## ARC-TRANSITIVE CAYLEY GRAPHS ON NONABELIAN SIMPLE GROUPS WITH PRIME VALENCY

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ABSTRACT. In 2011, Fang et al. in (J. Combin. Theory A 118 (2011) 1039-1051) posed the following problem: Classify non-normal locally primitive Cayley graphs of finite simple groups of valency d, where either  $d \leq 20$  or d is a prime number. The only case for which the complete solution of this problem is known is of d = 3. Except this, a lot of efforts have been made to attack this problem by considering the following problem: Characterize finite nonabelian simple groups which admit non-normal locally primitive Cayley graphs of certain valency  $d \geq 4$ . Even for this problem, it was only solved for the cases when either  $d \leq 5$  or d = 7 and the vertex stabilizer is solvable. In this paper, we make crucial progress towards the above problems by completely solving the second problem for the case when  $d \geq 11$  is a prime and the vertex stabilizer is solvable.

KEYWORDS. Cayley graph, simple group, arc-transitive graph.

#### 1. INTRODUCTION

Throughout this paper, graphs are assumed to be finite undirected graphs without loops and multiple edges, and groups are assumed to be finite. Let G be a permutation group on a set  $\Omega$ , and let  $\alpha \in \Omega$ . Denote by  $G_{\alpha}$  the stabiliser of  $\alpha$  in G, that is, the subgroup of G fixing the point  $\alpha$ . The group G is semiregular if  $G_{\alpha} = 1$  for every  $\alpha \in \Omega$ , and regular if G is transitive and semiregular.

For a graph  $\Gamma$ , denote by  $V(\Gamma)$ ,  $E(\Gamma)$  and  $\operatorname{Aut}(\Gamma)$  its vertex set, edge set and full automorphism group, respectively. For a vertex  $v \in V(\Gamma)$ , let  $\Gamma(v)$  be the neighbourhood of v in  $\Gamma$ . An *s*-arc in  $\Gamma$  is an ordered (s + 1)-tuple  $(v_0, v_1, ..., v_s)$  of vertices of  $\Gamma$ such that  $v_{i-1}$  is adjacent to  $v_i$  for  $1 \leq i \leq s$ , and  $v_{i-1} \neq v_{i+1}$  for  $1 \leq i < s$ . A graph  $\Gamma$ , with  $G \leq \operatorname{Aut}(\Gamma)$ , is said to be (G, s)-arc-transitive or G-regular if G is transitive on the *s*-arc set of  $\Gamma$  or G is regular on the vertex set  $V(\Gamma)$  of  $\Gamma$ , respectively. For short, a 1-arc means an arc, and (G, 1)-arc-transitive means G-arc-transitive. If a graph  $\Gamma$  is G-regular, then  $\Gamma$  is also called a Cayley graph of G, and the Cayley graph is normal if G is normal in  $\operatorname{Aut}(\Gamma)$ . A graph  $\Gamma$  is said to be *s*-arc-transitive if it is  $(\operatorname{Aut}(\Gamma), s)$ arc-transitive. In particular, 0-arc-transitive is vertex-transitive, and 1-arc-transitive is arc-transitive or symmetric.

A fair amount of work have been done on symmetric Cayley graphs on non-abeian simple groups in the literature. One of the remarkable achievements in this research field is the complete classification of cubic non-normal symmetric Cayley graphs of nonabelian simple groups, and it turns out that up to isomorphism, there are only two cubic non-normal symmetric Cayley graphs of non-abelian simple groups which are both cubic

<sup>1991</sup> Mathematics Subject Classification. 05C25, 20B25.

This work was supported by the National Natural Science Foundation of China (11731002,11671030) and by the 111 Project of China (B16002).

5-arc-transitive Cayley graphs on  $A_{47}$  (see [14, 26, 25]). Recall that a graph  $\Gamma$  is called locally primitive if for any  $v \in V(\Gamma)$ , the stabilizer  $\operatorname{Aut}(\Gamma)_v$  of v in  $\operatorname{Aut}(\Gamma)$  is primitive on  $\Gamma(v)$ . In view of the fact that every cubic symmetric graph is locally primitive, a natural question arises: What can we say about locally primitive non-normal symmetric Cayley graphs of non-abelian simple groups?

On locally primitive graphs, Weiss [23] conjectured that there is a function f defined on the positive integers such that, whenever  $\Gamma$  is a G-vertex-transitive locally primitive graph of valency d with  $G \leq \operatorname{Aut}(\Gamma)$  then, for any vertex  $v \in V(\Gamma)$ ,  $|G_v| \leq f(d)$ . By Conder et al. [1], Weiss conjecture is true for vertex-transitive locally primitive d-valent graphs if  $d \leq 20$  or d is a prime number, and by Spiga [21], Weiss conjecture is also true if the restriction  $G^{\Gamma(v)}$  of G on  $\Gamma(v)$  contains an abelian regular subgroup, that is, of affine type. In 2007, Fang et al. [8, Theorem 1.1] shown that for any valency d for which the Weiss conjecture holds, all but finitely many locally primitive Cayley graphs of valency d on the finite nonabelian simple groups are normal, and based on this, the following problem was proposed:

**Problem 1.1.** [8, Problem 1.2] Classify non-normal locally primitive Cayley graphs of finite simple groups of valency d, where either  $d \leq 20$  or d is a prime number.

As mentioned above, this problem has been completely solved by Li *et al.* for the case when d = 3. For the case when  $d \ge 4$ , however, it is quite difficult to give a complete solution of Problem 1.1. Because of this, researchers have focused on the following slightly easier problem.

# **Problem 1.2.** Characterize finite nonabelian simple groups which admit non-normal locally primitive Cayley graphs of certain valency $d \ge 4$ .

Clearly, a tetravalent graph is locally primitive if and only if the graph is 2-arctransitive. In 2004, Fang *et al* [7] proved that except 22 groups given in [7, Table 1], every tetravalent 2-arc-transitive Cayley graph  $\Gamma$  of a non-abelian simple group G is normal, and based on this, in 2018, Du and Feng [5] proved that there are exactly 7 non-abelian simple groups which admit at least one non-normal 2-arc-transitive Cayley graph, thus giving a complete solution of Problem 1.2 for the case when d = 4.

There are also some partial solutions of Problem 1.2 for the case when d is a prime number. It is easy to see that a graph with prime valency is locally primitive if and only if it is symmetric. Fang *et al* in [8] constructed an infinite family of *p*-valent non-normal symmetric Cayley graphs of the alternating groups for all prime  $p \ge 5$ , and using a result in [9] on the automorphism groups of Cayley graphs of non-abelian simple groups, they also gave all possible candidates of finite nonabelian simple groups which might have a pentavalent non-normal symmetric Cayley graph. This was recently improved by Du *et al* [6] by proving that there are only 13 finite nonabelian simple groups which admit a pentavalent non-normal symmetric Cayley graph.

More recently, Pan *et al* [17] considered Problem 1.2 for the case when d = 7, and they proved that for a 7-valent Cayley graph  $\Gamma$  of a non-abelian simple group G with solvable vertex stabilizer, either  $\Gamma$  is normal, or  $\operatorname{Aut}(\Gamma)$  has a normal arc-transitive nonabelian simple subgroup T such that G < T and  $(G, T) = (\mathsf{A}_6, \mathsf{A}_7)$ ,  $(\mathsf{A}_{20}, \mathsf{A}_{21})$ ,  $(\mathsf{A}_{62}, \mathsf{A}_{63})$ or  $(\mathsf{A}_{83}, \mathsf{A}_{84})$ , and for each of these 4 pairs (G, T), there do exist a 7-valent G-regular T-arc-transitive graph. In this paper, we shall prove the following theorem which generalizes the result in [17] to all prime valent cases, and hence gives a solution of Problem 1.2 for the case when d is a prime and the vertex-stabilizer is solvable.

**Theorem 1.3.** Let G be a non-abelian simple group and  $\Gamma$  a connected arc-transitive Cayley graph of G with prime valency  $p \geq 11$ . If  $\operatorname{Aut}(\Gamma)_v$  is solvable for  $v \in V(\Gamma)$ , then either  $G \leq \operatorname{Aut}(\Gamma)$ , or  $\operatorname{Aut}(\Gamma)$  has a normal subgroup T with G < T such that  $\Gamma$  is T-arc-transitive and (G, T, p) is one of the following four triples:

$$(\mathsf{A}_5, \mathsf{PSL}(2, 11), 11), (\mathsf{A}_5, \mathsf{PSL}(2, 29), 29), (\mathsf{M}_{22}, \mathsf{M}_{23}, 23), (\mathsf{A}_{n-1}, \mathsf{A}_n, p),$$

where  $n = pk\ell$  with  $k \mid \ell$  and  $\ell \mid (p-1)$ , and k and  $\ell$  have the same parity.

Conversely, we show that all the first three triples as well as the fourth triple in case of n = p can happen.

**Theorem 1.4.** Use the same notation as Theorem 1.3. If (G, T, p) is one of the following triples:

 $(A_5, PSL(2, 11), 11), (A_5, PSL(2, 29), 29), (M_{22}, M_{23}, 23), (A_{p-1}, A_p, p),$ 

then there exists a p-valent symmetric Cayley graph  $\Gamma$  of G such that  $\operatorname{Aut}(\Gamma)_v$  is solvable for some  $v \in V(\Gamma)$ .

Let p be a prime and  $\ell, k$  integers with  $k \mid \ell$  and  $\ell \mid (p-1)$  such that k and  $\ell$  have the same parity. The triple  $(p, \ell, k)$  is called *conceivable* if there exists an arc-transitive Cayley graph of the alternating group  $A_{pk\ell-1}$  with valency p and its automorphism group has solvable vertex stabilizer. We have been unable to determine all the conceivable triples  $(p, \ell, k)$ , and we would like to leave it as an open problem for future research.

**Problem 1.5.** Determine conceivable triples  $(p, \ell, k)$ .

By Theorem 1.4, (p, 1, 1) is conceivable for each prime  $p \ge 5$ , and by [6], (5, 4, 2) is conceivable, but not (5, 2, 2). For the case p = 7, it was shown in [17] that (7, 1, 1), (7, 3, 1), (7, 3, 3) and (7, 6, 2) are the only conceivable triples.

The paper is organized as follows. In Section 2 we introduce some preliminary results on nonabelian simple groups and arc-transitive graphs with prime valency. Then we prove Theorem 1.3 in Section 3 and Theorem 1.4 in Section 4.

## 2. Preliminary

In this section, we introduce some preliminary results that will be used latter.

For a positive integer n, we use  $\mathbb{Z}_n$  to denote the cyclic group of order n. For a group G and a subgroup H of G, denote by  $N_G(H)$  and  $C_G(H)$  the normalizer and the centralizer of H in G respectively. Given two groups N and H, denote by  $N \times H$  the direct product of N and H, by N.H an extension of N by H, and if such an extension is split, then we write N : H instead of N.H.

The following proposition is an exercise in Dixon and Mortimer's textbook [4, p.49].

**Proposition 2.1.** Let n be a positive integer and p a prime. Let  $p^{\nu(n)}$  be the largest power of p which divides n!. Then  $\nu(n) = \sum_{i=1} \lfloor \frac{n}{p^i} \rfloor < \frac{n}{p-1}$ .

The next proposition is called the *Frattini argument* on transitive permutation group, and we refer to [4, p.9].

**Proposition 2.2.** Let G be a transitive permutation group on  $\Omega$ , H a subgroup of G and  $v \in \Omega$ . Then H is transitive if and only if  $G = HG_v$ .

We denote by  $\operatorname{Aut}(G)$  the automorphism group of a group G, and by  $\operatorname{Inn}(G)$  the inner automorphism group of G consisting of these automorphisms of G induced by all element of G by conjugation on G. Then  $\operatorname{Inn}(G)$  is normal in  $\operatorname{Aut}(G)$ , and the quotient group  $\operatorname{Aut}(G)/\operatorname{Inn}(G)$  is called the *outer automorphism* of G, denoted by  $\operatorname{Out}(G)$ . The following proposition is a direct consequence of the classification of finite simple groups (see [13, Table 5.1.A-C] for example).

**Proposition 2.3.** Let T be a finite non-abelian simple group. Then Out(T) is solvable.

Let G and E be two groups. We call an extension E of G by N a central extension of G if E has a central subgroup N such that  $E/N \cong G$ , and if further E is perfect, that is, the derived group E' equals to E, we call E a covering group of G. A covering group E of G is called a *double cover* if |E| = 2|G|. Schur [20] proved that for every non-abelian simple group G there is a unique maximal covering group M such that every covering group of G is a factor group of M (see [12, Kapitel V, S23]). This group M is called the *full covering group* of G, and the center of M is the Schur multiplier of G, denoted by Mult(G). By Kleidman and Liebeck [13, Theorem 5.1.4] and Du *et al* [6, Proposition 2.6], we have the following proposition.

**Proposition 2.4.**  $\text{Mult}(A_n) = \mathbb{Z}_2$  with  $n \ge 8$ . For  $n \ge 5$ ,  $A_n$  has a unique double cover 2. $A_n$ , and for  $n \ge 7$ , all subgroups of index n of 2. $A_n$  are isomorphic to 2. $A_{n-1}$ .

By Kleidman and Liebeck [13, Proposition 5.3.7], we have the following proposition.

**Proposition 2.5.** Let r be a prime power and f a positive integer. If  $A_n \leq GL(f,r)$  with  $n \geq 9$ , then  $f \geq n-2$ .

Let  $\Gamma$  be a connected graph and G a group of automorphisms of  $\Gamma$ . For  $v \in V(\Gamma)$ , denote by  $G_v^{\Gamma(v)}$  the induced permutation group of the natural action of  $G_v$  on the neighbourhood  $\Gamma(v)$ . Let  $G_v^*$  be the subgroup of  $G_v$  fixing every vertex in  $\Gamma(v)$ . Then  $G_v^*$  is the kernel of the natural action of  $G_v$  on  $\Gamma(v)$ , and hence  $G_v/G_v^* \cong G_v^{\Gamma(v)}$ . By the connectivity of  $\Gamma$ , there exists a path  $v = v_0, v_1, v_2, \cdots, v_m$  such that  $G_{v_0v_1\cdots v_m}^* :=$  $G_{v_0}^* \cap G_{v_1}^* \cap \cdots \cap G_{v_m}^* = 1$ . Clearly,

$$1 = G^*_{v_0v_1\cdots v_m} \trianglelefteq G^*_{v_0v_1\cdots v_{m-1}} \trianglelefteq \cdots \trianglelefteq G^*_{v_0v_1} \trianglelefteq G^*_{v_0} = G^*_v \trianglelefteq G_v,$$

and for  $0 \leq i < m$ , we have  $G^*_{v_0v_1\cdots v_i}/G^*_{v_0v_1\cdots v_{i+1}} \cong (G^*_{v_0v_1\cdots v_i})^{\Gamma(v_{i+1})}$ . Then we can easily obtain the following proposition, and this was known from a series of lectures given by Cai Heng Li in Peking University in 2013.

**Proposition 2.6.** Let  $\Gamma$  be a connected graph and let G be a vertex-transitive group of automorphisms of  $\Gamma$ . Then  $G_v$  is nonsolvable if and only if  $G_v^{\Gamma(v)}$  is nonsolvable.

For self-containing, we give a short proof of the following proposition, which is mainly owed to an anonymous referee (also see [11] for another proof).

**Proposition 2.7.** Let  $\Gamma$  be a connected *G*-arc-transitive graph of prime valency  $p \geq 5$ , and let (u, v) be an arc of  $\Gamma$ . Assume that  $G_v$  is solvable. Then  $G_{uv}^* = 1$  and  $G_v \cong \mathbb{Z}_k \times (\mathbb{Z}_p : \mathbb{Z}_\ell)$  with  $k \mid \ell \mid (p-1)$ , where  $\mathbb{Z}_p : \mathbb{Z}_\ell \leq \mathsf{AGL}(1, p)$ . Proof. It follows from [23] that  $G_{uv}^* = 1$ . Let P be a Sylow p-subgroup of  $G_v$ . Note that  $G_v^{\Gamma(v)}$  is a transitive solvable group of prime degree. By the Burnside Theorem (also see [4, Theorem 3.5B]),  $G_v/G_v^* \cong G_v^{\Gamma(v)} \cong \mathbb{Z}_p : \mathbb{Z}_\ell \leq \mathsf{AGL}(1,p)$  with  $\ell \mid (p-1)$  and  $G_{uv}/G_v^* \cong \mathbb{Z}_\ell$ . In particular,  $PG_v^*/G_v^* \leq G_v/G_v^*$ , and so  $PG_v^* \leq G_v$ . Since  $G_u^* = G_u^*/G_{uv}^* = G_u^*/(G_u^* \cap G_v^*) \cong G_u^*G_v^*/G_v^* \leq G_{uv}/G_v^* \cong \mathbb{Z}_\ell$ , we have  $G_v^* \cong \mathbb{Z}_k$  with  $k \mid \ell$ , and then  $|G_v| = pk\ell$  with  $k \mid \ell \mid (p-1)$ . Since  $G_{uv} = G_{uv}/G_{uv}^* = G_{uv}/(G_u^* \cap G_v^*) \cong \mathbb{Z}_\ell$ . Then  $\langle a \rangle \cong \mathbb{Z}_\ell$  and  $\langle a \rangle \cap G_v^* = 1$ . It follows that  $G_{uv} = \langle a \rangle \times G_v^*$ .

Since  $|G_v^*| = \ell | (p-1)$ ,  $PG_v^*$  has a unique Sylow *p*-subgroup *P* and hence  $PG_v^* = P \times G_v^*$ . Then *P* is characteristic in  $PG_v^*$ , and since  $PG_v^* \trianglelefteq G_v$ , we have  $P \trianglelefteq G_v$ . It follows that  $G_v = P : G_{uv} = P : (\langle a \rangle \times G_v^*) = G_v^* \times (P : \langle a \rangle) \cong \mathbb{Z}_k \times (\mathbb{Z}_p : \mathbb{Z}_\ell)$ .  $\Box$ 

Taking normal quotient graphs is a useful method for studying arc-transitive graphs. Let  $\Gamma$  be an X-vertex-transitive graph, where  $X \leq \operatorname{Aut}(\Gamma)$  has an intransitive normal subgroup N. The normal quotient graph  $\Gamma_N$  of  $\Gamma$  induced by N is defined to be a graph with vertex set  $\{\alpha^N \mid \alpha \in V(\Gamma)\}$ , the set of all N-orbits on  $V(\Gamma)$ , such that two vertices  $B, C \in \{\alpha^N \mid \alpha \in V(\Gamma)\}$  are adjacent if and only if some vertex in B is adjacent in  $\Gamma$  to some vertex in C. If  $\Gamma$  and  $\Gamma_N$  have the same valency, then  $\Gamma$  is called a normal cover of  $\Gamma_N$ . The following proposition is a special case of [15, Lemma 2.5], which slightly improves a remarkable result of Praeger [18, Theorem 4.1].

**Proposition 2.8.** Let  $\Gamma$  be a connected X-arc-transitive graph of prime valency, with  $X \leq \operatorname{Aut}(\Gamma)$ , and let  $N \leq X$  have at least three orbits on  $V(\Gamma)$ . Then the following statements hold.

- (1) N is semi-regular on  $V(\Gamma)$ ,  $X/N \leq \operatorname{Aut}(\Gamma_N)$ ,  $\Gamma_N$  is a connected X/N-arctransitive graph, and  $\Gamma$  is a normal cover of  $\Gamma_N$ .
- (2)  $X_v \cong (X/N)_{\Delta}$  for any  $v \in V(\Gamma)$  and  $\Delta \in V(\Gamma_N)$ .

## 3. Proof of Theorem 1.3

Throughout this section we make the following assumption.

Assumption:  $\Gamma$  is a symmetric graph of prime valency  $p \ge 11$  with  $v \in V(\Gamma)$ ,  $\operatorname{Aut}(\Gamma)_v$  is solvable, and  $G \le \operatorname{Aut}(\Gamma)$  is a non-abelian simple group and transitive on  $V(\Gamma)$ .

The proof of the following lemma is straightforward, but will be used frequently latter.

**Lemma 3.1.** Let X = H : K be a transitive permutation group on  $\Omega$ . Let  $w \in \Omega$ . If H is transitive, then K is isomorphic to  $X_w/H_w$ .

*Proof.* Since H is transitive,  $X = HX_w$  by Proposition 2.2. So  $K \cong X/H = HX_w/H \cong X_w/(X_w \cap H) = X_w/H_w$ .

The product of all minimal normal subgroups of a group X is called the *socle* of X, denoted by soc(X), and the largest normal solvable subgroup of X is called the *radical* of X, denoted by rad(X).

**Lemma 3.2.** Let G,  $\Gamma$ , p and v be as given in Assumption. Let  $\Gamma$  be X-arc-transitive with  $G \leq X \leq \operatorname{Aut}(\Gamma)$ , and let  $\operatorname{rad}(X) = 1$ . Then either  $\operatorname{soc}(X) = G$ , or  $\Gamma$  is  $\operatorname{soc}(X)$ -arc-transitive with  $G < \operatorname{soc}(X)$  and one of the following holds:

- (1)  $(G, \operatorname{soc}(X)) = (A_{n-1}, A_n)$  with  $n \ge 6$ , and  $(\operatorname{soc}(X))_v$  is transitive on  $\{1, 2, \dots, n\}$ .
- (2)  $(G, \operatorname{soc}(X)) = (\mathsf{M}_{22}, \mathsf{M}_{23}), and (\operatorname{soc}(X))_v = \mathbb{Z}_{23}.$
- (3)  $(G, \operatorname{soc}(X)) = (A_5, \mathsf{PSL}(2, 11)), and (\operatorname{soc}(X))_v = \mathbb{Z}_{11}.$
- (4)  $(G, \operatorname{soc}(X)) = (\mathsf{A}_5, \mathsf{PSL}(2, 29)), and (\operatorname{soc}(X))_v = \mathbb{Z}_{29} : \mathbb{Z}_7.$

In particular,  $\Gamma$  is a Cayley graph of G for cases (2)-(4).

*Proof.* Let N be a minimal normal subgroup of X. Since rad(X) = 1, we have  $N = T_1 \times \cdots \times T_d \cong T^d$  for a non-abelian simple group T. Write K = NG.

Assume that  $G \leq X$ . If  $N \cap G = 1$ , applying Lemma 3.1 with K = G : N we have that  $N \cong (K)_v/G_v$  is solvable, a contradiction. Therefore,  $N \cap G \neq 1$ , forcing  $G \leq N$ , and since G is normal, the minimality of N implies N = G. By the arbitrariness of N, we have  $\operatorname{soc}(X) = G$ .

In what follows we assume that  $G \not\leq X$ . If  $\Gamma$  is bipartite, then the transitivity of G on  $V(\Gamma)$  implies that G has a normal subgroup of index 2, contradicting the simplicity of G. Thus,  $\Gamma$  is not bipartite. Therefore N has either one or at least three orbits on  $V(\Gamma)$ . We claim that the latter cannot occur.

We argue by contradiction and we suppose that N has at least three orbits on  $V(\Gamma)$ . By Proposition 2.8, N is semiregular on  $V(\Gamma)$ , and so  $|N| = |T|^d$  is a divisor of  $|V(\Gamma)|$ . In particular,  $|N| \mid |G|$ . Since N has at least three orbits,  $|G| \ge 3|N|$  and hence  $N \cap G = 1$ .

Consider the conjugate action of G on N, and since G is simple, the action is trivial or faithful. If it is trivial then  $K = N \times G$ , and by Lemma 3.1,  $N \cong K_v/G_v$  is solvable, a contradiction. It follows that the conjugate action of G on N is faithful, and hence we may assume  $G \leq \operatorname{Aut}(N)$ .

Note that  $\operatorname{Aut}(N) \cong \operatorname{Aut}(T)^d : S_d$ . Set  $M = \operatorname{Aut}(T)^d$  and  $M_1 = \operatorname{Inn}(N) \cong T^d$ . Then  $|M_1| = |N|, M_1 \trianglelefteq M, M \trianglelefteq \operatorname{Aut}(N)$  and  $M_1 \trianglelefteq \operatorname{Aut}(N)$ . Clearly,  $G \cap M_1 = 1$  as  $|G| \ge 3|N| = 3|M_1|$ . If  $G \cap M \neq 1$  then  $G \le M$  and hence  $G \cong G/(G \cap M_1) \cong GM_1/M_1 \le M/M_1 \cong \operatorname{Out}(T)^d$ , which is impossible because  $\operatorname{Out}(T)$  is solvable by Propostion 2.3. This means that  $G \cap M = 1$ , and therefore,  $G \cong G/(G \cap M) \cong GM/M \le \operatorname{Aut}(N)/M \cong S_d$ . Recall that  $|N| = |T|^d$  and  $|N| \mid |G|$ . Then for any prime p with  $p \mid |T|$ , we have  $p^d \mid d!$ , and by Proposition 2.1,  $d < \frac{d}{p-1}$ , a contradiction.

We have just shown that N has one orbit, that is, N is transitive on  $V(\Gamma)$ . If  $N \cap G = 1$ , Lemma 3.1 implies that  $G \cong K_v/N_v$  is solvable, a contradiction. Therefore,  $G \leq N$ , and by the arbitrariness of N, X has only one minimal normal subgroup, that is,  $\operatorname{soc}(X) = N$ .

Since G is not normal in X, we have G < N, and hence  $N_v \neq 1$  as  $\Gamma$  is G-vertextransitive. Clearly, we may chose v such that  $N_v^{\Gamma(v)} \neq 1$ . Since  $\Gamma$  has prime valency and  $N_v^{\Gamma(v)} \leq X_v^{\Gamma(v)}$ ,  $N_v^{\Gamma(v)}$  is transitive on  $\Gamma(v)$ , that is,  $\Gamma$  is N-arc-transitive.

Recall that  $N = T_1 \times T_2 \times \cdots \times T_d \cong T^d$ . Suppose  $d \ge 2$ . If  $T_1$  is transitive, then by Lemma 3.1,  $T_2 \times \cdots \times T_d \cong N_v/(T_1)_v$  is solvable, a contradiction. Thus,  $T_1$ has at least three orbits, and hence  $|G| \ge 3|T_1|$ . In particular,  $G \cap T_1 = 1$ . By the simplicity of G, the conjugate action of G on  $T_1$  is trivial or faithful. If it is trivial then  $GT_1 = G \times T_1$ , and by Lemma 3.1,  $T_1 \cong (GT_1)_v/G_v$  is solvable, a contradiction. Thus, the conjugate action of G on  $T_1$  is faithful and hence we may assume  $G \le \operatorname{Aut}(T_1)$ . Since  $|G| \ge 3|T_1| = 3|\operatorname{Inn}(T_1)|$ , we have  $G \cap \operatorname{Inn}(T_1) = 1$  and hence  $G = G/(G \cap \operatorname{Inn}(T_1)) \cong$  $G\operatorname{Inn}(T_1)/\operatorname{Inn}(T_1) \le \operatorname{Aut}(T_1)/\operatorname{Inn}(T_1) = \operatorname{Out}(T_1)$ , which is impossible because  $\operatorname{Out}(T_1)$  is solvable. Thus,  $\operatorname{soc}(X) = N = T$  is a non-abelian simple group. By the Frattini argument,  $T = GT_v$ . Then the triple  $(T, G, T_v)$  can be read out from [16], where  $T_v$  is a group given in Proposition 2.7. Note that  $p \ge 11$ .

By [16, Proposition 4.2], T cannot be any exceptional group of Lie type.

Assume that  $T = A_n$ . By [16, Proposition 4.3], one of the following occurs:

- (a)  $G = A_{n-1}, T = A_n$  with  $n \ge 6$  and  $T_v$  is transitive on  $\{1, 2, \dots, n\}$ , or
- (b)  $G = A_{n-2}, T = A_n$  with  $n = q^f$  for some prime q, and  $T_v \leq A\Gamma L(1, q^f)$  is 2-homogeneous on  $\{1, 2, \dots, n\}$ .

If (b) occurs, then  $T_v$  is primitive on  $\{1, 2, 3, \dots, q^f\}$  because it is 2-homogeneous. By Proposition 2.7,  $T_v$  has a normal subgroup  $\mathbb{Z}_p$ , and by the primitivity of  $T_v$ ,  $\mathbb{Z}_p$  is transitive and so regular on  $\{1, 2, 3, \dots, q^f\}$ . It follows  $q^f = p$  and  $T_v \leq \mathsf{AGL}(1, p) = \mathbb{Z}_p : \mathbb{Z}_{p-1}$ . Moreover, since  $|T_v| = \frac{|T||G_v|}{|G|} \geq \frac{|T|}{|G|} = p(p-1)$ , we have that  $T_v = \mathsf{AGL}(1, p) = \mathbb{Z}_p : \mathbb{Z}_{p-1}$ . Thus,  $\mathsf{A}_p$  contains a cyclic subgroup  $\mathbb{Z}_{p-1}$ , which is impossible because  $\mathbb{Z}_{p-1}$  contains odd permutations on  $\{1, 2, 3, \dots, p\}$ . It follows that  $T = \mathsf{A}_n$ ,  $G = \mathsf{A}_{n-1}$  and  $T_v$  is transitive on the n points, which is the case (1) of the lemma.

Assume that T is a sporadic simple group. By [16, Proposition 4.4],  $G = M_{22}$ ,  $T = M_{23}$ , and  $T_v = \mathbb{Z}_{23}$  or  $\mathbb{Z}_{23} : \mathbb{Z}_{11}$ . Suppose on the contrary that  $T_v = \mathbb{Z}_{23} : \mathbb{Z}_{11}$ . We may let  $T_{uv} = \mathbb{Z}_{11}$  for  $u \in \Gamma(v)$ . Since  $\Gamma$  is T-arc-transitive, there is an element  $g \in T$  interchanging u and v, and hence  $T_{uv}^g = T_{u^g v^g} = T_{uv}$ , that is,  $g \in N_T(T_{uv})$ . A computation with MAGMA [2] shows that there is only one conjugate class of  $\mathbb{Z}_{11}$  in  $M_{23}$ , and the normalizer of  $\mathbb{Z}_{11}$  in  $M_{23}$  is  $\mathbb{Z}_{11} : \mathbb{Z}_5$ . Thus,  $g \in \mathbb{Z}_{11} : \mathbb{Z}_5$  has odd order, which is impossible because g interchanges u and v. It follows that  $T_v = \mathbb{Z}_{23}$ , which is the case (2) of the lemma.

Assume that T is a classical simple group of Lie type. Note that  $T = GT_v$ , G is non-abelian simple and  $T_v$  is solvable. Let H is a maximal subgroup subject to that  $T_v \leq H$  and H is solvable. Then T = GH, and (T, G, H) is listed in [16, Table 1.1 and Table 1.2]. Clearly,  $|T : G| ||T_v| ||H|$ . For an integer m and a prime r, we use  $m_r$  to denote the largest r-power dividing m.

By Proposition 2.7,  $T_v = \mathbb{Z}_k \times (\mathbb{Z}_p : \mathbb{Z}_\ell)$  with  $k \mid \ell \mid p-1$ , where  $\mathbb{Z}_p : \mathbb{Z}_\ell \leq \mathsf{AGL}(1, p)$ . Let P and Q be the maximal normal r-subgroup of  $T_v$  and H respectively. Then  $Q \cap T_v \leq P$ , and since  $T_v/(T_v \cap Q) \cong QT_v/Q \leq H/Q$ , we have  $|T_v|_r \leq |T_v \cap Q| \cdot |H/Q|_r \leq |P||H/Q|_r$ . Clearly,  $|T_v|_p = p$  and hence  $|T:G|_p \leq p$ .

Suppose that  $r \neq p$  and  $r \mid |T_v|$ . If P is not contained in  $\mathbb{Z}_k$ , then  $1 \neq P\mathbb{Z}_k/\mathbb{Z}_k \leq T_v/\mathbb{Z}_k \cong \mathbb{Z}_p : \mathbb{Z}_\ell$ , which is impossible because  $\mathbb{Z}_p$  is the unique minimal normal subgroup of  $\mathbb{Z}_p : \mathbb{Z}_\ell$ . Therefore  $P \leq \mathbb{Z}_k$ . It follows from  $k \mid \ell$  that  $|P|^2 \leq |T_v|_r$ , and from  $|T_v|_r \leq |P||H/Q|_r$  that  $|P| \leq |H/Q|_r$ . Thus,  $|T:G|_r \leq |T_v|_r \leq (|H/Q|_r)^2$ .

Since G is a non-abelian simple group, we may exclude Row 1 of [16, Table 1.1] and Rows 7-10, 17 and 21 of [16, Table 1.2], and since  $p \ge 11$  and  $p \mid \mid H \mid$ , we may exclude Rows 6, 11-13, 16-20, 22 and 24-27 of [16, Table 1.2]. The remaining cases are Rows 2-9 of [16, Table 1.1], and Rows 1-5, 14, 15, 23 and 28 of [16, Table 1.2].

In what follows we write  $q = r^f$  for some prime r and positive integer f.

For Row 2 of [16, Table 1.1],  $T = \mathsf{PSL}(4,q)$ ,  $G = \mathsf{PSp}(4,q)$ , and  $H = q^3 : \frac{q^3-1}{(4,q-1)}$ .3. By [13, Table 5.1A],  $q^2 | |T : G|$ . Thus  $r \neq p$ . Note that  $|H/Q|_r = 1$  or 3. Since  $r^{2f} = q^2 \leq |T_v|_r \leq |H/Q|_r^2$ , we have r = 3 and f = 1, that is, q = 3. This is impossible because a computation with MAGMA shows  $T = \mathsf{PSL}(4,3)$  has no factorization T = GH. For Row 3 of [16, Table 1.1],  $T = \mathsf{PSp}(2m, q)$ ,  $G = \Omega^{-}(2m, q)$ , and  $H = q^{m(m+1)/2}$ :  $(q^m - 1).m$  with  $m \ge 2$  and q even. Then r = 2. By [13, Table 5.1A],  $q^m | |T : G|$ , implying  $r \ne p$ . Furthermore,  $r^{fm} = q^m \le |T_v|_r \le |H/Q|_r^2 = m_r^2$ . It follows  $r^{m_2} \le r^{fm} \le m_2^2$ , and this holds if and only if  $m_2 = 2$  or 4. If  $m_2 = 2$ , then  $2^{fm} \le m_2^2 = 4$  implies f = 1, m = 2, which is impossible because  $\mathsf{PSp}(4, 2) \cong \mathsf{S}_6$  is not a simple group. If  $m_2 = 4$ , then  $2^{fm} \le m_2^2 = 16$  implies that m = 4 and f = 1. In this case,  $|H| = 2^{12} \cdot 15$ , contradicting that p | |H| with  $p \ge 11$ .

For Rows 4 and 5 of [16, Table 1.1],  $T = \mathsf{PSp}(4, q)$ ,  $G = \mathsf{PSp}(2, q^2)$ , and  $H = q^3$ :  $\frac{q^2-1}{(2,q-1)}$ .2. By [13, Table 5.1A],  $q^2 | |T : G|$ , and so  $r \neq p$ . Note that  $|H/Q|_r = 1$  or 2. Since  $r^{2f} = q^2 \leq |T_v|_r \leq |H/Q|_r^2$ , we have that r = 2 and f = 1. This is impossible because  $T = \mathsf{PSp}(4, 2) \cong \mathsf{S}_6$  is not simple.

For Row 6 of [16, Table 1.1],  $T = \mathsf{PSU}(2m, q)$ ,  $G = \mathsf{SU}(2m - 1, q)$ , and  $H = q^{m^2}$ :  $\frac{q^{2m}-1}{q+1(2m,q+1)}$  m with  $m \ge 2$ . By [13, Table 5.1A],  $q^{2m-1} = r^{(2m-1)f} ||T : G|$  and  $r \ne p$ . Thus  $r^{(2m-1)f} = q^{2m-1} \le |T_v|_r \le H/Q|_r^2 = m_r^2$ , implying  $r^{2m_r-1} \le m_r^2$ , which is impossible.

For Row 7 of [16, Table 1.1],  $T = P\Omega(2m + 1, q)$ ,  $G = \Omega^{-}(2m, q)$ , and  $H = (q^{m(m-1)/2}.q^m) : \frac{q^m-1}{2}.m$  with  $m \ge 3$  and q odd. Then  $r, m_r \ge 3$ . By [13, Table 5.1A],  $q^m = 2^{fm} | |T:G|$  and hence  $r \ne p$ . Then  $r^{fm} = q^m \le |T_v|_r \le |H/Q|_r^2 = m_r^2$ , and so  $r^{m_r} \le m_r^2$ , which is impossible.

For Row 8 of [16, Table 1.1],  $T = P\Omega^+(2m, q)$ ,  $G = \Omega(2m - 1, q)$ , and  $H = q^{m(m-1)/2}$ :  $\frac{q^{m-1}}{(4,q^{m-1})} m$  with  $m \ge 5$ . By [13, Table 5.1A],  $q^{m-1} = r^{f(m-1)} | |T : G|$  and  $r \ne p$ . Then  $r^{f(m-1)} = q^{m-1} \le |T_v|_r \le |H/Q|_r^2 = m_r^2$ . Note that the inequality  $2^x > x^2$  always holds for  $x \ge 5$ . Thus  $m_r \le 4$ . Since  $r^{f(m-1)} \le m_r^2$  and  $m \ge 5$ , we have that  $r = 2, m_r = 4$  and m = 5, which is impossible because  $m_r = 5_2 = 1$ .

For Row 9 of [16, Table 1.1],  $T = \mathsf{P}\Omega^+(8,q)$ ,  $G = \Omega(7,q)$ , and  $H = q^6 : \frac{q^4-1}{(4,q^4-1)}$ .4. By [13, Table 5.1A],  $q^3 = r^{3f} | |T:G|$ , and  $r \neq p$ . Then  $r^{3f} = q^3 \leq |T_v|_r \leq |H/Q|_r^2 = (4_r)^2$ , implying r = 2 and f = 1. In this case,  $|H| = 2^8 \cdot 15$ , contradicting p | |H| with  $p \geq 11$ .

For Row 14 of [16, Table 1.2],  $T = \mathsf{PSp}(4, 11)$ ,  $H = 11^{1+2}_+$ : 10.A<sub>4</sub>, and  $G = \mathsf{PSL}(2, 11^2)$ . By [13, Table 5.1A],  $11^2 ||T : G|| |T_v|$  and hence  $p \neq 11$ , which is impossible because p is the largest prime divisor of  $|T_v|$ . Similarly, we may exclude Row 15 of [16, Table 1.2], because  $T = \mathsf{PSp}(4, 23)$ ,  $H = 23^{1+2}_+ : 22.S_4$ ,  $G = \mathsf{PSL}(2, 23^2)$ , and  $23^2 ||T : G|| |T_v|$  by [13, Table 5.1A].

For Row 23 of [16, Table 1.2],  $T = \Omega(7,3)$ ,  $H = 3^{3+3} : 13 : 3$  and  $G = \mathsf{Sp}(6,2)$ . Then p = 13, and since  $|T_v| = pk\ell$  with  $k \mid \ell \mid (p-1)$ , we have  $3^5 \nmid |T_v|$ . However,  $|T : G| = |\Omega(7,3)|/|\mathsf{Sp}(6,2)| = 13 \cdot 3^5$  implies  $3^5 \mid |T_v|$ , a contradiction. Similarly, we may exclude Row 28 of [16, Table 1.2] because  $T = \mathsf{P}\Omega^+(8,3)$ ,  $H = 3^6 : (3^3 : 13 : 3)$  or  $3^{3+6} : 13 : 3$ ,  $G = \Omega^+(8,2)$  and  $|T : G| = 13 \cdot 3^7$ .

For Rows 1-5 of [16, Table 1.2], by MAGMA we obtain the following:

- (a)  $(G, T, T_v) = (A_5, \mathsf{PSL}(2, 11), \mathbb{Z}_{11}),$
- (b)  $(G, T, T_v) = (\mathsf{A}_5, \mathsf{PSL}(2, 11), \mathbb{Z}_{11} : \mathbb{Z}_5),$
- (c)  $(G, T, T_v) = (\mathsf{A}_5, \mathsf{PSL}(2, 19), \mathbb{Z}_{19} : \mathbb{Z}_9),$
- (d)  $(G, T, T_v) = (\mathsf{A}_5, \mathsf{PSL}(2, 29), \mathbb{Z}_{29} : \mathbb{Z}_7),$
- (e)  $(G, T, T_v) = (\mathsf{A}_5, \mathsf{PSL}(2, 29), \mathbb{Z}_{29} : \mathbb{Z}_{14}),$
- (f)  $(G, T, T_v) = (A_5, \mathsf{PSL}(2, 59), \mathbb{Z}_{59} : \mathbb{Z}_{29}).$

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For case (b),  $|V(\Gamma)| = |T : T_v| = 12$  and hence  $\Gamma$  is a complete graph of order 12, contradicting that  $\operatorname{Aut}(\Gamma)_v$  is solvable. Similarly, cases (c),(e) and (f) cannot occur because  $\Gamma$  is a complete graph of order 20, 30 or 60, respectively. Thus, we have (a) or (d), which is the case (3) or (4) of the lemma.

For cases (2)-(4), it is easy to see that  $G_v = G \cap T_v = 1$ . Since G is transitive, it is regular, that is,  $\Gamma$  is Cayley graph of G.

**Lemma 3.3.** Let G,  $\Gamma$ , p and v be as given in Assumption and further assume that G is regular on  $V(\Gamma)$ . Then  $\operatorname{rad}(\operatorname{Aut}(\Gamma))$  has at least three orbits on  $V(\Gamma)$ , and if  $\operatorname{rad}(\operatorname{Aut}(\Gamma))G \trianglelefteq \operatorname{Aut}(\Gamma)$  then  $\operatorname{rad}(\operatorname{Aut}(\Gamma))G = \operatorname{rad}(\operatorname{Aut}(\Gamma)) \times G$ .

*Proof.* Set  $A = \operatorname{Aut}(\Gamma)$ ,  $R = \operatorname{rad}(A)$  and B = RG. If R is transitive on  $V(\Gamma)$ , then Lemma 3.1 implies that  $G \cong B_v/R_v$  is solvable, a contradiction. Since G is transitive,  $\Gamma$  is not bipartite and hence R has at least three orbits. Assume that  $B \leq A$ . To finish the proof, it suffices to show  $B = R \times G$ . This is clearly true for R = 1.

Assume  $R \neq 1$ . Then  $R \cap G = 1$ . Since G is regular,  $B_v \neq 1$ , and since  $B \leq A$ ,  $\Gamma$  is B-arc-transitive. By Proposition 2.7,  $B_v$  has a normal sylow p-subgroup  $\mathbb{Z}_p$ , and  $|B_v| = pm$  with (p, m) = 1. Note that  $RG = B = GB_v$ . Again by the regularity of G, we have  $|B_v| = |R| = pm$ . Let  $R_p$  be a Sylow p-subgroup of R. We claim  $R_p \leq B$ .

Suppose to the contrary that  $R_p \not \leq B$ . Since  $R \leq B$  is solvable, by the Jordan-Holder Theorem, B has a normal series:  $1 \leq R_1 \leq R_2 \leq \cdots \leq R \leq B$  such that  $R_1 \leq B$ ,  $R_2 \leq B$ ,  $R_2/R_1 \cong \mathbb{Z}_p$  and  $R_1 \neq 1$ . Since (p, m) = 1, we have  $p \not \mid \mid R_1 \mid$ . Note that  $R_2/R_1 \leq B/R_1$ and  $GR_1/R_1 \cong G/(G \cap R_1) = G$ . Since  $R_2/R_1 \cong \mathbb{Z}_p$ , the conjugate action of  $GR_1/R_1$  on  $R_2/R_1$  must be trivial by the simplicity of G. It follows that  $GR_2/R_1 = R_2/R_1 \times GR_1/R_1$ , and hence,  $GR_1/R_1 \leq GR_2/R_1$ , forcing  $GR_1 \leq GR_2$ . Since  $p \mid \mid R_2 \mid$ ,  $GR_2$  is arc-transitive on  $\Gamma$ , and hence  $GR_1$  is also arc-transitive because  $\mid (GR_1)_v \mid = \mid R_1 \mid \neq 1$ . It follows  $p \mid \mid R_1 \mid$ , a contradiction. Thus,  $R_p \leq B$ , as claimed.

Let  $C = C_B(R_p)$ . Since  $R_p \leq B$  and  $R_p \approx \mathbb{Z}_p$ , the conjugate action of G on  $R_p$  is trivial and so  $R_pG = R_p \times G$ . It follows that  $G \leq C$  and  $C = C \cap B = C \cap (RG) = (C \cap R)G$ . Clearly,  $R_p \leq C \cap R$  and hence  $R_p$  is a Sylow *p*-subgroup of  $C \cap R$ . This implies that  $C \cap R = R_p \times L$  where L is a *p*'-subgroup of  $C \cap R$ , and in particular, L is characteristic in  $C \cap R$  and so normal in B. Thus,  $C = (R_p \times L)G = R_p \times LG$ , and therefore,  $LG \leq C$ . Note that C is arc-transitive because  $G \leq C$  and  $R_p \leq C$ . If  $L \neq 1$  then  $(LG)_v \neq 1$  and then  $LG \leq C$  implies that LG is arc-transitive. This means that  $p \mid |(LG)_v|$ , and since  $LG = G(LG)_v$ , we have  $|(LG)_v| = |L|$  and  $p \mid |L|$ , which is impossible. It follows that L = 1 and  $C = R_p \times G$ . Furthermore,  $G \leq B$  and so  $B = R \times G$ .

**Proof of Theorem 1.3:** Let G,  $\Gamma$ , p and v as given in Assumption and further let G be regular on  $V(\Gamma)$ . Write  $A = \operatorname{Aut}(\Gamma)$ ,  $R = \operatorname{rad}(A)$  and B = RG. Then  $R \cap G = 1$  and  $B/R \cong G$ . By the Frattini argument,  $B = GR = GB_v$ , and so  $|R| = |B_v|$ .

Assume R = 1. By Lemma 3.2, either  $\operatorname{soc}(A) = G$ , or  $\Gamma$  is  $\operatorname{soc}(A)$ -arc-transitive and  $G < \operatorname{soc}(A)$  with  $(G, \operatorname{soc}(A)) = (A_{n-1}, A_n), (M_{22}, M_{23}), (A_5, \mathsf{PSL}(2, 11))$  or  $(A_5, \mathsf{PSL}(2, 29))$ .

Assume  $R \neq 1$ . By Lemma 3.3, R has at least three orbits, and by Proposition 2.8, the quotient graph  $\Gamma_R$  has valency p with A/R-arc-transitive and B/R-vertex-transitive. Moreover,  $(A/R)_{\Delta} \cong A_v$  is solvable for any  $\Delta \in V(\Gamma_R)$ . Write  $I/R = \operatorname{soc}(A/R)$ . Since  $B/R \cong G$ , Lemma 3.2 implies that either  $B/R = I/R \trianglelefteq A/R$ , or  $\Gamma_R$  is I/R-arctransitive with B/R < I/R and  $(B/R, I/R) = (\mathsf{A}_{n-1}, \mathsf{A}_n)$  with  $(I/R)_\Delta$  being transitive on  $\{1, 2, \dots, n\}$ , or  $(B/R, I/R, (I/R)_\Delta) = (\mathsf{M}_{22}, \mathsf{M}_{23}, \mathbb{Z}_{23}), (\mathsf{A}_5, \mathsf{PSL}(2, 11), \mathbb{Z}_{11})$  or  $(\mathsf{A}_5, \mathsf{PSL}(2, 29), \mathbb{Z}_{29} : \mathbb{Z}_7).$ 

Case 1:  $B/R = I/R \leq A/R$ .

In this case,  $B = GR \leq A$ , and by Lemma 3.3,  $B = G \times R$ . It follows that G is characteristic in B, and hence  $G \leq A$ .

**Case 2:**  $\Gamma_R$  is I/R-arc-transitive with B/R < I/R and  $(B/R, I/R) = (A_{n-1}, A_n)$ with  $(I/R)_{\Delta}$  being transitive on  $\{1, 2, \dots, n\}$ , or  $(B/R, I/R, (I/R)_{\Delta}) = (M_{22}, M_{23}, \mathbb{Z}_{23}),$  $(A_5, \mathsf{PSL}(2, 11), \mathbb{Z}_{11})$  or  $(A_5, \mathsf{PSL}(2, 29), \mathbb{Z}_{29} : \mathbb{Z}_7).$ 

Let  $(B/R, I/R, (I/R)_{\Delta}) = (\mathsf{M}_{22}, \mathsf{M}_{23}, \mathbb{Z}_{23}), (\mathsf{A}_5, \mathsf{PSL}(2, 11), \mathbb{Z}_{11})$  or  $(\mathsf{A}_5, \mathsf{PSL}(2, 29), \mathbb{Z}_{29} : \mathbb{Z}_7)$ . By Lemma 3.2,  $\Gamma$  is a Cayley graph on  $GB/R \cong G$ . Since  $\Gamma$  is a Cayley graph on G, we have that  $|V(\Gamma)| = |V(\Gamma_R)|$ , which contradicts the assumption  $R \neq 1$ . Thus  $(B/R, I/R) = (\mathsf{A}_{n-1}, \mathsf{A}_n)$  with  $(I/R)_{\Delta}$  being transitive on  $\{1, 2, \cdots, n\}$ .

First we claim  $B = R \times G$ . Suppose to the contrary that  $B \neq R \times G$ . Since R is solvable, there exists a series of normal subgroups of B:  $R_0 = 1 < R_1 < \cdots < R_s = B$  such that  $R_i \triangleleft B$  and  $R_{i+1}/R_i$  is an elementary abelian group for each  $0 \le i \le s - 1$ . Since  $RG \ne R \times G$ , there exists  $0 \le j \le s - 1$  such that  $GR_i = G \times R_i$  for any  $0 \le i \le j$ , but  $GR_{j+1} \ne G \times R_{j+1}$ .

Write  $R_{j+1}/R_j = \mathbb{Z}_r^f$  for some prime r and positive integer f. Note that  $G \cap R_i = 1$ for  $0 \leq i \leq s$  and so  $R_{i+1}G/R_i \cong G$  for  $0 \leq i \leq s-1$ . In particular, the conjugate action of  $R_{j+1}G/R_j$  on  $R_{j+1}/R_j$  is trivial or faithful. If it is trivial, then  $R_{j+1}G/R_j =$  $(R_{j+1}/R_j)(R_jG/R_j) = R_{j+1}/R_j \times R_jG/R_j$ , implying  $R_jG \triangleleft R_{j+1}G$ , and since  $GR_j =$  $G \times R_j$ , we have  $G \trianglelefteq R_{j+1}G$  and  $GR_{j+1} = G \times R_{j+1}$ , a contradiction. It follows that the conjugate action of  $R_{j+1}G/R_j$  on  $R_{j+1}/R_j$  is faithful, and we may assume  $G \leq \mathsf{GL}(f, r)$ .

Recall that  $|B_v| = |R|$  and  $R_{j+1}/R_j = \mathbb{Z}_r^f$ . Then  $r^f ||B_v|$ , and since  $\Gamma_R$  is I/Rarc-transitive,  $\Gamma$  is *I*-arc-transitive and Proposition 2.8 implies  $I_v \cong (I/R)_{\Delta}$ . Since B/R < I/R, we have  $|B_v| ||I_v|$  and so  $r^f ||(I/R)_{\Delta}|$ . If r = p then Proposition 2.7 implies  $r^2 \nmid |(I/R)_{\Delta}|$  and hence  $G \leq \mathsf{GL}(1, p)$ , a contradiction. It follows  $r \neq p$ , and again by Proposition 2.7,  $r^f | (p-1)^2$ .

Now  $B/R = A_{n-1} \leq \mathsf{GL}(f, r)$ . By assumption,  $p \geq 11$ . Since  $(I/R)_{\Delta}$  contains a normal subgroup  $\mathbb{Z}_p$ , we have  $p \mid n$  and so  $n-1 \geq 11-1 = 10$ . By Proposition 2.5,  $f \geq (n-1)-2 \geq p-3$  and so  $(p-1)^2 \geq r^f \geq 2^{p-3}$ . This is impossible because the function  $f(x) = 2^{x-3} - (x-1)^2 > 0$  always holds for  $x \geq 11$ . This completes the proof of the claim, and hence  $B = R \times G$ .

Set  $C = C_I(R)$ . Then  $G \leq C$ ,  $C \leq I$  and  $C \cap R \leq Z(C)$ . Recall that  $I/R = A_n$  or  $M_{23}$ . Since  $G \cong (R \times G)/R \leq CR/R \leq I/R$ , we have I = CR, and since  $Z(C)/(C \cap R) \leq C/C \cap R \cong CR/R = I/R$ , we have  $C \cap R = Z(C)$  and  $C/Z(C) \cong I/R$ . Furthermore,  $C'/(C' \cap Z(C)) \cong C'Z(C)/Z(C) = (C/Z(C))' = C/Z(C) \cong I/R$ , and so  $Z(C') = C' \cap Z(C)$ , C = C'Z(C) and  $C'/Z(C') \cong I/R$ . It follows C' = (C'Z(C))' = C'', and hence C' is a covering group of I/R.

Suppose  $Z(C') \neq 1$ . Then Proposition 2.4 implies that  $Z(C') = \mathbb{Z}_2$  and  $C' \cong 2.A_n$ . Since  $G \leq C$  and C/C' is abelian, we have  $G \leq C'$ . So  $G \times Z(C') \cong A_{n-1} \times \mathbb{Z}_2$  is a subgroup of  $C' \cong 2.A_n$ , which is impossible by Proposition 2.4. Thus, Z(C') = 1. It follows  $C' \cong I/R$ . Since G < C and C/C' is abelian, we have  $G < C' \trianglelefteq I$ , and since |I| = |I/R||R| = |C'||R| and  $C' \cap R = 1$ , we have  $I = C' \times R$ . Since C' is a nonabelian simple group, C' is characteristic in I, and hence  $C' \trianglelefteq A$  because  $I \trianglelefteq A$ . Since G is regular on  $\Gamma$  and  $G < C' \trianglelefteq I$ , C' has non-trivial stabilizer, and hence  $\Gamma$  is C'-arc-transitive on  $\Gamma$ . Note that  $C' \cong I/R = A_n$ .

Summing up, we have proved that either  $G \leq A$ , or A has a normal arc-transitive subgroup T such that G < T and  $(G, T) = (\mathsf{A}_5, \mathsf{PSL}(2, 11)), (\mathsf{A}_5, \mathsf{PSL}(2, 29)), (\mathsf{M}_{22}, \mathsf{M}_{23})$  or  $(\mathsf{A}_{n-1}, \mathsf{A}_n)$  (for  $R = 1, T = \mathsf{soc}(A)$ , and for  $R \neq 1, T = C'$ ). Let  $(G, T) = (\mathsf{A}_{n-1}, \mathsf{A}_n)$ . Since G is regular,  $|T_v| = n$ , and by Proposition 2.7,  $n = pk\ell$  with  $k \mid \ell \mid (p-1)$ . To finish the proof, we are left to show that k and  $\ell$  have the same parity.

Suppose to the contrary that k and  $\ell$  has different parity. Then k is odd and  $\ell$  is even as  $k \mid \ell$ . Since  $(G, T) = (A_{n-1}, A_n)$ , we have |T : G| = n and T can be viewed as the alternating permutation group by the well-known right multiplication action of Ton the set [T : G] of all right cosets of G in T, still denoted by  $A_n$ . By the regularity of G on  $\Gamma$ ,  $T = GT_v$  and  $G \cap T_v = 1$ , which implies that  $T_v \leq A_n$  is a regular permutation group on [T : G]. By Proposition 2.7,  $T_v = \mathbb{Z}_k \times (\mathbb{Z}_p : \mathbb{Z}_\ell)$ , and so  $T_v$  has a cyclic group  $\mathbb{Z}_\ell$  with odd index  $|T_v : \mathbb{Z}_\ell| = pk$ . Let  $\mathbb{Z}_\ell = \langle a \rangle$ . Since  $T_v$  is regular, a is a product of  $pk \ell$ -cycles on [T : G] in its distinct cycle decomposition, so an odd permutation as  $\ell$  is even and kp is odd, which is impossible because  $T_v \leq A_n$ . This completes the proof.  $\Box$ 

#### 4. Proof of Theorem 1.4

The goal of this section is to prove Theorem 1.4. To do that, we first describe a widely known construction for vertex-transitive and symmetric graphs, part of which is attributed to Sabidussi [19].

Let G be a group, H a subgroup of G, and D a union of some double cosets of H in G such that  $H \not\subseteq D$  and  $D^{-1} = D$ . Then the coset graph  $\Gamma = \mathsf{Cos}(G, H, D)$  is defined as the graph with vertex-set [G:H], the set of all right cosets of H in G, and edge-set  $E(\Gamma) = \{\{Hg, Hxg\} : g \in G, x \in D\}$ . This graph is regular with valency |D|/|H|, and is connected if and only if  $G = \langle D, H \rangle$ , that is, if and only if G is generated by D and H. The group G acts vertex-transitively on  $\Gamma$  by right multiplication. More precisely, for  $g \in G$ , the permutation  $\hat{g}_H : Hx \mapsto Hxg, x \in G$ , on [G:H] is an automorphism of  $\mathsf{Aut}(\Gamma)$ , and  $\hat{G}_H := \{\hat{g}_H \mid g \in G\}$  is a transitive subgroup of  $\mathsf{Aut}(\Gamma)$ . The map  $g \mapsto \hat{g}_H$ ,  $g \in G$ , is a homomorphism from G to  $S_{[G:H]}$ , the well-known coset action of G on H, and the kernel of this coset action is  $H_G = \bigcap_{g \in G} H^g$ , the largest normal subgroup of G contained in H. It follows that  $G/H_G \cong \hat{G}_H$ . Furthermore,  $\Gamma$  is  $\hat{G}_H$ -arc-transitive if and only if D consists of just one double coset HaH. If  $H_G = 1$ , we say that H is core-free in G, and in this case,  $G \cong \hat{G}_H$ .

If H = 1, denote Cos(G, H, D) and  $\hat{G}_H$  by Cay(G, D) and  $\hat{G}$ , respectively. In this case,  $\hat{G}$  is the right regular representation of G, and it is regular on the vertex set of Cay(G, D). By definition, Cay(G, D) is Cayley graph of  $\hat{G}$ , and for short, Cay(G, D) is also called a Cayley graph of G with respect to D.

Conversely, suppose  $\Gamma$  is any graph on which the group G acts faithfully and vertextransitively. Then it is easy to show that  $\Gamma$  is isomorphic to the coset graph  $\mathsf{Cos}(G, H, D)$ , where  $H = G_v$  is the stabiliser in G of the vertex  $v \in V(\Gamma)$ , and D is a union of double cosets of H, consisting of all elements of G taking v to one of its neighbours. Then  $H \not\subseteq D$  and  $D^{-1} = D$ . Moreover, if G is arc-transitive on  $\Gamma$  and g is an element of G that swaps v with one of its neighbours, then  $g^2 \in H$  and D = HgH, and the valency of  $\Gamma$  is  $|D|/|H| = |H : H \cap H^g|$ . Also a can be chosen as a 2-element in G. In particular, if  $L \leq G$  is regular on vertex set of  $\Gamma$ , then  $\Gamma$  is also isomorphic to Cay(L, S), where S consists of all elements of L taking v to one of its neighbours with  $S^{-1} = S$ , and by the regularity, we have  $S = D \cap L$ . Thus, we have the following proposition.

**Proposition 4.1.** Let  $\Gamma$  be a *G*-vertex-transitive graph and *L* be a regular subgroup of *G*. Then  $\Gamma \cong Cos(G, H, D) \cong Cay(L, S)$  with  $S = L \cap D$ , where  $H = G_v$  for  $v \in V(\Gamma)$ , *D* is a union of double cosets of *H*, consisting of all elements of *G* taking *v* to one of its neighbours, and *S* consists of all elements of *L* taking *v* to one of its neighbours. Moreover,  $\Gamma$  be *G*-arc-transitive if and only if *G* has a 2-element *g* such that D = HgH, and in this case,  $\Gamma$  has valency  $|H: H \cap H^g|$ .

Let  $\Gamma = \mathsf{Cos}(G, H, D)$  be a coset graph. We set  $\mathsf{Aut}(G, H, D) = \{\alpha \in \mathsf{Aut}(G) \mid H^{\alpha} = H, D^{\alpha} = D\}$ . For any  $\alpha \in \mathsf{Aut}(G, H, D)$ , the permutation  $\alpha_H : Hx \mapsto Hx^{\alpha}, x \in G$ , on [G : H] is an automorphism of  $\Gamma$ , and the map  $\alpha \mapsto \alpha_H$  is a natural action of  $\mathsf{Aut}(G, H, D)$  on  $V(\Gamma)$ . It follows that  $\mathsf{Aut}(G, H, D)/K \cong \mathsf{Aut}(G, H, D)_H$ , where  $\mathsf{Aut}(G, H, D)_H = \{\alpha_H \mid \alpha \in \mathsf{Aut}(G, H, D)\}$  and K is the kernel of the action. Furthermore,  $\mathsf{Aut}(G, H, D)_H \leq \mathsf{Aut}(\Gamma)$ . For  $h \in H$ , let  $\tilde{h}$  be the inner automorphism of Ginduced by h, that is,  $\tilde{h} : g \mapsto h^{-1}gh, g \in G$ . Then  $\tilde{H} := \{\tilde{h} \mid h \in H\} \leq \mathsf{Aut}(G, H, D)_H$ .

The following proposition was proved by Wang, Feng and Zhou [22, Lemma 2.10], which is important for computing automorphism groups of coset graphs.

**Proposition 4.2.** Let G be a finite group, H a core-free subgroup of G and D a union of several double-cosets HgH such that  $H \nsubseteq D$  and  $D = D^{-1}$ . Let  $\Gamma = \mathsf{Cos}(G, H, D)$ and  $A = \mathsf{Aut}(\Gamma)$ . Then  $\hat{G}_H \cong G$ ,  $\mathsf{Aut}(G, H, D)_H \cong \mathsf{Aut}(G, H, D)$ ,  $\tilde{H}_H \cong \tilde{H}$ , and  $\mathbf{N}_A(\hat{G}_H) = \hat{G}_H \mathsf{Aut}(G, H, D)_H$  with  $\hat{G}_H \cap \mathsf{Aut}(G, H, D)_H = \tilde{H}_H$ .

Now we are ready to prove Theorem 1.4 and this follows from Lemmas 4.3-4.6.

Let x, y, t be permutations in  $S_{11}$  as following:

$$x = (1, 11, 8, 3, 6, 9, 4, 10, 2, 7, 5)$$
  

$$y = (2, 10, 6)(3, 11, 4)(7, 8, 9)$$
  

$$t = (2, 5)(3, 9)(6, 11)(8, 10)$$

Let  $T = \langle x, t \rangle$ ,  $H = \langle x \rangle$ ,  $G = \langle y, t \rangle$ . Define

$$\Gamma = \mathsf{Cos}(T, H, HtH).$$

Then a computation with MAGMA [2] shows that  $T \cong \mathsf{PSL}(2, 11), H \cong \mathbb{Z}_{11}, |H \cap H^t| = 1$ , and  $G \cong \mathsf{A}_5$ . By Proposition 4.1,  $\Gamma$  has valency 11 and T acts arc-transitively on  $\Gamma$ . Since 11  $\not| |G|, G$  acts semiregularly on  $V(\Gamma)$ , and since  $|G| = |V(\Gamma)|, G$  is regular on  $V(\Gamma)$ . It follows that  $\Gamma$  is a non-normal Cayley group of  $\mathsf{A}_5$  with  $\mathsf{PSL}(2, 11)$ -arc-transitive. A direct computation with MAGMA shows that  $\mathsf{Aut}(\Gamma) \cong \mathsf{PGL}(2, 11)$  and this implies the following lemma.

**Lemma 4.3.** There exists an 11-valent symmetric Cayley graph  $\Gamma$  of  $A_5$  such that  $\operatorname{Aut}(\Gamma) \cong \operatorname{PGL}(2,11)$ . In particular,  $\operatorname{Aut}(\Gamma)_v$  is solvable for  $v \in V(\Gamma)$ .

Let x, y, t, z be permutations in  $S_{30}$  as following:

- x = (1, 21, 10, 9, 22, 28, 13, 15, 30, 6, 19, 18, 7, 27, 23, 4, 25, 17, 20, 2, 12, 29, 16, 26, 8, 11, 3, 24, 5)
- y = (1, 24, 9)(2, 6, 5)(3, 27, 21)(4, 12, 20)(7, 25, 26)(8, 10, 13)(11, 14, 16)(15, 30, 23)(17, 28, 29)(18, 22, 19)
- t = (1,3)(2,10)(4,11)(5,19)(6,24)(7,16)(8,17)(9,28)(12,27)(13,20)(14,22)(15,26)(18,30)(21,23)
- z = (2, 18, 23, 10, 29, 9, 17)(3, 7, 19, 20, 4, 24, 30)(5, 22, 27, 13, 28, 6, 16)(8, 12, 15, 21, 11, 25, 26)

Let  $T = \langle x, t \rangle$ ,  $H = \langle x, z \rangle$ ,  $G = \langle y, t \rangle$ . Define

$$\Gamma = \mathsf{Cos}(T, H, HtH).$$

Then a computation with MAGMA [2] shows that  $T \cong \mathsf{PSL}(2,29)$ ,  $H \cong \mathbb{Z}_{29} : \mathbb{Z}_7$ ,  $|H \cap H^t| = 7$ , and  $G \cong \mathsf{A}_5$ . By Proposition 4.1,  $\Gamma$  has valency 29 and T acts arctransitively on  $\Gamma$ . Since  $29 \not| |G|$ , G acts semiregularly on  $V(\Gamma)$ , and since  $|G| = |V(\Gamma)|$ , G is regular on  $V(\Gamma)$ . It follows that  $\Gamma$  is a non-normal Cayley group of  $\mathsf{A}_5$  with  $\mathsf{PSL}(2,29)$ -arc-transitive. A direct computation with MAGMA shows that  $\mathsf{Aut}(\Gamma) \cong$  $\mathsf{PGL}(2,29)$  and this implies the following lemma.

**Lemma 4.4.** There exists a 29-valent symmetric Cayley graph  $\Gamma$  of  $A_5$  such that  $\operatorname{Aut}(\Gamma) \cong \operatorname{PGL}(2,29)$ . In particular,  $\operatorname{Aut}(\Gamma)_v$  is solvable for  $v \in V(\Gamma)$ .

Let x, y, t be permutations in  $S_{23}$  as following:

x = (1, 4, 6, 7, 2, 19, 3, 11, 9, 20, 13, 23, 16, 8, 21, 5, 14, 22, 18, 15, 17, 10, 12)

y = (1, 14, 6, 5, 9, 2, 10, 3, 15, 13, 11)(4, 22, 16, 19, 17, 8, 21, 7, 12, 18, 23)

t = (1,17)(3,9)(5,18)(6,13)(7,12)(10,19)(14,22)(21,23)

Let  $T = \langle x, t \rangle$ ,  $H = \langle x \rangle$ ,  $G = \langle y, t \rangle$ . Define

$$\Gamma = \mathsf{Cos}(T, H, HtH).$$

**Lemma 4.5.** The above graph  $\Gamma$  is 23-valent symmetric Cayley graph of  $M_{22}$  and  $\operatorname{Aut}(\Gamma) = (\widehat{M}_{23})_H \cong M_{23}$ . In particular,  $\operatorname{Aut}(\Gamma)_v$  is solvable for  $v \in V(\Gamma)$ .

Proof. A computation with MAGMA [2] shows that  $T \cong M_{23}$ ,  $H \cong \mathbb{Z}_{23}$ ,  $|H \cap H^t| = 1$ , and  $G \cong M_{22}$ . By Proposition 4.1,  $\Gamma$  has valency 23 and T acts arc-transitively on  $\Gamma$ . Since 23  $/\!\!/|G|$ , G acts semiregularly on  $V(\Gamma)$ , and since  $|G| = |V(\Gamma)|$ , G is regular on  $V(\Gamma)$ . It follows that  $\Gamma$  is a non-normal Cayley group of  $M_{22}$  with  $M_{23}$ -arc-transitive. However, we cannot compute  $\operatorname{Aut}(\Gamma)$  with MAGMA because  $|V(\Gamma)|$  is too large. By Proposition 4.1, we may let  $\Gamma = \operatorname{Cay}(G, S)$  with  $S = G \cap HtH$ . Write  $A = \operatorname{Aut}(\Gamma)$ . By MAGMA,  $S = \{s_i \mid 1 \leq i \leq 23\}$ , where

 $s_1 = (1, 14, 6, 5, 9, 2, 10, 3, 15, 13, 11)(4, 22, 16, 19, 17, 8, 21, 7, 12, 18, 23),$  $s_2 = (1, 11, 13, 15, 3, 10, 2, 9, 5, 6, 14)(4, 23, 18, 12, 7, 21, 8, 17, 19, 16, 22),$  $s_3 = (1, 15, 5, 2, 12, 18, 16, 14, 21, 13, 7)(3, 6, 4, 22, 8, 19, 10, 17, 9, 23, 11),$  $s_4 = (1, 7, 13, 21, 14, 16, 18, 12, 2, 5, 15)(3, 11, 23, 9, 17, 10, 19, 8, 22, 4, 6),$  $s_5 = (1, 9, 14)(2, 19, 5, 4, 22, 12)(3, 21, 6)(7, 23, 15, 11, 8, 18)(10, 13)(16, 17),$  $s_6 = (1, 14, 9)(2, 12, 22, 4, 5, 19)(3, 6, 21)(7, 18, 8, 11, 15, 23)(10, 13)(16, 17),$   $s_7 = (1, 4, 3)(2, 6)(5, 8, 7, 10, 14, 21)(9, 12, 17, 22, 16, 13)(11, 19, 23)(15, 18),$  $s_8 = (1, 3, 4)(2, 6)(5, 21, 14, 10, 7, 8)(9, 13, 16, 22, 17, 12)(11, 23, 19)(15, 18),$  $s_9 = (1, 12)(2, 19, 3)(4, 6, 18, 5, 8, 10)(7, 11, 23, 16, 14, 22)(9, 13)(15, 17, 21),$  $s_{10} = (1, 12)(2, 3, 19)(4, 10, 8, 5, 18, 6)(7, 22, 14, 16, 23, 11)(9, 13)(15, 21, 17),$  $s_{11} = (1, 7, 3, 16, 12)(2, 11, 23, 22, 14)(4, 15, 5, 18, 10)(6, 9, 13, 8, 17),$  $s_{12} = (1, 12, 16, 3, 7)(2, 14, 22, 23, 11)(4, 10, 18, 5, 15)(6, 17, 8, 13, 9),$  $s_{13} = (3, 16, 23, 12, 6)(4, 11, 22, 18, 10)(5, 17, 7, 19, 9)(8, 14, 15, 21, 13),$  $s_{14} = (3, 6, 12, 23, 16)(4, 10, 18, 22, 11)(5, 9, 19, 7, 17)(8, 13, 21, 15, 14),$  $s_{15} = (1, 15, 12, 6, 19)(2, 11, 13, 14, 7)(3, 16, 21, 22, 4)(5, 10, 17, 9, 23),$  $s_{16} = (1, 19, 6, 12, 15)(2, 7, 14, 13, 11)(3, 4, 22, 21, 16)(5, 23, 9, 17, 10)$  $s_{17} = (1,7)(3,8)(4,6)(9,19)(11,23)(12,15)(13,18)(14,21),$  $s_{18} = (2,6)(3,10)(4,22)(8,16)(11,13)(12,18)(14,15)(21,23),$  $s_{19} = (1, 11)(2, 16)(4, 19)(6, 12)(8, 14)(9, 13)(15, 18)(17, 22),$  $s_{20} = (1, 17)(3, 9)(5, 18)(6, 13)(7, 12)(10, 19)(14, 22)(21, 23),$  $s_{21} = (1, 15)(5, 16)(6, 18)(7, 19)(8, 21)(9, 23)(11, 12)(17, 22),$  $s_{22} = (1, 17)(2, 9)(5, 11)(6, 19)(7, 13)(8, 23)(10, 12)(14, 15),$  $s_{23} = (1,5)(2,4)(3,11)(8,13)(9,19)(10,15)(14,16)(18,23).$ 

Let 1 be the identity in G. Then  $1 \in V(\Gamma)$ . Suppose to the contrary that  $A_1$  is nonsolvable. By Proposition 2.6, the restriction  $A_1^{\Gamma(1)}$  of  $A_1$  on the neighbourhood  $\Gamma(1)$ of 1 in  $\Gamma$  is nonsolvable, and since  $\Gamma$  has prime valency, the Burnside Theorem (also see [4, Theorem 3.5B]) implies that  $A_1^{\Gamma(1)}$  is 2-transitive on  $\Gamma(1)$ . This turns that there exists a 5-cycle passing though 1 and any two vertices in S because  $(1, s_{11}, s_{11}^2, s_{11}^3, s_{11}^4)$ is a 5-cycle in  $\Gamma$ . In particular, there is a 5-cycle passing through 1,  $s_1$  and  $s_2 = s_1^{-1}$ , and hence  $s_1^2 \in S^3 = \{s_{i_1}s_{i_2}s_{i_2} \mid s_{i_1}, s_{i_2}, s_{i_2} \in S\}$ , but this is not true by MAGMA [2]. Thus,  $A_1$  is solvable.

Now we let  $\Gamma = \mathsf{Cos}(T, H, HtH)$  and D = HtH. Since A has solvable stabilizer, Theorem 1.3 implies that  $\hat{T} = \hat{\mathsf{M}}_{23} \leq A$ . Note that H is core-free in T. By Proposition 4.2,  $A = \hat{T}_H \mathsf{Aut}(T, H, D)_H$  with  $\hat{T}_H \cap \mathsf{Aut}(T, H, D)_H = \tilde{H}_H$ , where  $\hat{T}_H \cong T$ ,  $\mathsf{Aut}(T, H, D)_H \cong$  $\mathsf{Aut}(T, H, D)$  and  $\tilde{H}_H \cong \tilde{H}$ . To prove  $A = \hat{T}_H$ , it suffices to show that  $\mathsf{Aut}(T, H, D) = \tilde{H}$ .

Suppose to the contrary that  $\alpha \in \operatorname{Aut}(T, H, D)$ , but  $\alpha \notin \tilde{H}$ . By [13, Table 5.1.C], Out(M<sub>23</sub>) = 1, that is, Aut(M<sub>23</sub>) = Inn(M<sub>23</sub>). Thus,  $\alpha$  is an automorphism of T induced by an element of  $b \in T$  by conjugation, namely  $g^{\alpha} = g^{b}$  for  $g \in T$ . Since  $\alpha \in \operatorname{Aut}(T, H, D)$ , we have  $H^{b} = H$  and  $D^{b} = D$ , and since  $\alpha \notin \tilde{H}$ , we have  $b \notin H$ . It follows that  $H\langle b \rangle$  is a subgroup of T containing H, and by Atlas [3],  $H\langle b \rangle \cong \mathbb{Z}_{23} : \mathbb{Z}_{11}$ . Since  $\tilde{H} \leq \operatorname{Aut}(T, H, D)$ , we may choose b such that b has order 11, and by MAGMA, we may let b = (2, 14, 18, 7, 16, 6, 9, 20, 8, 3, 4)(5, 21, 13, 22, 12, 15, 11, 19, 17, 23, 10) because  $H = \langle x \rangle$  with x = (1, 4, 6, 7, 2, 19, 3, 11, 9, 20, 13, 23, 16, 8, 21, 5, 14, 22, 18, 15, 17, 10, 12). However,  $D^{b} = (HtH)^{b} \neq HtH$  by MAGMA, a contradiction. Thus,  $A = \hat{T}_{H} \cong M_{23}$ .  $\Box$ 

Let  $p \ge 5$  be a prime, and let x, t and h be permutations in  $S_p$  as following:

$$x = (1, 2, \dots, p), \quad t = (1, 2)(3, 4), \quad h = (2, p)(3, p - 1) \cdots (\frac{p - 1}{2}, \frac{p + 5}{2})(\frac{p + 1}{2}, \frac{p + 3}{2})$$
  
Let  $T = \langle x, t \rangle$  and  $H = \langle x \rangle$ . By [8],  $T = \mathsf{A}_p, H \cong \mathbb{Z}_p$  and  $|H \cap H^t| = 1$ . Define  
 $\Gamma^p = \mathsf{Cos}(\mathsf{A}_p, H, HtH).$ 

**Lemma 4.6.** The above graph  $\Gamma^p$  is a p-valent symmetric Cayley graph of  $A_{p-1}$  such that  $\operatorname{Aut}(\Gamma^p) \cong S_p$  for  $p \equiv 3 \pmod{4}$  and  $\operatorname{Aut}(\Gamma^p) \cong A_p \times \mathbb{Z}_2$  for  $p \equiv 1 \pmod{4}$ . In particular,  $\operatorname{Aut}(\Gamma)_v$  is solvable for  $v \in V(\Gamma)$ .

*Proof.* By Proposition 4.1,  $\Gamma^p$  has valency p and  $A_p$  acts arc-transitively on  $\Gamma^p$ , with vertex stabilizer isomorphic to  $\mathbb{Z}_p$ . Let  $A_{p-1}$  be the subgroup of  $A_p$  fixing the point p. Since  $p \not| |A_{p-1}|$ ,  $A_{p-1}$  acts semiregularly on  $V(\Gamma^p)$ , and since  $|A_{p-1}| = |V(\Gamma^p)|$ ,  $A_{p-1}$  is regular on  $V(\Gamma^p)$ . It follows that  $\Gamma^p$  is a non-normal Cayley group of  $A_{p-1}$  with  $A_p$ -arc-transitive.

By Proposition 4.1, we may let  $\Gamma^p = \mathsf{Cay}(\mathsf{A}_{p-1}, S)$ , where  $S = \mathsf{A}_{p-1} \cap HtH$ . For p = 5 or p = 7, a computing with MAGMA shows that  $\mathsf{Aut}(\Gamma^5) \cong \mathsf{A}_5 \times \mathbb{Z}_2$  and  $\mathsf{Aut}(\Gamma^7) \cong \mathsf{S}_7$ . Write  $A = \mathsf{Aut}(\Gamma)$ . We may assume  $p \geq 11$ .

#### Claim: A has solvable stabilizer.

Recall that  $x = (1, 2, \dots, p), t = (1, 2)(3, 4)$  and  $H = \langle x \rangle$ . Let  $x^{-i}tx^j \in S = HtH \cap A_{p-1}$  for  $i, j \in \mathbb{Z}_p$ . Then  $p = p^{x^{-i}tx^j} = p^{x^{-i}tx^{i}x^{j-i}}$ . Note that  $x^{-i}tx^i = (1^{x^i}, 2^{x^i})(3^{x^i}, 4^{x^i})$ , and if  $j - i \neq 0$  then  $x^{j-i}$  is a *p*-cycle. For  $0 \le i \le p - 5$ ,  $p = p^{x^{-i}tx^{i}x^{j-i}} = p^{x^{j-i}}$  implies j = i. Furthermore, For i = p - 4, p - 3, p - 2 or  $p - 1, p = p^{x^{-i}tx^{i}x^{j-i}}$  implies that j = i + 1, i - 1, i + 1 or i - 1, respectively. Thus, we may set  $S = \{s_1, s_2, \dots, s_p\}$ , where  $s_{i+1} = x^{-i}tx^i = (1 + i, 2 + i)(3 + i, 4 + i)$  for  $0 \le i \le p - 5$ ,  $s_{p-2} = x^{-(p-3)}tx^{p-4} = (1, p - 1, p - 3, \dots, 3, 2), \quad s_{p-3} = x^{-(p-4)}tx^{p-3} = (s_{p-2})^{-1}, s_p = x^{-(p-1)}tx^{p-2} = (1, p - 1, p - 2, \dots, 4, 3), \quad s_{p-1} = x^{-(p-2)}tx^{p-1} = s_p^{-1}$ .

For  $z \in A_p$ , denote by o(z) the order of z and by supp(z) the support of z, that is, the number of points moving by z. Then  $o(s_i) = 2$  and  $supp(s_i) = 4$  for  $1 \le i \le p - 4$ , and  $o(s_i) = supp(s_i) = p - 2$  for  $p - 3 \le i \le p$ .

To prove the Claim, it suffices to show that  $A_1$  is solvable. We argue by contradiction and we suppose that  $A_1$  is nonsolvable. Note that  $\Gamma^p = \mathsf{Cay}(\mathsf{A}_{p-1}, S)$  and  $\Gamma^p(1) = S$ .

By Propostion 2.6,  $A_1^{\Gamma^p(1)}$  is nonsolvable, and the Burnside Theorem implies that  $A_1$  is 2-transitive on  $\Gamma^p(1)$ . Note that  $p \ge 11$ . Since  $s_1 = (1,2)(3,4)$  commutes with  $s_5 = (5,6)(7,8)$ , there is a 4-cycle passing through  $1, s_1$  and  $s_5$ . By the 2-transitivity of  $A_1$  on  $\Gamma^p(1)$ , there exists a 4-cycle through  $1, s_p$  and  $s_{p-1} = s_p^{-1}$ , and this implies  $|Ss_p \cap Ss_p^{-1}| \ge 2$ . Thus,  $|Ss_p^{-2} \cap S| \ge 2$ .

Let  $S_1 = \{s_i \mid 1 \leq i \leq p-4\}$  and  $S_2 = \{s_{p-2}, s_{p-2}^{-1}, s_p, s_p^{-1}\}$ . Then  $S = S_1 \cup S_2$ and  $S_1 \cap S_2 = \emptyset$ . Since  $s_p^{-1}$  is a (p-2)-cycle in  $A_p$  and p-2 is odd,  $s_p^{-2}$  is also a (p-2)-cycle, implying  $supp(s_p^{-2}) = p-2$ . Since  $supp(s_i) = 4$  for each  $1 \leq i \leq p-4$ , we have  $supp(s_i s_p^{-2}) \geq p-6 \geq 5$ , and  $s_i s_p^{-2}$  cannot be any involution in S. Thus,  $|S_1 s_p^{-2} \cap S_1| = 0$ .

Note that  $S_2 s_p^{-2} = \{s_p^{-1}, s_p^{-3}, s_{p-2} s_p^{-2}, s_{p-2}^{-1} s_p^{-2}\}$ . Then  $|S_2 s_p^{-2} \cap S_2| = 1$  by a simple checking one by one. If  $|S_2 s_p^{-2} \cap S_1| \neq 0$ , then  $z^2 = 1$  for some  $z \in S_2 s_p^{-2}$ , and we have  $s_p^{-2} = 1$  or  $s_p^{-6} = 1$  or  $(s_{p-2} s_p^{-2})^2 = 1$  or  $(s_{p-2}^{-1} s_p^{-2})^2 = 1$ , of which all are impossible because all these elements cannot fix 1. Thus,  $|S_2 s_p^{-2} \cap S_1| = 0$ . Similarly,  $|S_2 s_p^2 \cap S_1| = 0$ .

Recall that  $|Ss_p^{-2} \cap S| \ge 2$ . Since  $|S_2s_p^{-2} \cap S_2| = 1$  and  $|S_2s_p^{-2} \cap S_1| = 0$ , we have  $|S_1s_p^{-2} \cap S| = 1$ , and since  $|S_1s_p^{-2} \cap S_1| = 0$ , we have  $|S_1s_p^{-2} \cap S_2| = 1$ . It follows  $|S_2s_p^2 \cap S_1| = 1$ , a contradiction. Thus,  $A_1$  is solvable, as claimed.

From now on, we write  $\Gamma^p = \mathsf{Cos}(T, H, HtH)$ . Clearly, H is core-free in T. By Claim,  $A = \mathsf{Aut}(\Gamma^p)$  has solvable stabilizer. By Theorem 1.3,  $\hat{T}_H$  is normal in A, and by Proposition 4.2,  $A = N_A(\hat{T}_H) = \hat{T}_H \mathsf{Aut}(T, H, HtH)_H$  with  $\hat{T}_H \cap \mathsf{Aut}(T, H, HtH)_H = \tilde{H}_H$ . Furthermore,  $\hat{T}_H \cong T$ ,  $\tilde{H}_H \cong H$  and  $\mathsf{Aut}(T, H, HtH)_H \cong \mathsf{Aut}(T, H, HtH) = \{\alpha \in \mathsf{Aut}(T) | H^{\alpha} = H, (HtH)^{\alpha} = HtH\}.$ 

Let  $x^i tx^j \in HtH$  for some  $i, j \in \mathbb{Z}_p$ . If i + j = 0, then  $x^i tx^j = (1 + j, 2 + j)(3 + j, 4 + j)$  and  $supp(x^i tx^j) = 4$ . If  $i + j \neq 0$ , then  $x^{i+j}$  is a *p*-cycle and  $supp(x^i tx^j) = supp(x^{i+j}x^{-j}tx^j) \geq p - 4 > 4$  because  $supp(x^{-j}tx^j) = 4$ . Thus,  $I := \{x^{-i}tx^i \mid i \in \mathbb{Z}_p\}$  consists of all elements in HtH whose supports are 4.

Now we consider  $\operatorname{Aut}(T, H, HtH)$ . Let  $\beta \in \operatorname{Aut}(T, H, HtH)$ . Then  $\beta \in \operatorname{Aut}(T) = \operatorname{Aut}(\mathsf{A}_p) \cong \mathsf{S}_p$ , and  $\beta$  is an automorphism of T induced by some  $b \in \mathsf{S}_p$  by conjugation, that is,  $t^{\beta} = t^{b}$  for any  $t \in T$ . Since  $(HtH)^{\beta} = (HtH)^{b} = HtH$ , we have  $I^{\beta} = I$ , and in particular,  $supp(yz) = supp(y^{\beta}z^{\beta})$  for any  $y, z \in I$ . It is easy to see that for any  $x^{-i}tx^{i}, x^{-j}tx^{j} \in I$ ,  $supp(x^{-i}tx^{i}x^{-j}tx^{j}) = 5$  if and only if j = i + 1 or i - 1. In fact, if j = i + 2 or i - 2 then  $supp(x^{-i}tx^{i}x^{-j}tx^{j}) = 4$ , if j = i + 3 or i - 3 then  $supp(x^{-i}tx^{i}x^{-j}tx^{j}) = 8$ .

Let  $\Sigma$  be a graph with I as vertex set and with  $y, z \in I$  adjacent if and only if supp(yz) = 5. By the above paragraph,  $\Sigma$  is a cycle of length p, and  $\beta$  induces an automorphism of  $\Sigma$ . Thus,  $\operatorname{Aut}(T, H, HtH)$  acts on I, and since  $\Sigma$  is a p-cycle,  $\operatorname{Aut}(T, H, HtH)/K \leq D_{2p}$ , where K is the kernel of this action. Let  $\gamma \in K$ , and suppose  $\gamma$  is induced by  $c \in S_p$  by conjugation. Then  $\gamma$  fixes each element in I, that is,  $(x^{-i}tx^i)^c = x^{-i}tx^i$  for each  $i \in \mathbb{Z}_p$ . Since  $x^{-i}tx^i = (1^{x^i}, 2^{x^i})(3^{x^i}, 4^{x^i})$  and  $x^{-(i+3)}tx^{i+3} = (4^{x^i}, 5^{x^i})(6^{x^i}, 7^{x^i}), c$  fixes  $\{1^{x^i}, 2^{x^i}, 3^{x^i}, 4^{x^i}\}$  and  $\{4^{x^i}, 5^{x^i}, 6^{x^i}, 7^{x^i}\}$  setwise, and hence fixes  $4^{x^i} = \{1^{x^i}, 2^{x^i}, 3^{x^i}, 4^{x^i}\} \cap \{4^{x^i}, 5^{x^i}, 6^{x^i}, 7^{x^i}\}$  for each  $i \in \mathbb{Z}_p$ . It follows that c fixes  $\{1, 2, \dots, n\}$  pointwise, implying K = 1. Thus,  $|\operatorname{Aut}(T, H, HtH)| \leq |\operatorname{Aut}(\Sigma)| = 2p$ .

Recall that  $h = (2, p)(3, p-1) \cdots (\frac{p-1}{2}, \frac{p+5}{2})(\frac{p+1}{2}, \frac{p+3}{2})$ . For  $p = 1 \mod 4$ , h is an even permutation and  $h \in A_p$ , and for  $p = 3 \mod 4$ , h is an odd permutation and  $h \in S_p$ , but  $h \notin A_p$ . Since  $x = (1, 2, \cdots, p)$ , we have  $x^h = x^{-1}$  and so  $H^h = H$ , and since  $t^h = (1^h, 2^h)(3^h, 4^h) = (1, p)(p-1, p-2) = x^{-(p-3)}tx^{p-3} \in I \subset HtH$ , we have  $(HtH)^h = HtH$ . Clearly,  $H^x = H$  and  $(HtH)^x = H$ . For any  $z \in S_p$ , denote by  $\tilde{z}$  the induced automorphism of  $A_p$  by z by conjugation. Then  $\tilde{x}, \tilde{h} \in Aut(T, H, HtH)$  and  $\langle \tilde{x}, \tilde{h} \rangle \cong D_{2p}$ . Since  $|Aut(T, H, HtH)| \leq 2p$ , we have  $Aut(T, H, HtH) = \langle \tilde{x}, \tilde{h} \rangle \cong D_{2p}$ .

Recall that  $\tilde{x}_H : Hg \mapsto Hg^x$  for  $g \in A_p$ , and  $\tilde{h}_H : Hg \mapsto Hg^h$  for  $g \in A_p$ , are automorphisms of  $\Gamma^p$ , and  $\tilde{H}_H = \langle \tilde{x}_H \rangle$ . Since  $\operatorname{Aut}(T, H, HtH) \cong \operatorname{Aut}(T, H, HtH)_H$ , we have  $\operatorname{Aut}(T, H, HtH)_H = \langle \tilde{x}_H, \tilde{h}_H \rangle = \tilde{H}_H : \tilde{h}_H \cong \mathsf{D}_{2p}$ , and since  $\hat{T}_H \cap \operatorname{Aut}(T, H, HtH)_H =$  $\tilde{H}_H$  and  $A = \hat{T}_H \operatorname{Aut}(T, H, HtH)_H$ , we have  $|A : \hat{T}_H| = 2$  and hence  $A = \hat{T}_H : \langle \tilde{h}_H \rangle$ .

Set  $C = C_A(\hat{T}_H)$ , the centralizer of  $\hat{T}_H$  in A. Since  $\hat{T}_H \cong \mathsf{A}_p$ , we have  $C \cap \hat{T}_H = 1$ , and since  $A = \hat{T}_H : \langle \tilde{h}_H \rangle$ , we have C = 1 or  $C \cong \mathbb{Z}_2$ . For the former,  $A \cong \mathsf{S}_p$  by the N/C Theorem, and for the latter,  $A = \hat{T}_H \times C \cong \mathsf{A}_p \times \mathbb{Z}_2$ . To finish the proof, we only need to prove that  $C \cong \mathbb{Z}_2$  if and only if  $p = 1 \mod 4$ .

Assume  $C \cong \mathbb{Z}_2$ . Since  $A = \hat{T}_H \rtimes \langle \tilde{h}_H \rangle$ , we can let  $C = \langle \hat{y}_H \tilde{h}_H \rangle$  for some  $y \in T$ . This implies that for any  $z, g \in T$ , we have  $(Hz)^{\hat{y}_H \tilde{h}_H \hat{g}_H} = (Hz)^{\hat{g}_H \hat{y}_H \tilde{h}_H}$ , that is,  $H(zy)^h g =$  $H(zgy)^h$ , implying Hhzyhg = Hhzgyh. Set  $\ell = yhg(gyh)^{-1}$ . Then  $Hhz\ell(hz)^{-1} = H$ , that is,  $\ell \in H^{hz} = H^z$  for any  $z \in A_p$ . This implies that  $\ell \in \bigcap_{z \in A_p} H^z$ , and since  $\bigcap_{z \in A_p} H^z$  is the largest normal subgroup of  $A_p$  contained in H, we have  $\bigcap_{z \in A_p} H^z = 1$ and hence  $\ell = 1$ . This means that yhg = gyh, and by the arbitrary of  $g \in A_p$ , we have  $yh \in C_{A_p}(S_p) = 1$ . It follows that  $h = y \in A_p$  and hence  $p = 1 \mod 4$ . On the other hand, if  $p = 1 \mod 4$  then it is easy to check that  $\hat{h}\tilde{h} \in C$ . Thus,  $C \cong \mathbb{Z}_2$  if and only if  $p = 1 \mod 4$ , as required.  $\Box$ 

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