1}^{\Gamma(1)}$ of $A_{1}$ on the neighbourhood $\Gamma(1)$ of 1 in $\Gamma$ is nonsolvable, and since $\Gamma$ has prime valency, the Burnside Theorem (also see $\left[4\right.$, Theorem 3.5B]) implies that $A_{1}^{\Gamma(1)}$ is 2-transitive on $\Gamma(1)$. This turns that there exists a 5 -cycle passing though 1 and any two vertices in $S$ because ( $1, s_{11}, s_{11}^{2}, s_{11}^{3}, s_{11}^{4}$ ) is a 5 -cycle in $\Gamma$. In particular, there is a 5 -cycle passing through $1, s_{1}$ and $s_{2}=s_{1}^{-1}$, and hence $s_{1}^{2} \in S^{3}=\left\{s_{i_{1}} s_{i_{2}} s_{i_{2}} \mid s_{i_{1}}, s_{i_{2}}, s_{i_{2}} \in S\right\}$, but this is not true by MAGMA [2]. Thus, $A_{1}$ is solvable.

Now we let $\Gamma=\operatorname{Cos}(T, H, H t H)$ and $D=H t H$. Since $A$ has solvable stabilizer, Theorem 1.3 implies that $\hat{T}=\hat{\mathrm{M}}_{23} \unlhd A$. Note that $H$ is core-free in $T$. By Proposition 4.2, $A=\hat{T}_{H} \operatorname{Aut}(T, H, D)_{H}$ with $\hat{T}_{H} \cap \operatorname{Aut}(T, H, D)_{H}=\tilde{H}_{H}$, where $\hat{T}_{H} \cong T, \operatorname{Aut}(T, H, D)_{H} \cong$ $\operatorname{Aut}(T, H, D)$ and $\tilde{H}_{H} \cong \tilde{H}$. To prove $A=\hat{T}_{H}$, it suffices to show that $\operatorname{Aut}(T, H, D)=\tilde{H}$.

Suppose to the contrary that $\alpha \in \operatorname{Aut}(T, H, D)$, but $\alpha \notin \tilde{H}$. By [13, Table 5.1.C], $\operatorname{Out}\left(\mathrm{M}_{23}\right)=1$, that is, $\operatorname{Aut}\left(\mathrm{M}_{23}\right)=\operatorname{Inn}\left(\mathrm{M}_{23}\right)$. Thus, $\alpha$ is an automorphism of $T$ induced by an element of $b \in T$ by conjugation, namely $g^{\alpha}=g^{b}$ for $g \in T$. Since $\alpha \in \operatorname{Aut}(T, H, D)$, we have $H^{b}=H$ and $D^{b}=D$, and since $\alpha \notin \tilde{H}$, we have $b \notin H$. It follows that $H\langle b\rangle$ is a subgroup of $T$ containing $H$, and by Atlas $[3], H\langle b\rangle \cong \mathbb{Z}_{23}: \mathbb{Z}_{11}$. Since $\tilde{H} \leq \operatorname{Aut}(T, H, D)$, we may choose $b$ such that $b$ has order 11, and by MAGMA, we may let $\bar{b}=(2,14,18,7,16,6,9,20,8,3,4)(5,21,13,22,12,15,11,19,17,23,10)$ because $H=\langle x\rangle$ with $x=(1,4,6,7,2,19,3,11,9,20,13,23,16,8,21,5,14,22,18,15,17,10,12)$. However, $D^{b}=(H t H)^{b} \neq H t H$ by MAGMA, a contradiction. Thus, $A=\hat{T}_{H} \cong \mathrm{M}_{23}$.

Let $p \geq 5$ be a prime, and let $x, t$ and $h$ be permutations in $\mathrm{S}_{p}$ as following:

$$
x=(1,2, \cdots, p), \quad t=(1,2)(3,4), \quad h=(2, p)(3, p-1) \cdots\left(\frac{p-1}{2}, \frac{p+5}{2}\right)\left(\frac{p+1}{2}, \frac{p+3}{2}\right) .
$$

Let $T=\langle x, t\rangle$ and $H=\langle x\rangle$. By [8], $T=\mathrm{A}_{p}, H \cong \mathbb{Z}_{p}$ and $\left|H \cap H^{t}\right|=1$. Define

$$
\Gamma^{p}=\operatorname{Cos}\left(\mathrm{A}_{p}, H, H t H\right)
$$

Lemma 4.6. The above graph $\Gamma^{p}$ is a $p$-valent symmetric Cayley graph of $A_{p-1}$ such that $\operatorname{Aut}\left(\Gamma^{p}\right) \cong \mathrm{S}_{p}$ for $p \equiv 3(\bmod 4)$ and $\operatorname{Aut}\left(\Gamma^{p}\right) \cong \mathrm{A}_{p} \times \mathbb{Z}_{2}$ for $p \equiv 1(\bmod 4)$. In particular, $\operatorname{Aut}(\Gamma)_{v}$ is solvable for $v \in V(\Gamma)$.

Proof. By Proposition 4.1, $\Gamma^{p}$ has valency $p$ and $\mathrm{A}_{p}$ acts arc-transitively on $\Gamma^{p}$, with vertex stabilizer isomorphic to $\mathbb{Z}_{p}$. Let $A_{p-1}$ be the subgroup of $A_{p}$ fixing the point $p$. Since $p \backslash\left|\mathrm{~A}_{p-1}\right|, \mathrm{A}_{p-1}$ acts semiregularly on $V\left(\Gamma^{p}\right)$, and since $\left|\mathrm{A}_{p-1}\right|=\left|V\left(\Gamma^{p}\right)\right|, \mathrm{A}_{p-1}$ is regular on $V\left(\Gamma^{p}\right)$. It follows that $\Gamma^{p}$ is a non-normal Cayley group of $\mathrm{A}_{p-1}$ with $\mathrm{A}_{p}$-arc-transitive.

By Proposition 4.1, we may let $\Gamma^{p}=\operatorname{Cay}\left(\mathrm{A}_{p-1}, S\right)$, where $S=\mathrm{A}_{p-1} \cap H t H$. For $p=5$ or $p=7$, a computing with Magma shows that $\operatorname{Aut}\left(\Gamma^{5}\right) \cong \mathrm{A}_{5} \times \mathbb{Z}_{2}$ and $\operatorname{Aut}\left(\Gamma^{7}\right) \cong \mathrm{S}_{7}$. Write $A=\operatorname{Aut}(\Gamma)$. We may assume $p \geq 11$.
Claim: $A$ has solvable stabilizer.
Recall that $x=(1,2, \cdots, p), t=(1,2)(3,4)$ and $H=\langle x\rangle$. Let $x^{-i} t x^{j} \in S=H t H \cap$ $\mathrm{A}_{p-1}$ for $i, j \in \mathbb{Z}_{p}$. Then $p=p^{x^{-i} t x^{j}}=p^{x^{-i} t x^{i} x^{j-i}}$. Note that $x^{-i} t x^{i}=\left(1^{x^{i}}, 2^{x^{i}}\right)\left(3^{x^{i}}, 4^{x^{i}}\right)$, and if $j-i \neq 0$ then $x^{j-i}$ is a $p$-cycle. For $0 \leq i \leq p-5, p=p^{x^{-i} t x^{i} x^{j-i}}=p^{x^{j-i}}$ implies $j=i$. Furthermore, For $i=p-4, p-3, p-2$ or $p-1, p=p^{x^{-i} t x^{i} x^{j-i}}$ implies that $j=i+1, i-1, i+1$ or $i-1$, respectively. Thus, we may set $S=\left\{s_{1}, s_{2}, \cdots, s_{p}\right\}$, where $s_{i+1}=x^{-i} t x^{i}=(1+i, 2+i)(3+i, 4+i)$ for $0 \leq i \leq p-5$, $s_{p-2}=x^{-(p-3)} t x^{p-4}=(1, p-1, p-3, \cdots, 3,2), \quad s_{p-3}=x^{-(p-4)} t x^{p-3}=\left(s_{p-2}\right)^{-1}$, $s_{p}=x^{-(p-1)} t x^{p-2}=(1, p-1, p-2, \cdots, 4,3), \quad s_{p-1}=x^{-(p-2)} t x^{p-1}=s_{p}^{-1}$.

For $z \in \mathrm{~A}_{p}$, denote by $o(z)$ the order of $z$ and by $\operatorname{supp}(z)$ the support of $z$, that is, the number of points moving by $z$. Then $o\left(s_{i}\right)=2$ and $\operatorname{supp}\left(s_{i}\right)=4$ for $1 \leq i \leq p-4$, and $o\left(s_{i}\right)=\operatorname{supp}\left(s_{i}\right)=p-2$ for $p-3 \leq i \leq p$.

To prove the Claim, it suffices to show that $A_{1}$ is solvable. We argue by contradiction and we suppose that $A_{1}$ is nonsolvable. Note that $\Gamma^{p}=\operatorname{Cay}\left(\mathrm{A}_{p-1}, S\right)$ and $\Gamma^{p}(1)=S$.

By Propostion 2.6, $A_{1}^{\Gamma^{p}(1)}$ is nonsolvable, and the Burnside Theorem implies that $A_{1}$ is 2 -transitive on $\Gamma^{p}(1)$. Note that $p \geq 11$. Since $s_{1}=(1,2)(3,4)$ commutes with $s_{5}=(5,6)(7,8)$, there is a 4 -cycle passing through $1, s_{1}$ and $s_{5}$. By the 2 -transitivity of $A_{1}$ on $\Gamma^{p}(1)$, there exists a 4 -cycle through $1, s_{p}$ and $s_{p-1}=s_{p}^{-1}$, and this implies $\left|S s_{p} \cap S s_{p}^{-1}\right| \geq 2$. Thus, $\left|S s_{p}^{-2} \cap S\right| \geq 2$.

Let $S_{1}=\left\{s_{i} \mid 1 \leq i \leq p-4\right\}$ and $S_{2}=\left\{s_{p-2}, s_{p-2}^{-1}, s_{p}, s_{p}^{-1}\right\}$. Then $S=S_{1} \cup S_{2}$ and $S_{1} \cap S_{2}=\emptyset$. Since $s_{p}^{-1}$ is a $(p-2)$-cycle in $\mathrm{A}_{p}$ and $p-2$ is odd, $s_{p}^{-2}$ is also a $(p-2)$-cycle, implying $\operatorname{supp}\left(s_{p}^{-2}\right)=p-2$. Since $\operatorname{supp}\left(s_{i}\right)=4$ for each $1 \leq i \leq p-4$, we have $\operatorname{supp}\left(s_{i} s_{p}^{-2}\right) \geq p-6 \geq 5$, and $s_{i} s_{p}^{-2}$ cannot be any involution in $S$. Thus, $\left|S_{1} s_{p}^{-2} \cap S_{1}\right|=0$.

Note that $S_{2} s_{p}^{-2}=\left\{s_{p}^{-1}, s_{p}^{-3}, s_{p-2} s_{p}^{-2}, s_{p-2}^{-1} s_{p}^{-2}\right\}$. Then $\left|S_{2} s_{p}^{-2} \cap S_{2}\right|=1$ by a simple checking one by one. If $\left|S_{2} s_{p}^{-2} \cap S_{1}\right| \neq 0$, then $z^{2}=1$ for some $z \in S_{2} s_{p}^{-2}$, and we have $s_{p}^{-2}=1$ or $s_{p}^{-6}=1$ or $\left(s_{p-2} s_{p}^{-2}\right)^{2}=1$ or $\left(s_{p-2}^{-1} s_{p}^{-2}\right)^{2}=1$, of which all are impossible because all these elements cannot fix 1. Thus, $\left|S_{2} s_{p}^{-2} \cap S_{1}\right|=0$. Similarly, $\left|S_{2} s_{p}^{2} \cap S_{1}\right|=0$.

Recall that $\left|S s_{p}^{-2} \cap S\right| \geq 2$. Since $\left|S_{2} s_{p}^{-2} \cap S_{2}\right|=1$ and $\left|S_{2} s_{p}^{-2} \cap S_{1}\right|=0$, we have $\left|S_{1} S_{p}^{-2} \cap S\right|=1$, and since $\left|S_{1} s_{p}^{-2} \cap S_{1}\right|=0$, we have $\left|S_{1} S_{p}^{-2} \cap S_{2}\right|=1$. It follows $\left|S_{2} s_{p}^{2} \cap S_{1}\right|=1$, a contradiction. Thus, $A_{1}$ is solvable, as claimed.

From now on, we write $\Gamma^{p}=\operatorname{Cos}(T, H, H t H)$. Clearly, $H$ is core-free in $T$. By Claim, $A=\operatorname{Aut}\left(\Gamma^{p}\right)$ has solvable stabilizer. By Theorem 1.3, $\hat{T}_{H}$ is normal in $A$, and by Proposition 4.2, $A=N_{A}\left(\hat{T}_{H}\right)=\hat{T}_{H} \operatorname{Aut}(T, H, H t H)_{H}$ with $\hat{T}_{H} \cap \operatorname{Aut}(T, H, H t H)_{H}=\tilde{H}_{H}$. Furthermore, $\hat{T}_{H} \cong T, \tilde{H}_{H} \cong H$ and $\operatorname{Aut}(T, H, H t H)_{H} \cong \operatorname{Aut}(T, H, H t H)=\{\alpha \in$ $\left.\operatorname{Aut}(T) \mid H^{\alpha}=H,(H t H)^{\alpha}=H t H\right\}$.

Let $x^{i} t x^{j} \in H t H$ for some $i, j \in \mathbb{Z}_{p}$. If $i+j=0$, then $x^{i} t x^{j}=(1+j, 2+j)(3+$ $j, 4+j)$ and $\operatorname{supp}\left(x^{i} t x^{j}\right)=4$. If $i+j \neq 0$, then $x^{i+j}$ is a $p$-cycle and $\operatorname{supp}\left(x^{i} t x^{j}\right)=$ $\operatorname{supp}\left(x^{i+j} x^{-j} t x^{j}\right) \geq p-4>4$ because $\operatorname{supp}\left(x^{-j} t x^{j}\right)=4$. Thus, $I:=\left\{x^{-i} t x^{i} \mid i \in \mathbb{Z}_{p}\right\}$ consists of all elements in HtH whose supports are 4.

Now we consider $\operatorname{Aut}(T, H, H t H)$. Let $\beta \in \operatorname{Aut}(T, H, H t H)$. Then $\beta \in \operatorname{Aut}(T)=$ $\operatorname{Aut}\left(\mathrm{A}_{p}\right) \cong \mathrm{S}_{p}$, and $\beta$ is an automorphism of $T$ induced by some $b \in \mathrm{~S}_{p}$ by conjugation, that is, $t^{\beta}=t^{b}$ for any $t \in T$. Since $(H t H)^{\beta}=(H t H)^{b}=H t H$, we have $I^{\beta}=I$, and in particular, $\operatorname{supp}(y z)=\operatorname{supp}\left(y^{\beta} z^{\beta}\right)$ for any $y, z \in I$. It is easy to see that for any $x^{-i} t x^{i}, x^{-j} t x^{j} \in I, \operatorname{supp}\left(x^{-i} t x^{i} x^{-j} t x^{j}\right)=5$ if and only if $j=i+1$ or $i-1$. In fact, if $j=i+2$ or $i-2$ then $\operatorname{supp}\left(x^{-i} t x^{i} x^{-j} t x^{j}\right)=4$, if $j=i+3$ or $i-3$ then $\operatorname{supp}\left(x^{-i} t x^{i} x^{-j} t x^{j}\right)=7$, and if $|i-j| \geq 4$ then $\operatorname{supp}\left(x^{-i} t x^{i} x^{-j} t x^{j}\right)=8$.

Let $\Sigma$ be a graph with $I$ as vertex set and with $y, z \in I$ adjacent if and only if $\operatorname{supp}(y z)=5$. By the above paragraph, $\Sigma$ is a cycle of length $p$, and $\beta$ induces an automorphism of $\Sigma$. Thus, $\operatorname{Aut}(T, H, H t H)$ acts on $I$, and since $\Sigma$ is a $p$-cycle, $\operatorname{Aut}(T, H, H t H) / K \leq D_{2 p}$, where $K$ is the kernel of this action. Let $\gamma \in K$, and suppose $\gamma$ is induced by $c \in \mathrm{~S}_{p}$ by conjugation. Then $\gamma$ fixes each element in $I$, that is, $\left(x^{-i} t x^{i}\right)^{c}=x^{-i} t x^{i}$ for each $i \in \mathbb{Z}_{p}$. Since $x^{-i} t x^{i}=\left(1^{x^{i}}, 2^{x^{i}}\right)\left(3^{x^{i}}, 4^{x^{i}}\right)$ and $x^{-(i+3)} t x^{i+3}=$ $\left(4^{x^{i}}, 5^{x^{i}}\right)\left(6^{x^{i}}, 7^{x^{i}}\right), c$ fixes $\left\{1^{x^{i}}, 2^{x^{i}}, 3^{x^{i}}, 4^{x^{i}}\right\}$ and $\left\{4^{x^{i}}, 5^{x^{i}}, 6^{x^{i}}, 7^{x^{i}}\right\}$ setwise, and hence fixes $\left.\left.4^{x^{i}}=\left\{1^{x^{i}}, 2^{x^{i}}, 3^{x^{i}}, 4^{x^{i}}\right)\right\} \cap\left\{4^{x^{i}}, 5^{x^{i}}, 6^{x^{i}}, 7^{x^{i}}\right)\right\}$ for each $i \in \mathbb{Z}_{p}$. It follows that $c$ fixes $\{1,2, \cdots, n\}$ pointwise, implying $K=1$. Thus, $|\operatorname{Aut}(T, H, H t H)| \leq|\operatorname{Aut}(\Sigma)|=2 p$.

Recall that $h=(2, p)(3, p-1) \cdots\left(\frac{p-1}{2}, \frac{p+5}{2}\right)\left(\frac{p+1}{2}, \frac{p+3}{2}\right)$. For $p=1 \bmod 4, h$ is an even permutation and $h \in \mathrm{~A}_{p}$, and for $p=3 \bmod 4, h$ is an odd permutation and $h \in \mathrm{~S}_{p}$, but $h \notin \mathrm{~A}_{p}$. Since $x=(1,2, \cdots, p)$, we have $x^{h}=x^{-1}$ and so $H^{h}=H$, and since $t^{h}=\left(1^{h}, 2^{h}\right)\left(3^{h}, 4^{h}\right)=(1, p)(p-1, p-2)=x^{-(p-3)} t x^{p-3} \in I \subset H t H$, we have $(H t H)^{h}=H t H$. Clearly, $H^{x}=H$ and $(H t H)^{x}=H$. For any $z \in \mathrm{~S}_{p}$, denote by $\tilde{z}$ the induced automorphism of $\mathrm{A}_{p}$ by $z$ by conjugation. Then $\tilde{x}, \tilde{h} \in \operatorname{Aut}(T, H, H t H)$ and $\langle\tilde{x}, \tilde{h}\rangle \cong D_{2 p}$. Since $|\operatorname{Aut}(T, H, H t H)| \leq 2 p$, we have $\operatorname{Aut}(T, H, H t H)=\langle\tilde{x}, \tilde{h}\rangle \cong D_{2 p}$.

Recall that $\tilde{x}_{H}: H g \mapsto H g^{x}$ for $g \in \mathrm{~A}_{p}$, and $\tilde{h}_{H}: H g \mapsto H g^{h}$ for $g \in \mathrm{~A}_{p}$, are automorphisms of $\Gamma^{p}$, and $\tilde{H}_{H}=\left\langle\tilde{x}_{H}\right\rangle$. Since $\operatorname{Aut}(T, H, H t H) \cong \operatorname{Aut}(T, H, H t H)_{H}$, we have $\operatorname{Aut}(T, H, H t H)_{H}=\left\langle\tilde{x}_{H}, \tilde{h}_{H}\right\rangle=\tilde{H}_{H}: \tilde{h}_{H} \cong \mathrm{D}_{2 p}$, and since $\hat{T}_{H} \cap \operatorname{Aut}(T, H, H t H)_{H}=$ $\tilde{H}_{H}$ and $A=\hat{T}_{H} \operatorname{Aut}(T, H, H t H)_{H}$, we have $\left|A: \hat{T}_{H}\right|=2$ and hence $A=\hat{T}_{H}:\left\langle\tilde{h}_{H}\right\rangle$.

Set $C=C_{A}\left(\hat{T}_{H}\right)$, the centralizer of $\hat{T}_{H}$ in $A$. Since $\hat{T}_{H} \cong \mathrm{~A}_{p}$, we have $C \cap \hat{T}_{H}=1$, and since $A=\hat{T}_{H}:\left\langle\tilde{h}_{H}\right\rangle$, we have $C=1$ or $C \cong \mathbb{Z}_{2}$. For the former, $A \cong \mathrm{~S}_{p}$ by the N/C Theorem, and for the latter, $A=\hat{T}_{H} \times C \cong \mathrm{~A}_{p} \times \mathbb{Z}_{2}$. To finish the proof, we only need to prove that $C \cong \mathbb{Z}_{2}$ if and only if $p=1 \bmod 4$.

Assume $C \cong \mathbb{Z}_{2}$. Since $A=\hat{T}_{H} \rtimes\left\langle\tilde{h}_{H}\right\rangle$, we can let $C=\left\langle\hat{y}_{H} \tilde{h}_{H}\right\rangle$ for some $y \in T$. This implies that for any $z, g \in T$, we have $(H z)^{\hat{y}_{H} \tilde{h}_{H} \hat{g}_{H}}=(H z)^{\hat{g}_{H} \hat{y}_{H} \tilde{h}_{H}}$, that is, $H(z y)^{h} g=$ $H(z g y)^{h}$, implying Hhzyhg $=H h z g y h$. Set $\ell=y h g(g y h)^{-1}$. Then $H h z \ell(h z)^{-1}=H$, that is, $\ell \in H^{h z}=H^{z}$ for any $z \in \mathrm{~A}_{p}$. This implies that $\ell \in \bigcap_{z \in \mathrm{~A}_{p}} H^{z}$, and since
$\bigcap_{z \in \mathrm{~A}_{p}} H^{z}$ is the largest normal subgroup of of $\mathrm{A}_{p}$ contained in $H$, we have $\bigcap_{z \in \mathrm{~A}_{p}} H^{z}=1$ and hence $\ell=1$. This means that $y h g=g y h$, and by the arbitrary of $g \in \mathrm{~A}_{p}$, we have $y h \in C_{\mathrm{A}_{p}}\left(\mathrm{~S}_{p}\right)=1$. It follows that $h=y \in \mathrm{~A}_{p}$ and hence $p=1 \bmod 4$. On the other hand, if $p=1 \bmod 4$ then it is easy to check that $\hat{h} \tilde{h} \in C$. Thus, $C \cong \mathbb{Z}_{2}$ if and only if $p=1 \bmod 4$, as required.

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