Bounds on the spectrum of nonsingular triangular (0, 1)-matrices

V. Kaarnioja[†]

May 28, 2022

Abstract

Let K_n be the set of all nonsingular $n \times n$ lower triangular (0, 1)-matrices. Hong and Loewy (2004) introduced the numbers

 $c_n = \min\{\lambda \mid \lambda \text{ is an eigenvalue of } XX^{\mathrm{T}}, X \in K_n\}, n \in \mathbb{Z}_+.$

A related family of numbers was considered by Ilmonen, Haukkanen, and Merikoski (2008):

$$C_n = \max\{\lambda \mid \lambda \text{ is an eigenvalue of } XX^{\mathrm{T}}, X \in K_n\}, n \in \mathbb{Z}_+.$$

These numbers can be used to bound the singular values of matrices belonging to K_n and they appear, e.g., in eigenvalue bounds for power GCD matrices, lattice-theoretic meet and join matrices, and related number-theoretic matrices. In this paper, it is shown that for n odd, one has the lower bound

$$c_n \ge \frac{1}{\sqrt{\frac{1}{25}\varphi^{-4n} + \frac{2}{25}\varphi^{-2n} - \frac{2}{5\sqrt{5}}n\varphi^{-2n} - \frac{23}{25} + n + \frac{2}{25}\varphi^{2n} + \frac{2}{5\sqrt{5}}n\varphi^{2n} + \frac{1}{25}\varphi^{4n}}}$$

and for n even, one has

$$c_n \ge \frac{1}{\sqrt{\frac{1}{25}\varphi^{-4n} + \frac{4}{25}\varphi^{-2n} - \frac{2}{5\sqrt{5}}n\varphi^{-2n} - \frac{2}{5} + n + \frac{4}{25}\varphi^{2n} + \frac{2}{5\sqrt{5}}n\varphi^{2n} + \frac{1}{25}\varphi^{4n}}},$$

where φ denotes the golden ratio. These lower bounds improve the estimates derived previously by Mattila (2015) and Altmisik et al. (2016). The sharpness of these lower bounds is assessed numerically and it is conjectured that $c_n \sim 5\varphi^{-2n}$ as $n \to \infty$. In addition, a new closed form expression is derived for the numbers C_n , viz.

$$C_n = \frac{1}{4}\csc^2\left(\frac{\pi}{4n+2}\right) = \frac{4n^2}{\pi^2} + \frac{4n}{\pi^2} + \left(\frac{1}{12} + \frac{1}{\pi^2}\right) + \mathcal{O}\left(\frac{1}{n^2}\right), \quad n \in \mathbb{Z}_+.$$

1 Introduction

Let K_n denote the set of all nonsingular $n \times n$ lower triangular (0, 1)-matrices. For example, K_3 consists of the elements

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

[†]School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia (vesa.kaarnioja@iki.fi). Present address: School of Engineering Science, LUT University, P.O. Box 20, FI-53851 Lappeenranta, Finland.

and it is easy to see that $\#K_n = 2^{n(n-1)/2}$ for all $n \in \mathbb{Z}_+$.

Hong and Loewy [11] introduced the numbers

 $c_n = \min\{\lambda \mid \lambda \text{ is an eigenvalue of } XX^{\mathrm{T}}, \ X \in K_n\}, \quad n \in \mathbb{Z}_+,$

as a means to give a lower bound for the smallest eigenvalue of power GCD matrices defined on any set of positive integers. A closely related sequence of numbers

 $C_n = \max\{\lambda \mid \lambda \text{ is an eigenvalue of } XX^{\mathrm{T}}, X \in K_n\}, n \in \mathbb{Z}_+,$

was introduced by Ilmonen, Haukkanen, and Merikoski [14] in order to derive upper bounds for the largest eigenvalues of lattice-theoretic meet and join matrices.

The numbers c_n and C_n have an intimate connection with the extremal singular values of matrices belonging to K_n . Let $\sigma_{\min}(X)$ and $\sigma_{\max}(X)$ denote the smallest and largest singular values of matrix X, respectively. Then

$$\min_{X \in K_n} \sigma_{\min}(X) = \sqrt{c_n} \quad \text{and} \quad \max_{X \in K_n} \sigma_{\max}(X) = \sqrt{C_n}, \quad n \in \mathbb{Z}_+.$$

The following example illustrates how the numbers c_n and C_n can be used to give bounds for the smallest and largest eigenvalues of certain number-theoretic matrices.

Example 1.1 (cf. [14]). Let $(P, \leq, \wedge, \hat{0})$ be a locally finite meet semilattice, where \leq is a partial ordering on the set P, \wedge denotes the *meet* (or *greatest lower bound*) of two elements in P, and $\hat{0} \in P$ is the least element such that $\hat{0} \leq x$ for all $x \in P$. Let $S = \{x_1, \ldots, x_n\} \subset P$ be a lower closed set such that $x_i \leq x_j$ only if $i \leq j$. Let $f: P \to \mathbb{R}$ be a function and define the $n \times n$ meet matrix A elementwise by setting $A_{i,j} = f(x_i \wedge x_j)$ for $i, j \in \{1, \ldots, n\}$. Define the function

$$J_{P,f}(x) = \sum_{\hat{0} \preceq z \preceq x} f(z)\mu(z,x) \text{ for all } x \in P,$$

where μ denotes the Möbius function of P. If $J_{P,f}(x) > 0$ for all $x \in S$, then

$$\lambda_{\min}(A) \ge c_n \min_{x \in S} J_{P,f}(x) \quad \text{and} \quad \lambda_{\max}(A) \le C_n \max_{x \in S} J_{P,f}(x). \tag{1}$$

For example, in the case of the divisor lattice $(\mathbb{Z}_+, |, \text{gcd}), f(x) = x^{\alpha}$ for $x \in \mathbb{Z}_+$, and $\alpha > 0$, the function $J_{P,f}$ coincides with the Jordan totient function and the matrix A is called a power GCD matrix. (If $\alpha = 1$, then $J_{P,f}$ is Euler's totient function and the matrix A is called a GCD matrix.) See [11, 14] for further discussion and generalizations of this result.

The eigenvalues of power GCD and LCM matrices, meet and join matrices, and a variety of closely related number-theoretic matrices have been considered by many authors. Balatoni [4] gave lower and upper bounds on the extremal eigenvalues of Smith's matrix, i.e., the matrix with its (i, j) element equal to gcd(i, j), which was considered in 1875/76 by the eponymous Smith [20]. Beslin and Ligh showed that GCD matrices defined on an arbitrary set of distinct positive integers are positive definite [5]. Bourque and Ligh investigated GCD matrices composed elementwise with arithmetic functions and determined conditions which ensure the positive definiteness of these matrices [6]. Further progress on this line of research was made by Hong [9], who studied the behavior of the largest eigenvalue of GCD matrices associated with certain arithmetic functions, while research on the behavior of the corresponding smallest eigenvalue was conducted by Hong and Loewy [12]. Hong and Lee obtained results on the asymptotic behavior of the eigenvalues of reciprocal power LCM matrices [10]. Recently, Merikoski revisited the

lower bound of the smallest eigenvalue of Smith's matrix [17], Mattila and Haukkanen derived eigenvalue bounds for "mixed" power GCD and LCM matrices [16], Altinişik and Büyükköse derived new eigenvalue bounds for GCD and LCM matrices [2], and Ilmonen obtained bounds for the eigenvalues of meet hypermatrices [13]. Haukkanen et al. gave an interesting lattice-theoretic interpretation for the inertia of LCM matrices [8].

In their work [11], Hong and Loewy did not give a lower bound or other estimates for c_n and there do not appear to have been any developments on estimating c_n in the literature until Altınışık and Büyükköse [1] analyzed a related quantity, which could be used to obtain an upper bound on the numbers c_n . Mattila [15] derived the following lower bounds for c_n :

$$c_n \ge \left(\frac{48}{n^4 + 56n^2 + 48n}\right)^{(n-1)/2}$$
 for even n , (2)

$$c_n \ge \left(\frac{48}{n^4 + 50n^2 + 48n - 51}\right)^{(n-1)/2}$$
 for odd n . (3)

The lower bounds (2) and (3) were subsequently improved in [3]:

$$c_n \ge \frac{2}{2F_nF_{n+1} + (-1)^n + 1}$$
 for $n \in \mathbb{Z}_+,$ (4)

where $(F_n)_{n=1}^{\infty}$ denotes the Fibonacci sequence. However, a straightforward numerical investigation shows that the bounds (2)–(4) are *not* sharp. It is the goal of this article to remedy this situation by developing an improved lower bound for the numbers c_n . In addition, a new characterization for the numbers C_n is also derived in this paper.

This paper is structured as follows. Section 2 begins with the development of a new lower bound on Hong and Loewy's numbers c_n . In Subsection 2.1, it is shown that this lower bound can be expressed in a much simplified form, which is the main contribution of this paper. The sharpness of this new lower bound is assessed by numerical experiments in Subsection 2.2 along with numerical comparisons involving the bounds (2)–(4). A novel characterization for the closely related sequence of Ilmonen–Haukkanen–Merikoski numbers C_n is proved in Section 3. Finally, some conclusions and thoughts about future work are given at the end of the paper.

2 Hong and Loewy's numbers c_n

Altınışık et al. [3] proved the following characterization

$$c_n = \lambda_{\min}(Z_n^{-1}) \quad \text{for all } n \in \mathbb{Z}_+,$$
(5)

where Z_n is the symmetric $n \times n$ matrix defined elementwise by

$$(Z_n)_{i,j} = \begin{cases} 1 + \sum_{k=i+1}^n F_{k-i}^2 & \text{if } i = j, \\ (-1)^{j-i} (F_{j-i} + \sum_{k=j+1}^n F_{k-i} F_{k-j}) & \text{if } i < j, \\ (-1)^{i-j} (F_{i-j} + \sum_{k=i+1}^n F_{k-i} F_{k-j}) & \text{if } i > j, \end{cases}$$

for $i, j \in \{1, ..., n\}$ and the sequence of *Fibonacci numbers* is defined by the recurrence relation $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k \ge 2$.

The following technical result will serve as the basis for the analysis in Subsection 2.1.

Lemma 2.1. It holds for all $n \in \mathbb{Z}_+$ that

$$c_n \ge \frac{1}{\sqrt{1 + \sum_{i=2}^n (1 + F_i F_{i-1})^2 + 2\sum_{i=2}^n \sum_{j=2}^i (F_{j-1} + \sum_{k=j+1}^i F_{k-1} F_{k-j})^2}}.$$

Proof. Let $n \in \mathbb{Z}_+$. By the characterization (5), it holds that

$$c_n = \lambda_{\max}(Z_n)^{-1} = \|Z_n\|_2^{-1} \ge \|Z_n\|_{\mathrm{F}}^{-1},\tag{6}$$

where $\|\cdot\|_2$ denotes the spectral norm, $\|\cdot\|_F$ is the Frobenius norm, and the final inequality is due to $\|\cdot\|_2 \leq \|\cdot\|_F$. To prove the claim, it is sufficient to compute the value of the Frobenius norm appearing in (6).

Making use of the block structure of the matrices Z_n , it is possible to write

$$Z_1 = (1)$$
 and $Z_n = \begin{pmatrix} a_n & b_n^{\mathrm{T}} \\ b_n & Z_{n-1} \end{pmatrix}$ for $n \ge 1$,

where $a_n = (Z_n)_{1,1} = 1 + F_n F_{n-1}$ and the (n-1)-vector $b_n = [(Z_n)_{2,1}, \dots, (Z_n)_{n,1}]^{\mathrm{T}} \in \mathbb{R}^{n-1}$ clearly satisfies

$$||b_n||^2 = \sum_{j=2}^n \left(F_{j-1} + \sum_{k=j+1}^n F_{k-1}F_{k-j}\right)^2.$$

Hence $||Z_n||_{\rm F}^2 = a_n^2 + 2||b_n||^2 + ||Z_{n-1}||_{\rm F}^2$, which yields the recurrence relation

$$||Z_1||_{\mathbf{F}}^2 = 1,$$

$$||Z_n||_{\mathbf{F}}^2 = ||Z_{n-1}||_{\mathbf{F}}^2 + (1 + F_n F_{n-1})^2 + 2\sum_{j=2}^n \left(F_{j-1} + \sum_{k=j+1}^n F_{k-1} F_{k-j}\right)^2, \quad n \ge 2.$$

This recurrence can be used to produce the expression

$$||Z_n||_{\mathbf{F}}^2 = 1 + \sum_{i=2}^n (1 + F_i F_{i-1})^2 + 2\sum_{i=2}^n \sum_{j=2}^i \left(F_{j-1} + \sum_{k=j+1}^i F_{k-1} F_{k-j}\right)^2, \quad n \in \mathbb{Z}_+,$$

which, together with the inequality (6), proves the assertion.

Lemma 2.1 gives a computable, albeit rather unwieldy, lower bound on the numbers c_n . However, it is shown in the following section that this lower bound can be recast into a much simpler closed form expression.

2.1 Simplifying the lower bound on c_n

In this section, a closed form expression for the term inside the square root in Lemma 2.1 is derived. To this end, recall that the sequence of *Lucas numbers* can be defined by the recursion $L_0 = 2$, $L_1 = 1$, and $L_k = L_{k-1} + L_{k-2}$ for $k \ge 2$. The Fibonacci–Binet formula and the Lucas–Binet formula can be used to write the Fibonacci and Lucas numbers explicitly as

$$F_k = \frac{\varphi^k - (-\varphi)^{-k}}{\sqrt{5}} \quad \text{and} \quad L_k = \varphi^k + (-\varphi)^{-k} \quad \text{for all } k \in \mathbb{Z}_{\ge 0}, \tag{7}$$

where φ denotes the golden ratio.

The main result of this paper is given by the following theorem. It is a simplified version of the lower bound given in Lemma 2.1.

Theorem 2.2. It holds for all $n \in \mathbb{Z}_+$ that

$$c_n \ge \frac{1}{\sqrt{\frac{1}{25}\varphi^{-4n} + \frac{3+(-1)^n}{25}\varphi^{-2n} - \frac{2}{5\sqrt{5}}n\varphi^{-2n} + \frac{13(-1)^n - 33}{50} + n + \frac{3+(-1)^n}{25}\varphi^{2n} + \frac{2}{5\sqrt{5}}n\varphi^{2n} + \frac{1}{25}\varphi^{4n}}}$$

Proof. The proof is based on simplifying the term inside the square root in Lemma 2.1.

The claim is clearly true with equality for n = 1. In the following analysis, let $n \ge i \ge j \ge 2$ be integers. Using the formulae (7), it is straightforward to check that

$$F_{j-1} + \sum_{k=j+1}^{i} F_{k-1}F_{k-j} = \frac{1}{5} \left(L_{2i-j} + \frac{5}{2}F_{j-1} - \frac{1}{2}(-1)^{i-j}L_{j-1} \right),$$

where the sum is taken to be 0 if the index set is empty. In consequence, it follows that

$$\left(F_{j-1} + \sum_{k=j+1}^{i} F_{k-1}F_{k-j}\right)^2 = \frac{1}{25}L_{2i-j}^2 + \frac{1}{5}F_{j-1}L_{2i-j} - \frac{1}{25}(-1)^{i-j}L_{j-1}L_{2i-j} + \frac{1}{4}F_{j-1}^2 - \frac{1}{10}(-1)^{i-j}F_{j-1}L_{j-1} + \frac{1}{100}L_{j-1}^2.$$
(8)

Using the summation formulae

$$\sum_{j=2}^{i} L_{2i-j}^{2} = L_{2i-2}L_{2i-1} - L_{i}L_{i-1}$$

$$\sum_{j=2}^{i} F_{j-1}^{2} = F_{i-1}F_{i}$$

$$\sum_{j=2}^{i} F_{j-1}L_{2i-j} = (i-1)F_{2i-1} + \frac{L_{2i-2}}{5} + \frac{2}{5}(-1)^{i}$$

$$\sum_{j=2}^{i} (-1)^{j}F_{j-1}L_{j-1} = (-1)^{i}\frac{F_{i+1}^{2} - F_{i-2}^{2}}{4}$$

$$\sum_{j=2}^{i} (-1)^{j}L_{j-1}L_{2i-j} = 2 + \frac{1 + (-1)^{i}}{2}L_{2i-1} - L_{2i-2}$$

$$\sum_{j=2}^{i} L_{j-1}^{2} = L_{i-1}L_{i} - 2$$

in conjunction with (8) yields that

$$\begin{split} \sum_{j=2}^{i} \left(F_{j-1} + \sum_{k=j+1}^{i} F_{k-1} F_{k-j} \right)^2 &= \frac{1}{25} L_{2i-2} L_{2i-1} - \frac{3}{100} L_i L_{i-1} + \frac{1}{5} i F_{2i-1} - \frac{1}{5} F_{2i-1} \\ &+ \frac{1}{25} L_{2i-2} + \frac{1}{25} (-1)^i L_{2i-2} - \frac{1}{50} L_{2i-1} - \frac{1}{50} (-1)^i L_{2i-1} \\ &+ \frac{1}{4} F_i F_{i-1} + \frac{1}{40} F_{i-2}^2 - \frac{1}{40} F_{i+1}^2 - \frac{1}{50}. \end{split}$$

Applying the summation formulae

$$\sum_{i=2}^{n} L_{2i-2}L_{2i-1} = n - 3 + F_{4n-1}$$

$$\sum_{i=2}^{n} L_{i}L_{i-1} = L_{2n} - \frac{7 + (-1)^{n}}{2}$$

$$\sum_{i=2}^{n} L_{i}L_{i-1} = L_{2n} - \frac{7 + (-1)^{n}}{2}$$

$$\sum_{i=2}^{n} L_{2i-1} = nF_{2n-1} - (n+1)F_{2n-1}$$

$$\sum_{i=2}^{n} F_{2i-1} = F_{2n} - 1$$

$$\sum_{i=2}^{n} F_{2i-1} = F_{2n} - 1$$

$$\sum_{i=2}^{n} L_{2i-2} = L_{2n-1} - 1$$

$$\sum_{i=2}^{n} L_{2i-2} = L_{2n-1} - 1$$

$$\sum_{i=2}^{n} F_{i-1}^{2} = F_{n-1}F_{n-2}$$

$$\sum_{i=2}^{n} F_{i+1}^{2} = F_{n+1}F_{n+2} - 2$$

leads to the identity

$$\sum_{i=2}^{n} \sum_{j=2}^{i} \left(F_{j-1} + \sum_{k=j+1}^{i} F_{k-1} F_{k-j} \right)^{2}$$

= $\frac{3}{20} + \frac{(-1)^{n}}{8} + \frac{3}{200} (-1)^{n} + \frac{n}{50} + \frac{1}{40} F_{n-2} F_{n-1} + \frac{1}{4} F_{n}^{2} - \frac{1}{5} F_{2n}$
- $\frac{1}{40} F_{n+1} F_{n+2} - \frac{1}{5} F_{2n-1} - \frac{1}{5} n F_{2n-1} + \frac{1}{5} n F_{2n+1} + \frac{1}{25} F_{4n-1}$
+ $\frac{1}{25} (-1)^{n} F_{n-1} L_{n} - \frac{1}{20} L_{2n} - \frac{1}{50} (-1)^{n} F_{n-1} L_{n+1} + \frac{1}{25} L_{2n-1}.$

Meanwhile, it holds that

$$1 + \sum_{i=2}^{n} (1 + F_i F_{i-1})^2 = n + 2 \sum_{i=2}^{n} F_i F_{i-1} + \sum_{i=2}^{n} F_i^2 F_{i-1}^2$$
$$= 2F_n^2 + (-1)^n - 1 + \frac{24}{25}n + \frac{1}{25}F_{4n} + \frac{2}{25}(-1)^n F_n L_n,$$

since $\sum_{i=2}^{n} F_i F_{i-1} = F_n^2 + \frac{(-1)^{n-1}}{2}$ and $\sum_{i=2}^{n} F_i^2 F_{i-1}^2 = -\frac{n}{25} + \frac{1}{25} F_{4n} + \frac{2}{25} (-1)^n F_n L_n$. Putting the previous formulae together results in the equation

$$1 + \sum_{i=2}^{n} (1 + F_i F_{i-1})^2 + 2 \sum_{i=2}^{n} \sum_{j=2}^{i} \left(F_{j-1} + \sum_{k=j+1}^{i} F_{k-1} F_{k-j} \right)^2$$

= $\frac{1}{25} F_{4n} + \frac{2}{25} F_{4n-1}$ (9)

$$+n - \frac{2}{5}nF_{2n-1} + \frac{2}{5}nF_{2n+1} \tag{10}$$

$$-\frac{7}{10} + \frac{2}{25}(-1)^n F_{n-1}L_n + \frac{2}{25}(-1)^n F_n L_n - \frac{1}{25}(-1)^n F_{n-1}L_{n+1}$$
(11)

$$+\frac{32}{25}(-1)^{n}+\frac{1}{20}F_{n-2}F_{n-1}+\frac{5}{2}F_{n}^{2}-\frac{2}{5}F_{2n}-\frac{1}{20}F_{n+1}F_{n+2}-\frac{2}{5}F_{2n-1}-\frac{1}{10}L_{2n}+\frac{2}{25}L_{2n-1}.$$
(12)

At this juncture, one can proceed as follows.

• To simplify row (9), use the identity

$$\frac{1}{25}F_{4n} + \frac{2}{25}F_{4n-1} = \frac{1}{25}L_{4n}.$$

• To simplify row (10), apply

$$\frac{2}{5}nF_{2n+1} = \frac{2}{5}n(F_{2n} + F_{2n-1}).$$

• To cope with row (11), use the identity

$$-\frac{7}{10} + \frac{2}{25}(-1)^n F_{n-1}L_n + \frac{2}{25}(-1)^n F_nL_n - \frac{1}{25}(-1)^n F_{n-1}L_{n+1} = (-1)^n \frac{L_{2n}}{25} - \frac{33}{50}.$$

• Finally, to handle row (12), utilize the identity

$$\frac{32}{25}(-1)^n + \frac{1}{20}F_{n-2}F_{n-1} + \frac{5}{2}F_n^2 - \frac{2}{5}F_{2n} - \frac{1}{20}F_{n+1}F_{n+2} - \frac{2}{5}F_{2n-1} - \frac{1}{10}L_{2n} + \frac{2}{25}L_{2n-1} - \frac{1}{10}L_{2n} + \frac{2}{25}L_{2n-1} - \frac{1}{25}L_{2n} - \frac{$$

It is straightforward to verify the validity of each of these formulae. Altogether, the above formulae yield

$$1 + \sum_{i=2}^{n} (1 + F_i F_{i-1})^2 + 2 \sum_{i=2}^{n} \sum_{j=2}^{i} \left(F_{j-1} + \sum_{k=j+1}^{i} F_{k-1} F_{k-j} \right)^2$$
$$= \frac{1}{25} L_{4n} + \frac{3 + (-1)^n}{25} L_{2n} + \frac{2}{5} n F_{2n} + \frac{13(-1)^n - 33}{50} + n.$$

The claim follows by expanding the Fibonacci and Lucas numbers in terms of the golden ratio using (7). $\hfill \Box$

It is evident that Theorem 2.2 can be recast in the following way.

Corollary 2.3. For n odd, it holds that

$$c_n \ge \frac{1}{\sqrt{\frac{1}{25}\varphi^{-4n} + \frac{2}{25}\varphi^{-2n} - \frac{2}{5\sqrt{5}}n\varphi^{-2n} - \frac{23}{25} + n + \frac{2}{25}\varphi^{2n} + \frac{2}{5\sqrt{5}}n\varphi^{2n} + \frac{1}{25}\varphi^{4n}}}$$

and for n even, one has

$$c_n \ge \frac{1}{\sqrt{\frac{1}{25}\varphi^{-4n} + \frac{4}{25}\varphi^{-2n} - \frac{2}{5\sqrt{5}}n\varphi^{-2n} - \frac{2}{5} + n + \frac{4}{25}\varphi^{2n} + \frac{2}{5\sqrt{5}}n\varphi^{2n} + \frac{1}{25}\varphi^{4n}}}$$

2.2 Numerical experiments

The sharpness of the lower bound presented in Theorem 2.2 is assessed by numerical experiments. The characterization (5) provides an easy way of computing the numerical value of c_n for $n \in \mathbb{Z}_+$. The value of the lower bound corresponding to c_n is denoted by $\|Z_n\|_{\mathrm{F}}^{-1}$.

In Table 1, the values of both c_n and the lower bound of Theorem 2.2 have been tabulated for $n \in \{1, \ldots, 10\}$. The results suggest that the lower bound becomes *sharper* with increasing n. This observation is further backed by the results illustrated in Figure 1, where the absolute errors, relative errors as well as the number of common significant digits between c_n and the lower bound in Theorem 2.2 have been tabulated for $n \in \{2, \ldots, 100\}$.

The absolute and relative errors were also computed between

- c_n and the lower bounds (2)–(3) derived by Mattila [15];
- c_n and the lower bound (4) derived by Altinişik et al. [3].

These results are displayed in Figure 2 for $n \in \{2, ..., 40\}$ alongside the corresponding errors between the lower bound of Theorem 2.2 and c_n . The lower bounds (2)–(3) and (4) tend to zero at a faster rate than both c_n and the lower bound given by Theorem 2.2, explaining why the relative errors for the bounds (2)–(3) and (4) do not tend to zero as well as the poor rates of convergence in the left-hand side image of Figure 2.

All numerical experiments were carried out by using 150 digit precision computations in Mathematica 11.2.

| n | c_n | Theorem 2.2 | Lower bound (4) | Lower bounds (2) – (3) |
|----|-------------|---------------|-------------------|----------------------------|
| 1 | 1.000000000 | 1.000000000 | 1.000000000 | 1.00000000 |
| 2 | 0.381966011 | 0.377964473 | 0.3333333333 | 0.377964473 |
| 3 | 0.198062264 | 0.196116135 | 0.166666667 | 0.076923077 |
| 4 | 0.087003112 | 0.086710997 | 0.062500000 | 0.006749366 |
| 5 | 0.037068335 | 0.037037037 | 0.025000000 | 0.000540833 |
| 6 | 0.014827585 | 0.014824986 | 0.009523810 | 0.000020528 |
| 7 | 0.005816999 | 0.005816805 | 0.003663004 | $8.16298e{-7}$ |
| 8 | 0.002245345 | 0.002245332 | 0.001398601 | 1.62711e - 8 |
| 9 | 0.000862203 | 0.000862202 | 0.000534759 | $3.63629 \mathrm{e}{-10}$ |
| 10 | 0.000330004 | 0.000330004 | 0.000204248 | $4.33809e{-12}$ |

Table 1: Tabulated values of the constant c_n , the lower bound of Theorem 2.2, the lower bound (4) derived by Altınışık et al. [3], and the lower bounds (2)–(3) derived by Mattila [15] for $n \in \{1, ..., 10\}$. The numerical values suggest that the lower bound of Theorem 2.2 becomes sharper as n increases, while the other bounds become less sharp with increasing n. Note that the lower bound given by Theorem 2.2 coincides with Mattila's bound for $n \in \{1, 2\}$.

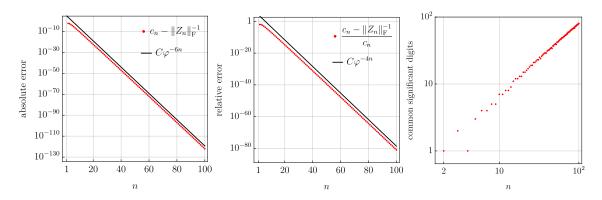


Figure 1: Left and middle images: both the absolute error and the relative error between c_n and the lower bound of Theorem 2.2 decay at an exponential rate. Right image: the number of common significant digits between c_n and the lower bound of Theorem 2.2 is displayed for increasing n.

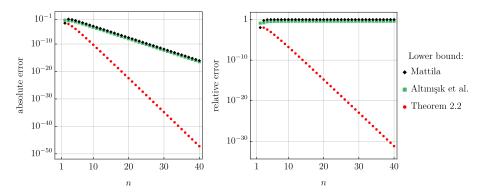


Figure 2: Left image: the absolute errors between c_n and Mattila's the lower bounds (2)–(3), c_n and the lower bound (4) by Altinişik et al., and c_n and the lower bound of Theorem 2.2 for increasing n. Right image: the corresponding relative errors.

3 The Ilmonen–Haukkanen–Merikoski numbers C_n

To conclude this paper, the following new characterization is proved for the Ilmonen–Haukkanen–Merikoski numbers C_n .

Theorem 3.1. It holds for all $n \in \mathbb{Z}_+$ that

$$C_n = \frac{1}{4}\csc^2\left(\frac{\pi}{4n+2}\right) = \frac{4n^2}{\pi^2} + \frac{4n}{\pi^2} + \left(\frac{1}{12} + \frac{1}{\pi^2}\right) + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Proof. The second identity is a consequence of the Laurent expansion of the first expression developed at infinity. It is therefore enough to focus on proving the first identity.

It is easy to check that the claim holds for n = 1. Let $n \ge 2$. By [14], it is known that

$$C_n = \lambda_{\max}(W_n),$$

where W_n is the $n \times n$ matrix defined elementwise by

$$(W_n)_{i,j} = \min\{i, j\}, \quad i, j \in \{1, \dots, n\}.$$

It is easy to see that its matrix inverse $B_n = W_n^{-1}$ is the tridiagonal matrix

$$B_n = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix}$$

and hence $C_n = \lambda_{\min}(B_n)^{-1}$. In other words, it is sufficient to find the reciprocal of the minimal eigenvalue of B_n . Noting that B_n is a special case of a second order finite difference matrix subject to mixed Dirichlet–Neumann boundary conditions (see also the Remark following this proof), it is well known that the eigenvalues of B_n are roots of certain Chebyshev polynomials and, as such, the roots have closed form solutions. A brief derivation is presented in the following for completeness.

Let $A_n = B_n + e_n e_n^{\mathrm{T}}$, where $e_n = [0, 0, \dots, 0, 1]^{\mathrm{T}} \in \mathbb{R}^n$. Let $p_n(\lambda) = \det(A_n - \lambda I_n)$ and $q_n(\lambda) = \det(B_n - \lambda I_n)$ be the characteristic polynomials of A_n and B_n , respectively. The matrix A_n is a tridiagonal Toeplitz matrix and it has the eigenvalues (cf., e.g., [19])

$$\mu_k = 2 - 2\cos\left(\frac{\pi k}{n+1}\right), \quad k \in \{1, \dots, n\}.$$

It is useful to consider Chebyshev polynomials of the second kind U_n , which are characterized by the three-term recurrence

$$U_0(x) = 1$$
, $U_1(x) = 2x$, and $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$ for $n \ge 2$. (13)

If |x| < 1, then it additionally holds that (cf., e.g., [7])

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}}.$$
(14)

Equation (14) implies that the roots of U_n lie in the open interval (-1, 1) and they are given explicitly by the formula

$$x_k = \cos\left(\frac{\pi k}{n+1}\right), \quad k \in \{1, \dots, n\}.$$

Hence $\mu_k = 2 - 2x_k$ for all $k \in \{1, \ldots, n\}$. Since p_n is a monic polynomial, it follows that

$$p_n(\lambda) = \prod_{k=1}^n (\mu_k - \lambda) = \prod_{k=1}^n (2 - 2x_k - \lambda) = 2^n \prod_{k=1}^n \left(\left(1 - \frac{\lambda}{2} \right) - x_k \right) = U_n \left(1 - \frac{\lambda}{2} \right),$$

where the final equality follows from the identity $U_n(x) = 2^n \prod_{k=1}^n (x - x_k)$ derived using the fundamental theorem of algebra, the leading coefficient being a consequence of the three-term recurrence (13).

Developing the Laplace cofactor expansion of $det(B_n - \lambda I_n)$ across the final column and using the properties (13) and (14) yields that

$$q_n(\lambda) = (1-\lambda)p_{n-1}(\lambda) - p_{n-2}(\lambda)$$

$$= 2\left(1-\frac{\lambda}{2}\right)U_{n-1}\left(1-\frac{\lambda}{2}\right) - U_{n-2}\left(1-\frac{\lambda}{2}\right) - U_{n-1}\left(1-\frac{\lambda}{2}\right)$$

$$= U_n\left(1-\frac{\lambda}{2}\right) - U_{n-1}\left(1-\frac{\lambda}{2}\right)$$

$$= \frac{\sin\left((n+1)\arccos\left(1-\frac{\lambda}{2}\right)\right) - \sin\left(n\arccos\left(1-\frac{\lambda}{2}\right)\right)}{\sqrt{1-(1-\frac{\lambda}{2})^2}}.$$

The previous expression can be used to solve the roots of q_n by elementary means, i.e.,

$$\lambda_j = 4\cos^2\left(\frac{j\pi}{2n+1}\right), \quad j \in \{1,\ldots,n\}.$$

Since the smallest root of q_n is λ_n for all $n \in \mathbb{Z}_+$, it follows that

$$C_n = \frac{1}{\lambda_{\min}(B_n)} = \frac{1}{4}\sec^2\left(\frac{n\pi}{2n+1}\right) = \frac{1}{4}\csc^2\left(\frac{\pi}{4n+2}\right)$$

completing the proof.

Remark. The matrix B_n is (up to a scalar multiple) precisely the finite difference matrix corresponding to the Dirichlet–Neumann problem

$$-u''(x) = f(x)$$
 for $x \in (a,b)$, $u(a) = 0$, $u'(b) = 0$.

The properties of finite difference matrices for this problem are very well known in the literature; see, e.g., [18] for a comprehensive treatment of the topic.

It was shown in [14] that the numbers C_n can be bounded by

$$C_n \le \sqrt{(2n-1) + 4(2n-3) + 9(2n-5) + \dots + 3(n-1)^2 + n^2}, \quad n \in \mathbb{Z}_+,$$

but the closed form solution stated in Theorem 3.1 appears to have eluded the authors of the aforementioned paper.

Conclusions

The numerical experiments presented in this paper suggest that the lower bound obtained for the numbers c_n is extremely sharp as n tends to infinity. Theorems 2.2 and 3.1 can be used in conjunction with eigenvalue bounds such as (1) to obtain new explicit lower and upper bounds for the smallest and largest eigenvalues of power GCD matrices as well as lattice-theoretic meet and join matrices. See [14, Sections 3–6] for further discussion on eigenvalue bounds involving c_n and C_n as well as [11, Theorem 4.2] for the special case of power GCD matrices.

The numerical evidence leads the author to further conjecture that $c_n \sim 5\varphi^{-2n}$ as $n \to \infty$, based on the dominating term that appears in Theorem 2.2. Proving this asymptotic result appears to require developing mathematical techniques which are beyond the scope of this paper, posing an interesting challenge for researchers working in this area.

Acknowledgements

The author gratefully acknowledges the financial support from the Australian Research Council (grant number DP180101356). The author thanks the anonymous referees for their valuable comments and suggestions which helped to improve this paper.

References

- E. Altınışık and Ş. Büyükköse. A proof of a conjecture on monotonic behavior of the smallest and the largest eigenvalues of a number theoretic matrix. *Linear Algebra Appl.*, 471:141–149, 2015.
- [2] E. Altınışık and Ş. Büyükköse. On bounds for the smallest and the largest eigenvalues of GCD and LCM matrices. *Math. Inequal. Appl.*, 19(1):117–125, 2016.
- [3] E. Altınışık, A. Keskin, M. Yıldız, and M. Demirbüken. On a conjecture of Ilmonen, Haukkanen and Merikoski concerning the smallest eigenvalues of certain GCD related matrices. *Linear Algebra Appl.*, 493:1–13, 2016.
- [4] F. Balatoni. On the eigenvalues of the matrix of the Smith determinant. Mat. Lapok, 20:397–403, 1969. In Hungarian.
- [5] S. Beslin and S. Ligh. Greatest common divisor matrices. *Linear Algebra Appl.*, 118:69–76, 1989.
- [6] K. Bourque and S. Ligh. Matrices associated with arithmetical functions. *Linear Multilinear Algebra*, 34:261–267, 1993.
- [7] W. Gautschi. Orthogonal Polynomials: Computation and Approximation. Oxford University Press, 2004.
- [8] P. Haukkanen, M. Mattila, and J. Mäntysalo. Studying the inertias of LCM matrices and revisiting the Bourque-Ligh conjecture. J. Combin. Theory Ser. A, 171:105161, 2020.
- [9] S. Hong. Asymptotic behavior of largest eigenvalue of matrices associated with completely even functions (mod r). Asian-Eur. J. Math., 1(2):225–235, 2008.
- [10] S. Hong and K. S. Enoch Lee. Asymptotic behavior of eigenvalues of reciprocal power LCM matrices. *Glasgow Math. J.*, 50:163–174, 2008.
- [11] S. Hong and R. Loewy. Asymptotic behavior of eigenvalues of greatest common divisor matrices. *Glasgow Math. J.*, 46(3):551–569, 2004.
- [12] S. Hong and R. Loewy. Asymptotic behavior of the smallest eigenvalue of matrices associated with completely even functions (mod r). Int. J. Number Theory, 7(6):1681–1704, 2011.
- [13] P. Ilmonen. On meet hypermatrices and their eigenvalues. *Linear Multilinear Algebra*, 64(5):842–855, 2016.
- [14] P. Ilmonen, P. Haukkanen, and J. K. Merikoski. On eigenvalues of meet and join matrices associated with incidence functions. *Linear Algebra Appl.*, 429:859–874, 2008.

- [15] M. Mattila. On the eigenvalues of combined meet and join matrices. *Linear Algebra Appl.*, 466:1–20, 2015.
- [16] M. Mattila and P. Haukkanen. On the eigenvalues of certain number-theoretic matrices. East-West J. Math., 14(2):121–130, 2012.
- [17] J. K. Merikoski. Lower bounds for the largest eigenvalue of the GCD matrix on $\{1, 2, ..., n\}$. Czechoslovak Math. J., 66:1027–1038, 2016.
- [18] A. R. Mitchell and D. F. Griffiths. The Finite Difference Method in Partial Differential Equations. John Wiley & Sons, 1980.
- [19] S. Noschese, L. Pasquini, and L. Reichel. Tridiagonal Toeplitz matrices: properties and novel applications. *Numer. Linear Algebra Appl.*, 20(2):302–326, 2013.
- [20] H. J. S. Smith. On the value of a certain arithmetical determinant. Proc. London Math. Soc., 7:208–212, 1875/76.