# On flag-transitive 2- $(v, k, 2)$ designs 

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#### Abstract

This paper is devoted to the classification of flag-transitive $2-(v, k, 2)$ designs. We show that apart from two known symmetric $2-(16,6,2)$ designs, every flag-transitive subgroup $G$ of the automorphism group of a nontrivial $2-(v, k, 2)$ design is primitive of affine or almost simple type. Moreover, we classify the $2-(v, k, 2)$ designs admitting a flag transitive almost simple group $G$ with socle $\operatorname{PSL}(n, q)$ for some $n \geqslant 3$. Alongside this analysis we give a construction for a flag-transitive 2- $(v, k-1, k-2)$ design from a given flag-transitive $2-(v, k, 1)$ design which induces a 2 -transitive action on a line. Taking the design of points and lines of the projective space $\operatorname{PG}(n-1,3)$ as input to this construction yields a $G$-flag-transitive $2-(v, 3,2)$ design where $G$ has socle $\operatorname{PSL}(n, 3)$ and $v=\left(3^{n}-1\right) / 2$. Apart from these designs, our PSL-classification yields exactly one other example, namely the complement of the Fano plane.


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## 1 Introduction

A $2-(v, k, \lambda)$ design $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$ with a set $\mathcal{P}$ of $v$ points and a set $\mathcal{B}$ of blocks such that each block is a $k$-subset of $\mathcal{P}$ and each two distinct points are contained in $\lambda$ blocks. We say $\mathcal{D}$ is nontrivial if $2<k<v$, and symmetric if $v=b$. All $2-(v, k, \lambda)$ designs in this paper are assumed to be nontrivial. An automorphism of $\mathcal{D}$ is a permutation of the point set which preserves the block set. The set of all automorphisms of $\mathcal{D}$ with the composition of permutations forms a group, denoted by $\operatorname{Aut}(\mathcal{D})$. For a subgroup $G$ of $\operatorname{Aut}(\mathcal{D}), G$ is said to be point-primitive if $G$ acts primitively on $\mathcal{P}$, and said to be point-imprimitive otherwise. A flag of $\mathcal{D}$ is a point-block pair $(\alpha, B)$ where $\alpha$ is a point and $B$ is a block incident with $\alpha$. A subgroup $G$ of $\operatorname{Aut}(\mathcal{D})$ is said to be flag-transitive if $G$ acts transitively on the set of flags of $\mathcal{D}$.

A 2- $(v, k, \lambda)$ design with $\lambda=1$ is also called a finite linear space. In 1990, Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl [5] classified all flag-transitive linear spaces apart from those with a one-dimensional affine automorphism group. Since then, there have been efforts to classify $2-(v, k, 2)$ designs $\mathcal{D}$ admitting a flag-transitive group $G$ of automorphisms. Through a series of papers [24, 25, 26, 27], Regueiro proved that, if $\mathcal{D}$ is

[^0]symmetric, then either $(v, k) \in\{(7,4),(11,5),(16,6)\}$, or $G \leqslant \mathrm{~A} \Gamma \mathrm{~L}(1, q)$ for some odd prime power $q$. Recently, Zhou and the second author [15] proved that, if $\mathcal{D}$ is not symmetric and $G$ is point-primitive, then $G$ is affine or almost simple. In each of these cases $G$ has a unique minimal normal subgroup, its socle $\operatorname{Soc}(G)$, which is elementary abelian or a nonabelian simple group, respectively.

Our first objective in this paper is to fill in a missing piece in this story, namely to treat the case where $G$ is flag-transitive and point-imprimitive and $\mathcal{D}$ is a not-necessarily-symmetric $2-(v, k, 2)$ design. Such flag-transitive, point-imprimitive designs exist: it was shown in 1945 by Hussain [12], and independently in 1946 by Nandi [19], that there are exactly three 2(16, 6, 2)-designs. O'Reilly Regueiro [24, Examples 1.2] showed that exactly two of these designs are flag-transitive, and each admits a point-imprimitive, flag-transitive subgroup of automorphisms (one with automorphism group $2^{4} \mathrm{~S}_{6}$ and point stabiliser $\left(\mathbb{Z}_{2} \times \mathbb{Z}_{8}\right)\left(\mathrm{S}_{4} \cdot 2\right)$ and the other with automorphism group $S_{6}$ and point stabiliser $S_{4} .2$, see also [23, Remark $1.4(1)]$ ). We prove that these are the only point-imprimitive examples, and thus, together with [15, Theorem 1.1] and [24, Theorem 2], we obtain the following result.

Theorem 1.1. Let $\mathcal{D}$ be a 2-( $v, k, 2)$ design with a flag-transitive group $G$ of automorphisms. Then either
(i) $\mathcal{D}$ is one of two known symmetric $2-(16,6,2)$ designs with $G$ point-imprimitive; or
(ii) $G$ is point-primitive of affine or almost simple type.

Theorem 1.1 reduces the study of flag-transitive $2-(v, k, 2)$ designs to those whose automorphism group $G$ is point-primitive of affine or almost simple type. Regueiro [24, 25, 26, 27] has classified all such examples where the design is symmetric (up to those admitting a onedimensional affine group). In the non-symmetric case, the second author and Zhou have dealt with the cases where the socle $\operatorname{Soc}(G)$ is a sporadic simple group or an alternating group, identifying three possibilities: namely $(v, k)=(176,8)$ with $G=$ HS, the HigmanSims group in [15], and $(v, k)=(6,3)$ or $(10,4)$ with $\operatorname{Soc}(G)=A_{v}$ in [16]. Our contribution is the case where $\operatorname{Soc}(G)=\operatorname{PSL}(n, q)$ for some $n \geqslant 3$ and $q$ a prime power. In contrast to the cases considered previously, an infinite family of examples occurs, which may be obtained from the following general construction method for flag-transitive designs from linear spaces.

Construction 1.1. For a $2-(v, k, 1)$ design $\mathcal{S}=(\mathcal{P}, \mathcal{L})$ with $k \geqslant 3$, let

$$
\mathcal{B}=\{\ell \backslash\{\alpha\} \mid \ell \in \mathcal{L}, \alpha \in \ell\}
$$

and $\mathcal{D}(\mathcal{S})=(\mathcal{P}, \mathcal{B})$.
We show in Proposition 4.1 that $\mathcal{D}(\mathcal{S})$ is a $2-(v, k-1, k-2)$ design, and moreover, that $\mathcal{D}(\mathcal{S})$ is $G$-flag-transitive whenever $G \leqslant \operatorname{Aut}(\mathcal{S})$ is flag-transitive on $\mathcal{S}$ and induces a 2 -transitive action on each line of $\mathcal{S}$. In particular, these conditions hold if $\mathcal{S}$ is the design of points and lines of $\operatorname{PG}(n-1,3)$, for some $n \geqslant 3$, and $\operatorname{Soc}(G)=\operatorname{PSL}(n, 3)$ (Proposition 4.1). Apart from these designs, our analysis shows that there is only one other $G$-flag-transitive $2-(v, k, 2)$ design with $\operatorname{Soc}(G)=\operatorname{PSL}(n, q), n \geqslant 3$.

Theorem 1.2. Let $\mathcal{D}$ be a $2-(v, k, 2)$ design admitting a flag-transitive group $G$ of automorphisms, such that $\operatorname{Soc}(G)=\operatorname{PSL}(n, q)$ for some $n \geqslant 3$ and prime power $q$. Then either
(a) $\mathcal{D}=\mathcal{D}(\mathcal{S})$ is as in Construction 1.1, where $\mathcal{S}$ is the design of points and lines of $\mathrm{PG}(n-$ $1,3)$; or
(b) $\mathcal{D}$ is the complement of the Fano plane (that is, blocks are the complements of the lines of $\mathrm{PG}(2,2)$ ).

The designs in part (a) are non-symmetric (Proposition4.1), while the complement of the Fano plane is symmetric, and arises also in Regueiro's classification [26, Theorem 1] (noting that the group $\operatorname{PSL}(3,2)$ is isomorphic to the group $\operatorname{PSL}(2,7)$ in her result).

The proofs of Theorems 1.1 and 1.2 will be given in Sections 3 and 4 , respectively.

## 2 Preliminaries

We first collect some useful results on flag-transitive designs and groups of Lie type.
Lemma 2.1. Let $\mathcal{D}$ be a $2-(v, k, \lambda)$ design and let $b$ be the number of blocks of $\mathcal{D}$. Then the number of blocks containing each point of $\mathcal{D}$ is a constant $r$ satisfying the following:
(i) $r(k-1)=\lambda(v-1)$;
(ii) $b k=v r$;
(iii) $b \geqslant v$ and $r \geqslant k$;
(iv) $r^{2}>\lambda v$.

In particular, if $\mathcal{D}$ is non-symmetric then $b>v$ and $r>k$.
Proof. Parts (i) and (ii) follow immediately by simple counting. Part (iii) is Fisher's Inequality [28, p.99]. By (i) and (iii) we have

$$
r(r-1) \geqslant r(k-1)=\lambda(v-1)
$$

and so $r^{2} \geqslant \lambda v+r-\lambda$. Since $\mathcal{D}$ is nontrivial, we deduce from (i) that $r>\lambda$. Hence $r^{2}>\lambda v$, as stated in part (iv).

For a permutation group $G$ on a set $\mathcal{P}$ and an element $\alpha$ of $\mathcal{P}$, denote by $G_{\alpha}$ the stabiliser of $\alpha$ in $G$, that is, the subgroup of $G$ fixing $\alpha$. A subdegree $s$ of a transitive permutation group $G$ is the length of some orbit of $G_{\alpha}$. We say that $s$ is non-trivial if the orbit is not $\{\alpha\}$, and $s$ is unique if $G_{\alpha}$ has only one orbit of size $s$.

Lemma 2.2. Let $\mathcal{D}$ be a $2-(v, k, \lambda)$ design, let $G$ be a flag-transitive subgroup of $\operatorname{Aut}(\mathcal{D})$, and let $\alpha$ be a point of $\mathcal{D}$. Then the following statements hold:
(i) $\left|G_{\alpha}\right|^{3}>\lambda|G|$;
(ii) $r$ divides $\operatorname{gcd}\left(\lambda(v-1),\left|G_{\alpha}\right|\right)$;
(iii) $r$ divides $\lambda \operatorname{gcd}\left(v-1,\left|G_{\alpha}\right|\right)$;
(iv) $r$ divides $s \operatorname{gcd}(r, \lambda)$ for every nontrivial subdegree $s$ of $G$.

Proof. By Lemma 2.1 we have $r^{2}>\lambda v$. Moreover, the flag-transitivity of $G$ implies that $v=|G| /\left|G_{\alpha}\right|$ and $r$ divides $\left|G_{\alpha}\right|$, and in particular, $\left|G_{\alpha}\right| \geqslant r$. It follows that

$$
\left|G_{\alpha}\right|^{2} \geqslant r^{2}>\lambda v=\frac{\lambda|G|}{\left|G_{\alpha}\right|}
$$

and so $\left|G_{\alpha}\right|^{3}>\lambda|G|$. This proves statement (i).
Since $r$ divides $r(k-1)=\lambda(v-1)$ and $r$ divides $\left|G_{\alpha}\right|$, we conclude that $r$ divides

$$
\begin{equation*}
\operatorname{gcd}\left(\lambda(v-1),\left|G_{\alpha}\right|\right), \tag{2.1}
\end{equation*}
$$

as statement (ii) asserts. Note that the quantity in (2.1) divides

$$
\operatorname{gcd}\left(\lambda(v-1), \lambda\left|G_{\alpha}\right|\right)=\lambda \operatorname{gcd}\left(v-1,\left|G_{\alpha}\right|\right)
$$

We then conclude that $r$ divides $\lambda \operatorname{gcd}\left(v-1,\left|G_{\alpha}\right|\right)$, proving statement (iii).
Finally, statement (iv) is proved in [8, p.91] and [9].
For a positive integer $n$ and prime number $p$, let $n_{p}$ denote the $p$-part of $n$ and let $n_{p^{\prime}}$ denote the $p^{\prime}$-part of $n$, that is, $n_{p}=p^{t}$ such that $p^{t} \mid n$ but $p^{t+1} \nmid n$ and $n_{p^{\prime}}=n / n_{p}$. We will denote by $d$ the greatest common divisor of $n$ and $q-1$.

Lemma 2.3. Suppose that $\mathcal{D}$ is a $2-(v, k, 2)$ design admitting a flag-transitive point-primitive group $G$ of automorphisms with socle $X=\operatorname{PSL}(n, q)$, where $n \geqslant 3$ and $q=p^{f}$ for some prime $p$ and positive integer $f$, and $d=\operatorname{gcd}(n, q-1)$. Then for any point $\alpha$ of $\mathcal{D}$ the following statements hold:
(i) $|X|<2(d f)^{2}\left|X_{\alpha}\right|^{3}$;
(ii) $r$ divides $2 d f\left|X_{\alpha}\right|$;
(iii) if $p \mid v$, then $r_{p}$ divides 2 , $r$ divides $2 d f\left|X_{\alpha}\right|_{p^{\prime}}$, and $|X|<2(d f)^{2}\left|X_{\alpha}\right|_{p^{\prime}}^{2}\left|X_{\alpha}\right|$.

Proof. Since $G$ is point-primitive and $X$ is normal in $G$, the group $X$ is transitive on the point set. Hence $G=X G_{\alpha}$ and so

$$
\frac{\left|G_{\alpha}\right|}{\left|X_{\alpha}\right|}=\frac{\left|G_{\alpha}\right|}{\left|X \cap G_{\alpha}\right|}=\frac{\left|X G_{\alpha}\right|}{|X|}=\frac{|G|}{|X|}
$$

Moreover, as $\operatorname{Soc}(G)=X=\operatorname{PSL}(n, q)$, we have $G \leqslant \operatorname{Aut}(X)$. Hence $\left|G_{\alpha}\right| /\left|X_{\alpha}\right|=|G| /|X|$ divides $|\operatorname{Out}(X)|=2 d f$. Consequently, $\left|G_{\alpha}\right| /\left|X_{\alpha}\right| \leqslant 2 d f$. Since Lemma 2.2(i) yields

$$
\left|G_{\alpha}\right|^{3}>2|G|=\frac{2|X|\left|G_{\alpha}\right|}{\left|X_{\alpha}\right|}
$$

it follows that

$$
2|X|<\left|X_{\alpha}\right|\left|G_{\alpha}\right|^{2}=\left(\frac{\left|G_{\alpha}\right|}{\left|X_{\alpha}\right|}\right)^{2}\left|X_{\alpha}\right|^{3} \leqslant(2 d f)^{2}\left|X_{\alpha}\right|^{3}
$$

This leads to statement (i). Since $\left|G_{\alpha}\right| /\left|X_{\alpha}\right|$ divides $|\operatorname{Out}(X)|=2 d f$ and the flag-transitivity of $G$ implies that $r$ divides $\left|G_{\alpha}\right|$, we derive that $r$ divides $2 d f\left|X_{\alpha}\right|$, as in statement (ii).

Now suppose that $p$ divides $v$. Then the equality $2(v-1)=r(k-1)$ implies that $r_{p}$ divides 2. As a consequence of this and part (ii) we see that $r$ divides $2 d f\left|X_{\alpha}\right|_{p^{\prime}}$. Since $r^{2}>2 v$ by Lemma 2.1(iv), and $v=|X| /\left|X_{\alpha}\right|$ by the point-transitivity of $X$, it then follows that

$$
\left(2 d f\left|X_{\alpha}\right|_{p^{\prime}}\right)^{2}>2 v=\frac{2|X|}{\left|X_{\alpha}\right|}
$$

This implies that $2(d f)^{2}\left|X_{\alpha}\right|_{p^{\prime}}^{2}\left|X_{\alpha}\right|>|X|$, completing the proof of part (iii).
Lemma 2.4. Suppose that $\mathcal{D}$ is a $2-(v, k, 2)$ design admitting a flag-transitive point-primitive group $G$ of automorphisms with socle $X=\operatorname{PSL}(n, q)$, where $n \geqslant 3$ and $q=p^{f}$ for some prime $p$ and positive integer $f$, and $d=\operatorname{gcd}(n, q-1)$. Let $\alpha$ and $\beta$ be distinct points of $\mathcal{D}$, and suppose $H \leqslant G_{\alpha, \beta}$. Then $r$ divides $4 d f\left|X_{\alpha}\right| /|H|$.
Proof. By Lemma 2.2 (iv), $r$ divides $2\left|\beta^{G_{\alpha}}\right|=2\left|G_{\alpha}\right| /\left|G_{\alpha \beta}\right|$. Since $\left|G_{\alpha}\right|$ divides $2 d f\left|X_{\alpha}\right|$ (see proof of Lemma (2.3) and $H$ divides $\left|G_{\alpha, \beta}\right|$, it follows that $r$ divides $4 d f\left|X_{\alpha}\right| /|H|$.

We will need the following results on finite groups of Lie type.
Lemma 2.5. Suppose that $\mathcal{D}$ is a $2-(v, k, 2)$ design admitting a flag-transitive point-primitive group $G$ of automorphisms with socle $X=\operatorname{PSL}(n, q)$, where $n \geqslant 3$ and $q=p^{f}$ for some prime $p$ and positive integer $f$, and $r$ is the number of blocks incident with a given point. Let $\alpha$ be a point of $\mathcal{D}$. Suppose that $X_{\alpha}$ has a normal subgroup $Y$, which is a finite simple group of Lie type in characteristic $p$, and $Y$ is not isomorphic to $\mathrm{A}_{5}$ or $\mathrm{A}_{6}$ if $p=2$. If $r_{p} \mid 2_{p}$, then $r$ is divisible by the index of a proper parabolic subgroup of $Y$.

Proof. Since $G$ is flag-transitive, we have $r=\left|G_{\alpha}\right| /\left|G_{\alpha, B}\right|$, where $B$ is a block through $\alpha$. Since $X_{\alpha} \unlhd G_{\alpha},\left|X_{\alpha}\right| /\left|X_{\alpha, B}\right|$ divides $r$. Now since $Y \unlhd X_{\alpha}$, we also have that $|Y| /\left|Y_{B}\right|$ divides $r$. Let $H:=Y_{B}$. Since $r_{p} \mid 2_{p}$, we have that $|Y: H|_{p} \leqslant 2_{p}$. We claim that $H$ is contained in a proper parabolic subgroup of $Y$. First assume $|Y: H|_{p}=1$. Then by [29, Lemma 2.3], $H$ is contained in a proper parabolic subgroup of $Y$. Now suppose $|Y: H|_{p}=2$. Then $p=2$ and $4 \nmid Y: H \mid$, and so by [26, Lemma 7], $H$ is contained in a proper parabolic subgroup of $Y$. So the claim is proved in both cases. It follows that $r$ is divisible by the index of a parabolic subgroup of $Y$.

Lemma 2.6. ([1, Lemma 4.2, Corollary 4.3]) Table 1 gives upper bounds and lower bounds for the orders of certain $n$-dimensional classical groups defined over a field of order $q$, where $n$ satisfies the conditions in the last column.

We finish this section with an arithmetic result.
Lemma 2.7. ([4, Lemma 1.13.5]) Let $p$ be a prime, let $n, e$ and $f$ be positive integers such that $n>1$ and $e \mid f$, and let $q_{0}=p^{e}$ and $q=p^{f}$. Then
(i) $\frac{q-1}{\operatorname{lcm}\left(q_{0}-1,(q-1) / \operatorname{gcd}(n, q-1)\right)}=\operatorname{gcd}\left(n, \frac{q-1}{q_{0}-1}\right)$;
(ii) $\frac{q+1}{\operatorname{lcm}\left(q_{0}+1,(q+1) / \operatorname{gcd}(n, q+1)\right)}=\operatorname{gcd}\left(n, \frac{q+1}{q_{0}+1}\right)$;

Table 1: Bounds for the order of some classical groups

| Group $G$ | Lower bound on $\|G\|$ | Upper bound on $\|G\|$ | Conditions on $n$ |
| :--- | :--- | :--- | :---: |
| $\operatorname{GL}(n, q)$ | $>\left(1-q^{-1}-q^{-2}\right) q^{n^{2}}$ | $\leqslant\left(1-q^{-1}\right)\left(1-q^{-2}\right) q^{n^{2}}$ | $n \geqslant 2$ |
| $\operatorname{PSL}(n, q)$ | $>q^{n^{2}-2}$ | $\leqslant\left(1-q^{-2}\right) q^{n^{2}-1}$ | $n \geqslant 2$ |
| $\operatorname{GU}(n, q)$ | $\geqslant\left(1+q^{-1}\right)\left(1-q^{-2}\right) q^{n^{2}}$ | $\leqslant\left(1+q^{-1}\right)\left(1-q^{-2}\right)\left(1+q^{-3}\right) q^{n^{2}}$ | $n \geqslant 2$ |
| $\operatorname{PSU}(n, q)$ | $>\left(1-q^{-1}\right) q^{n^{2}-2}$ | $\leqslant\left(1-q^{-2}\right)\left(1+q^{-3}\right) q^{n^{2}-1}$ | $n \geqslant 3$ |
| $\operatorname{Sp}(n, q)$ | $>\left(1-q^{-2}-q^{-4}\right) q^{\frac{1}{2} n(n+1)}$ | $\leqslant\left(1-q^{-2}\right)\left(1-q^{-4}\right) q^{\frac{1}{2} n(n+1)}$ | $n \geqslant 4$ |

(iii) If $f$ is even, then $q^{1 / 2}=p^{f / 2}$ and $\frac{q-1}{\operatorname{lcm}\left(q^{1 / 2}+1,(q-1) / \operatorname{gcd}(n, q-1)\right)}=\operatorname{gcd}\left(n, q^{1 / 2}-1\right)$.

## 3 Proof of Theorem 1.1

Let $\mathcal{D}=(\mathcal{P}, \mathcal{B})$ be a $2-(v, k, 2)$ design admitting a flag-transitive group $G$ of automorphisms. If $G$ is point-primitive, then by [15] and [24], $G$ is of affine or almost simple type. Thus we may assume that $G$ leaves invariant a non-trivial partition $\mathcal{C}=\left\{\Delta_{1}, \Delta_{2}, \ldots, \Delta_{y}\right\}$ of $\mathcal{P}$, where

$$
\begin{equation*}
v=x y . \tag{3.2}
\end{equation*}
$$

with $1<y<v$ and $\left|\Delta_{i}\right|=x$ for each $i$. If $(v, k)=(16,6)$ then by Lemma 2.1, it follows that $\mathcal{D}$ is symmetric and hence, in the light of the discussion before the statement of Theorem 1.1, in this case Theorem 1.1(i) holds. Hence we may assume further that $(v, k) \neq(16,6)$. Our objective now is to derive a contradiction to these assumptions. Our proof uses the facts, which can easily be verified by Magma [3], that for each 2-transitive permutation group of degree $2 p=10$ or 22 there is a unique class of subgroups of index $2 p$ and each such group is almost simple with a 2 -transitive unique minimal normal subgroup (its socle). In fact the socle is one of $\operatorname{PSL}(2,9)$ or $\mathrm{A}_{10}$ (for degree 10), or $M_{22}$ or $\mathrm{A}_{22}$ (for degree 22).

First we introduce a new parameter $\ell$ : let $\alpha \in \mathcal{P}$ and $\Delta \in \mathcal{C}$ such that $\alpha \in \Delta$; choose $B \in \mathcal{B}$ containing $\alpha$, and let $\ell=|B \cap \Delta|$. It follows from [23, Lemma 2.1] that, for each $B^{\prime} \in \mathcal{B}$ and $\Delta^{\prime} \in \mathcal{C}$ such that $B^{\prime} \cap \Delta^{\prime} \neq \emptyset$, the intersection size $\left|B^{\prime} \cap \Delta^{\prime}\right|=\ell$, so that $B^{\prime}$ meets each of exactly $k / \ell$ parts of $\mathcal{C}$ in $\ell$ points and is disjoint from the other parts. Moreover,

$$
\begin{equation*}
\ell \mid k \quad \text { and } 1<\ell<k \tag{3.3}
\end{equation*}
$$

(Note that the proof of [23, Lemma 2.1] uses flag-transitivity of $\mathcal{D}$, but is valid for all 2designs, not only symmetric ones.)
Claim 1: $\quad(v, b, r, k, \ell)=\left(x^{2}, \frac{2 x^{2}(x-1)}{x+2}, 2 x-2, x+2,2\right)$, and $x=2 p$ with $p \in\{5,11\}$.

Proof of Claim: Counting the point-block pairs $\left(\alpha^{\prime}, B^{\prime}\right)$ with $\alpha^{\prime} \in \Delta \backslash\{\alpha\}$ and $B^{\prime}$ containing $\alpha$ and $\alpha^{\prime}$, we obtain

$$
\begin{equation*}
2(x-1)=r(\ell-1) . \tag{3.4}
\end{equation*}
$$

It follows from (3.2) and Lemma 2.1(i) that

$$
r(k-1)=2(x y-1)=2 y(x-1)+2(y-1)
$$

which together with (3.4) yields

$$
\begin{equation*}
r(k-1)=y r(\ell-1)+2(y-1) . \tag{3.5}
\end{equation*}
$$

Let $z=k-1-y(\ell-1)$. Then $z$ is an integer and, by (3.5), $r z=2(y-1)>0$ so $z$ is a positive integer and

$$
\begin{equation*}
y=\frac{r z+2}{2} . \tag{3.6}
\end{equation*}
$$

This in conjunction with (3.5) leads to

$$
r(k-1)+2=y(r(\ell-1)+2)=\frac{(r z+2)(r(\ell-1)+2)}{2}
$$

Hence

$$
\begin{equation*}
2(k-\ell-z)=r z(\ell-1) . \tag{3.7}
\end{equation*}
$$

Since $k \leqslant r$ (Lemma 2.1(iii)), we have

$$
k z(\ell-1) \leqslant r z(\ell-1)=2(k-\ell-z)<2 k,
$$

and hence $z=1$ and $\ell=2$. Then (3.4) becomes $r=2 x-2$, and so (3.6) gives $y=x$ (and hence $v=x^{2}$ ) and the definition of $z$ gives $k=x+2$. It then follows from $r \geqslant k$ that $x \geqslant 4$, and from (3.3) that $k$, and hence also $x$, is even. Finally by Lemma 2.1(ii),

$$
b=\frac{v r}{k}=\frac{x^{2}(2 x-2)}{x+2}=2 x^{2}-6 x+12-\frac{24}{x+2},
$$

and hence $(x+2) \mid 24$. Therefore, $x=4,6,10$ or 22 , but since we are assuming that $(v, k) \neq(16,6)$ the parameter $x \neq 4$. If $x=6$, then $(v, b, r, k)=(36,45,10,8)$, but one can see from [6, II.1.35] that there is no $2-(36,8,2)$ design. Thus $x=10$ or 22 , and Claim 1 is proved.
Claim 2: For $\Delta \in \mathcal{C}$, the induced group $G_{\Delta}^{\Delta}$ is 2-transitive. Moreover the kernel $K:=$ $G_{(\mathcal{C})} \neq 1, \mathcal{C}$ is the set of $K$-orbits in $\mathcal{P}$, and $K^{\Delta}$ and its socle $\operatorname{Soc}(K)^{\Delta}$ are 2 -transitive with 2-transitive socle $\operatorname{PSL}(2,9)$ or $\mathrm{A}_{10}$ for degree 10 , and $M_{22}$ or $\mathrm{A}_{22}$ for degree 22.
Proof of Claim: Since each element of $G$ fixing $\alpha$ stabilises $\Delta$, we have the inclusion $G_{\alpha} \leqslant$ $G_{\Delta}$. Let $\beta, \gamma$ be arbitrary points in $\Delta \backslash\{\alpha\}$, and consider $B_{1} \in \mathcal{B}$ containing $\alpha$ and $\beta$, and $B_{2} \in \mathcal{B}$ containing $\alpha$ and $\gamma$. Since $G$ is flag-transitive, there exists $h \in G_{\alpha}$ such that $B_{1}^{h}=B_{2}$, and in particular, $\beta^{h} \in B_{2}$. As $\ell=2$ (by Claim 1), each block of $\mathcal{D}$ through $\alpha$ contains exactly one point in $\Delta \backslash\{\alpha\}$. Since $\beta^{h} \in(\Delta \backslash\{\alpha\})^{h}=\Delta \backslash\{\alpha\}$, it then follows that $\beta^{h}=\gamma$. This shows that $G_{\alpha}$ is transitive on $\Delta \backslash\{\alpha\}$, and hence $G_{\Delta}^{\Delta}$ is 2-transitive and hence primitive.

By Claim 1, each non-trivial block of imprimitivity for $G$ in $\mathcal{P}$ has size $x=\sqrt{v}=2 p$ (with $p=5$ or 11 ), and hence the induced permutation group $G^{\mathcal{C}}$ on $\mathcal{C}$ is primitive. Suppose that $K=1$, so $G^{\mathcal{C}} \cong G$. Since $G$ is point-transitive and $v=4 p^{2}$, it follows that $|G|=\left|G^{\mathcal{C}}\right|$ is divisible by $p^{2}$, and hence $G_{\Delta}^{\mathcal{C}} \cong G_{\Delta}$ has order divisible by $p$ (since $\left|G: G_{\Delta}\right|=2 p$ ). Thus $G_{\Delta}^{\mathcal{C}}$ contains an element of order $p$ which acts on $\mathcal{C}$ as a $p$-cycle fixing $p$ of the parts. Then by a result of Jordan [30, Theorem 13.9] we have $G^{\mathcal{C}}=\mathrm{A}_{2 p}$ or $\mathrm{S}_{2 p}$ and thus $G_{\Delta} \cong G_{\Delta}^{\mathcal{C}}=\mathrm{A}_{2 p-1}$ or $\mathrm{S}_{2 p-1}$. The kernel of the action of $G_{\Delta}$ on $\Delta$ is normal in $G_{\Delta}$ and so can only be $1, \mathrm{~A}_{2 p-1}$ or $\mathrm{S}_{2 p-1}$. Since $G_{\Delta}^{\Delta}$ is transitive of degree $2 p>2$, this kernel must be trivial. Hence $G_{\Delta} \cong G_{\Delta}^{\Delta}$ is primitive of degree $2 p$ and neither $\mathrm{A}_{2 p-1}$ nor $\mathrm{S}_{2 p-1}$ has such an action, for $p \in\{5,11\}$. This contradiction implies that $K \neq 1$.

Since $K \neq 1$ and $K$ is normal in $G$, its orbits are nontrivial blocks of imprimitivity for $G$ in $\mathcal{P}$, and by Claim 1, they must have size $x=2 p$. Hence the set of $K$-orbits in $\mathcal{P}$ is the partition $\mathcal{C}$. Since $1 \neq \operatorname{Soc}(K) \unlhd G$ it follows that $\operatorname{Soc}(K)^{\Delta} \neq 1$ and hence $\operatorname{Soc}(K)^{\Delta}$ contains the socle of $G_{\Delta}^{\Delta}$, which is 2-transitive on $\Delta$ (see above). Therefore $\operatorname{Soc}(K)^{\Delta}$ is 2transitive, and so also $K^{\Delta}$ is 2-transitive. By Burnside's Theorem (see [22, Theorem 3.21]), since $|\Delta|=2 p$ is not a prime power, $G_{\Delta}^{\Delta}, K^{\Delta}$ and $\operatorname{Soc}(K)^{\Delta}$ are almost simple with 2transitive nonabelian simple socle. As mentioned above these 2-transitive groups must have socle $\operatorname{PSL}(2,9)$ or $\mathrm{A}_{10}$ for degree 10 , and $M_{22}$ or $\mathrm{A}_{22}$ for degree 22 , and that socle is also 2 -transitive on $\Delta$.

Claim 3: The group $K$ is faithful on $\Delta$, so $K$ is almost simple with nonabelian simple socle.
Proof of Claim: Let $\Delta \in \mathcal{C}$ and suppose that $A=K_{(\Delta)} \neq 1$. Let $F$ denote the set of fixed points of $A$, so $\Delta \subseteq F$. If $\beta \in F$ and $\beta \in \Delta^{\prime} \in \mathcal{C}$, then since $K$ is transitive on $\Delta^{\prime}$ (Claim 2) and $A \unlhd K$, it follows that $A$ fixes $\Delta^{\prime}$ pointwise. Thus $A \leqslant K_{\left(\Delta^{\prime}\right)}$, and since $K_{(\Delta)}, K_{\left(\Delta^{\prime}\right)}$ are conjugate in $G$ we have $A=K_{\left(\Delta^{\prime}\right)}$. Therefore $F$ is a union of parts of $\mathcal{C}$.

If $g \in G$, then $A^{g}$ has fixed point set $F^{g}$ and $F^{g}$ is a union of some parts of $\mathcal{C}$. Thus if $F \cap F^{g}$ contains a point $\beta$ and $\beta \in \Delta^{\prime} \in \mathcal{C}$, then by the previous paragraph $A=K_{\left(\Delta^{\prime}\right)}=A^{g}$ and so $F=F^{g}$. It follows that $F$ is a block of imprimitivity for $G$ in $\mathcal{P}$, and $F$ is non-trivial since $A \neq 1$. Thus $\mathcal{C}^{\prime}:=\left\{F^{g} \mid g \in G\right\}$ is a non-trivial $G$-invariant partition of $\mathcal{P}$. By Claim $1,|F|=x$, and since $F$ contains $\Delta$ we conclude that $F=\Delta$. This means that $A^{\Delta^{\prime}} \neq 1$ for each $\Delta^{\prime} \in \mathcal{C} \backslash\{\Delta\}$, and since $K^{\Delta^{\prime}}$ is 2-transitive (Claim 2), it follows that $A^{\Delta^{\prime}}$ is transitive. Now choose $\alpha, \beta \in F=\Delta$ and let $B_{1}, B_{2} \in \mathcal{B}$ be the two blocks containing $\{\alpha, \beta\}$. Then $A \leqslant G_{\alpha \beta}$, and $G_{\alpha \beta}$ fixes $B_{1} \cup B_{2}$ setwise. By Claim 1, there exists $\Delta^{\prime} \in \mathcal{C} \backslash\{\Delta\}$ such that $\left|B_{1} \cap \Delta^{\prime}\right|=\ell=2$, and $\left|B_{2} \cap \Delta^{\prime}\right|=0$ or 2 . Thus $\left(B_{1} \cup B_{2}\right) \cap \Delta^{\prime}$ has size between 2 and 4 and is fixed setwise by $A$. This is a contradiction since $A$ is transitive on $\Delta^{\prime}$ and $\left|\Delta^{\prime}\right|=2 p \geqslant 10$. Therefore $A=1$ so $K$ is faithful on $\Delta$. By Claim $2, K \cong K^{\Delta}$ is almost simple with nonabelian simple socle.

Since $K$ is 2 -transitive of degree $c=2 p$, as mentioned above, $K$ has only one conjugacy class of subgroups of index $2 p$, and so $K$ has a unique 2-transitive representation of degree $c$, up to permutational equivalence. It follows that, for $\alpha \in \Delta$, the stabiliser $K_{\alpha}$ fixes exactly one point in each part of $\mathcal{C}$. Let $\beta$ be another point fixed by $K_{\alpha}$. Let $B_{1}, B_{2} \in \mathcal{B}$ be the two blocks containing $\{\alpha, \beta\}$. By Claim 1, $\left|B_{i} \cap \Delta\right|=2$ for each $i$ and hence $K_{\alpha \beta}$ fixes setwise $\left(B_{1} \cup B_{2}\right) \cap \Delta$, a set of size 2 or 3 . On the other hand $K_{\alpha \beta}=K_{\alpha}$ since $\beta$ is a fixed point of $K_{\alpha}$, and by Claim $2, K$ is 2 -transitive on $\Delta$, so the $K_{\alpha}$-orbits in $\Delta$ have sizes $1, c-1$. This final contradiction completes the proof of Theorem 1.1.

## 4 Proof of Theorem 1.2

Our first result in this section proves that the designs arising from Construction 1.1 are all 2-designs, and inherit certain symmetry properties from those of the input design. In particular we show that the designs coming from projective geometries over a field of three elements give examples for Theorem 1.2 ,

Proposition 4.1. Let $\mathcal{S}=(\mathcal{P}, \mathcal{L})$ be a $2-(v, k, 1)$ design, $\ell \in \mathcal{L}$ and $G \leqslant \operatorname{Aut}(\mathcal{S})$.
(i) Then the design $\mathcal{D}(\mathcal{S})$ given in Construction 1.1 is a non-symmetric $2-(v, k-1, k-2)$ design and $G$ is a subgroup of $\operatorname{Aut}(\mathcal{D}(\mathcal{S}))$;
(ii) Moreover, if $G$ is flag-transitive on $\mathcal{S}$ and $G_{\ell}$ is 2 -transitive on $\ell$, then $G$ is flagtransitive and point-primitive on $\mathcal{D}(\mathcal{S})$;
(iii) In particular, if $\mathcal{S}$ is the design of points and lines of the projective space $\operatorname{PG}(n-1,3)$ ( $n \geqslant 3$ ), and $G \geqslant \operatorname{PSL}(n, 3)$, then $\mathcal{D}(\mathcal{S})$ is a non-symmetric $G$-flag-transitive, $G$-pointprimitive 2-( $v, 3,2)$ design.

Proof. Let $\mathcal{D}=\mathcal{D}(\mathcal{S})$ with block set $\mathcal{B}=\{\ell \backslash\{\alpha\} \mid \ell \in \mathcal{L}, \alpha \in \ell\}$, so $\mathcal{D}=(\mathcal{P}, \mathcal{B})$. Let $\alpha, \beta$ be distinct points of $\mathcal{P}$. Then there exists a unique line $\ell \in \mathcal{L}$, such that $\alpha, \beta \in \ell$. As $|\ell|=k$, exactly $k-2$ blocks of $\mathcal{B}$ contain $\alpha$ and $\beta$. Thus, $\mathcal{D}$ is a $2-(v, k-1, k-2)$ design, which is nontrivial provided that $3<k$. By Lemma 2.1 applied to $\mathcal{S},|\mathcal{L}| \geqslant v$, and since $|\mathcal{B}|=k|\mathcal{L}|>|\mathcal{L}|$ it follows that $\mathcal{D}$ is not symmetric. Moreover, for all $B=\ell \backslash\{\alpha\} \in \mathcal{B}$ and for all $g \in G \leqslant \operatorname{Aut}(\mathcal{S})$, we have $\ell^{g} \in \mathcal{L}$ and $\alpha^{g} \in \ell^{g}$, and so $B^{g}=(\ell \backslash\{\alpha\})^{g}=\ell^{g} \backslash\left\{\alpha^{g}\right\} \in \mathcal{B}$. Thus, $G \leqslant \operatorname{Aut}(\mathcal{D})$ and part (i) is proved.

Now assume that $G$ is flag-transitive on $\mathcal{S}$ and $G_{\ell}$ is 2 -transitive on $\ell$. Let $\alpha \in \ell$ and $B=\ell \backslash\{\alpha\}$. From the flag-transitivity of $G$, we know that $G$ acts primitively on the point set $\mathcal{P}$ by [11, Propositions 1-3], and $G$ acts transitively on the block set $\mathcal{B}$ of $\mathcal{D}$. Furthermore, $G_{\ell, \alpha} \leqslant G_{B}$. Since $G_{\ell}$ is 2-transitive on $\ell, G_{\ell, \alpha}$ is transitive on $B$. Hence $G_{B}$ is transitive on $B$, and so $G$ is flag-transitive on $\mathcal{D}$ and part (ii) is proved.

In the special case where $\mathcal{S}$ is the design of points and lines of the projective space $\operatorname{PG}(n-1,3)(n \geqslant 3)$, and $H=\operatorname{PSL}(n, 3), H$ is flag-transitive on $\mathcal{S}$ and $H_{\ell}$ induces the 2-transitive group $\operatorname{PGL}(2,3) \cong \mathrm{S}_{4}$ on $\ell$. Thus part (iii) follows from part (i) and (ii) for any group $G$ such that $H \leqslant G \leqslant \operatorname{Aut}(G)$.

### 4.1 Broad proof strategy and the natural projective action

In the remainder of the paper we assume the following hypothesis:
Hypothesis 4.1. Let $\mathcal{D}=(\mathcal{P}, \mathcal{B})$ be a $2-(v, k, 2)$ design admitting a flag-transitive pointprimitive group $G$ of automorphisms with socle $X=\operatorname{PSL}(n, q)$ for some $n \geqslant 3$, where $q=p^{f}$ with prime $p$ and positive integer $f$.

Observe that $G \cap \operatorname{P\Gamma L}(n, q)$ has a natural projective action on a vector space $V$ of dimension $n$ over the field $\mathbb{F}_{q}$. Consider a point $\alpha$ of $\mathcal{D}$ and a basis $v_{1}, v_{2}, \ldots, v_{n}$ of the vector space $V$. Since $G$ is primitive on $\mathcal{P}$, the stabiliser $G_{\alpha}$ is maximal in $G$, and so by Aschbacher's Theorem [2](see also [14]), $G_{\alpha}$ lies in one of the geometric subgroup families $\mathcal{C}_{i}(1 \leqslant i \leqslant 8)$, or in the family $\mathcal{C}_{9}$ of almost simple subgroups not contained in any of these families. When investigating the subgroups in the Aschbacher families, we make frequent use of the information on their structures in [14, Chap. 4]. We will sometimes use the symbol $\tilde{H}$ to indicate that we are giving the structure of the pre-image of $H$ in the corresponding (semi)linear group.

In the next proposition we treat the case where $\mathcal{P}$ is the point set of the projective space $\mathrm{PG}(n-1, q)$ associated with $V$.

Proposition 4.2. Assume Hypothesis 4.1, and that $\mathcal{P}$ is the point set of the projective space $\operatorname{PG}(n-1, q)$, with $G$ acting naturally on $\mathcal{P}$. Then either
(a) $q=3, k=3, v=\left(3^{n}-1\right) / 2$ and $\mathcal{D}=\mathcal{D}(\mathcal{S})$ from Construction 1.1, where $\mathcal{S}$ is the design of points and lines of $\mathrm{PG}(n-1,3)$; or
(b) $q=2, k=4, v=2$ and $\mathcal{D}$ is the complement of the Fano plane (that is, blocks are the complements of the lines in $\operatorname{PG}(2,2)$ ).

Proof. Let $\alpha, \beta$ be distinct points. Since $\lambda=2$, there are exactly two blocks $B_{1}$ and $B_{2}$ containing $\alpha$ and $\beta$. Moreover, $G_{\alpha \beta}$ fixes $B_{1} \cup B_{2}$ setwise, so $B_{1} \cup B_{2}$ is a union of $G_{\alpha, \beta^{-}}$ orbits. Let $\ell$ be the unique projective line containing $\alpha$ and $\beta$. Then $G_{\alpha, \beta}$ is transitive on the $v-(q+1)$ points $\mathcal{P} \backslash \ell$ and on $\ell \backslash\{\alpha, \beta\}$. Hence, either

1. $\left(B_{1} \cup B_{2}\right) \backslash\{\alpha, \beta\} \supseteq \mathcal{P} \backslash \ell$, or
2. $B_{1} \cup B_{2}=\ell$.

Suppose first that $\left(B_{1} \cup B_{2}\right) \backslash\{\alpha, \beta\} \supseteq \mathcal{P} \backslash \ell$. Then $2 k-2 \geqslant\left|B_{1} \cup B_{2}\right| \geqslant 2+v-(q+1)$, that is $k-1 \geqslant(v-q+1) / 2$. Now $r(k-1)=2(v-1)\left(\right.$ Lemma 2.1) and $v=\left(q^{n}-1\right) /(q-1)$, so that

$$
\begin{equation*}
r=\frac{2(v-1)}{k-1} \leqslant \frac{4(v-1)}{v-q+1}=4 \cdot\left(1+\frac{q-2}{q^{n-1}+\cdots+q^{2}+2}\right)<8 \tag{4.8}
\end{equation*}
$$

Since $r \geqslant k$, we have that $k \leqslant 7$. Now combining this with $k-1 \geqslant(v-q+1) / 2$, we have that $12 \geqslant 2(k-1) \geqslant q^{n-1}+\cdots+q^{2}+2$. If $n \geqslant 4$, then $12 \geqslant q^{n-1}+\cdots+q^{2}+2 \geqslant q^{3}+q^{2}+2 \geqslant$ $2^{3}+2^{2}+2=14$, a contradiction. So $n=3$ and $12 \geqslant q^{2}+2$, which implies that $q \leqslant 3$. If $q=3$, then $v=13$, and $6 \geqslant k-1 \geqslant(v-q+1) / 2$ implies that $k=7$. Now $r(k-1)=2(v-1)$ implies that $r=4$, contradicting $r \geqslant k$. Hence $(n, q)=(3,2)$. Then $v=7, k-1 \geqslant 3$, and $r=2(v-1) /(k-1) \leqslant 4$, and so $r \leqslant 4 \leqslant k$. Since $r \geqslant k$, we get that $r=k=4$, and thus $b=(v r) / k=7$. Thus, $\mathcal{D}$ is a symmetric $2-(7,4,2)$ design with $X=\operatorname{PSL}(3,2)$. Since $k=4$, and $G_{B}$ is transitive on the block $B$, it follows that $B$ does not contain a line of $\mathrm{PG}(2,2)$. The only possibility is that $B=\mathcal{P} \backslash \ell^{\prime}$, where $\ell^{\prime}$ is a line of $\operatorname{PG}(2,2)$, that is, the blocks are complements of the lines of $\operatorname{PG}(2,2)$. Hence $\mathcal{D}$ is the complement of the Fano projective plane and (b) holds.

Now assume that $B_{1} \cup B_{2}=\ell$, and every block is contained in a line of the projectice space. We get $2 k-2 \geqslant\left|B_{1} \cup B_{2}\right|=q+1$, while $q+1=|\ell|>\left|B_{i}\right|=k$. Hence $q>k-1 \geqslant$ $(q+1) / 2>q / 2$.

Assume that there are $s$ blocks of $\mathcal{D}$ through $\alpha$ contained in the projective line $\ell$. Since $G$ acts flag-transitively on the projective space $\operatorname{PG}(n-1, q)$, for any projective line $\ell^{\prime}$ and any point $\alpha^{\prime} \in \ell^{\prime}$, there are $s$ blocks containing $\alpha^{\prime}$ that are contained in $\ell^{\prime}$. Since for any two distinct points, there is a unique projective line containing them, the sets of blocks on $\alpha$ that are contained in distinct lines $\ell, \ell^{\prime}$ through $\alpha$ are disjoint. Note that there are $\left(q^{n-1}-1\right) /(q-1)$ projective lines through $\alpha$, so the number of blocks through $\alpha$ is $r=s\left(q^{n-1}-1\right) /(q-1)$.

As $r(k-1)=2(v-1)$, it follows that $s(k-1)\left(q^{n-1}-1\right) /(q-1)=2\left(\left(q^{n}-1\right) /(q-1)-1\right)$, so $s(k-1)=2 q$. Then it follows from $q>k-1>q / 2$ that $1>2 / s>1 / 2$, and so $s=3$. Thus there are 3 blocks through $\alpha$ contained in $\ell$, and $k-1=2 q / 3$, so $q=3^{f}$ for some $f$, and $k=2 \cdot 3^{f-1}+1$.

Assume that there are $c$ blocks of $\mathcal{D}$ contained in the projective line $\ell$. Since $G$ acts transitively on the projective lines, for any projective line $\ell^{\prime}$, there are $c$ blocks contained in $\ell^{\prime}$. Now, counting the number of flags $(\gamma, B)$ in two ways, where $\gamma \in \ell$ and $B \subseteq \ell$ for a fixed line $\ell$, we have that $3(q+1)=c k$, so $3\left(3^{f}+1\right)=c\left(2 \cdot 3^{f-1}+1\right)$, which can be rewritten as $3^{f-1}(9-2 c)=c-3$. Suppose $f \geqslant 2$. Then 3 divides $c$ : when $c=3$, the equation cannot hold, and when $c \geqslant 6$ the left hand side is negative while the right hand side is positive. Hence $f=1, q=3, k=3$, and $c=4$. Therefore, the blocks contained in $\ell$ are all the sets $\ell \backslash\{\gamma\}$, for $\gamma \in \ell$, and this implies that $\mathcal{B}=\{\ell \backslash\{\gamma\} \mid \ell \in \mathcal{L}, \gamma \in \ell\}$. Therefore, $\mathcal{D}=\mathcal{D}(\mathcal{S})$ is the design in Construction 1.1, where $\mathcal{S}$ is the design of points and lines of $\operatorname{PG}(n-1,3)$.

In what follows, we analyse each of the families $\mathcal{C}_{1}-\mathcal{C}_{9}$ for $G_{\alpha}$.

## $4.2 \quad \mathcal{C}_{1}$-subgroups

In this analysis we repeatedly use the Gaussian binomial coefficient $\left[\begin{array}{c}m \\ i\end{array}\right]_{q}$ for the number of $i$-spaces in an $m$-dimensional space $\mathbb{F}_{q}^{m}$, where $0 \leqslant i \leqslant m$. A straightforward argument counting bases of $\mathbb{F}_{q}^{m}$ and its subspaces shows that, for $i \geqslant 1$,

$$
\left[\begin{array}{c}
m \\
i
\end{array}\right]_{q}=\frac{\left(q^{m}-1\right)\left(q^{m}-q\right) \cdots\left(q^{m}-q^{i-1}\right)}{\left(q^{i}-1\right)\left(q^{i}-q\right) \cdots\left(q^{i}-q^{i-1}\right)}=\frac{\prod_{j=1}^{i}\left(q^{m-i+j}-1\right)}{\prod_{j=1}^{i}\left(q^{j}-1\right)}=\prod_{j=1}^{i} \frac{q^{m-i+j}-1}{q^{j}-1} .
$$

We use this equality without further comment. We also use the facts that $\left[\begin{array}{c}m \\ i\end{array}\right]_{q}=\left[\begin{array}{c}m \\ m-i\end{array}\right]_{q}$, that the number of complements in $\mathbb{F}_{q}^{m}$ of a given $i$-space is $q^{i(m-i)}$, and hence that the number of decompositions $U \oplus W$ of $\mathbb{F}_{q}^{m}$ with $\operatorname{dim}(U)=i$ is $\left[\begin{array}{c}m \\ i\end{array}\right]_{q} \cdot q^{i(m-i)}$.

Lemma 4.1. Assume Hypothesis 4.1. If the point-stabilizer $G_{\alpha} \in \mathcal{C}_{1}$, then $G_{\alpha}$ is the stabiliser in $G$ of an $i$-space and $G \leqslant \operatorname{P\Gamma L}(n, q)$.

Proof. If $G \leqslant \operatorname{P\Gamma L}(n, q)$ then $G_{\alpha}$ is the stabiliser in $G$ of an $i$-space, for some $i$, so assume that $G \not \leq \mathrm{P} \Gamma \mathrm{L}(n, q)$. Then $G$ contains a graph automorphism of $\operatorname{PSL}(n, q)$, so in particular $n \geqslant 3$, and $G_{\alpha}$ stabilizes a pair $\{U, W\}$ of subspaces $U$ and $W$, where $U$ has dimension $i$ and
$W$ has dimension $n-i$ with $1 \leqslant i<n / 2$. It follows that $G^{*}:=G \cap \operatorname{P\Gamma L}(n, q)$ has index 2 in $G$. Moreover, either $U \subseteq W$ or $U \cap W=0$.

Case 1: $U \subset W$.
In this case, $v$ is the number $\left[\begin{array}{c}n \\ n-i\end{array}\right]_{q}$ of $(n-i)$-spaces $W$ in $V$, times the number $\left[\begin{array}{c}n-i \\ i\end{array}\right]_{q}$ of $i$-spaces $U$ in $W$, so

$$
\begin{aligned}
v & =\left[\begin{array}{c}
n \\
n-i
\end{array}\right]_{q} \cdot\left[\begin{array}{c}
n-i \\
i
\end{array}\right]_{q}=\prod_{j=1}^{n-i} \frac{q^{n-(n-i)+j}-1}{q^{j}-1} \cdot \prod_{j=1}^{i} \frac{q^{(n-i)-i+j}-1}{q^{j}-1} \\
& =\prod_{j=1}^{n-i}\left(q^{i+j}-1\right) /\left(\prod_{j=1}^{n-2 i}\left(q^{j}-1\right) \cdot \prod_{j=1}^{i}\left(q^{j}-1\right)\right) \\
& =\prod_{j=1}^{i} \frac{q^{i+j}-1}{q^{j}-1} \cdot \prod_{j=1}^{n-2 i} \frac{q^{2 i+j}-1}{q^{j}-1} .
\end{aligned}
$$

Then, using the fact that $q^{m}-1>q^{m-j}\left(q^{j}-1\right)$, for integers $1 \leqslant j<m$,

$$
v>\prod_{j=1}^{i} q^{i} \cdot \prod_{j=1}^{n-2 i} q^{2 i}=q^{i^{2}+2 i(n-2 i)}=q^{i(2 n-3 i)}
$$

Consider the following points of $\mathcal{D}: \alpha=\{U, W\}$, where $W=\left\langle v_{1}, v_{2}, \ldots, v_{n-i}\right\rangle$ and $U=\left\langle v_{1}, v_{2}, \ldots, v_{i}\right\rangle$, and $\beta=\left\{U^{\prime}, W\right\}$, where $U^{\prime}=\left\langle v_{1}, v_{2}, \ldots, v_{i-1}, v_{i+1}\right\rangle$. Then the $G_{\alpha}^{*}$ orbit $\Delta$ containing $\beta$ consists of all the points $\left\{U^{\prime \prime}, W\right\}$ such that the $i$-space $U^{\prime \prime} \subset W$ and $\operatorname{dim}\left(U \cap U^{\prime \prime}\right)=i-1$. Thus the cardinality $|\Delta|$ is the number $\left[\begin{array}{c} \\ i \\ i-1\end{array}\right]_{q}$ of $(i-1)$-spaces $U \cap U^{\prime \prime}$ in $U$, times the number $\left[\begin{array}{c}n-2 i+1 \\ 1\end{array}\right]_{q}-1$ of 1-spaces in $W /\left(U \cap U^{\prime \prime}\right)$ distinct from $U /\left(U \cap U^{\prime \prime}\right)$. Therefore, since $\left[\begin{array}{c}i \\ i-1\end{array}\right]_{q}=\left[\begin{array}{c}i \\ 1\end{array}\right]_{q}$,

$$
\left|G_{\alpha}^{*}: G_{\alpha \beta}^{*}\right|=|\Delta|=\left[\begin{array}{l}
i \\
1
\end{array}\right]_{q} \cdot\left(\left[\begin{array}{c}
n-2 i+1 \\
1
\end{array}\right]_{q}-1\right)=\frac{q^{i}-1}{q-1} \cdot \frac{q\left(q^{n-2 i}-1\right)}{q-1} .
$$

Note that $G_{\alpha}$ contains a graph automorphism, and each such graph automorphism interchanges $U$ and $W$, and hence does not leave $\Delta$ invariant. Thus the $G_{\alpha}$-orbit containing $\beta$ has cardinality $2|\Delta|$ (a subdegree of $G$ ), so by Lemma 2.2 (iv), $r$ divides

$$
4|\Delta|=\frac{4 q\left(q^{n-2 i}-1\right)\left(q^{i}-1\right)}{(q-1)^{2}}
$$

Note that $\left(q^{j}-1\right) /(q-1)<2 q^{j-1}$ for each integer $j>0$. It follows that

$$
r \leqslant \frac{4 q\left(q^{n-2 i}-1\right)\left(q^{i}-1\right)}{(q-1)^{2}}<4 q \cdot 2 q^{n-2 i-1} \cdot 2 q^{i-1}=16 q^{n-i-1}
$$

Combining this with $r^{2}>2 v$ and $v>q^{i(2 n-3 i)}$, we see that $16^{2} q^{2(n-i-1)}>2 q^{i(2 n-3 i)}$, that is,

$$
\begin{equation*}
2^{7}>q^{2(i-1) n-3 i^{2}+2 i+2} \geqslant 2^{2(i-1) n-3 i^{2}+2 i+2} . \tag{4.9}
\end{equation*}
$$

Since $n>2 i$, it follows that $2(i-1) n-3 i^{2}+2 i+2>4 i(i-1)-3 i^{2}+2 i+2=i^{2}-2 i+2$, and so $i^{2}-2 i-5<0$, which implies $i \leqslant 3$.

Subcase 1.1: $i=3$.
Then $n>2 i=6$. From (4.9) we have $2^{7}>q^{4 n-19} \geqslant 2^{4 n-19}$, which implies $n \leqslant 6$, a contradiction.

Subcase 1.2: $i=2$.
Then $n>4$. From (4.9) we have $2^{7}>q^{2 n-6} \geqslant 2^{2 n-6}$, which implies $n=5$ or 6 . Then $r \mid 4 q(q+1)^{n-4}$ (for $n=5$ or 6 ) and $v>q^{4 n-12}$. Combining this with $r^{2}>2 v$, we deduce $16 q^{2}(q+1)^{2 n-8}>2 q^{4 n-12}$, that is, $8(q+1)^{2 n-8}>q^{4 n-14}$. For $n=6$, this gives $8(q+1)^{4}>q^{10}$, which is impossible. Thus $n=5$ and $8(q+1)^{2}>q^{6}$, so $q=2$ and $v=5 \cdot 7 \cdot 31$. On the one hand $r \mid 24$ and on the other hand the condition $r^{2}>2 v$ implies $r \geqslant 47$, a contradiction.

Subcase 1.3: $i=1$.
Then $n>2$, $r$ divides $4 q\left(q^{n-2}-1\right) /(q-1)$, and

$$
v=\frac{\left(q^{n}-1\right)\left(q^{n-1}-1\right)}{(q-1)^{2}}
$$

Combining this with the condition $r \mid 2(v-1)$, we seee that $r$ divides

$$
\begin{aligned}
R: & =\operatorname{gcd}\left(2(v-1), \frac{4 q\left(q^{n-2}-1\right)}{q-1}\right) \\
& =2 \operatorname{gcd}\left(\frac{\left(q^{n}-1\right)\left(q^{n-1}-1\right)}{(q-1)^{2}}-1, \frac{2 q\left(q^{n-2}-1\right)}{q-1}\right) \\
& =\frac{2 q}{(q-1)^{2}} \cdot \operatorname{gcd}\left(q^{2 n-2}-q^{n-1}-q^{n-2}-q+2,2(q-1)\left(q^{n-2}-1\right)\right)
\end{aligned}
$$

Since

$$
\left(q^{2 n-2}-q^{n-1}-q^{n-2}-q+2\right)-(q-1)^{2}=\left(q^{n}+q^{2}-q-1\right)\left(q^{n-2}-1\right)
$$

is divisible by $(q-1)\left(q^{n-2}-1\right)$, we see that

$$
\operatorname{gcd}\left(q^{2 n-2}-q^{n-1}-q^{n-2}-q+2,(q-1)\left(q^{n-2}-1\right)\right) \text { divides }(q-1)^{2}
$$

and so

$$
\operatorname{gcd}\left(q^{2 n-2}-q^{n-1}-q^{n-2}-q+2,2(q-1)\left(q^{n-2}-1\right)\right) \text { divides } 2(q-1)^{2} .
$$

Therefore, $R$ divides

$$
\frac{2 q}{(q-1)^{2}} \cdot 2(q-1)^{2}=4 q
$$

Combining this with $r \mid R, r^{2}>2 v$ and $v>q^{2 n-3}$, we deduce $16 q^{2}>2 q^{2 n-3}$. Therefore, $8>q^{2 n-5} \geqslant 2^{2 n-5}$, which leads to $n=3$, and $q<8$. Note that $v=\left(q^{2}+q+1\right)(q+1)$, so $R=\operatorname{gcd}(2(v-1), 4 q)=2 \operatorname{gcd}\left(q\left(q^{2}+2 q+2\right), 2 q\right)=2 q \operatorname{gcd}\left(q^{2}+2 q+2,2\right)$. When $q$ is odd we see that $R=2 q$. Then $r^{2}>2 v$ leads to $2\left(q^{2}+q+1\right)(q+1)<r^{2} \leqslant R^{2}=4 q^{2}$, which is not possible. Hence $q \in\{2,4\}$ and $R=4 q$.

First assume that $q=4$. Then $v=105$ and $R=16$. Combining this with $r \mid R$ and $r^{2}>2 v$, we conclude that $r=16$. Then it follows from $r(k-1)=2(v-1)$ and $b k=v r$ that
$k=14$ and $b=120$. Since $G$ is block-transitive, it follows that $X:=\operatorname{Soc}(G)=\operatorname{PSL}(3,4)$ has equal length orbits on blocks, of length dividing $b=120$. This implies that $X$ has a maximal subgroup of index dividing 120 , and hence by [7, page 23], we conclude that $X$ is primitive on blocks, that the stabiliser $X_{B}$ of a block $B$ is a maximal $\mathcal{C}_{5}$-subgroup stabilising an $\mathbb{F}_{2}$-structure $V_{0}=\mathbb{F}_{2}^{3}<V$, and $X_{B}$ has two orbits on 1 -spaces, and on 2-spaces in $V$. An easy computation shows that $X_{B}$ has precisely four orbits on the point set $\mathcal{P}$, of lengths 14, 14, 21, 56: these are subsets of flags $\{U, W\}$ determined by whether $U \cap V_{0}$ contains a non-zero vector or not, and whether $W \cap V_{0}$ is a 2 -space of $V_{0}$ or not. Since $X_{B}$ preserves the $k=14$ points of $B$, it follows that $B$ is equal to one of the $X_{B}$-orbits of length 14 , so that $X$ acts flag-transitively and point-imprimitively on $\mathcal{D}$, contradicting Theorem 1.1. (In fact $G_{B}$ interchanges the two $X_{B}$-orbits of length 14 and so $G_{B}$ does not leave invariant a point-subset of size 14.)

Thus $q=2$. Then $v=21$ and $R=8$ and $G=\operatorname{PSL}(3,2) .2 \cong \operatorname{PGL}(2,7)$. This together with $r \mid R$ and $r^{2}>2 v$ implies $r=8$. Then we derive from $r(k-1)=2(v-1)$ and $b k=v r$ that $k=6$ and $b=28$. However, one can see from [6, II.1.35] that there is no $2-(21,6,2)$ design, a contradiction. We also checked with Magma that considering every subgroup of index 28 as a block stabiliser, and each of its orbits of size 6 as a possible block, the orbit of that block under $G$ does not yield a 2-design.

Case 2: $V=U \oplus W$.
In this case the number $v$ of points is the number $\left[\begin{array}{c}n \\ i\end{array}\right]_{q}$ of $i$-spaces $U$ of $V$, times the number $q^{i(n-i)}$ of complements $W$ to $U$ in $V$, so

$$
v=q^{i(n-i)} \prod_{j=1}^{i} \frac{q^{n-i+j}-1}{q^{j}-1}
$$

so in particular $p \mid v$, and by Lemma 2.3(iii), $r_{p}$ divides 2 .
Note that $q^{i}-1>q^{i-j}\left(q^{j}-1\right)$, for integers $i>j$. Thus

$$
v>q^{i(n-i)} \prod_{j=1}^{i} q^{n-i}=q^{i(n-i)}\left(q^{n-i}\right)^{i}=q^{2 i(n-i)} .
$$

We consider the point $\alpha=\{U, W\}$ with $U=\left\langle v_{1}, \ldots, v_{i}\right\rangle, W=\left\langle v_{i+1}, \ldots, v_{n}\right\rangle$ and the $G_{\alpha}^{*}-$ orbit $\Delta$ containing $\beta=\left\{U^{\prime}, W^{\prime}\right\}$ with $U^{\prime}=\left\langle v_{1}, \ldots, v_{i-1}, v_{i+1}\right\rangle, W^{\prime}=\left\langle v_{i}, v_{i+2}, \ldots, v_{n}\right\rangle$. Then $\Delta$ consists of all $\left\{U^{\prime \prime}, W^{\prime \prime}\right\}$ with $\operatorname{dim}\left(U^{\prime \prime} \cap U\right)=i-1, \operatorname{dim}\left(W^{\prime \prime} \cap W\right)=n-i-1, \operatorname{dim}\left(U^{\prime \prime} \cap W\right)=$ $\operatorname{dim}\left(W^{\prime \prime} \cap U\right)=1$, so $|\Delta|$ is the number $\left[\begin{array}{l}i \\ 1\end{array}\right]_{q} \cdot q^{i-1}$ of decompositions $U=\left(U^{\prime \prime} \cap U\right) \oplus\left(W^{\prime \prime} \cap U\right)$, times the number $\left[\begin{array}{c}n-i \\ 1\end{array}\right]_{q} \cdot q^{n-i-1}$ of decompositions $W=\left(U^{\prime \prime} \cap W\right) \oplus\left(W^{\prime \prime} \cap W\right)$. Thus

$$
\left|G_{\alpha}^{*}: G_{\alpha \beta}^{*}\right|=|\Delta|=q^{i-1} \frac{q^{i}-1}{q-1} \cdot q^{n-i-1} \frac{q^{n-i}-1}{q-1}=q^{n-2} \frac{\left(q^{i}-1\right)\left(q^{n-i}-1\right)}{(q-1)^{2}},
$$

and $G$ has a subdegree $|\Delta|$ or $2|\Delta|$. By Lemma 2.2(iv), $r$ divides $4|\Delta|$. Since $r_{p} \mid 2$, we deduce that $r$ divides $4\left(q^{i}-1\right)\left(q^{n-i}-1\right) /(q-1)^{2}$ (and even $2\left(q^{i}-1\right)\left(q^{n-i}-1\right) /(q-1)^{2}$ if $q$ is even). Let $a=1$ if $q$ is even and 2 otherwise. Then $r$ divides $2^{a}\left(q^{i}-1\right)\left(q^{n-i}-1\right) /(q-1)^{2}$

Considering the inequality $r^{2}>2 v>2 q^{2 i(n-i)}$ and the fact that $\left(q^{j}-1\right) /(q-1)<2 q^{j-1}$ for each integer $j>0$, it follows that

$$
\begin{equation*}
q^{2 i(n-i)}<2^{2 a-1} \frac{\left(q^{i}-1\right)^{2}\left(q^{n-i}-1\right)^{2}}{(q-1)^{4}}<2^{2 a-1}\left(2 q^{i-1}\right)^{2}\left(2 q^{n-i-1}\right)^{2}=2^{2 a+3} q^{2 n-4} \tag{4.10}
\end{equation*}
$$

Thus $2^{2 a+3}>q^{2 n(i-1)-2 i^{2}+4} \geqslant 2^{2 n(i-1)-2 i^{2}+4}$, so (since $n>2 i$ )

$$
2 a-2 \geqslant 2 n(i-1)-2 i^{2}>4 i(i-1)-2 i^{2}=2 i(i-2) .
$$

Hence $i=1$ or 2 , and the case $i=2$ only happens if $a=2$, that is if $q$ is odd.
Assume $i=2$, so $q$ is odd. Then $2 \geqslant 2 n(i-1)-2 i^{2}=2 n-8$, so $n \leqslant 5$. On the other hand $n>2 i$, so $n=5$. By (4.10) $q^{12}<2^{7} q^{6}$, so $q^{6}<2^{7}$, a contradiction since $q \geqslant 3$. Therefore $i=1$. In this case $v=q^{n-1} \frac{q^{n}-1}{q-1}$ and we compute that $v-1=\frac{q^{n-1}-1}{q-1} \cdot\left(q^{n}+q-1\right)$. Since $r \mid 2(v-1), r$ divides $\operatorname{gcd}\left(2 \frac{q^{n-1}-1}{q-1} \cdot\left(q^{n}+q-1\right), 4 \frac{q^{n-1}-1}{q-1}\right)=2 \frac{q^{n-1}-1}{q-1} \operatorname{gcd}\left(q^{n}+q-1,2\right)=2 \frac{q^{n-1}-1}{q-1}$. In other words $a=1$ in the computation above whether $q$ is odd or even. Then by (4.10) $q^{2(n-1)}<2\left(q^{n-1}-1\right)^{2} /(q-1)^{2}<2\left(2 q^{n-2}\right)^{2}=2^{3} q^{2 n-4}$, which can be rewritten as $q^{2}<2^{3}$, so $q=2$. Thus $v=2^{n-1}\left(2^{n}-1\right)$ and $r$ divides $2\left(2^{n-1}-1\right)$, so $r^{2}>2 v$ implies that $2^{2 n-1}-2^{n-1}<2\left(2^{n-1}-1\right)^{2}=2^{2 n-1}-2^{n+1}+2$, which is impossible.

Lemma 4.2. Assume Hypothesis 4.1, and that the point-stabilizer $G_{\alpha} \in \mathcal{C}_{1}$. Then either
(a) $\mathcal{D}=\mathcal{D}(\mathcal{S})$ is as in Construction 1.1, where $\mathcal{S}$ is the design of points and lines of $\mathrm{PG}(n-$ $1,3)$; or
(b) $\mathcal{D}$ is the complement of the Fano plane.

Proof. By Lemma 4.1, $G \leqslant \operatorname{P\Gamma L}(n, q)$, and $G_{\alpha} \cong \mathrm{P}_{\mathrm{i}}$ is the stabiliser of a subspace $W$ of $V$ of dimension $i$, for some $i$. As we will work with the action on the underlying space $V$ we will usually consider a linear group $\tilde{G}$ satisfying $\tilde{X}=\operatorname{SL}(n, q) \leqslant \tilde{G} \leqslant \Gamma \mathrm{~L}(n, q)$, acting unfaithfully on $\mathcal{P}$ with kernel a subgroup of scalars. By Proposition 4.2 we may assume that $i \geqslant 2$. Also, on applying a graph automorphism that interchanges $i$-spaces and $(n-i)$-spaces (and replacing $\mathcal{D}$ by an isomorphic design) we may assume further that $i \leqslant n / 2$. Then $v$ is the number of $i$-spaces:

$$
v=\left[\begin{array}{c}
n \\
i
\end{array}\right]_{q}=\prod_{j=1}^{i} \frac{q^{n-i+j}-1}{q^{j}-1} .
$$

Using the fact that $q^{i}-1>q^{i-j}\left(q^{j}-1\right)$, for integers $i>j$, it follows that $v>q^{i(n-i)}$.
Consider the following points of $\mathcal{D}: \alpha=W$, where $W=\left\langle v_{1}, v_{2}, \ldots, v_{i}\right\rangle$, and $\beta=W^{\prime}$, where $W^{\prime}=\left\langle v_{1}, v_{2}, \ldots, v_{i-1}, v_{i+1}\right\rangle$. Then the $\tilde{G}_{\alpha}$-orbit $\Delta$ containing $\beta$ consists of all the points $W^{\prime \prime}$ such that $\operatorname{dim}\left(W \cap W^{\prime \prime}\right)=i-1$. Thus the cardinality $|\Delta|$ is the number $\left[\begin{array}{c}i \\ i-1\end{array}\right]_{q}$ of $(i-1)$-spaces $W \cap W^{\prime \prime}$ in $W$, times the number $\left[\begin{array}{c}n-i+1 \\ 1\end{array}\right]_{q}-1$ of 1 -spaces in $V /\left(W \cap W^{\prime \prime}\right)$ distinct from $W /\left(W \cap W^{\prime \prime}\right)$. Therefore, since $\left[\begin{array}{c}i \\ i-1\end{array}\right]_{q}=\left[\begin{array}{c}i \\ 1\end{array}\right]_{q}$,

$$
|\Delta|=\left[\begin{array}{l}
i \\
1
\end{array}\right]_{q} \cdot\left(\left[\begin{array}{c}
n-i+1 \\
1
\end{array}\right]_{q}-1\right)=\frac{q\left(q^{i}-1\right)\left(q^{n-i}-1\right)}{(q-1)^{2}} .
$$

Since $\tilde{G}$ is flag-transitive, $r$ divides $2|\Delta|$ (by Lemma 2.2(iv)). Combining this with $r^{2}>2 v$ (Lemma 2.1(iv)) we have that

$$
\frac{2 q^{2}\left(q^{i}-1\right)^{2}\left(q^{n-i}-1\right)^{2}}{(q-1)^{4}}>\frac{\left(q^{n}-1\right) \cdots\left(q^{n-i+1}-1\right)}{\left(q^{i}-1\right) \cdots(q-1)}>q^{i(n-i)}
$$

Since $2 q^{j-1}>\left(q^{j}-1\right) /(q-1)$ for all $j \in \mathbb{N}$, it follows that

$$
\begin{equation*}
q^{i(n-i)}<\frac{2 q^{2}\left(q^{i}-1\right)^{2}\left(q^{n-i}-1\right)^{2}}{(q-1)^{4}}<2 q^{2}\left(2 q^{i-1}\right)^{2}\left(2 q^{n-i-1}\right)^{2}=32 q^{2 n-2} \leqslant q^{2 n+3} \tag{4.11}
\end{equation*}
$$

Hence

$$
\begin{equation*}
2 n+3>i(n-i) \tag{4.12}
\end{equation*}
$$

and so $i^{2}+3>n(i-2) \geqslant 2 i(i-2)$, which implies that $i \leqslant 4$. Note from Lemma 2.1 that $r \mid 2(v-1)$. Let $R=2 \operatorname{gcd}(|\Delta|, v-1)$. As $r$ divides $2|\Delta|$, it follows that $r$ divides $R$ and hence $r \leqslant R$.

Case 1: $i=4$.
In this case, we derive from (4.12) that $n \leqslant 9$. This together with the restriction $n \geqslant$ $2 i=8$ leads to $n=8$ or 9 . We also deduce from (4.11) that $32 q^{2 n-2}>q^{4(n-4)}$, that is $32>q^{2 n-14}$. First assume that $n=8$. Then $32>q^{2}$, so $q \leqslant 5$. We get

$$
|\Delta|=\frac{q\left(q^{4}-1\right)^{2}}{(q-1)^{2}}
$$

and

$$
v=\frac{\left(q^{8}-1\right)\left(q^{7}-1\right)\left(q^{6}-1\right)\left(q^{5}-1\right)}{\left(q^{4}-1\right)\left(q^{3}-1\right)\left(q^{2}-1\right)(q-1)}
$$

We easily compute that $R=4,6,40,10$ when $q=2,3,4,5$ respectively, in each case contradicting $r^{2}>2 v$, since $r \leqslant R$.

Next assume $n=9$. Then $32>q^{4}$, so $q=2$. We get

$$
|\Delta|=\frac{q\left(q^{4}-1\right)\left(q^{5}-1\right)}{(q-1)^{2}}=930
$$

and

$$
v=\frac{\left(q^{9}-1\right)\left(q^{8}-1\right)\left(q^{7}-1\right)\left(q^{6}-1\right)}{\left(q^{4}-1\right)\left(q^{3}-1\right)\left(q^{2}-1\right)(q-1)}=3309747
$$

Therefore $R=124$, again contradicting $r^{2}>2 v$.
Case 2: $i=3$.
In this case, we derive from (4.12) that $n \leqslant 11$. Together with the restriction $n \geqslant 2 i=6$ leads to $n \in\{6,7,8,9,10,11\}$.

For $n=6,|\Delta|=\frac{q\left(q^{3}-1\right)^{2}}{(q-1)^{2}}=q\left(q^{2}+q+1\right)^{2}$, while

$$
v-1=\frac{\left(q^{6}-1\right)\left(q^{5}-1\right)\left(q^{4}-1\right)}{\left(q^{3}-1\right)\left(q^{2}-1\right)(q-1)}-1=q\left(q^{8}+q^{7}+2 q^{6}+3 q^{5}+3 q^{4}+3 q^{3}+3 q^{2}+2 q+1\right)
$$

Thus $R=2 \operatorname{gcd}(|\Delta|, v-1)=2 q \operatorname{gcd}\left(\left(q^{2}+q+1\right)^{2}, q^{8}+q^{7}+2 q^{6}+3 q^{5}+3 q^{4}+3 q^{3}+3 q^{2}+2 q+1\right)$. Using the Euclidean algorithm, we easily see that

$$
\operatorname{gcd}\left(q^{2}+q+1, q^{8}+q^{7}+2 q^{6}+3 q^{5}+3 q^{4}+3 q^{3}+3 q^{2}+2 q+1\right)=1
$$

so $R=2 q$, contradicting $r^{2}>2 v$.
For $n=7,|\Delta|=\frac{q\left(q^{3}-1\right)\left(q^{4}-1\right)}{(q-1)^{2}}=q\left(q^{2}+q+1\right)(q+1)\left(q^{2}+1\right)$, while
$v-1=\frac{\left(q^{7}-1\right)\left(q^{6}-1\right)\left(q^{5}-1\right)}{\left(q^{3}-1\right)\left(q^{2}-1\right)(q-1)}-1=q\left(q^{2}+1\right)\left(q^{9}+q^{8}+q^{7}+2 q^{6}+3 q^{5}+2 q^{4}+2 q^{3}+2 q^{2}+2 q+1\right)$.
Thus
$R=2 \operatorname{gcd}(|\Delta|, v-1)=2 q\left(q^{2}+1\right) \operatorname{gcd}\left(\left(q^{2}+q+1\right)(q+1), q^{9}+q^{8}+q^{7}+2 q^{6}+3 q^{5}+2 q^{4}+2 q^{3}+2 q^{2}+2 q+1\right)$.
Using the Euclidean algorithm, we easily see that

$$
\operatorname{gcd}\left(q^{2}+q+1, q^{9}+q^{8}+q^{7}+2 q^{6}+3 q^{5}+2 q^{4}+2 q^{3}+2 q^{2}+2 q+1\right)=1
$$

and

$$
\operatorname{gcd}\left(q+1, q^{9}+q^{8}+q^{7}+2 q^{6}+3 q^{5}+2 q^{4}+2 q^{3}+2 q^{2}+2 q+1\right)=1
$$

so $R=2 q\left(q^{2}+1\right)$, contradicting $r^{2}>2 v$.
Assume now that $8 \leqslant n \leqslant 11$. We deduce from (4.11) that $32 q^{2 n-2}>q^{3(n-3)}$, that is $32>q^{n-7}$. So there are only a finite number of cases to consider and we easily check that for all of them, $R^{2}<2 v$, a contradiction.

Case 3: $i=2$.
In this case, the point set is the set of 2 -spaces and $n \geqslant 4$, but the above restrictions on $r$ do not lead easily to contradictions as they do for larger values of $i$. So we have a different approach. Recall that $\tilde{X}=\operatorname{SL}(n, q) \leqslant \tilde{G} \leqslant \Gamma \mathrm{~L}(n, q)$, acting unfaithfully on $\mathcal{P}$ (with kernel a scalar subgroup of $\tilde{G})$. First we deal with $n=4$. In this case

$$
v=\frac{\left(q^{4}-1\right)\left(q^{3}-1\right)}{\left(q^{2}-1\right)(q-1)}=\left(q^{2}+1\right)\left(q^{2}+q+1\right), \quad|\Delta|=\frac{q\left(q^{2}-1\right)^{2}}{(q-1)^{2}}=q(q+1)^{2}
$$

and by Lemmas 2.1 and 2.2, $r^{2}>2 v$ and $r$ divides

$$
\begin{aligned}
2 \operatorname{gcd}(v-1,|\Delta|) & =2 \operatorname{gcd}\left(q^{4}+q^{3}+2 q^{2}+q, q(q+1)^{2}\right) \\
& =2 q \operatorname{gcd}\left(q^{3}+q^{2}+2 q+1,(q+1)^{2}\right) \\
& =2 q \operatorname{gcd}\left((q+1)^{2}(q-1)+3 q+2,(q+1)^{2}\right) \\
& =2 q \operatorname{gcd}\left(3 q+2,(q+1)^{2}\right)=2 q
\end{aligned}
$$

which implies $4 q^{2} \geqslant r^{2}>2 v>q^{4}$, a contradiction. Thus $n \geqslant 5$.
Let $H:=\tilde{G} \cap \mathrm{GL}(n, q)$. Then setwise stabiliser $H_{\{\alpha, \beta\}}$ of the points $\alpha=W=\left\langle v_{1}, v_{2}\right\rangle$ and $\beta=W^{\prime}=\left\langle v_{1}, v_{3}\right\rangle$, fixes setwise the two blocks $B_{1}, B_{2}$ of $\mathcal{D}$ containing $\{\alpha, \beta\}$. Also $H_{\{\alpha, \beta\}}$ leaves invariant the spaces $Y=W+W^{\prime}=\left\langle v_{1}, v_{2}, v_{3}\right\rangle$ and $Y^{\prime}=W \cap W^{\prime}=\left\langle v_{1}\right\rangle$, induces $\mathrm{GL}(n-3, q)$ on $V / Y\left(\right.$ since even $\mathrm{SL}(V) \cap\left(\mathrm{GL}\left(\left\langle v_{1}\right\rangle\right) \times \operatorname{GL}\left(\left\langle v_{4}, \ldots, v_{n}\right\rangle\right)\right)$ induces $\mathrm{GL}(n-3, q)$
on $V / Y)$. Moreover $H_{\{\alpha, \beta\}}$ is transitive on $V \backslash Y$, and has orbits of lengths $1,2 q, q^{2}-q$ on the 1-spaces in $Y$. Since $H_{\{\alpha, \beta\}} \cap \tilde{G}_{B_{1}}=H_{\{\alpha, \beta\}} \cap H_{B_{1}}$ is normal of index 1 or 2 in $H_{\{\alpha, \beta\}}$, it follows that $H_{\{\alpha, \beta\}} \cap H_{B_{1}}$ also induces at least $\operatorname{SL}(n-3, q)$ on $V / Y$ and is transitive on $V \backslash Y$. Hence the only non-zero proper subspaces of $V$ left invariant by $H_{\{\alpha, \beta\}} \cap H_{B_{1}}$ are $Y, W, W^{\prime}, Y^{\prime}$, and if $q=2,3$ then possibly also the $q-1$ other 2 -spaces of $Y$ containing $Y^{\prime}$.

We claim that $H_{B_{1}}$ is irreducible on $V$. Suppose to the contrary that $H_{B_{1}}$ leaves invariant a nonzero proper subspace $U$. Then also $H_{\{\alpha, \beta\}} \cap H_{B_{1}}$ leaves $U$ invariant. We see from the previous paragraph that $U$ must be contained in $Y$. If $U=Y$, then as $\tilde{G}_{B_{1}}$ is transitive on the set $\left[B_{1}\right]$ of points of $\mathcal{D}$ incident with $B_{1}$, it follows that all such points must be 2 spaces contained in $Y$. This is impossible since $\operatorname{dim}(Y)=3$, while some block, and hence all blocks, must be incident with a pair of 2-spaces which intersect trivially. Thus $U$ is a proper subspace of $Y$. The only 1-space invariant under $H_{\{\alpha, \beta\}} \cap H_{B_{1}}$ is $Y^{\prime}$, and if $U=Y^{\prime}$ then the same argument would yield that all 2-spaces incident with $B_{1}$ would contain $Y^{\prime}$, which is not true since some block, and hence all blocks, must be incident with a pair of 2-spaces which intersect trivially. Thus $\operatorname{dim}(U)=2$, and $U$ is a 2-space of $Y$ containing $Y^{\prime}$. Since $H_{B_{1}}$ does not fix $\alpha$ or $\beta$, it follows that $U \neq W$ or $W^{\prime}$, and hence $q=2$ or 3 , and $U$ is one of the $q-1$ other 2 -spaces containing $Y^{\prime}$. Again, since $\tilde{G}_{B_{1}}$ is transitive on $\left[B_{1}\right]$, each 2-space $\alpha^{\prime} \in\left[B_{1}\right]$ intersects $U$ in a 1-space. Let $\gamma=W^{\prime \prime}$ be a 2 -space which intersects $\alpha=W$ trivially, and let $B$ be a block of $\mathcal{D}$ containing $\{\alpha, \gamma\}$. Then $H_{B}$ leaves invariant a 2-space, say $U^{\prime}$, and we have shown that both $W \cap U^{\prime}$ and $W^{\prime \prime} \cap U^{\prime}$ have dimension 1 , so $U^{\prime}$ is contained in the 4-space $W \oplus W^{\prime \prime}$. Now the subgroup induced by $H_{\{\alpha, \gamma\}}$ on $W \oplus W^{\prime \prime}$ contains $\mathrm{GL}(W) \times \mathrm{GL}\left(W^{\prime \prime}\right)$. The orbit of $U^{\prime}$ under this group has size $(q+1)^{2}$. However the group $H_{\{\alpha, \gamma\}} \cap H_{B}$ has index at most 2 in $H_{\{\alpha, \gamma\}}$ and fixes $U^{\prime}$, so we have a contradiction. Thus we conclude that $H_{B_{1}}$ is irreducible.

The irreducible group $H_{B_{1}}$ has a subgroup $H_{\{\alpha, \beta\}} \cap H_{B_{1}}$ inducing at least $\operatorname{SL}(n-3, q)$ on $V / Y$. We will apply a deep theorem from [21] which relies on the presence of various prime divisors of the subgroup order $\left|H_{B_{1}}\right|$. For $b, e \geqslant 2$, a primitive prime divisor (ppd) of $b^{e}-1$ is a prime $r$ which divides $b^{e}-1$ but which does not divide $b^{i}-1$ for any $i<e$. Such ppd's are known to exist unless either $(b, e)=(2,6)$, or $e=2$ and $b=2^{s}-1$ for some $s$, (a theorem of Zsigmondy, see [21, Theorem 2.1]). Each ppd $r$ of $b^{e}-1$ satisfies $r \equiv 1(\bmod e)$, and if $r>e+1$ then $r$ is said to be large; usually $b^{e}-1$ has a large ppd and the rare exceptions are known explicitly, see [21, Theorem 2.2]. Also, if $b=p^{f}$ for a prime $p$ then each ppd of $p^{f e}-1$ is a ppd of $b^{e}-1$ (but not conversely) and this type of ppd of $b^{e}-1$ is called basic. We will apply [21, Theorem 4.8] which, in particular, classifies all subgroups $H_{B_{1}}$ with the following properties:

1. for some integer $e$ such that $n / 2<e \leqslant n-4,\left|H_{B_{1}}\right|$ is divisible by a ppd of $q^{e}-1$ and also by a ppd of $q^{e+1}-1$;
2. for some (not necessarily different) integers $e^{\prime}, e^{\prime \prime}$ such that $n / 2<e^{\prime} \leqslant n-3$ and $n / 2<e^{\prime \prime} \leqslant n-3,\left|H_{B_{1}}\right|$ is divisible by a large ppd of $q^{e^{\prime}}-1$ and a basic ppd of $q^{e^{\prime \prime}}-1$.

Since $\left|H_{B_{1}}\right|$ is divisible by $|\operatorname{SL}(n-3, q)|$, it is straightforward to check, using [21, Theorems 2.1 and 2.2], that $H_{B_{1}}$ has these properties whenever either $n \geqslant 11$ with arbitrary $q$, or $n \in\{9,10\}$ with $q>2$. In these cases we can apply [21, Theorem 4.8] to the irreducible subgroup $H_{B_{1}}$ of $\operatorname{GL}(n, q)$. Note that $H$ does not contain $\operatorname{SL}(n, q)$ since it fixes [ $B_{1}$ ] setwise;
also, since $e, e+1$ differ by 1 and $e+1 \leqslant n-3, H$ is not one of the 'Extension field examples' from [21, Theorem 4.8 (b), see Lemma 4.2], and finally since $n \geqslant 9$ and $e+1 \leqslant n-3, H$ is not one of the 'Nearly simple examples' from [21, Theorem 4.8 (c)]. Thus we conclude that either $n \in\{9,10\}$ with $q=2$, or $n \in\{5,6,7,8\}$.

Finally we deal with the remaining values of $n$. Since $H_{\{\alpha, \beta\}} \cap H_{B_{1}}$ has index at most 2 in $H_{\{\alpha, \beta\}}$ it follows that $H_{B_{1}}$ has a subgroup of the form $\left[q^{3 \times(n-3)}\right]$. $\operatorname{SL}(n-3, q)$ which is transitive on $V \backslash Y$, and hence $H_{B_{1}}$ has order divisible by $q^{x}$ with $x=x(n)=3(n-3)+\binom{n-3}{2}=$ $(n-3)(n+2) / 2$; also $H_{B_{1}}$ does not contain $\operatorname{SL}(n, q)$ since it fixes [ $B_{1}$ ] setwise. It follows that $H_{B_{1}} \cap \mathrm{SL}(n, q)$ is contained in a maximal subgroup of $\operatorname{SL}(n, q)$ which is irreducible (that is, not in class $\mathcal{C}_{1}$ in [4]) and has order divisible by $q^{x(n)}$. A careful check of the possible maximal subgroups in the relevant tables in [4], as listed in Table 2, shows that no such subgroup exists. This completes the proof.

Table 2: Tables from [4] to check for the proof of Lemma 4.2, Case $i=2$

| $n$ | $x(n)$ | Tables from [4] for $n$ |
| :---: | :---: | :--- |
| 5 | 7 | Tables 8.18 and 8.19 |
| 6 | 12 | Tables 8.24 and 8.25 |
| 7 | 18 | Tables 8.35 and 8.36 |
| 8 | 25 | Tables 8.44 and 8.45 |
| 9 | 33 | Tables 8.54 and 8.55 |
| 10 | 42 | Tables 8.60 and 8.61 |

## $4.3 \quad \mathcal{C}_{2}$-subgroups

Here $G_{\alpha}$ is a subgroup of type $\mathrm{GL}(m, q)$ ¿ $\mathrm{S}_{\mathrm{t}}$, preserving a decomposition $V=V_{1} \oplus \cdots \oplus V_{t}$ with each $V_{i}$ of the same dimension $m$, where $n=m t, t \geqslant 2$. We can think of the pointset of $\mathcal{D}$ as the set of these decompositions (for a fixed $m$ and $t$ ). Note that graph automorphisms swap $i$-spaces with $n-i$-spaces, so $G \leqslant \operatorname{P\Gamma L}(n, q)$ unless $t=2$. When $t=2$ we have to consider that $G$ could contain graph automorphisms, and so could $G_{\alpha}$.

Lemma 4.3. Assume Hypothesis 4.1. Then the point-stabilizer $G_{\alpha} \notin \mathcal{C}_{2}$.
Proof. Recall that we denote $\operatorname{gcd}(n, q-1)$ by $d$. By Lemma 2.6, $|X|=|\operatorname{PSL}(n, q)|>q^{n^{2}-2}$, and by [14, Proposition 4.2.9],

$$
v=\frac{|\mathrm{GL}(m t, q)|}{|\operatorname{GL}(m, q)|^{t} t!} \quad \text { so } \quad\left|X_{\alpha}\right|=\frac{|X|}{v}=\frac{t!|\mathrm{GL}(m, q)|^{t}}{d(q-1)}
$$

Case 1: $m=1$.
Then $n=t \geqslant 3$, so $\tilde{G} \leqslant \Gamma \mathrm{~L}(n, q)$. Take $\alpha$ as the decomposition $\oplus_{i=1}^{n}\left\langle e_{i}\right\rangle$ and $\beta$ as the decomposition $\left\langle e_{1}+e_{2}\right\rangle \oplus\left(\oplus_{i=2}^{n}\left\langle e_{i}\right\rangle\right)$. The orbit of $\beta$ under $G_{\alpha}$ consists of the decomposition
$\left\langle e_{i}+\lambda e_{j}\right\rangle \oplus\left(\oplus_{\ell \neq i}\left\langle e_{\ell}\right\rangle\right)$, which has size $s:=n(n-1)(q-1)$. Thus by Lemmas 2.1(iv) and 2.2(iv), and Table 1.

$$
4 n^{2}(n-1)^{2}(q-1)^{2} \geqslant(2 s)^{2} \geqslant r^{2}>2 v=2 \frac{|\mathrm{GL}(n, q)|}{(q-1)^{n} n!}>2 \frac{q^{n^{2}}}{4(q-1)^{n} n!}
$$

so $8 n^{2}(n-1)^{2} n!>q^{n^{2}} /(q-1)^{n+2}>q^{n^{2}-n-2}$. This implies that either $(n, q)=(5,2)$ or $(4,2)$, or $n=3$ and $q \leqslant 5$.

Suppose first that $n=3$. Then $v=q^{3}\left(q^{2}+q+1\right)(q+1) / 6$ and $r \leqslant 2 s=12(q-1)$. Since $r^{2}>2 v$ we conclude that $q=2$ or 3 . In either case $v$ is divisible by $q$, and since $r$ divides $2(v-1)$ (Lemma 2.1), $r$ is not divisible by 4 if $q=2$, and not divisible by 3 if $q=3$. Hence $r$ divides $6(q-1)$ if $q=2$, or $4(q-1)$ if $q=3$ (Lemma [2.2), and then $r^{2}>2 v$ leads to a contradiction. Thus $q=2$ and $n$ is 4 or 5 . In either case, $v$ is divisible by 4 , so 4 does not divide $r$ (Lemma 2.3). Then, since $r$ divides $2 s=2 n(n-1)$, we see that $r$ divides 6 or 10 for $n=4,5$ respectively, giving a contradiction to $r^{2}>2 v$. Thus we may assume that $m \geqslant 2$.

Case 2: $t=2$.
Next we deal with the case where $G$ may contain a graph automorphism, namely the case $t=2$, so $n=m t \geqslant 4$, and $G$ acts on decomposition into two subspaces of dimension $m=n / 2$. Let $\alpha$ be the decomposition $V_{1} \oplus V_{2}$ where

$$
V_{1}=\left\langle v_{1}, \ldots, v_{m}\right\rangle, \quad V_{2}=\left\langle v_{m+1}, \ldots, v_{2 m}\right\rangle .
$$

Leet $\beta$ be the decomposition $V_{1}^{\prime} \oplus V_{2}^{\prime}$, where $V_{1}^{\prime}=\left\langle v_{1}, \ldots, v_{m-1}, v_{m+1}\right\rangle$ and $V_{2}^{\prime}=\left\langle v_{m}, v_{m+2}, \ldots, v_{2 m}\right\rangle$. Let $G^{*}:=G \cap \operatorname{P\Gamma L}(n, q)$, so $\left|G: G^{*}\right| \leqslant 2$. Since $G$ is point-primitive, $G$ is point-transitive, and so $\left|G_{\alpha}: G_{\alpha}^{*}\right|=\left|G: G^{*}\right| \leqslant 2$.

Moreover, let $G_{V_{1}, V_{2}}^{*}$ be the subgroup of $G_{\alpha}^{*}$ fixing $V_{1}$ and $V_{2}$, so $G_{V_{1}, V_{2}}^{*}$ has index at most 2 in $G_{\alpha}^{*}$. If $m>2$, then we are in the same situation as in Lemma 4.1 (Case 2) with $i=m=n / 2$ and

$$
\left|\beta^{G_{V_{1}, V_{2}}^{*}}\right|=q^{n-2} \frac{\left(q^{m}-1\right)^{2}}{(q-1)^{2}}=q^{2(m-1)} \frac{\left(q^{m}-1\right)^{2}}{(q-1)^{2}} .
$$

If $m=2$, then we have double counted (as $G_{V_{1}, V_{2}}^{*}$ does not fix each of the spaces $V_{i} \cap V_{j}^{\prime}$; in fact it contains an element $\left.x: v_{1} \leftrightarrow v_{2}, v_{3} \leftrightarrow v_{4}\right)$, and $\left|\beta^{G_{V_{1}, V_{2}}^{*}}\right|=q^{2(m-1)} \frac{\left(q^{m}-1\right)^{2}}{2(q-1)^{2}}$. In both cases, $\left|\beta^{G_{\alpha}}\right| \left\lvert\, 4 q^{2(m-1)} \frac{\left(q^{m}-1\right)^{2}}{(q-1)^{2}}\right.$. By Lemma $2.2(\mathrm{iv}), r$ divides $2\left|\beta^{G_{\alpha}}\right|$, and hence

$$
r \left\lvert\, 8 q^{2(m-1)} \frac{\left(q^{m}-1\right)^{2}}{(q-1)^{2}}\right.
$$

Note that

$$
v=\frac{|X|}{\left|X_{\alpha}\right|}=\frac{q^{m^{2}}\left(q^{2 m}-1\right) \cdots\left(q^{m+1}-1\right)}{2\left(q^{m}-1\right) \cdots(q-1)}>\frac{q^{2 m^{2}}}{2}
$$

and in particular $p \mid v$. By Lemma 2.3(iii), $r_{p}$ divides 2, and hence $r$ divides $\frac{8\left(q^{m}-1\right)^{2}}{(q-1)^{2}}$. This together with $r^{2}>2 v$ leads to

$$
\begin{equation*}
\frac{64\left(q^{m}-1\right)^{4}}{(q-1)^{4}}>q^{2 m^{2}} \tag{4.13}
\end{equation*}
$$

It follows that $64 \cdot\left(2 q^{m-1}\right)^{4}>q^{2 m^{2}}$ and so

$$
2^{10}>q^{2\left(m^{2}-2 m+2\right)} \geqslant 2^{2\left(m^{2}-2 m+2\right)} .
$$

Hence $10>2\left(m^{2}-2 m+2\right)$ and so $m=2$ and $r \mid 8(q+1)^{2}$. Then we deduce from (4.13) that $64(q+1)^{4}>q^{8}$, which implies that $q=2$ or 3 . Assume $q=2$. Then $r_{2} \mid 2$, so $r$ divides $2(q+1)^{2}=18$, contradicting the condition $r^{2}>2 v=560$. Hence $q=3, r \mid 2^{7}$ and $v=5265$. Combining this with $r \mid 2(v-1)$ we conclude that $r$ divides $2^{5}$, again contradicting the condition $r^{2}>2 v$. Thus $t \geqslant 3$ and in particular $n=m t \geqslant 6$ and $G \leqslant \Gamma \mathrm{~L}(n, q)$.

Case 3: $t \geqslant 3$.
Since $|\mathrm{GL}(m, q)|<q^{m^{2}}$, we have

$$
\left|X_{\alpha}\right|=\frac{t!|\mathrm{GL}(m, q)|^{t}}{d(q-1)}<\frac{t!q^{n^{2} / t}}{d(q-1)}
$$

Combining this with the assertion $|X|<2(d f)^{2}\left|X_{\alpha}\right|^{3}$ from Lemma 2.3(i), we obtain

$$
|X|<\frac{2 f^{2}(t!)^{3} q^{3 n^{2} / t}}{d(q-1)^{3}}<2(t!)^{3} q^{3 n^{2} / t}
$$

It then follows from $|X|>q^{n^{2}-2}$ that $q^{n^{2}-2}<2(t!)^{3} q^{3 n^{2} / t}$, that is,

$$
\begin{equation*}
q^{n^{2}\left(1-\frac{3}{t}\right)-2}<2(t!)^{3} . \tag{4.14}
\end{equation*}
$$

Since $n \geqslant 2 t$, we derive from (4.14) that

$$
\begin{equation*}
2^{4 t(t-3)-2} \leqslant q^{4 t(t-3)-2} \leqslant q^{n^{2}\left(1-\frac{3}{t}\right)-2}<2(t!)^{3} . \tag{4.15}
\end{equation*}
$$

Hence either $t=3$ or $(t, q)=(4,2)$. Consider the latter case. Here (4.14) becomes $2^{n^{2} / 4-2}<$ $2 \cdot(4!)^{3}$ and hence $n \leqslant 8$. As $n \geqslant 2 t=8$, we conclude that $n=8$ and $m=2$. However, then $|X|=|\mathrm{PSL}(8,2)|$ and $\left|X_{\alpha}\right|=24|\mathrm{GL}(2,2)|^{4}$, contradicting the condition $|X|<2(d f)^{2}\left|X_{\alpha}\right|^{3}=$ $2\left|X_{\alpha}\right|^{3}$ from Lemma 2.3(i).

Thus $t=3$, and $\alpha$ is a decomposition $V_{1} \oplus V_{2} \oplus V_{3}$ with $\operatorname{dim}\left(V_{1}\right)=\operatorname{dim}\left(V_{2}\right)=\operatorname{dim}\left(V_{3}\right)=$ $m=n / 3$. Say

$$
V_{1}=\left\langle v_{1}, \ldots, v_{m}\right\rangle, \quad V_{2}=\left\langle v_{m+1}, \ldots, v_{2 m}\right\rangle, \quad V_{3}=\left\langle v_{2 m+1}, \ldots, v_{3 m}\right\rangle .
$$

Let $\beta$ be the decomposition $\left\langle v_{1}, \ldots, v_{m-1}, v_{m+1}\right\rangle \oplus\left\langle v_{m}, v_{m+2}, \ldots, v_{2 m}\right\rangle \oplus V_{3}$. Arguing as in Case 2 we find that $\left|\beta^{G_{V_{1}}, V_{2}, V_{3}}\right|=q^{2(m-1) \frac{\left(q^{m}-1\right)^{2}}{(q-1)^{2}}}$ if $m \geqslant 3$, or $\left|\beta^{G_{V_{1}, V_{2}, V_{3}}}\right|=q^{2(m-1)} \frac{\left(q^{m}-1\right)^{2}}{2(q-1)^{2}}$ if $m=2$. Now $G_{V_{1}, V_{2}, V_{3}}$ has index dividing 6 in $G_{\alpha}$, so $\left|\beta^{G_{\alpha}}\right|$ divides $6 q^{2(m-1)} \frac{\left(q^{m}-1\right)^{2}}{(q-1)^{2}}$. By Lemma 2.2(iv), $r$ divides $2\left|\beta^{G_{\alpha}}\right|$. Since $v=\frac{|\operatorname{GL}(3 m, q)|}{|\operatorname{GL}(m, q)|^{3}!\text { ! }}$, it follows that $p$ divides $v$ and so by Lemma 2.3, $r_{p}$ divides 2, and hence

$$
r \left\lvert\, 12 \frac{\left(q^{m}-1\right)^{2}}{(q-1)^{2}}\right., \quad \text { so } \quad r^{2}<144\left(2 q^{m-1}\right)^{4}=2304 q^{4 m-4}
$$

Note that

$$
v=\frac{|\mathrm{GL}(3 m, q)|}{|\mathrm{GL}(m, q)|^{3} 3!}=\frac{q^{3 m^{2}}}{6} \prod_{i=1}^{m} \frac{q^{2 m+i}-1}{q^{i}-1} \cdot \prod_{i=1}^{m} \frac{q^{m+i}-1}{q^{i}-1}>\frac{1}{6} q^{3 m^{2}+2 m \cdot m+m \cdot m}=\frac{q^{6 m^{2}}}{6},
$$

and since $r^{2}>2 v$, we get

$$
\frac{q^{6 m^{2}}}{3}<2 v<r^{2}<2304 q^{4 m-4}
$$

and so $6912>q^{6 m^{2}-4 m+4} \geqslant 2^{6 m^{2}-4 m+4} \geqslant 2^{20}$, a contradiction.

## $4.4 \quad \mathcal{C}_{3}$-subgroups

Here $G_{\alpha}$ is an extension field subgroup.
Lemma 4.4. Assume Hypothesis 4.1. Then the point-stabilizer $G_{\alpha} \notin \mathcal{C}_{3}$.
Proof. By Lemma [2.6 we have $|X|>q^{n^{2}-2}$, and by [14, Proposition 4.3.6],

$$
X_{\alpha} \cong \mathbb{Z}_{a} \cdot \operatorname{PSL}\left(n / s, q^{s}\right) \cdot \mathbb{Z}_{b} \cdot \mathbb{Z}_{s}
$$

where $s$ is a prime divisor of $n, d=\operatorname{gcd}(n, q-1), a=\operatorname{gcd}(n / s, q-1)\left(q^{s}-1\right) /(d(q-1))$, and $b=\operatorname{gcd}\left(n / s, q^{s}-1\right) / \operatorname{gcd}(n / s, q-1)$. Thus,

$$
\left|X_{\alpha}\right|=\frac{s\left|\operatorname{GL}\left(n / s, q^{s}\right)\right|}{d(q-1)} .
$$

Case 1: $n=s$.
Here $n$ is a prime, $\left|X_{\alpha}\right|=n\left(q^{n}-1\right) /(d(q-1))$, and by Lemma 2.3(i),

$$
|X|<2(d f)^{2}\left|X_{\alpha}\right|^{3}=\frac{2 f^{2} n^{3}}{d}\left(\frac{q^{n}-1}{q-1}\right)^{3}<2 q^{2} n^{3} \cdot\left(2 q^{n-1}\right)^{3}=16 n^{3} q^{3 n-1}
$$

Combining this with $|X|>q^{n^{2}-2}$ we obtain

$$
\begin{equation*}
q^{n^{2}-3 n-1}<16 n^{3} \tag{4.16}
\end{equation*}
$$

and so $2^{n^{2}-3 n-1}<16 n^{3}$, which implies $n \leqslant 5$.
Subcase 1.1: $n=5$.
In this case (4.16) implies that $q^{9}<16 \cdot 5^{3}$, which leads to $q=2$. However, this means that $|X|=|\operatorname{PSL}(5,2)|$ and $\left|X_{\alpha}\right|=5 \cdot 31$, contradicting the condition $|X|<2(d f)^{2}\left|X_{\alpha}\right|^{3}=2\left|X_{\alpha}\right|^{3}$ from Lemma 2.3(i).

Subcase 1.2: $n=3$.
Then $X=\operatorname{PSL}(3, q),\left|X_{\alpha}\right|=3\left(q^{2}+q+1\right) / d$ and so $v=q^{3}\left(q^{2}-1\right)(q-1) / 3$. It follows from Lemma [2.3(ii) that $r$ divides $2 d f\left|X_{\alpha}\right|=6 f\left(q^{2}+q+1\right)$. Combining this with $r^{2}>2 v$, we obtain that $54 f^{2}\left(q^{2}+q+1\right)^{2}>q^{3}\left(q^{2}-1\right)(q-1)$, that is,

$$
54 f^{2}>\frac{q^{6}-q^{5}-q^{4}+q^{3}}{\left(q^{2}+q+1\right)^{2}}
$$

Table 3: Possible values of $q, v$ and $R$

| $q$ | $v$ | $R$ | $q$ | $v$ | $R$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 2 | 8 | 14 | 8 | 75264 | 146 |
| 3 | 144 | 26 | 9 | 155520 | 182 |
| 4 | 960 | 14 | 16 | 5222400 | 182 |
| 5 | 4000 | 186 | 32 | 346390528 | 6342 |
| 7 | 32928 | 38 |  |  |  |

This inequality holds only when

$$
q \in\{2,3,4,5,7,8,9,16,32\}
$$

Let $R=\operatorname{gcd}\left(6 f\left(q^{2}+q+1\right), 2(v-1)\right)$. Then $r$ is a divisor of $R$. For each $q$ and $f$ as above, the possible values of $v$ and $R$ are listed in Table 3. Hence the condition $r^{2}>2 v$ implies that $q \in\{2,3,5\}$.

Assume $q=2$. Then $v=8$ and $r$ and divides 14. From $r(k-1)=2(v-1)$ and $r \geqslant k \geqslant 3$ we deduce that $r=7$ and $k=3$, which contradicts the condition that $b k=v r$. Similarly, we have $q \neq 5$ (two cases to check: $(r, k) \in\{(186,44),(93,87)\}$.) Hence $q=3$. By Table 3, $v=144$ and $r$ divides 26. Then from $r(k-1)=2(v-1), b k=v r$ and $r \geqslant k \geqslant 3$, we deduce that $r=26, k=12$ and $b=312$. Since $\left|X_{\alpha}\right|=39$, Lemma 2.2(ii) implies that $G>X$. Since $\operatorname{Out}(X)$ has size 2, we must have $G=X .2$ (with graph automorphism). By flag-transitivity, a block stabiliser must have index 312 and have an orbit of size 12. We checked with Magma, considering every subgroup of index 312, and only one has an orbit of size 12 (which is unique), and the orbit of that block under $G$ does not yield a 2-design.

Case 2: $n \geqslant 2 s$.
By Lemma 2.6 we have

$$
\left|X_{\alpha}\right|=\frac{s\left|\mathrm{GL}\left(n / s, q^{s}\right)\right|}{d(q-1)} \leqslant \frac{s\left(1-q^{-s}\right)\left(1-q^{-2 s}\right) q^{n^{2} / s}}{d(q-1)}<\frac{s q^{n^{2} / s}}{d(q-1)} .
$$

Moreover $\left|X_{\alpha}\right|_{p}=s_{p} \cdot q^{n(n-s) / 2 s}$ and $|X|_{p}=q^{n(n-1) / 2}$. We deduce that $p$ divides $v=\left|X: X_{\alpha}\right|$, so by Lemma 2.3(iii), $r_{p}$ divides 2 , and

$$
|X|<2(d f)^{2}\left|X_{\alpha}\right|_{p^{\prime}}^{2}\left|X_{\alpha}\right|=2(d f)^{2}\left|X_{\alpha}\right|^{3} /\left|X_{\alpha}\right|_{p}^{2}<\frac{2 f^{2} s^{3} q^{\left(3 n^{2} / s\right)-n(n-s) / s}}{\left(s_{p}\right)^{2} d(q-1)^{3}} \leqslant \frac{n^{3}}{4} q^{\left(2 n^{2} / s\right)+n} .
$$

For the last inequality, we used that $s \leqslant n / 2$ and $f^{2} \leqslant(q-1)^{3}$. Combining this with $|X|>q^{n^{2}-2}$ we obtain

$$
\begin{equation*}
4 q^{(1-2 / s) n^{2}-n-2} \leqslant n^{3} \tag{4.17}
\end{equation*}
$$

Subcase 2.1: $s \geqslant 3$.
Then $n \geqslant 2 s \geqslant 6$ and (4.17) implies that

$$
n^{3} \geqslant 4 q^{(1-2 / s) n^{2}-n-2} \geqslant 4 q^{\left(n^{2} / 3\right)-n-2} \geqslant 2^{\left(n^{2} / 3\right)-n} .
$$

We easily see that this inequality only holds for $n \leqslant 6$. Therefore $n=2 s=6$, and so (4.17) implies that $q=2$. It follows that $X=\operatorname{PSL}(6,2)$ and $\left|X_{\alpha}\right|=3|\operatorname{GL}(2,8)|=2^{3} \cdot 3^{3} \cdot 7^{2}$, so we can compute $v=|X| /\left|X_{\alpha}\right|=2^{12} \cdot 3 \cdot 5 \cdot 31$ and $v-1=11 \cdot 173149$. We know that $r \mid 2(v-1)$. By Lemma 2.3(iii), we also know that $\left.r|2 d f| X_{\alpha}\right|_{p^{\prime}}=2 \cdot 3^{3} \cdot 7^{2}$, thus $r \mid 2$, contradicting $r^{2}>2 v$.

Subcase 2.2: $s=2$.
Then $n=2 m \geqslant 4$ and $n$ is even,

$$
\left|X_{\alpha}\right|=\frac{2\left|\mathrm{GL}\left(n / 2, q^{2}\right)\right|}{d(q-1)}=\frac{2 q^{n(n-2) / 4}\left(q^{n}-1\right)\left(q^{n-2}-1\right) \cdots\left(q^{2}-1\right)}{d(q-1)}
$$

and

$$
v=\frac{q^{n^{2} / 4}\left(q^{n-1}-1\right)\left(q^{n-3}-1\right) \cdots\left(q^{3}-1\right)(q-1)}{2} .
$$

As we observed, $r_{p} \mid 2$. Also $v$ is even, and so, from $r(k-1)=2(v-1)$ we deduce that $4 \nmid r$.
First assume that $n=4$. Then

$$
\left|X_{\alpha}\right|=\frac{2 q^{2}\left(q^{4}-1\right)(q+1)}{d} \quad \text { and } \quad v=\frac{q^{4}\left(q^{3}-1\right)(q-1)}{2} .
$$

By Lemma 2.3(iii), $r$ divides $2 d f\left|X_{\alpha}\right|_{p^{\prime}}$ and hence $r \mid 2 f\left(q^{4}-1\right)(q+1)$, which can be rewritten as $r \mid 2 f\left(q^{2}+1\right)(q-1)(q+1)^{2}$. Note that

$$
v-1=\frac{(q+1)\left(q^{7}-2 q^{6}+2 q^{5}-3 q^{4}+4 q^{3}-4 q^{2}+4 q-4\right)}{2}+1
$$

so that $\operatorname{gcd}(v-1, q+1)=1$. Hence, since $r \mid 2(v-1)$, it follows that $\operatorname{gcd}(r, q+1) \mid 2$. Moreover, it follows from $(q-1) \mid v$ that $\operatorname{gcd}(r, q-1) \mid 2$. Combining this with $4 \nmid r$ and $r \mid 2 f\left(q^{4}-1\right)(q+1)$, we obtain $r \mid 2 f\left(q^{2}+1\right)$. Therefore, using Lemma 2.1(iv),

$$
4 f^{2}\left(q^{2}+1\right)^{2} \geqslant r^{2}>2 v=q^{4}\left(q^{3}-1\right)(q-1)
$$

However, there is no $q=p^{f}$ satisfying $4 f^{2}\left(q^{2}+1\right)^{2}>q^{4}\left(q^{3}-1\right)(q-1)$, a contradiction.
Thus $n \geqslant 6$. Recall that $\tilde{X}=\operatorname{SL}(n, q) \leqslant \tilde{G} \leqslant \Gamma \mathrm{~L}(n, q)$, acting unfaithfully on $\mathcal{P}$ (with kernel a scalar subgroup of $\tilde{G})$. We regard $V$ as an $m$-dimensional vector space over $\mathbb{F}_{q^{2}}$ with basis $\left\{e_{1}, e_{2}, \ldots, e_{m}\right\}$ and $\tilde{G}_{\alpha}$ the subgroup of $\tilde{G}$ preserving this vector space structure. Take $w \in \mathbb{F}_{q^{2}} \backslash \mathbb{F}_{q}$. Then

$$
V=\left\langle e_{1}, e_{2}, \ldots, e_{m}\right\rangle_{\mathbb{F}_{q^{2}}}=\left\langle e_{1}, w e_{1}, e_{2}, w e_{2}, \ldots, e_{m}, w e_{m}\right\rangle_{\mathbb{F}_{q}} .
$$

Let

$$
W=\left\langle e_{1}, e_{2}\right\rangle_{\mathbb{F}_{q^{2}}}=\left\langle e_{1}, w e_{1}, e_{2}, w e_{2}\right\rangle_{\mathbb{F}_{q}} .
$$

Consider $g \in \operatorname{SL}(n, q)$ defined by

$$
\begin{cases}e_{1}^{g}=e_{1}, e_{2}^{g}=-e_{2},\left(w e_{1}\right)^{g}=w e_{2},\left(w e_{2}\right)^{g}=w e_{1} & \text { for } 1 \leqslant i \leqslant 2 \\ \left(e_{i}\right)^{g}=e_{i},\left(w e_{i}\right)^{g}=w e_{i} & \text { for } 3 \leqslant i \leqslant m\end{cases}
$$

Then $g$ does not fix $\alpha$. Let $\beta=\alpha^{g}$ and let $\tilde{G}_{\alpha,(W)}$ be the subgroup of $\tilde{G}_{\alpha}$ fixing every vector of $W$. Note that $W^{g}=\left\langle e_{1}, w e_{1},-e_{2}, w e_{2}\right\rangle_{\mathbb{F}_{q}}=W$ and so $\tilde{G}_{\alpha,(W)} \leqslant \tilde{G}_{\alpha, \beta}$. Now $\operatorname{SL}(n, q)_{\alpha,(W)}$ contains $I_{4} \times \operatorname{SL}\left(n / 2-2, q^{2}\right)$, and since this subgroup intersects the scalar subgroup trivially it follows that $X_{\alpha,(W)}$ contains a subgroup isomorphic to $\operatorname{SL}\left(n / 2-2, q^{2}\right)$ (and so do $G_{\alpha,(W)}, G_{\alpha, \beta}$, and $\left.X_{\alpha, \beta}\right)$. By Lemma [2.4, $r$ divides $4 d f\left|X_{\alpha}\right| /\left|\operatorname{SL}\left(\frac{n}{2}-2, q^{2}\right)\right|=8 f q^{2 n-6}\left(q^{n}-1\right)\left(q^{n-2}-1\right)(q+1)$. Combining this with $r_{p} \mid 2$ and $4 \nmid r$, we obtain

$$
\begin{equation*}
r \mid 2 f\left(q^{n}-1\right)\left(q^{n-2}-1\right)(q+1) \tag{4.18}
\end{equation*}
$$

Then from $r^{2}>2 v$ and

$$
2 v=q^{n^{2} / 4}\left(q^{n-1}-1\right)\left(q^{n-3}-1\right) \cdots\left(q^{3}-1\right)(q-1)
$$

we deduce that

$$
\begin{equation*}
4 f^{2}\left(q^{n}-1\right)^{2}\left(q^{n-2}-1\right)^{2}(q+1)^{2}>q^{n^{2} / 4}\left(q^{n-1}-1\right)\left(q^{n-3}-1\right) \cdots\left(q^{3}-1\right)(q-1) \tag{4.19}
\end{equation*}
$$

and so

$$
4 q^{2}\left(q^{n}\right)^{2}\left(q^{n-2}\right)^{2}(2 q)^{2}>q^{n^{2} / 4} q^{n-2} q^{n-4} \cdots q^{4} q^{2}=q^{\left(n^{2}-n\right) / 2} .
$$

Therefore,

$$
2^{4} q^{4 n}>q^{\left(n^{2}-n\right) / 2}
$$

This implies that

$$
2^{4}>q^{n(n-9) / 2} \geq 2^{n(n-9) / 2}
$$

and hence $n \leq 8$ (since $n$ is even).
Assume that $n=8$. By (4.19) we have that

$$
4 f^{2}\left(q^{8}-1\right)^{2}\left(q^{6}-1\right)^{2}(q+1)^{2}>q^{16}\left(q^{7}-1\right)\left(q^{5}-1\right)\left(q^{3}-1\right)(q-1)
$$

and this implies that $q \in\{2,3,4\}$. By (4.18), $r$ divides $u:=2 f\left(q^{8}-1\right)\left(q^{6}-1\right)(q+1)$, and hence $r$ divides $R:=\operatorname{gcd}(2(v-1), u)$. However, for each $q \in\{2,3,4\}$, we find $R^{2}<2 v$, contradicting the fact that $r^{2}>2 v$.

Hence $n=6$, and here $r \mid 2 f\left(q^{6}-1\right)\left(q^{4}-1\right)(q+1)$ by (4.18), which can be rewritten as $r \mid 2 f\left(q^{2}-q+1\right)\left(q^{2}+1\right)\left(q^{3}-1\right)(q-1)(q+1)^{3}$. Recall that $r \mid 2(v-1)$, and in this case $2(v-1)=q^{9}\left(q^{5}-1\right)\left(q^{3}-1\right)(q-1)-2$, which is congruent to 6 module $q+1$. Thus $\operatorname{gcd}(2(v-1), q+1)=6$, and so $\operatorname{gcd}(r, q+1)$ divides 6 . On the other hand, $\left(q^{3}-1\right)(q-1)$ divides $v$, so $\operatorname{gcd}\left(r,\left(q^{3}-1\right)(q-1)\right)$ divides 2. Recall that $4 \nmid r$. We conclude that $r \mid$ $54 f\left(q^{2}-q+1\right)\left(q^{2}+1\right)$. Thus

$$
2916 f^{2}\left(q^{2}-q+1\right)^{2}\left(q^{2}+1\right)^{2} \geqslant r^{2}>2 v=q^{9}\left(q^{5}-1\right)\left(q^{3}-1\right)(q-1)
$$

which implies that $q=2$. It then follows that $v=55,552$ and $r \mid 810$. However, as $r \mid 2(v-1)$, we conclude that $r \mid 6$, contradicting $r^{2}>2 v$.

## $4.5 \mathcal{C}_{4}$-subgroups

Here $G_{\alpha}$ stabilises a tensor product $V_{1} \otimes V_{2}$, where $V_{1}$ has dimension $a$, for some divisor $a$ of $n$, and $V_{2}$ has dimension $n / a$, with $2 \leqslant a<n / a$, that is $2 \leqslant a<\sqrt{n}$. In particular $n \geqslant 6$. Recall that $d=\operatorname{gcd}(n, q-1)$.

Lemma 4.5. Assume Hypothesis 4.1. Then the point-stabilizer $G_{\alpha} \notin \mathcal{C}_{4}$.
Proof. According to [14, Proposition 4.4.10], we have

$$
\left|X_{\alpha}\right|=\frac{\operatorname{gcd}(a, n / a, q-1)}{d} \cdot|\operatorname{PGL}(a, q)| \cdot|\operatorname{PGL}(n / a, q)|
$$

By Lemma 2.6,

$$
\left|X_{\alpha}\right| \leqslant|\operatorname{PGL}(a, q)| \cdot|\operatorname{PGL}(n / a, q)|<\frac{\left(1-q^{-1}\right) q^{a^{2}}}{q-1} \cdot \frac{\left(1-q^{-1}\right) q^{n^{2} / a^{2}}}{q-1}=q^{a^{2}+\left(n^{2} / a^{2}\right)-2}
$$

Let $f(a)=a^{2}+\frac{n^{2}}{a^{2}}-2=\left(a+\frac{n}{a}\right)^{2}-2-2 n$. This is a decreasing function of $a$ on the interval $(2, \sqrt{n})$, and hence $f(a) \leqslant f(2)=\left(n^{2} / 4\right)+2$. Hence $\left|X_{\alpha}\right|<q^{a^{2}+\left(n^{2} / a^{2}\right)-2} \leqslant q^{\left(n^{2} / 4\right)+2}$. By Lemma 2.3(i),

$$
|X|<2(d f)^{2}\left|X_{\alpha}\right|^{3}<2 d^{2} f^{2} q^{\left(3 n^{2} / 4\right)+6}<2 q^{\left(3 n^{2} / 4\right)+10}
$$

Combining this with the fact that $|X|>q^{n^{2}-2}$ (from Lemma 2.6), we obtain

$$
q^{\left(n^{2} / 4\right)-12}<2
$$

Therefore, $n^{2} / 4 \leqslant 12$, which implies that $n=6$, and hence that $a=2$. Thus

$$
\left|X_{\alpha}\right|=\frac{q^{4}\left(q^{3}-1\right)\left(q^{2}-1\right)^{2}}{d} \quad \text { and } \quad v=q^{11}\left(q^{6}-1\right)\left(q^{5}-1\right)\left(q^{2}+1\right)
$$

Consequently, $p \mid v$ and $v$ is even. By Lemma [2.3(iii), $r_{p}$ divides 2, $4 \nmid r$, and $r$ divides $2 d f\left|X_{\alpha}\right|_{p^{\prime}}$ and hence $r$ divides $2 f\left(q^{3}-1\right)\left(q^{2}-1\right)^{2}$. Note that $\left(q^{3}-1\right)(q+1) \mid q^{6}-1$ and $q-1 \mid q^{5}-1$, so $\left(q^{3}-1\right)\left(q^{2}-1\right)$ divides $v$. We conclude that $\operatorname{gcd}\left(r,\left(q^{3}-1\right)\left(q^{2}-1\right)\right)$ divides 2. Hence, $r \mid 2 f\left(q^{2}-1\right)$, contradicting the condition $r^{2}>2 v$.

## $4.6 \quad \mathcal{C}_{5}$-subgroups

Here $G_{\alpha}$ is a subfield subgroup of $G$ of type $\operatorname{GL}\left(n, q_{0}\right)$, where $q=p^{f}=q_{0}^{s}$ for some prime divisor $s$ of $f$.

Lemma 4.6. Assume Hypothesis 4.1. Then the point-stabilizer $G_{\alpha} \notin \mathcal{C}_{5}$.
Proof. According to [14. Proposition 4.5.3],

$$
\left|X_{\alpha}\right| \cong \frac{q-1}{d \cdot \operatorname{lcm}\left(q_{0}-1,(q-1) / \operatorname{gcd}(n, q-1)\right)} \operatorname{PGL}\left(n, q_{0}\right)
$$

and, setting $d_{0}=\operatorname{gcd}\left(n,(q-1) /\left(q_{0}-1\right)\right)$ (a divisor of $d$ ), by Lemma 2.7(i) we have

$$
\begin{equation*}
\left|X_{\alpha}\right|=\frac{d_{0}}{d} \cdot\left|\operatorname{PGL}\left(n, q_{0}\right)\right|=\frac{d_{0}}{d} \cdot q_{0}^{n(n-1) / 2}\left(q_{0}^{n}-1\right)\left(q_{0}^{n-1}-1\right) \cdots\left(q_{0}^{2}-1\right) \tag{4.20}
\end{equation*}
$$

In particular, the $p$-part $\left|X_{\alpha}\right|_{p}=q_{0}^{n(n-1) / 2}$ is strictly less than $|X|_{p}=q^{n(n-1) / 2}$, so $v=$ $|X| /\left|X_{\alpha}\right|$ is divisible by $p$, and hence, by Lemma 2.3(iii), $r_{p}$ divides 2 , and $2(d f)^{2}\left|X_{\alpha}\right|_{p^{\prime}}^{2}\left|X_{\alpha}\right|>$ $|X|$. Hence

$$
q^{n^{2}-2}<|X|<2 d^{2} f^{2} q_{0}^{n(n-1) / 2} \cdot \frac{d_{0}^{3}}{d^{3}} \cdot\left(\left(q_{0}^{n}-1\right)\left(q_{0}^{n-1}-1\right) \cdots\left(q_{0}^{2}-1\right)\right)^{3}
$$

Since $d_{0} \leqslant d<q, f<q$ and $2 \leqslant q_{0}$, this implies that

$$
\begin{equation*}
q^{n^{2}-2}<2 d^{2} f^{2} \cdot q_{0}^{n(n-1) / 2} \cdot q_{0}^{3(n+2)(n-1) / 2}<q_{0} \cdot q^{4} \cdot q_{0}^{2 n^{2}+n-3} \tag{4.21}
\end{equation*}
$$

As $q=q_{0}^{s}$, we have $s\left(n^{2}-2\right)<4 s+2 n^{2}+n-2$, so

$$
2 n^{2}+n-3 \geqslant s\left(n^{2}-6\right)
$$

Case 1: $s \geqslant 5$.
Then $2 n^{2}+n-3 \geqslant 5\left(n^{2}-6\right)$, and so $n=3$. However, the first inequality in (4.21) then implies

$$
q^{7}<2 \cdot 3^{2} \cdot q^{2} \cdot q_{0}^{18}
$$

that is, $q_{0}^{5 s-18}<18$. This is not possible as $q_{0}^{5 s-18} \geqslant q_{0}^{7} \geqslant 2^{7}$.
Case 2: $s=3$, that is $q=q_{0}^{3}$.
Then $2 n^{2}+n-3 \geqslant 3\left(n^{2}-6\right)$, and so $n=3$ or 4 . Suppose $n=4$. Then the first inequality in (4.21) implies

$$
q^{14}<2 \cdot 4^{2} \cdot q^{2} \cdot q_{0}^{33}
$$

that is, $32>q_{0}^{3}$. This leads to $q_{0}=2$ or 3 , and so $q=q_{0}^{3}=8$ or 27 , which does not satisfy the first inequality in (4.21), a contradiction. Therefore, $n=3=s$, and examining $d=\operatorname{gcd}(3, q-1)$ and $d_{0}=\operatorname{gcd}\left(3, q_{0}^{2}+q_{0}+1\right)$, we see that $d_{0}=d \in\{1,3\}$. The inequality $|X|<2(d f)^{2}\left|X_{\alpha}\right|_{p^{\prime}}^{2}\left|X_{\alpha}\right|$ from Lemma [2.3(iii) becomes (using (4.20))

$$
q^{3}\left(q^{3}-1\right)\left(q^{2}-1\right) / d=|X|<2 d^{2} f^{2} q_{0}^{3} \cdot\left(q_{0}^{3}-1\right)^{3}\left(q_{0}^{2}-1\right)^{3}
$$

or equivalently, since $q=q_{0}^{3}$,

$$
q_{0}^{6}\left(q_{0}^{9}-1\right)\left(q_{0}^{6}-1\right)<2 d^{3} f^{2} \cdot\left(q_{0}^{3}-1\right)^{3}\left(q_{0}^{2}-1\right)^{3} .
$$

Since $\left(q_{0}^{3}-1\right)^{3}\left(q_{0}^{2}-1\right)^{3}<\left(q_{0}^{9}-1\right)\left(q_{0}^{6}-1\right)$ and $d \leqslant n=3$, it follows that

$$
\begin{equation*}
q_{0}^{6}<2 d^{3} f^{2} \leqslant 54 f^{2} \tag{4.22}
\end{equation*}
$$

As $3 \mid f$ and $q_{0}=p^{f / 3}$, we then conclude that $f=3$ and $q_{0}=2$, but this means that $d=1$, contradicting the first inequality of (4.22).

Case 3: $s=2$, that is $q=q_{0}^{2}$.

In this case, $d_{0}=\operatorname{gcd}\left(n, q_{0}+1\right)$ in the expression for $\left|X_{\alpha}\right|$ in (4.20). Let $a \in \mathbb{F}_{q} \backslash \mathbb{F}_{q_{0}}$ and consider

$$
g=\left(\begin{array}{lll}
a & & \\
& a^{-1} & \\
& & I_{n-2}
\end{array}\right) \in \tilde{X}=\operatorname{SL}(n, q)
$$

Now $g$ does not preserve $\alpha$. Let $\beta=\alpha^{g} \neq \alpha$. Then

$$
\left\{\left.\left(\begin{array}{ccc}
1 & & \\
& 1 & \\
& & B
\end{array}\right) \right\rvert\, B \in \mathrm{SL}\left(n-2, q_{0}\right)\right\} \leqslant \tilde{X}_{\alpha} \cap\left(\tilde{X}_{\alpha}\right)^{g}=\tilde{X}_{\alpha \beta}
$$

Since this subgroup intersects the scalar subgroup trivially, $X_{\alpha \beta}$ contains a subgroup isomorphic to $\operatorname{SL}\left(n-2, q_{0}\right)$, and hence so does $G_{\alpha \beta}$. By Lemma [2.4, $r$ divides $4 d f\left|X_{\alpha}\right| /\left|\operatorname{SL}\left(n-2, q_{0}\right)\right|$. Thus, using (4.20),

$$
r \mid 4 f d_{0} q_{0}^{2 n-3}\left(q_{0}^{n}-1\right)\left(q_{0}^{n-1}-1\right) .
$$

Recall that $r_{p} \mid 2$. Moreover,

$$
v=\frac{|X|}{\left|X_{\alpha}\right|}=\frac{q_{0}^{n(n-1) / 2}\left(q_{0}^{n}+1\right)\left(q_{0}^{n-1}+1\right) \cdots\left(q_{0}^{2}+1\right)}{d_{0}}
$$

is even, and so $4 \nmid r$. Therefore,

$$
\begin{equation*}
r \mid 2 f d_{0}\left(q_{0}^{n}-1\right)\left(q_{0}^{n-1}-1\right) \tag{4.23}
\end{equation*}
$$

From $r^{2}>2 v$, that is to say, $r^{2} / 2>v$, we see that

$$
\begin{equation*}
2 f^{2} d_{0}^{2}\left(q_{0}^{n}-1\right)^{2}\left(q_{0}^{n-1}-1\right)^{2}>\frac{q_{0}^{n(n-1) / 2}\left(q_{0}^{n}+1\right)\left(q_{0}^{n-1}+1\right) \cdots\left(q_{0}^{2}+1\right)}{d_{0}} \tag{4.24}
\end{equation*}
$$

and so, using $f<q=q_{0}^{2}$,

$$
2 d_{0}^{3} q_{0}^{4 n+2}>q_{0}^{n^{2}-1}
$$

that is, $2 \operatorname{gcd}\left(n, q_{0}+1\right)^{3}=2 d_{0}^{3}>q_{0}^{n^{2}-4 n-3}$. If $n \geqslant 6$, then it follows that $2\left(q_{0}+1\right)^{3}>q_{0}^{9}$, a contradiction. Thus $3 \leqslant n \leqslant 5$.

Assume that $n=5$, so $2 d_{0}^{3}>q_{0}^{2}$. It follows that $d_{0} \neq 1$, and so $d_{0}=\operatorname{gcd}\left(5, q_{0}+1\right)=5$. This together with $250>q_{0}^{2}$ implies that $q_{0} \in\{4,9\}$. In either case $f=4$, and the inequality (4.24) does not hold, a contradiction. Hence $n \leqslant 4$.

Since $\operatorname{PSL}\left(n, q_{0}\right) \triangleleft X_{\alpha}$ and $r_{p} \mid 2_{p}$, by Lemma 2.5, $r$ is divisible by the index of a parabolic subgroup of $\operatorname{PSL}\left(n, q_{0}\right)$, that is, the number of $i$-spaces for some $i \leqslant n / 2$.

Subcase 3.1: $n=4$. There are $\left(q_{0}+1\right)\left(q_{0}^{2}+1\right) 1$-spaces and $\left(q_{0}^{2}+1\right)\left(q_{0}^{2}+q_{0}+1\right) 2$-spaces, so $q_{0}^{2}+1$ divides $r$. Moreover, it follows from $v=q_{0}^{6}\left(q_{0}^{4}+1\right)\left(q_{0}^{3}+1\right)\left(q_{0}^{2}+1\right) / \operatorname{gcd}\left(4, q_{0}+1\right)$ that $q_{0}^{2}+1$ divides $v$, since $\operatorname{gcd}\left(4, q_{0}+1\right)$ is a divisor of $q_{0}^{3}+1$. Therefore, $q_{0}^{2}+1$ divides $\operatorname{gcd}(r, v)$. However $r \mid 2(v-1)$ and hence $\operatorname{gcd}(r, v) \mid 2$, and this implies that $q_{0}^{2}+1$ divides 2, a contradiction.

Subcase 3.2: $n=3$. Here the number $q_{0}^{2}+q_{0}+1$ of 1 -spaces must divide $r$. Since $r \mid 2(v-1)$ and $q_{0}^{2}+q_{0}+1$ is odd, it follows that $q_{0}^{2}+q_{0}+1$ divides $v-1$. On the other hand $v=q_{0}^{3}\left(q_{0}^{3}+1\right)\left(q_{0}^{2}+1\right) / d_{0}$, and it follows that $\operatorname{gcd}\left(v-1, q_{0}^{2}+q_{0}+1\right)=q_{0}^{2}+q_{0}+1$ must divide $2 q_{0}+d_{0}$. This implies that $q_{0}=2$ and $d_{0}=\operatorname{gcd}\left(3, q_{0}+1\right)=3$. Therefore, $7 \mid r, f=2$ and $v=120$. However, from (4.23) and $r \mid 2(v-1)$ we obtain $r=7$ or 14, contradicting $r^{2}>2 v$.

## $4.7 \quad \mathcal{C}_{6}$-subgroups

Here $G_{\alpha}$ is of type $t^{2 m} \cdot \operatorname{Sp}_{2 m}(t)$, where $n=t^{m}$ for some prime $t \neq p$ and positive integer $m$, and moreover $f$ is odd and is minimal such that $t \operatorname{gcd}(2, t)$ divides $q-1=p^{f}-1$ (see [14, Table 3.5.A]).

Lemma 4.7. Assume Hypothesis 4.1. Then the point-stabilizer $G_{\alpha} \notin \mathcal{C}_{6}$.
Proof. From [14, Propositions 4.6.5 and 4.6.6] we have $\left|X_{\alpha}\right| \leqslant t^{2 m}\left|\operatorname{Sp}_{2 m}(t)\right|$, and from Lemma 2.6 we have $\left|\operatorname{Sp}_{2 m}(t)\right|<t^{m(2 m+1)}$. Moreover $t<q$, since $t \operatorname{gcd}(2, t)$ divides $q-1$. Hence $\left|X_{\alpha}\right|<t^{2 m+m(2 m+1)}<q^{2 m^{2}+3 m}$. By Lemma 2.3(i), recalling that $d=\operatorname{gcd}(n, q-1)$,

$$
|X|<2(d f)^{2}\left|X_{\alpha}\right|^{3}<2 d^{2} f^{2} q^{6 m^{2}+9 m}<2 q^{6 m^{2}+9 m+4}
$$

Combining this with the fact that $|X|>q^{n^{2}-2}=q^{t^{2 m}-2}$ (by Lemma (2.6), we obtain

$$
q^{t^{2^{m}}-\left(6 m^{2}+9 m+6\right)}<2 .
$$

Therefore,

$$
\begin{equation*}
t^{2 m} \leqslant 6 m^{2}+9 m+6 \tag{4.25}
\end{equation*}
$$

As $t \geqslant 2$, we deduce that $2^{2 m} \leqslant 6 m^{2}+9 m+6$, and hence $m \leqslant 3$.
Case 1: $m=1$. Here $t=n \geqslant 3$, so $t$ is an odd prime, and from (4.25) we have $t^{2} \leqslant 21$. Hence $t=n=3$, so that $t \operatorname{gcd}(2, t)=3$ divides $q-1$, and $d=\operatorname{gcd}(n, q-1)=3$. Also $\left|X_{\alpha}\right| \leqslant t^{2 m}\left|\operatorname{Sp}_{2 m}(t)\right|=3^{2}\left|\operatorname{Sp}_{2}(3)\right|=2^{3} \cdot 3^{3}$, and then it follows from $q^{n^{2}-2}<|X|<2(d f)^{2}\left|X_{\alpha}\right|^{3}$ that

$$
q^{7}<2(3 f)^{2}\left|X_{\alpha}\right|^{3} \leqslant 2 \cdot(3 f)^{2} \cdot\left(2^{3} \cdot 3^{3}\right)^{3}=f^{2} \cdot 2^{10} \cdot 3^{11}
$$

This inequality, together with the fact that $f$ is odd and is minimal such that $t \operatorname{gcd}(2, t)=3$ divides $p^{f}-1$, implies that $q \in\{7,13\}$, and hence also that $f=1$. In particular, $q \equiv 4$ or $7(\bmod 9)$, so that, by [14, Proposition 4.6.5], we have $X_{\alpha} \cong 3^{2} . Q_{8}$. According to Lemma 2.3(ii), $r$ divides $2 d f\left|X_{\alpha}\right|=432$. Thus $r$ divides $R:=\operatorname{gcd}(432,2(v-1))$. If $q=7$ then $v=2^{2} \cdot 7^{3} \cdot 19$, and so $R=6$; and if $q=13$, then $v=2^{2} \cdot 7 \cdot 13^{3} \cdot 61$, and again $R=6$. Then $R^{2}<2 v$, contradicting $r^{2}>2 v$.

Case 2: $m=2$. In this case (4.25) shows that $t^{4} \leqslant 48$ and so $t=2$ and $n=4$. Thus $\left|X_{\alpha}\right| \leqslant t^{2 m}\left|\operatorname{Sp}_{2 m}(t)\right|=2^{4}\left|\operatorname{Sp}_{4}(2)\right|<2^{14}$. From [14, Proposition 4.6.6] we see that $q=p \equiv 1$ $(\bmod 4)$. In particular, $f=1$ and $d=4$. Then the condition $q^{n^{2}-2}<|X|<2(d f)^{2}\left|X_{\alpha}\right|^{3}$ implies that

$$
q^{14}<2 \cdot 4^{2} \cdot\left(2^{14}\right)^{3}=2^{47}
$$

which yields $q=5$. Then by [14, Proposition 4.6.6] we have $X_{\alpha} \cong 2^{4} . \mathrm{A}_{6}$. Therefore, $v=|X| /\left|X_{\alpha}\right|=5^{5} \cdot 13 \cdot 31$. By Lemma [2.3(ii), $r$ divides $2 d f\left|X_{\alpha}\right|=2^{10} \cdot 3^{2} \cdot 5$. This together with $r \mid 2(v-1)$ implies that $r \mid 4$, contradicting the condition $r^{2}>2 v$.

Case 3: $m=3$. We conclude similarly (using [14, Proposition 4.6.6]) that $t=2$, $n=8, q=p \equiv 1(\bmod 4)($ so $f=1)$ and $\left|X_{\alpha}\right|<2^{27}$. However, this together with $q^{n^{2}-2}<|X|<2(d f)^{2}\left|X_{\alpha}\right|^{3}$ implies that $q^{62}<2^{82} \operatorname{gcd}(8, q-1)^{2}<2^{88}$. Thus $q=2$, a contradiction.

## $4.8 \quad \mathcal{C}_{7}$-subgroups

Here $G_{\alpha}$ is a tensor product subgroup of type $\operatorname{GL}(m, q)\left\langle\mathrm{S}_{t}\right.$, where $t \geqslant 2, m \geqslant 3$ and $n=m^{t}$ (see [14, Table 3.5.A]).

Lemma 4.8. Assume Hypothesis 4.1. Then the point-stabilizer $G_{\alpha} \notin \mathcal{C}_{7}$.
Proof. From [14, Proposition 4.7.3] we deduce that $\left|X_{\alpha}\right| \leqslant|\operatorname{PGL}(m, q)|^{t} \cdot t$ !. This together with Lemma 2.6 implies that $\left|X_{\alpha}\right|<q^{t\left(m^{2}-1\right)} \cdot t$ !. Then by Lemma 2.3(i),

$$
|X|<2(d f)^{2}\left|X_{\alpha}\right|^{3}<2 d^{2} f^{2} q^{3 t\left(m^{2}-1\right)} \cdot(t!)^{3}<q^{3 t\left(m^{2}-1\right)+5} \cdot(t!)^{3}
$$

Combining this with the fact that $|X|>q^{n^{2}-2}=q^{m^{2 t}-2}$ (by Lemma 2.6), we obtain

$$
\begin{equation*}
(t!)^{3}>q^{m^{2 t}-3 t\left(m^{2}-1\right)-7} \geqslant 2^{m^{2 t}-3 t\left(m^{2}-1\right)-7} . \tag{4.26}
\end{equation*}
$$

Let $f(m)=m^{2 t}-3 t\left(m^{2}-1\right)-7$. It is straightforward to check that $f(m)$ is an increasing function of $m$, for $m \geqslant 3$, and hence $f(m) \geqslant f(3)=3^{2 t}-24 t-7$. Thus (4.26) implies that

$$
2^{3^{2 t}-24 t-7}<(t!)^{3} \leqslant t^{3 t}
$$

Taking logarithms to base 2 we have $3^{2 t}-24 t-7<3 t \log _{2}(t)$, which has no solutions for $t \geqslant 2$.

## $4.9 \quad \mathcal{C}_{8}$-subgroups

Here $G_{\alpha}$ is a classical group in its natural representation.
Lemma 4.9. Assume Hypothesis 4.1. If the point-stabilizer $G_{\alpha} \in \mathcal{C}_{8}$, then $G_{\alpha}$ cannot be symplectic.

Proof. Suppose for a contradiction that $G_{\alpha}$ is a symplectic group in $\mathcal{C}_{8}$. Then by [14, Proposition 4.8.3], $n$ is even, $n \geqslant 4$, and

$$
X_{\alpha} \cong \operatorname{PSp}(n, q) \cdot\left[\frac{\operatorname{gcd}(2, q-1) \operatorname{gcd}(n / 2, q-1)}{d}\right]
$$

where $d=\operatorname{gcd}(n, q-1)$. For convenience we will also use the notation $d^{\prime}=\operatorname{gcd}(n / 2, q-1)$ in this proof. Therefore,

$$
\left|X_{\alpha}\right|=q^{n^{2} / 4}\left(q^{n}-1\right)\left(q^{n-2}-1\right) \cdots\left(q^{2}-1\right) d^{\prime} / d
$$

and so

$$
v=\frac{|X|}{\left|X_{\alpha}\right|}=\frac{q^{\left(n^{2}-2 n\right) / 4}\left(q^{n-1}-1\right)\left(q^{n-3}-1\right) \cdots\left(q^{3}-1\right)}{d^{\prime}}
$$

so in particular $p \mid v$. By Lemma 2.3(iii), $r_{p}$ divides 2. Since $\operatorname{PSp}(n, q) \unlhd X_{\alpha}$, except for $(n, q)=(4,2)$, we can apply Lemma [2.5, and so in these cases $r$ is divisible by the index of a parabolic subgroup of $\operatorname{PSp}(n, q)$. We first treat the case $n=4$.

Case 1: $n=4$.

In this case,

$$
X_{\alpha} \cong \operatorname{PSp}(4, q) \cdot\left[\frac{\operatorname{gcd}(2, q-1)^{2}}{\operatorname{gcd}(4, q-1)}\right] \quad \text { and } \quad v=\frac{q^{2}\left(q^{3}-1\right)}{\operatorname{gcd}(2, q-1)}
$$

If $(n, q)=(4,2)$, then a Magma computation shows that the subdegrees of $G$ are 12 and 15 , so by Lemma 2.2 (iv), $r \mid \operatorname{gcd}(24,30)=6$, contradicting $r^{2}>2 v$. Since $X \cong \mathrm{~A}_{8}$, using [25, Theorem 1] for symmetric designs and [16, Theorem 1.1] for non-symmetric designs also rules out this case. Hence $(n, q) \neq(4,2)$. Then, since the indices of the parabolic subgroups $\mathrm{P}_{1}$ and $\mathrm{P}_{2}$ in $\operatorname{PSp}(4, q)$ are both equal to $(q+1)\left(q^{2}+1\right)$, it follows that $(q+1)\left(q^{2}+1\right) \mid r$ and, since $r \mid 2(v-1)$, that $(q+1)\left(q^{2}+1\right)$ divides $2(v-1)$. Suppose first that $q$ is even. Then

$$
2(v-1)=2 q^{2}\left(q^{3}-1\right)-2=2\left(q^{2}+1\right)\left(q^{3}-q-1\right)+2 q,
$$

which is not divisible by $q^{2}+1$. Thus $q$ is odd, and we have

$$
2(v-1)=q^{2}\left(q^{3}-1\right)-2=\left(q^{2}+1\right)\left(q^{3}-q-1\right)+q-1,
$$

and again this is not divisible by $q^{2}+1$. Thus $n \neq 4$.
Case 2: $n \geqslant 6$.
Let $\tilde{X}=\operatorname{SL}(n, q)$, the preimage of $X$ in $\operatorname{GL}(n, q)$, and let $\left\{e_{1}, \ldots, e_{n / 2}, f_{1}, \ldots, f_{n / 2}\right\}$ be a basis for $V$ such that the nondegenerate alternating form preserved by $\tilde{X}_{\alpha}$ satisfies

$$
\left(e_{i}, e_{j}\right)=\left(f_{i}, f_{j}\right)=0 \quad \text { and } \quad\left(e_{i}, f_{j}\right)=\delta_{i j} \quad \text { for all } i, j
$$

Let $\operatorname{SL}(4, q)$ denote the subgroup of $\tilde{X}$ acting naturally on $U:=\left\langle e_{1}, e_{2}, f_{1}, f_{2}\right\rangle$ and fixing $W:=\left\langle e_{3}, \ldots, e_{n / 2}, f_{3}, \ldots, f_{n / 2}\right\rangle$ pointwise, and let $\operatorname{Sp}(4, q)=\operatorname{SL}(4, q) \cap \tilde{X}_{\alpha}$, namely the pointwise stabiliser of $W$ in $\tilde{X}_{\alpha}$. Let $g \in \operatorname{SL}(4, q) \backslash \mathbf{N}_{\mathrm{SL}(4, q)}(\operatorname{Sp}(4, q))$ so $g \notin \tilde{X}_{\alpha}$, and let $\beta=\alpha^{g} \neq \alpha$. Since $g$ fixes $W$ pointwise, it follows that the alternating forms preserved by $\alpha$ and $\beta$ agree on $W$ and hence that $\tilde{X}_{\alpha \beta}=\tilde{X}_{\alpha} \cap\left(\tilde{X}_{\alpha}\right)^{g}$ contains the pointwise stabiliser $\operatorname{Sp}(n-4, q)$ of $U$ in $\tilde{X}_{\alpha}$.

Since this subgroup $\operatorname{Sp}(n-4, q)$ intersects the scalar subgroup trivially, $X_{\alpha \beta}$ contains a subgroup isomorphic to $\operatorname{Sp}(n-4, q)$, and hence so does $G_{\alpha \beta}$. By Lemma 2.4, $r$ divides $4 d f\left|X_{\alpha}\right| /|\operatorname{Sp}(n-4, q)|$, that is,

$$
r \mid 4 d^{\prime} f q^{2 n-4}\left(q^{n}-1\right)\left(q^{n-2}-1\right)
$$

Recall that $r_{p} \mid 2$. Also, since $n \geqslant 6, v$ is even, and hence $4 \nmid r$. Similarly, it follows from $(q-1) \mid v$ that $r_{t} \mid 2$ for each prime divisor $t$ of $q-1$. Therefore,

$$
r \left\lvert\, 2 f \cdot \frac{q^{n}-1}{q-1} \cdot \frac{q^{n-2}-1}{q-1}\right.
$$

As $f<q$ and $\left(q^{j}-1\right) /(q-1)<2 q^{j-1}$ for all $j$, it follows that $r<8 q^{2 n-3}$. From $r^{2}>2 v$ we derive that

$$
\begin{aligned}
64 q^{4 n-6} & >2 q^{\left(n^{2}-2 n\right) / 4}\left(q^{n-1}-1\right)\left(q^{n-3}-1\right) \cdots\left(q^{3}-1\right) / d^{\prime} \\
& >2 q^{\left(n^{2}-2 n\right) / 4}\left(q^{n-2} q^{n-4} \cdots q^{2}\right) / q=2 q^{\left(n^{2} / 2\right)-n-1},
\end{aligned}
$$

and so $32>q^{\left(n^{2} / 2\right)-5 n+5} \geqslant 2^{\left(n^{2} / 2\right)-5 n+5}$, that is, $n^{2}-10 n<0$. This implies that $n \leqslant 8$.
Suppose that $n=8$. Here $d^{\prime}=\operatorname{gcd}(4, q-1)$. In this case the index of each of the parabolic subgroups $P_{i}$, for $1 \leqslant i \leqslant 4$, is divisible by $q^{4}+1$, and hence $q^{4}+1$ divides $r$, which in turn divides $2(v-1)$ by Lemma 2.2. Then

$$
q^{4}+1 \mid 2 d^{\prime}(v-1)=2 q^{12}\left(q^{7}-1\right)\left(q^{5}-1\right)\left(q^{3}-1\right)-2 d^{\prime}
$$

Since the remainders on dividing $q^{12}, q^{7}-1, q^{5}-1$ by $q^{4}+1$ are $-1,-q^{3}-1$ and $-q-1$, respectively, it follows that

$$
q^{4}+1 \mid-2\left(q^{3}+1\right)(q+1)\left(q^{3}-1\right)-2 d^{\prime}=-2\left(q^{6}-1\right)(q+1)-2 d^{\prime} .
$$

The remainder on dividing $q^{6}-1$ by $q^{4}+1$ is $-q^{2}-1$, and hence

$$
q^{4}+1 \left\lvert\, 2\left(q^{2}+1\right)(q+1)-2 d^{\prime}=2\left(\frac{q^{4}-1}{q-1}-d^{\prime}\right)\right.
$$

This implies that

$$
q^{4}+1 \mid 2\left(q^{4}-1\right)-2 d^{\prime}(q-1)=2\left(q^{4}+1\right)-4-2 d^{\prime}(q-1)
$$

and hence $q^{4}+1 \leqslant 2 d^{\prime}(q-1)+4 \leqslant 8 q-4\left(\right.$ since $\left.d^{\prime} \leqslant 4\right)$, a contradiction.
Thus $n=6$. Here $d^{\prime}=\operatorname{gcd}(3, q-1)$. The indices of the parabolic subgroups $\mathrm{P}_{1}, \mathrm{P}_{2}$ and $\mathrm{P}_{3}$ in $\operatorname{PSp}(6, q)$ are $\left(q^{3}+1\right)\left(q^{2}+q+1\right),\left(q^{3}+1\right)\left(q^{2}+q+1\right)\left(q^{2}+1\right)$ and $\left(q^{3}+1\right)\left(q^{2}+1\right)(q+1)$, and since one of these numbers divides $r$, we deduce that $\left(q^{3}+1\right) \mid r$, and so $\left(q^{3}+1\right)$ divides $2 d^{\prime}(v-1)=2\left(q^{6}\left(q^{5}-1\right)\left(q^{3}-1\right)-d^{\prime}\right)$. Since the remainders on dividing $q^{6}, q^{5}-1, q^{3}-1$ by $q^{3}+1$ are $1,-q^{2}-1$ and -2 , respectively, it follows that $q^{3}+1$ divides $2\left(2\left(q^{2}+1\right)-d^{\prime}\right)$. Hence $q^{3}+1 \leqslant 2\left(2 q^{2}+2-d^{\prime}\right) \leqslant 2\left(2 q^{2}+1\right)$, which implies that $q \leqslant 4$. If $q=3$, but then $d^{\prime}=1$ and $q^{3}+1=28$ does not divide $2\left(2\left(q^{2}+1\right)-d^{\prime}\right)=38$. Thus $q$ is even and the divisibility condition implies that $q^{3}+1$ divides $2\left(q^{2}+1\right)-d^{\prime} \leqslant 2 q^{2}+1$, which forces $q=2$ and $d^{\prime}=1$. Hence $v=2^{6} \cdot 7 \cdot 31$, and therefore $v-1$ is coprime to 5 and 7 . However $r$, and hence also $2(v-1)$ is divisible by the index of one of the parabolic subgroups $\mathrm{P}_{1}, \mathrm{P}_{2}$ or $\mathrm{P}_{3}$ of $\operatorname{PSp}(6,2)$, and these are $3^{2} \cdot 7,3^{2} \cdot 5 \cdot 7,3^{3} \cdot 5$. This is a contradiction.

Lemma 4.10. Assume Hypothesis 4.1. If the point-stabilizer $G_{\alpha} \in \mathcal{C}_{8}$, then $G_{\alpha}$ cannot be orthogonal.

Proof. Suppose for a contradiction that $G_{\alpha}$ is an orthogonal group in $\mathcal{C}_{8}$. Then by [14, Proposition 4.8.4], $q$ is odd, $n \geqslant 3$, and

$$
X_{\alpha} \cong \mathrm{PSO}^{\epsilon}(n, q) \cdot \operatorname{gcd}(n, 2)
$$

where $\epsilon \in\{0,+,-\}$. Let $\tilde{X}=\operatorname{SL}(n, q)$ and let $\tilde{X}_{\alpha}$ denote the full preimage of $X_{\alpha}$ in $\tilde{X}$.
Let $\varphi$ be the non-degenerate symmetric bilinear form on $V$ preserved by $\tilde{X}_{\alpha}$, and let $e_{1}, f_{1} \in V$ be a hyperbolic pair, that is $e_{1}, f_{1}$ are isotropic vectors and $\varphi\left(e_{1}, f_{1}\right)=1$. Let $U=\left\langle e_{1}, f_{1}\right\rangle$, and consider the decomposition $V=U \oplus U^{\perp}$. Let $g \in \tilde{X}$ fixing $U^{\perp}$ pointwise and mapping $e_{1}$ onto itself and $f_{1}$ onto $e_{1}+f_{1}$. Then $g$ maps the isotropic vector $f_{1}$ onto
the non-isotropic vector $e_{1}+f_{1}$, and so $g \notin \tilde{X}_{\alpha}$. Let $\beta=\alpha^{g}$, so that $\tilde{X}_{\beta}$ leaves invariant the form $\varphi^{g}$. Then, since $\varphi$ and $\varphi^{g}$ restrict to the same form on $U^{\perp}$, we have that

$$
\left\{\left.\left(\begin{array}{ll}
I_{2} & \\
& B
\end{array}\right) \right\rvert\, B \in \mathrm{SO}^{\epsilon}(n-2, q)\right\} \leqslant \tilde{X}_{\alpha} \cap \tilde{X}_{\alpha}^{g}=\tilde{X}_{\alpha \beta} .
$$

Since this group intersects the scalar subgroup trivially, $X_{\alpha \beta}$ contains a subgroup isomorphic to $\mathrm{SO}^{\epsilon}(n-2, q)$, and hence so does $G_{\alpha \beta}$. By Lemma 2.4,

$$
\begin{equation*}
r|4 d f| X_{\alpha}\left|/\left|\mathrm{SO}^{\epsilon}(n-2, q)\right| .\right. \tag{4.27}
\end{equation*}
$$

We now split into cases where $n$ is odd or even.
Case 1: $n=2 m+1$ is odd, so $\epsilon=\circ$ and is usually omitted.
In this case, $X_{\alpha} \cong \operatorname{PSO}(2 m+1, q)$. Thus

$$
\left|X_{\alpha}\right|=q^{m^{2}}\left(q^{2 m}-1\right)\left(q^{2 m-2}-1\right) \cdots\left(q^{2}-1\right)
$$

and so

$$
v=|X| /\left|X_{\alpha}\right|=q^{m^{2}+m}\left(q^{2 m+1}-1\right)\left(q^{2 m-1}-1\right) \cdots\left(q^{3}-1\right) / d,
$$

where $d=\operatorname{gcd}(2 m+1, q-1)$, and this implies that $v$ is even and $p \mid v$. By Lemma 2.3(iii), $r_{p}$ divides 2, so $r_{p}=1$ since $q$ is odd. Moreover, since $r \mid 2(v-1)$, it follows that $4 \nmid r$.

## Subcase 1.1: $m=1$.

Then

$$
\left|X_{\alpha}\right|=q\left(q^{2}-1\right) \quad \text { and } \quad v=q^{2}\left(q^{3}-1\right) / d
$$

As $p \mid v$, it follows from Lemma 2.3(iii) that $r$ divides $2 d f\left|X_{\alpha}\right|_{p^{\prime}}$ and hence $r$ divides $2 d f\left(q^{2}-\right.$ 1). Combining this with $r \mid 2(v-1)$, we deduce that $r$ divides

$$
2 \operatorname{gcd}\left(d(v-1), d f\left(q^{2}-1\right)\right)=2 \operatorname{gcd}\left(q^{2}\left(q^{3}-1\right)-d, d f\left(q^{2}-1\right)\right)
$$

Noting that $\operatorname{gcd}\left(q^{2}\left(q^{3}-1\right)-d, q^{2}-1\right)$ divides

$$
q^{2}\left(q^{3}-1\right)-d-\left(q^{2}-1\right)\left(q^{3}+q-1\right)=q-1-d,
$$

we conclude that $r$ divides $2 d f(q-1-d)$. If $d=\operatorname{gcd}(3, q-1)=3$, then $q \geqslant 7$ (since $q$ is odd) and $r \mid 6 f(q-4)$. From $r^{2}>2 v=2 q^{2}\left(q^{3}-1\right) / 3$ we derive that $54 f^{2}(q-4)^{2}>q^{2}\left(q^{3}-1\right)$, which yields a contradiction. Consequently, $d=1$. Then $r \mid 2 f(q-2)$, and from $r^{2}>2 v=$ $2 q^{2}\left(q^{3}-1\right)$ we derive that $2 f^{2}(q-2)^{2}>q^{2}\left(q^{3}-1\right)$, which is not possible.

Subcase 1.2: $m \geqslant 2$. By (4.27), $r|4 d f| X_{\alpha}\left|/\left|\operatorname{SO}^{\epsilon}(n-2, q)\right|\right.$, that is,

$$
r \mid 4 d f q^{2 m-1}\left(q^{2 m}-1\right)
$$

Recall that $r_{p}=1$ and $4 \nmid r$. We conclude that

$$
r \mid 2 d f\left(q^{2 m}-1\right)
$$

Therefore, as $r^{2}>2 v$, we have

$$
4 d^{2} f^{2}\left(q^{2 m}-1\right)^{2}>\frac{2 q^{m^{2}+m}\left(q^{2 m+1}-1\right)\left(q^{2 m-1}-1\right) \cdots\left(q^{3}-1\right)}{d}
$$

and hence

$$
\begin{aligned}
2 q^{3} \cdot q^{2} \cdot q^{4 m} & >2 d^{3} f^{2}\left(q^{2 m}-1\right)^{2} \\
& >q^{m^{2}+m}\left(q^{2 m+1}-1\right)\left(q^{2 m-1}-1\right) \cdots\left(q^{3}-1\right) \\
& >q^{m^{2}+m}\left(q^{2 m} q^{2 m-2} \cdots q^{2}\right) \\
& =q^{2 m^{2}+2 m},
\end{aligned}
$$

This implies that $q^{2 m^{2}-2 m-5}<2$ and so $2 m^{2}-2 m-5 \leqslant 0$. Thus $m=2$ and $d \leqslant 5$. Therefore $q^{6}\left(q^{5}-1\right)\left(q^{3}-1\right)<2 d^{3} f^{2}\left(q^{4}-1\right)^{2}<250 f^{2}\left(q^{4}-1\right)^{2}$, which implies $q=2$, a contradiction.

Case 2: $n=2 m$ is even, where $m \geqslant 2$ since $2 m=n \geqslant 3$.
In this case, $X_{\alpha} \cong \operatorname{PSO}^{\epsilon}(2 m, q) \cdot 2$ with $\epsilon= \pm$ (we identify $\pm$ with $\pm 1$ for superscripts). Hence

$$
\left|X_{\alpha}\right|=q^{m(m-1)}\left(q^{m}-\epsilon\right)\left(q^{2 m-2}-1\right)\left(q^{2 m-4}-1\right) \cdots\left(q^{2}-1\right),
$$

and so

$$
v=\frac{|X|}{\left|X_{\alpha}\right|}=\frac{q^{m^{2}}\left(q^{m}+\epsilon\right)\left(q^{2 m-1}-1\right)\left(q^{2 m-3}-1\right) \cdots\left(q^{3}-1\right)}{d}
$$

where $d=\operatorname{gcd}(2 m, q-1)$, and this implies that $v$ is even and $p \mid v$. By Lemma 2.3(iii), $r_{p}$ divides 2 , so $r_{p}=1$ since $q$ is odd. Moreover, since $r \mid 2(v-1)$, it follows that $4 \nmid r$.

By (4.27), $r|4 d f| X_{\alpha}\left|/\left|\operatorname{SO}^{\epsilon}(n-2, q)\right|\right.$, that is, $r$ divides

$$
4 d f q^{2 m-2}\left(q^{m}-\epsilon\right) \frac{q^{2 m-2}-1}{q^{m-1}-\epsilon}=4 d f q^{2 m-2}\left(q^{m}-\epsilon\right)\left(q^{m-1}+\epsilon\right) .
$$

As $r_{p}=1$ and $4 \nmid r$, it follows that

$$
\begin{equation*}
r \mid 2 d f\left(q^{m}-\epsilon\right)\left(q^{m-1}+\epsilon\right) . \tag{4.28}
\end{equation*}
$$

Then we deduce from $r^{2}>2 v$ that

$$
\begin{align*}
& 2 d^{3} f^{2}\left(q^{m}-\epsilon\right)^{2}\left(q^{m-1}+\epsilon\right)^{2} \\
> & q^{m^{2}}\left(q^{m}+\epsilon\right)\left(q^{2 m-1}-1\right)\left(q^{2 m-3}-1\right) \cdots\left(q^{3}-1\right), \tag{4.29}
\end{align*}
$$

and so

$$
\begin{aligned}
2 q^{3} \cdot q^{2}\left(2 q^{2 m-1}\right)^{2} & >2 d^{3} f^{2}\left(q^{m}-\epsilon\right)^{2}\left(q^{m-1}+\epsilon\right)^{2} \\
& >q^{m^{2}}\left(q^{m}+\epsilon\right)\left(q^{2 m-1}-1\right)\left(q^{2 m-3}-1\right) \cdots\left(q^{3}-1\right) \\
& >q^{m^{2}}\left(2 q^{m-1}\right)\left(q^{2 m-2} \cdots q^{2}\right) \\
& =2 q^{2 m^{2}-1}
\end{aligned}
$$

Hence $q^{2 m^{2}-4 m-4}<4$ and so $2 m^{2}-4 m-4<2$, which implies $m=2$ and $d \leqslant 4$. Thus $X_{\alpha} \cong \operatorname{PSO}^{\epsilon}(4, q) \cdot 2$.

Suppose $\epsilon=-$, so that $X_{\alpha} \cong \mathrm{PSO}^{-}(4, q) \cdot 2$. Then (4.29) gives

$$
2 d^{3} f^{2}\left(q^{2}+1\right)^{2}(q-1)^{2}>q^{4}\left(q^{2}-1\right)\left(q^{3}-1\right)
$$

which can be simplified to

$$
\begin{equation*}
2 d^{3} f^{2}\left(q^{2}+1\right)^{2}>q^{4}(q+1)\left(q^{2}+q+1\right) \tag{4.30}
\end{equation*}
$$

Thus $128 f^{2}\left(q^{2}+1\right)^{2}>q^{4}(q+1)\left(q^{2}+q+1\right)$. Since $q$ is odd, this implies that $q=3$ so that $d=2$, but then (4.30) is not satisfied.

Therefore $\epsilon=+$, so that $X_{\alpha} \cong \operatorname{PSO}^{+}(4, q) \cdot 2$. Then (4.29) gives

$$
\begin{equation*}
2 d^{3} f^{2}\left(q^{2}-1\right)^{2}(q+1)^{2}>q^{4}\left(q^{2}+1\right)\left(q^{3}-1\right) \tag{4.31}
\end{equation*}
$$

and thus

$$
128 f^{2}(q+1)^{2}>\left(q^{2}+1\right)\left(q^{3}-1\right)
$$

Since $q$ is odd, we conclude that $q=3$ or 5 . However, $q=3$ does not satisfy (4.31), thus $q=5, f=1$ and $d=4$. Then $v=|X| /\left|X_{\alpha}\right|=503750$. By (??), $r \mid 2 d f\left(q^{2}-1\right)(q+1)=2^{7} * 3^{3}$. This together with $r \mid 2(v-1)$ (Lemma 2.1(i)) leads to $r \mid 2$, contradicting $r^{2}>2 v$.

Lemma 4.11. Assume Hypothesis 4.1. If the point-stabilizer $G_{\alpha} \in \mathcal{C}_{8}$, then $G_{\alpha}$ cannot be unitary.

Proof. Suppose that $G_{\alpha}$ is a unitary group in $\mathcal{C}_{8}$. Then by [14, Proposition 4.8.5], $n \geqslant 3$, $q=q_{0}^{2}$, and

$$
X_{\alpha} \cong \operatorname{PSU}\left(n, q_{0}\right) \cdot\left[\frac{\operatorname{gcd}\left(n, q_{0}+1\right) c}{d}\right]
$$

where $d=\operatorname{gcd}(n, q-1)$ and $c=(q-1) / \operatorname{lcm}\left(q_{0}+1,(q-1) / d\right)$. By Lemma 2.7(iii), $c=$ $\operatorname{gcd}\left(n, q_{0}-1\right)$. Hence

$$
\begin{aligned}
\left|X_{\alpha}\right| & =\left|\operatorname{PSU}\left(n, q_{0}\right)\right| \cdot \frac{\operatorname{gcd}\left(n, q_{0}+1\right) \operatorname{gcd}\left(n, q_{0}-1\right)}{\operatorname{gcd}\left(n, q_{0}^{2}-1\right)} \\
& =\frac{c}{d} \cdot q_{0}^{n(n-1) / 2} \prod_{i=2}^{n}\left(q_{0}^{i}-(-1)^{i}\right)
\end{aligned}
$$

and

$$
v=\frac{|X|}{\left|X_{\alpha}\right|}=\frac{1}{c} \cdot q_{0}^{n(n-1) / 2} \prod_{i=2}^{n}\left(q_{0}^{i}+(-1)^{i}\right)
$$

which implies that $p \mid v$ and $v$ is even. Since $r \mid 2(v-1)$, it follows that $r_{p} \mid 2$ and $4 \nmid r$.
Case 1: $n=3$.
In this case,

$$
\left|X_{\alpha}\right|=\frac{c q_{0}^{3}\left(q_{0}^{3}+1\right)\left(q_{0}^{2}-1\right)}{d} \quad \text { and } \quad v=\frac{q_{0}^{3}\left(q_{0}^{3}-1\right)\left(q_{0}^{2}+1\right)}{c}
$$

where $c=\operatorname{gcd}\left(3, q_{0}-1\right)$ and $d=\operatorname{gcd}\left(3, q_{0}^{2}-1\right)$. Since $\operatorname{PSU}\left(n, q_{0}\right) \unlhd X_{\alpha}$, by Lemma 2.5, $r$ is divisible by the index of a parabolic subgroup of $\operatorname{PSU}\left(3, q_{0}\right)$, that is, $q_{0}^{3}+1$. Hence $\left(q_{0}^{3}+1\right) \mid r$, which implies that $\left(q_{0}^{3}+1\right)$ divides $2(v-1)$ and hence also $2 c(v-1)=2 q_{0}^{3}\left(q_{0}^{3}-1\right)\left(q_{0}^{2}+1\right)-2 c$. Since the remainders on dividing $q_{0}^{3}, q_{0}^{3}-1$ by $q_{0}^{3}+1$ are $-1,-2$, respectively, it follows that

$$
q_{0}^{3}+1 \mid 4\left(q_{0}^{2}+1\right)-2 c
$$

which implies that $q_{0}=2, d=3, c=1$, and $f=2$. Thus $v=q_{0}^{3}\left(q_{0}^{3}-1\right)\left(q_{0}^{2}+1\right)=280$ and $\left|X_{\alpha}\right|=q_{0}^{3}\left(q_{0}^{3}+1\right)\left(q_{0}^{2}-1\right) / 3=72$. Since $r \mid 2(v-1)$ and $\left.r|2 d f| X_{\alpha}\right|_{p^{\prime}}$ by Lemma 2.3(iii), we conclude that $r$ divides 18, contradicting the condition $r^{2}>2 v$.

Case 2: $n \geqslant 4$.
Let $\tilde{X}=\operatorname{SL}(n, q)$ and let $\tilde{X}_{\alpha}$ denote the full preimage of $X_{\alpha}$ in $\tilde{X}$. Let $U=\left\langle e_{1}, f_{1}\right\rangle$ be a nondegenerate 2 -subspace of $V$ relative to the unitary form $\varphi$ preserved by $\tilde{X}_{\alpha}$. Let $A \in \mathrm{SL}(U)$ such that $A$ does not preserve modulo scalars the restriction of $\varphi$ to $U$. Then the element $g=\left(\begin{array}{cc}A & \\ & I\end{array}\right) \in \tilde{X}$ but $g$ does not lie in $\tilde{X}_{\alpha}$. Hence $\beta:=\alpha^{g} \neq \alpha$. On the other hand

$$
\left\{\left.\left(\begin{array}{ll}
I & \\
& B
\end{array}\right) \right\rvert\, B \in \mathrm{SU}\left(n-2, q_{0}\right)\right\} \leqslant \tilde{X}_{\alpha} \cap \tilde{X}_{\alpha}^{g}=\tilde{X}_{\alpha \beta} .
$$

Since this group intersects the scalar subgroup trivially, $X_{\alpha \beta}$ contains a subgroup isomorphic to $\mathrm{SU}(n-2, q)$, and hence so does $G_{\alpha \beta}$. By Lemma 2.4, $r$ divides $4 d f\left|X_{\alpha}\right| /\left|\operatorname{SU}\left(n-2, q_{0}\right)\right|$, that is,

$$
r \mid 4 c f q_{0}^{2 n-3}\left(q_{0}^{n}-(-1)^{n}\right)\left(q_{0}^{n-1}-(-1)^{n-1}\right)
$$

Since $r_{p} \mid 2$ and $4 \nmid r$, we derive that

$$
r \mid 2 c f\left(q_{0}^{n}-(-1)^{n}\right)\left(q_{0}^{n-1}-(-1)^{n-1}\right)
$$

This together with $r^{2}>2 v$ and $v=|X| /\left|X_{\alpha}\right|$ leads to $r^{2}\left|X_{\alpha}\right|>2|X|$. By Lemma [2.6 we have

$$
|X|>q_{0}^{2 n^{2}-4} \quad \text { and } \quad\left|X_{\alpha}\right|<\frac{q_{0}^{n^{2}-1} c \operatorname{gcd}\left(n, q_{0}+1\right)}{d}
$$

Consequently, noting that $\operatorname{gcd}\left(n, q_{0}+1\right) \leqslant d=\operatorname{gcd}\left(n, q_{0}^{2}-1\right), c=\operatorname{gcd}\left(n, q_{0}-1\right)<q_{0}$, and $f<q=q_{0}^{2}$, we get

$$
\begin{aligned}
2 q_{0}^{2 n^{2}-4} & <4 c^{3} f^{2}\left(q_{0}^{n}-(-1)^{n}\right)^{2}\left(q_{0}^{n-1}-(-1)^{n-1}\right)^{2} \cdot \frac{q_{0}^{n^{2}-1} \operatorname{gcd}\left(n, q_{0}+1\right)}{d} \\
& <4 q_{0}^{7}\left(q_{0}^{n}-(-1)^{n}\right)^{2}\left(q_{0}^{n-1}-(-1)^{n-1}\right)^{2} \cdot q_{0}^{n^{2}-1} \\
& <4 q_{0}^{n^{2}+6}\left(2 q_{0}^{n+n-1}\right)^{2}=16 q_{0}^{n^{2}+4 n+4}
\end{aligned}
$$

and hence

$$
q_{0}^{n^{2}-4 n-8}<8
$$

It follows that $n^{2}-4 n-8<3$, which implies $n=4$ or 5 .
Subcase 2.1: $n=4$.
Then

$$
v=q_{0}^{6}\left(q_{0}^{4}+1\right)\left(q_{0}^{3}-1\right)\left(q_{0}^{2}+1\right) / c
$$

where $c=\operatorname{gcd}\left(4, q_{0}-1\right)$. Since $r$ is divisible by the index of a parabolic subgroup of $\operatorname{PSU}\left(4, q_{0}\right)$, which is either $\left(q_{0}^{2}+1\right)\left(q_{0}^{3}+1\right)$ or $\left(q_{0}+1\right)\left(q_{0}^{3}+1\right)$, we derive that $\left(q_{0}^{3}+1\right) \mid r$. Hence $\left(q_{0}^{3}+1\right)$ divides $2(v-1)$, and hence also $2 c(v-1)=2 q_{0}^{6}\left(q_{0}^{4}+1\right)\left(q_{0}^{3}-1\right)\left(q_{0}^{2}+1\right)-2 c$. Since the remainders on dividing $q_{0}^{6}, q_{0}^{4}+1, q_{0}^{3}-1$ by $q_{0}^{3}+1$ are $1,-q_{0}+1$ and -2 , respectively, it follows that $q_{0}^{3}+1$ divides $2\left(-q_{0}+1\right)(-2)\left(q_{0}^{2}+1\right)-2 c=4\left(q_{0}-1\right)\left(q_{0}^{2}+1\right)-2 c$, which
equals $4\left(q_{0}^{3}+1\right)-4\left(q_{0}^{2}-q_{0}+2\right)-2 c$. It follows that $q_{0}^{3}+1$ divides $4\left(q_{0}^{2}-q_{0}+2\right)+2 c$, which implies $q_{0}=2$. Thus $v=2^{6} \cdot 5 \cdot 7 \cdot 17$, and the index of a parabolic subgroup of $\operatorname{PSU}\left(4, q_{0}\right)$ is either 45 or 27 . However, neither 45 nor 27 divides $2(v-1)$, a contradiction.

Subcase 2.2: $n=5$.
Then

$$
v=q_{0}^{10}\left(q_{0}^{5}-1\right)\left(q_{0}^{4}+1\right)\left(q_{0}^{3}-1\right)\left(q_{0}^{2}+1\right) / c,
$$

where $c=\operatorname{gcd}\left(5, q_{0}-1\right)$. Since $r$ is divisible by the index of a parabolic subgroup of $\operatorname{PSU}\left(5, q_{0}\right)$, which is either $\left(q_{0}^{2}+1\right)\left(q_{0}^{5}+1\right)$ or $\left(q_{0}^{3}+1\right)\left(q_{0}^{5}+1\right)$, we derive that $\left(q_{0}^{5}+1\right) \mid r$. Hence $\left(q_{0}^{5}+1\right)$ divides $2(v-1)$, and hence also $2 c(v-1)=2 q_{0}^{10}\left(q_{0}^{5}-1\right)\left(q_{0}^{4}+1\right)\left(q_{0}^{3}-1\right)\left(q_{0}^{2}+1\right)-2 c$. Since the remainders on dividing $q_{0}^{10}, q_{0}^{5}-1,\left(q_{0}^{3}-1\right)\left(q_{0}^{2}+1\right)$ by $q_{0}^{5}+1$ are $1,-2$ and $q_{0}^{3}-q_{0}^{2}-2$, respectively, it follows that $q_{0}^{5}+1$ divides $-4\left(q_{0}^{4}+1\right)\left(q_{0}^{3}-q_{0}^{2}-2\right)-2 c$, which equals

$$
-4\left(q_{0}^{5}+1\right)\left(q_{0}^{2}-q_{0}\right)-4\left(-2 q_{0}^{4}+q_{0}^{3}-2 q_{0}^{2}+q_{0}-2\right)-2 c .
$$

Thus $q_{0}^{5}+1$ divides $8 q_{0}^{4}-4 q_{0}^{3}+8 q_{0}^{2}-4 q_{0}+8-2 c$. However, there is no prime power $q_{0}$ satisfying this condition, a contradiction.

## $4.10 \quad \mathcal{C}_{9}$-subgroups

Here $G_{\alpha}$ is an almost simple group not contained in any of the subgroups in $\mathcal{C}_{1}-\mathcal{C}_{8}$.
Lemma 4.12. Assume Hypothesis 4.1. Then the point-stabilizer $G_{\alpha} \notin \mathcal{C}_{9}$.
Proof. By Lemma 2.2(i) and Lemma 2.6, we have $\left|G_{\alpha}\right|^{3}>|G| \geqslant|X|=|\operatorname{PSL}(n, q)|>q^{n^{2}-2}$. Moreover, by [17, Theorem 4.1], we have that $\left|G_{\alpha}\right|<q^{3 n}$. Hence $q^{n^{2}-2}<\left|G_{\alpha}\right|^{3}<q^{9 n}$, which yields $n^{2}-2<9 n$ and so $3 \leqslant n \leqslant 9$. Further, it follows from [17, Corollary 4.3] that either $n=y(y-1) / 2$ for some integer $y$ or $\left|G_{\alpha}\right|<q^{2 n+4}$. If $n=y(y-1) / 2$, then as $3 \leqslant n \leqslant 9$ we have $n=3$ or 6 . If $\left|G_{\alpha}\right|<q^{2 n+4}$, then we deduce from $\left|G_{\alpha}\right|^{3}>q^{n^{2}-2}$ that $q^{6 n+12}>q^{n^{2}-2}$, which implies $6 n+12>n^{2}-2$ and so $3 \leqslant n \leqslant 7$. Therefore, we always have $3 \leqslant n \leqslant 7$. The possibilities for $X_{\alpha}$ can be read off from [4, Tables 8.4, 8.9, 8.19, 8.25, 8.36]. In Table 4 we list all possibilities, sometimes fusing some cases together. Not all conditions from [4] are listed, but we list what is necessary for our proof. Note that in some listed cases $X_{\alpha}$ is not maximal in $X$ but there is a group $G$ with $X<G \leqslant \operatorname{Aut}(X)$ such that $G_{\alpha}$ is maximal in $G$ and $G_{\alpha} \cap X$ is equal to this non-maximal subgroup $X_{\alpha}$.

Table 4: Possible groups $X$ and $X_{\alpha}$

| Case | $X$ | $X_{\alpha}$ | Conditions on $q$ from [4] | Bound (4.32) |
| :---: | :---: | :--- | :--- | :--- |
| 1 | $\operatorname{PSL}(3, q)$ | $\operatorname{PSL}(2,7)$ | $q=p \equiv 1,2,4(\bmod 7), q \neq 2$ | $q<14$ |
| 2 |  | $\mathrm{~A}_{6}$ | $q=p \equiv 1,4(\bmod 15)$ | $q<19$ |
| 3 |  | $\mathrm{~A}_{6}$ | $q=p^{2}, p \equiv 2,3(\bmod 5), p \neq 3$ | $q<23$ |
| 4 | $\operatorname{PSL}(4, q)$ | $\operatorname{PSL}(2,7)$ | $q=p \equiv 1,2,4(\bmod 7), q \neq 2$ | $q<4$ |
| 5 |  | $\mathrm{~A}_{7}$ | $q=p \equiv 1,2,4(\bmod 7)$ | $q<7$ |
| 6 |  | $\operatorname{PSU}(4,2)$ | $q=p \equiv 1(\bmod 6)$ | $q<12$ |


| 7 | $\operatorname{PSL}(5, q)$ | $\operatorname{PSL}(2,11)$ | $q=p$ odd | $q<3$ |
| :---: | :--- | :--- | :--- | :--- |
| 8 |  | $\mathrm{M}_{11}$ | $q=3$ | $q<4$ |
| 9 |  | $\operatorname{PSU}(4,2)$ | $q=p \equiv 1(\bmod 6)$ | $q<5$ |
| 10 | $\operatorname{PSL}(6, q)$ | $\mathrm{A}_{6} \cdot 2_{3}$ | $q=p$ odd | $q<3$ |
| 11 |  | $\mathrm{~A}_{6}$ | $q=p$ or $p^{2}$ odd | $q<2$ |
| 12 |  | $\operatorname{PSL}(2,11)$ | $q=p$ odd | $q<3$ |
| 13 |  | $\mathrm{~A}_{7}$ | $q=p$ or $p^{2}$ odd | $q<3$ |
| 14 |  | $\operatorname{PSL}(3,4) \cdot 2_{1}^{-}$ | $q=p$ odd | $q<3$ |
| 15 |  | $\operatorname{PSL}(3,4)$ | $q=p$ odd | $q<3$ |
| 16 |  | $\mathrm{M}_{12}$ | $q=3$ | $q<4$ |
| 17 |  | $\operatorname{PSU}(4,3) \cdot 2_{2}^{-}$ | $q=p \equiv 1(\bmod 12)$ | $q<5$ |
| 18 |  | $\operatorname{PSU}(4,3)$ | $q=p \equiv 7(\bmod 12)$ | $q<5$ |
| 19 |  | $\operatorname{PSL}(3, q)$ | $q$ odd |  |
| 20 | $\operatorname{PSL}(7, q)$ | $\operatorname{PSU}(3,3)$ | $q=p$ odd | $q<2$ |

By Lemma 2.3(i) and Lemma 2.6, we have $2 d^{2} f^{2}\left|X_{\alpha}\right|^{3}>|X|>q^{n^{2}-2}$. Using the fact that $d=\operatorname{gcd}(n, q-1) \leqslant n$, it follows that

$$
\begin{equation*}
q<\left(2 n^{2} f^{2}\left|X_{\alpha}\right|^{3}\right)^{1 /\left(n^{2}-2\right)} \tag{4.32}
\end{equation*}
$$

Note that, except for case (19), we know that $f=1$ or 2 . This inequality gives us, in each case except (19), an upper bound for $q$, which is listed in the last column in Table 4 . Comparing the last two columns of the table we see the condition and bound are satisfied only in the following cases: (1) for $q=11$, (3) for $q=4$, (5) for $q=2$, (6) for $q=7$, (8) and (16). For case (19), we know that $f<q$ and $|\operatorname{PSL}(3, q)|<q^{8}$ by Lemma [2.6, so $72 q^{2} q^{24}>q^{34}$, that is $q^{8}<72$, which is not satisfied for any $q$. In case (1) for $q=11, d=1$ and the inequality $2 d^{2} f^{2}\left|X_{\alpha}\right|^{3}>q^{n^{2}-2}$ is not satisfied.

For each of the remaining cases, we compute $v$ and $2 d f\left|X_{\alpha}\right|$. By Lemma 2.3(ii), $r \mid$ $2 d f\left|X_{\alpha}\right|$. On the other hand $r \mid 2(v-1)$, so $r$ divides $R:=\operatorname{gcd}\left(2(v-1), 2 d f\left|X_{\alpha}\right|\right)$. Now using $R^{2} \geqslant r^{2}>2 v$, this argument rules out cases (3) for $q=4,(6)$ for $q=7,(8)$ and (16). This leaves the single remaining case (5) with $q=2$. Then this argument yields $r \mid 14, v=8$. As $r^{2}>2 v, r=7$ or 14. By Lemma 2.1(i), $r(k-1)=14$, so the condition $k \geqslant 3$ implies that $r=7$ and $k=3$. Now Lemma 2.1(ii) yields a contradiction since $k \nmid v r$. Hence, we rule out case (5) for $q=2$, completing the proof.

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