A proof of Lin's conjecture on inversion sequences avoiding patterns of relation triples

George E. Andrews and Shane Chern

Abstract. A sequence $e = e_1e_2\cdots e_n$ of natural numbers is called an inversion sequence if $0 \le e_i \le i-1$ for all $i \in \{1,2,\ldots,n\}$. Recently, Martinez and Savage initiated an investigation of inversion sequences that avoid patterns of relation triples. Let ρ_1, ρ_2 and ρ_3 be among the binary relations $\{<,>,\le,\ge,=,\ne,-\}$. Martinez and Savage defined $\mathbf{I}_n(\rho_1,\rho_2,\rho_3)$ as the set of inversion sequences of length n such that there are no indices $1 \le i < j < k \le n$ with $e_i\,\rho_1\,e_j$, $e_j\,\rho_2\,e_k$ and $e_i\,\rho_3\,e_k$. In this paper, we will prove a curious identity concerning the ascent statistic over the sets $\mathbf{I}_n(>,\ne,\ge)$ and $\mathbf{I}_n(\ge,\ne,>)$. This confirms a recent conjecture of Zhicong Lin.

Keywords. Inversion sequence, pattern avoidance, relation triple, generating function, kernel method.

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1. Introduction

A sequence $e = e_1e_2\cdots e_n$ of natural numbers is called an *inversion sequence* if $0 \le e_i \le i-1$ for all $i \in [n] := \{1,2,\ldots,n\}$. We usually denote by $\mathbf{I}_n$ the set of inversion sequences of length n.

Inversion sequences have close connections with many combinatorial objects. For instance, they play an important role in various codings for $\mathfrak{S}_n$, the set of permutations of $[n]$. It is known that permutations that avoid given patterns have extensive applications in computer science, biology and many other fields; see the monograph of Kitaev [12]. Considering the close connection between permutations and inversion sequences, there are also flourish trends on the study of pattern avoidance in inversion sequences [1–11, 14–20, 22, 23]. Here we say that a sequence $e = e_1e_2\cdots e_n$ *avoids* a given pattern $P = p_1p_2\cdots p_k$ if none of the subsequences of e are order isomorphic to P. For example, the sequence $e = e_1e_2\cdots e_6 = 002030$ does not avoid the pattern 100 since the subsequence $e_3e_4e_6 = 200$ is order isomorphic to 100, but avoids 011 since none of the subsequences of e are order isomorphic to 011.

Recently, Martinez and Savage [20] generalized the pattern avoidance to a given triple of binary relations (ρ_1,ρ_2,ρ_3) with $\rho_1,\rho_2,\rho_3 \in \{<,>,\le,\ge,=,\ne,-\}$ in which the last binary relation "$-$" stands for "no restriction", that is, if $e_i - e_j$, then we assume that there is no restriction on the order of e_i and e_j.

Definition 1.1 (Martinez and Savage [20]). We denote by $\mathbf{I}_n(\rho_1,\rho_2,\rho_3)$ where $\rho_1,\rho_2,\rho_3 \in \{<,>,\le,\ge,=,\ne,-\}$ the set of inversion sequences $e = e_1e_2\cdots e_n \in \mathbf{I}_n$ such that there are no indices $1 \le i < j < k \le n$ with

$$e_i\,\rho_1\,e_j,\quad e_j\,\rho_2\,e_k \quad\text{and}\quad e_i\,\rho_3\,e_k.$$

Since the work of Martinez and Savage, the enumerations of such sequences have been investigated extensively. In particular, a handful of Wilf equivalences among the 343 possible sets of inversion sequences avoiding patterns of relation triples were conjectured in [20] and proved later in [5–7, 10, 11, 14, 15, 23].

A further direction for the study of pattern avoidance is to take account of various statistics and investigate their distribution over pattern avoiding sequences; see, for instance, [7, 11, 15, 20]. Along this road, in [15], Lin conjectured a curious identity concerning the ascent statistic over $\mathbf{I}_n(\geq, \neq, >)$ and $\mathbf{I}_n(>, \neq, \geq)$.

We first recall that the *ascent* statistic is defined as follows.

Definition 1.2. Let $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n$. We define, $\operatorname{asc}(e) := \#\{i \in [n-1] : e_i < e_{i+1}\}$, that is, the number of *ascents* of e.

Conjecture 1.1 (Lin [15, Conjecture 2.4])**.** For $n \geq 1$,

$$\sum_{e \in \mathbf{I}_n(\geq, \neq, >)} z^{\operatorname{asc}(e)} = \sum_{e \in \mathbf{I}_n(>, \neq, \geq)} z^{n-1-\operatorname{asc}(e)}. \tag{1.1}$$

Below are the expressions of (1.1) for $1 \leq n \leq 6$:

$$\begin{gathered} 1, \\ 1+z, \\ 1+4z+z^2, \\ 1+10z+11z^2+z^3, \\ 1+20z+55z^2+25z^3+z^4, \\ 1+35z+188z^2+220z^3+50z^4+z^5. \end{gathered}$$

It is notable that the Wilf equivalence of $\mathbf{I}_n(\geq, \neq, >)$ and $\mathbf{I}_n(>, \neq, \geq)$ was first conjectured by Martinez and Savage [20] and later proved bijectively by Lin [15]. However, Lin's bijection, although preserves other statistics, does not imply his conjecture.

Our objective of this paper is to confirm Conjecture 1.1. More precisely, what we are going to show is the following equivalent form.

Theorem 1.1. *For $n \geq 1$,*

$$\sum_{e \in \mathbf{I}_n(>, \neq, \geq)} z^{\operatorname{asc}(e)} = \sum_{e \in \mathbf{I}_n(\geq, \neq, >)} z^{n-1-\operatorname{asc}(e)}. \tag{1.2}$$

One will see that by replacing z with z^{-1} in (1.2) and then multiplying z^{n-1} on both sides, the identity (1.1) follows.

Our proof of Theorem 1.1 is algebraic with the application of the kernel method. But as commented in [15], a bijective proof of Conjecture 1.1 would be more intriguing. Such a proof still remains mysterious.

2. Sequences in $\mathbf{I}_n(>, \neq, \geq)$ and $\mathbf{I}_n(\geq, \neq, >)$

In this section, we prove some combinatorial properties of sequences in $\mathbf{I}_n(>, \neq, \geq)$ and $\mathbf{I}_n(\geq, \neq, >)$. In particular, we are interested in the behavior of the subsequence from the left-most appearance of the largest entry to the last entry. The study of such subsequences will lead to useful recurrences concerning these inversion sequences which will be presented in the next section.

Definition 2.1. Let $e = e_1e_2\cdots e_n$ be a sequence of natural numbers in which e_ℓ is the left-most appearance of the largest entry. We call the subsequence $e_\ell e_{\ell+1}\cdots e_n$ the tail of e, denoted by $\tau(e)$. For example,

$$\tau(0,1,0,3,1,3,5,3,3,3,6,5,7,8,8,6,8,6,8) = (8,8,6,8,6,8).$$

Definition 2.2. We use $a_{\geq k}$ to denote a sequence of consecutive a's appearing at least k times, that is,

$$\underbrace{aa\cdots a}_{\geq k \text{ times}}.$$

2.1. Sequences in $\mathbf{I}_n(>,\neq,\geq)$.

Lemma 2.1. *Let $e \in \mathbf{I}_n(>,\neq,\geq)$. Then the tail of e has the form*

$$a_{\geq 1}\, b_{\geq 0} \qquad (\textit{with } a > b),$$

that is, a sequence of at least one a followed by several b's while the subsequence of b might be empty.

Proof. Recall that for any $e \in \mathbf{I}_n(>,\neq,\geq)$, we cannot find indices $i < j < k$ such that

$$e_i > e_j, \quad e_j \neq e_k \quad \text{and} \quad e_i \geq e_k. \tag{2.1}$$

Let $e_\ell = a$ be the left-most appearance of the largest entry in e. We first claim that among $e_{\ell+1},\ldots,e_n$, there do not exist two distinct entries both of which are smaller than a. Otherwise, if we have such two entries e_j and e_k (with $\ell+1 \leq j < k \leq n$), then $e_\ell e_j e_k$ satisfies (2.1), which is not allowed. The above indicates that $e_{\ell+1},\ldots,e_n \in \{a,b\}$ for some $b < a$.

Further, if we have $e_{\ell'} = a$ for some $\ell+1 \leq \ell' \leq n$, then we must have $e_j = a$ for all $\ell+1 \leq j \leq \ell'$. Otherwise, if there exists one such index j with $e_j = b$, then $e_\ell e_j e_{\ell'} = aba$ satisfies (2.1).

The desired lemma therefore follows. □

Equipped with Lemma 2.1, we may categorize $\mathbf{I}_n(>,\neq,\geq)$ into four disjoint types. (Below we always assume that $a > b$.)

- ▸ Type I.
  The tail of $e \in \mathbf{I}_n(>,\neq,\geq)$ is of the form
  $$a_{\geq 2} \quad \text{or} \quad a_{\geq 1}\, b_{\geq 2};$$
- ▸ Type II.
  The tail of e is of the form
  $$a;$$
- ▸ Type III.
  The tail of e is of the form
  $$a_{\geq 2}\, b;$$

▸ TYPE IV.
The tail of e is of the form

$$a\,b.$$

2.2. Sequences in $\mathbf{I}_n(\geq,\neq,>)$.

Lemma 2.2. *Let $e\in\mathbf{I}_n(\geq,\neq,>)$. Then the tail of e has the form*

$$a\,b_{\geq 0}\,a_{\geq 0}\qquad (\textit{with } a>b),$$

that is, a sequence of one a followed by several b's and then by several a's while the subsequence of b and the second subsequence of a might be empty.

Proof. Recall that for any $e\in\mathbf{I}_n(\geq,\neq,>)$, we cannot find indices $i<j<k$ such that

$$e_i\geq e_j,\quad e_j\neq e_k\quad\text{and}\quad e_i>e_k. \tag{2.2}$$

Let $e_\ell=a$ be the left-most appearance of the largest entry in e. We first claim that among $e_{\ell+1},\ldots,e_n$, there do not exist two distinct entries both of which are smaller than a. Otherwise, if we have such two entries e_j and e_k (with $\ell+1\leq j<k\leq n$), then $e_\ell e_j e_k$ satisfies (2.1), which is not allowed. The above indicates that $e_{\ell+1},\ldots,e_n\in\{a,b\}$ for some $b<a$.

Further, if we have $e_{\ell'}=b$ for some $\ell+1\leq\ell'\leq n$, then we must have $e_j=b$ for all $\ell+1\leq j\leq\ell'$. Otherwise, if there exists one such index j with $e_j=a$, then $e_\ell e_j e_{\ell'}=aab$ satisfies (2.1).

Also, if we have $e_{\ell''}=a$ for some $\ell+1\leq\ell''\leq n$, then we must have $e_k=a$ for all $\ell''\leq k\leq n$. Otherwise, if there exists one such index k with $e_k=b$, then $e_\ell e_{\ell''}e_k=aab$ satisfies (2.1).

The desired lemma therefore follows. □

Analogously, we categorize $\mathbf{I}_n(\geq,\neq,>)$ into four disjoint types. (Below we also assume that $a>b$.)

▸ TYPE I.
The tail of $e\in\mathbf{I}_n(\geq,\neq,>)$ is of the form

$$a_{\geq 2}\quad\text{or}\quad a\,b_{\geq 2}\quad\text{or}\quad a\,b_{\geq 1}\,a_{\geq 2};$$

▸ TYPE II.
The tail of e is of the form

$$a;$$

▸ TYPE III.
The tail of e is of the form

$$a\,b_{\geq 1}\,a;$$

▸ TYPE IV.
The tail of e is of the form

$$a\,b.$$

3. Recurrences and generating functions

3.1. Recurrences. For $1 \le i \le 4$, let

$$\mathbf{I}_{n,i}(>,\neq,\geq) := \{e \in \mathbf{I}_n(>,\neq,\geq) : e \text{ is of Type } i\}$$

and

$$\mathbf{I}_{n,i}^{(\Lambda)}(>,\neq,\geq) := \{e \in \mathbf{I}_{n,i}(>,\neq,\geq) : \text{the largest entry of } e \text{ is } \Lambda\}.$$

We further write

$$f_i(n,\Lambda) := \sum_{e \in \mathbf{I}_{n,i}^{(\Lambda)}(>,\neq,\geq)} z^{\mathrm{asc}(e)}. \tag{3.1}$$

Notice that the initial values of the f_i's are

$$f_1(1,\Lambda) = f_3(1,\Lambda) = f_4(1,\Lambda) = 0 \quad \text{for all } \Lambda \geq 0, \tag{3.2}$$

$$f_2(1,\Lambda) = \begin{cases} 1 & \text{for } \Lambda = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

and

$$f_3(2,\Lambda) = 0 \quad \text{for all } \Lambda \geq 0. \tag{3.4}$$

Lemma 3.1. *For $n \geq 2$, we have*

(a). *for $\Lambda \geq 0$,*

$$f_1(n,\Lambda) = f_1(n-1,\Lambda) + f_2(n-1,\Lambda) + f_3(n-1,\Lambda) + f_4(n-1,\Lambda);$$

(b). *for $\Lambda = 0$ and $\Lambda \geq n$,*

$$f_2(n,\Lambda) = 0,$$

and for $1 \leq \Lambda \leq n-1$,

$$f_2(n,\Lambda) = \sum_{0 \leq \Lambda' < \Lambda} \big(z f_1(n-1,\Lambda') + z f_2(n-1,\Lambda') + z f_3(n-1,\Lambda') + z f_4(n-1,\Lambda') \big);$$

(c). *for $\Lambda \geq 0$,*

$$f_3(n,\Lambda) = f_3(n-1,\Lambda) + f_4(n-1,\Lambda);$$

(d). *for $\Lambda = 0$ and $\Lambda \geq n-1$,*

$$f_4(n,\Lambda) = 0,$$

and for $1 \leq \Lambda \leq n-2$,

$$f_4(n,\Lambda) = \sum_{0 \leq \Lambda' < \Lambda} \big(z f_1(n-1,\Lambda') + f_2(n-1,\Lambda') + z f_3(n-1,\Lambda') + z f_4(n-1,\Lambda') \big).$$

Further,

(b'). *for $1 \leq \Lambda \leq n-1$,*

$$f_2(n,\Lambda) - f_2(n,\Lambda-1) = z f_1(n,\Lambda-1);$$

(d'). *for $1 \leq \Lambda \leq n-2$,*

$$f_4(n,\Lambda) - f_4(n,\Lambda-1) = f_2(n-1,\Lambda) + z f_3(n,\Lambda-1).$$

Proof. To prove (a), (b), (c) and (d) of the lemma, we need to bijectively construct sequences in the desired subset of $\mathbf{I}_{n-1}(>,\neq,\geq)$ for each type of sequences in $\mathbf{I}_n^{(\Lambda)}(>,\neq,\geq)$. Such constructions will be presented explicitly below by deleting one paticular element from each sequence in $\mathbf{I}_n^{(\Lambda)}(>,\neq,\geq)$ of a fixed type. The inverse constructions from the desired subset of $\mathbf{I}_{n-1}(>,\neq,\geq)$ to each type of sequences in $\mathbf{I}_n^{(\Lambda)}(>,\neq,\geq)$ will not be explicitly given but they are simply done by adding the paticular element to where it is deleted. In the sequel, we always write $e = e_1e_2\cdots e_n \in \mathbf{I}_n^{(\Lambda)}(>,\neq,\geq)$.

Case 1. If e is of Type I, then we observe that $e_{n-1} = e_n$. By deleting the last entry e_n, we obtain an inversion sequence e' of length $n-1$. Apparently, $e' \in \mathbf{I}_{n-1}(>,\neq,\geq)$. Also, we claim that e' can be any of the four types. For example, if $\tau(e) = \Lambda\,\Lambda$, then $\tau(e') = \Lambda$ and hence e' is of Type II. For other cases, we may carry on similar arguments. Further, the largest entry in e' is still Λ. Finally, we observe that $\operatorname{asc}(e) = \operatorname{asc}(e')$.

Case 2. If e is of Type II, then by deleting the last entry $e_n = \Lambda$, we obtain an inversion sequence e' of length $n-1$. Again, we notice that $e' \in \mathbf{I}_{n-1}(>,\neq,\geq)$ can be any of the four types. However, in this case, the largest entry in e' is smaller than Λ. This is because $e_n = \Lambda$ is the only largest entry in e, but it is deleted. Finally, we observe that $\operatorname{asc}(e) = \operatorname{asc}(e') + 1$.

Case 3. If e is of Type III, then $\tau(e)$ is of the form $\Lambda_{\geq 2}\, b$ for some $b < \Lambda$. By deleting one of the Λ's, the resulting sequences e' is in $\mathbf{I}_{n-1}^{(\Lambda)}(>,\neq,\geq)$ with largest entry still equal to Λ. Also, $\tau(e')$ is either of the form $\Lambda_{\geq 2}\, b$ or of the form $\Lambda\, b$. Therefore, e' is of either Type III or Type IV. Finally, we observe that $\operatorname{asc}(e) = \operatorname{asc}(e')$.

Case 4. If e is of Type IV, then $\tau(e) = \Lambda\, b$ for some $b < \Lambda$. We delete Λ from e to get e'. It is not hard to verify that $e' \in \mathbf{I}_{n-1}(>,\neq,\geq)$ with largest entry smaller than Λ. We have three subcases as follows.

- $b = e_n = e_{n-2}$. Then e' is of Type I and in this case $\operatorname{asc}(e) = \operatorname{asc}(e') + 1$.
- $b = e_n > e_{n-2}$. Then $b > \max\{e_1, e_2, \ldots, e_{n-2}\}$. Otherwise, there exists some e_i in $e_1e_2\cdots e_{n-3}$ such that $e_i \geq b = e_n$. Now the subsequence $e_ie_{n-2}e_n$ satisfies $e_i \geq e_n > e_{n-2}$, which is not allowed. It is then obvious that e' is of Type II and $\operatorname{asc}(e) = \operatorname{asc}(e')$.
- $b = e_n < e_{n-2}$. Then $\tau(e_1e_2\cdots e_{n-2})$ must be of the form $a_{\geq 1}$. Otherwise, there exists some $e_i > e_{n-2}$ with $i < n-2$. Hence, the subsequence $e_ie_{n-2}e_n$ satisfies $e_i > e_{n-2} > e_n$ and thus satisfies (2.1). But this is not allowed. Now if $\tau(e_1e_2\cdots e_{n-2})$ is of the form $a_{\geq 2}$, then e' is of Type III and $\operatorname{asc}(e) = \operatorname{asc}(e')+1$; if $\tau(e_1e_2\cdots e_{n-2})$ is of the form a, then e' is of Type IV and as well $\operatorname{asc}(e) = \operatorname{asc}(e') + 1$.

Now (a), (b), (c) and (d) of the lemma are proved. Next, we show (b') and (d'). For (b'), we simply notice that

$$\begin{aligned}
&f_2(n,\Lambda) - f_2(n,\Lambda-1)\\
&= zf_1(n-1,\Lambda-1) + zf_2(n-1,\Lambda-1) + zf_3(n-1,\Lambda-1) + zf_4(n-1,\Lambda-1)\\
&= zf_1(n,\Lambda-1),
\end{aligned}$$

where we make use of (a) in the last equality. For (d'),

$$f_4(n,\Lambda) - f_4(n,\Lambda-1)$$

$$\begin{aligned}&= zf_1(n-1,\Lambda-1) + f_2(n-1,\Lambda-1) + zf_3(n-1,\Lambda-1) + zf_4(n-1,\Lambda-1)\\&= \big(zf_1(n-1,\Lambda-1) + f_2(n-1,\Lambda-1)\big) + z\big(f_3(n-1,\Lambda-1) + f_4(n-1,\Lambda-1)\big)\\&= f_2(n-1,\Lambda) + zf_3(n,\Lambda-1),\end{aligned}$$

where we utilize (b') and (c) in the last equality. □

Proposition 3.2. *For $n \geq 1$,*

$$\begin{cases} f_1(n,\Lambda) = 0, & \text{if } \Lambda > n-2,\\ f_2(n,\Lambda) = 0, & \text{if } \Lambda > n-1,\\ f_3(n,\Lambda) = 0, & \text{if } \Lambda > n-3,\\ f_4(n,\Lambda) = 0, & \text{if } \Lambda > n-2. \end{cases}$$

Proof. The equalities for f_2 and f_4 come from Lemma 3.1 (b) and (d). The equalities for f_1 and f_3 can be proved jointly by a simple induction on n. □

On the other hand, for $1 \leq i \leq 4$, let

$$\mathbf{I}_{n,i}(\geq,\neq,>) := \{e \in \mathbf{I}_n(\geq,\neq,>) : e \text{ is of Type } i\}$$

and

$$\mathbf{I}_{n,i}^{(\Lambda)}(\geq,\neq,>) := \{e \in \mathbf{I}_{n,i}(\geq,\neq,>) : \text{the largest entry of } e \text{ is } \Lambda\}.$$

We further write

$$h_i(n,\Lambda) := \sum_{e \in \mathbf{I}_{n,i}^{(\Lambda)}(\geq,\neq,>)} z^{\operatorname{asc}(e)}. \tag{3.5}$$

Notice that the initial values of the h_i's are

$$h_1(1,\Lambda) = h_3(1,\Lambda) = h_4(1,\Lambda) = 0 \quad \text{for all } \Lambda \geq 0,$$

$$h_2(1,\Lambda) = \begin{cases} 1 & \text{for } \Lambda = 0,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$h_3(2,\Lambda) = 0 \quad \text{for all } \Lambda \geq 0.$$

Lemma 3.3. *For $n \geq 2$, we have*

(a). *for $\Lambda \geq 0$,*

$$h_1(n,\Lambda) = h_1(n-1,\Lambda) + h_2(n-1,\Lambda) + h_3(n-1,\Lambda) + h_4(n-1,\Lambda);$$

(b). *for $\Lambda = 0$ and $\Lambda \geq n$,*

$$h_2(n,\Lambda) = 0,$$

and for $1 \leq \Lambda \leq n-1$,

$$\begin{aligned}h_2(n,\Lambda) = \sum_{0 \leq \Lambda' < \Lambda} \big(&zh_1(n-1,\Lambda') + zh_2(n-1,\Lambda')\\&+ zh_3(n-1,\Lambda') + zh_4(n-1,\Lambda')\big);\end{aligned}$$

(c). *for $\Lambda \geq 0$,*

$$h_3(n,\Lambda) = h_3(n-1,\Lambda) + zh_4(n-1,\Lambda);$$

(d). *for* $\Lambda = 0$ *and* $\Lambda \geq n-1$,

$$h_4(n,\Lambda) = 0,$$

and for $1 \leq \Lambda \leq n-2$,

$$h_4(n,\Lambda) = \sum_{0\leq \Lambda' < \Lambda} \big(zh_1(n-1,\Lambda') + h_2(n-1,\Lambda') \\ + h_3(n-1,\Lambda') + zh_4(n-1,\Lambda')\big).$$

Further,

(b'). *for* $1 \leq \Lambda \leq n-1$,

$$h_2(n,\Lambda) - h_2(n,\Lambda-1) = zh_1(n,\Lambda-1);$$

(d'). *for* $1 \leq \Lambda \leq n-2$,

$$h_4(n,\Lambda) - h_4(n,\Lambda-1) = h_2(n-1,\Lambda) + h_3(n,\Lambda-1).$$

Proof. In analogy to the proof of Lemma 3.1, we construct bijective maps between each type of sequences in $\mathbf{I}_n^{(\Lambda)}(\geq,\neq,>)$ and the desired subset of $\mathbf{I}_{n-1}(\geq,\neq,>)$ while still only one side of the maps will be explicitly stated. For (a) and (b), we use the same way as that for Lemma 3.1(a) and (b) to reduce $e \in \mathbf{I}_n^{(\Lambda)}(\geq,\neq,>)$ to $e' \in \mathbf{I}_{n-1}(\geq,\neq,>)$ and hence the details are omitted. Now let us treat the rest two cases. We as well write $e = e_1e_2\cdots e_n \in \mathbf{I}_n^{(\Lambda)}(\geq,\neq,>)$.

Case 3. If e is of Type III, then $\tau(e)$ is of the form $\Lambda\, b_{\geq 1}\, \Lambda$ for some $b < \Lambda$. We distinguish it into two subcases. It should be pointed out in advance that the largest entry of the resulting sequence e' in both cases is still Λ.

- $\tau(e) = \Lambda\, b\, \Lambda$. Then we delete the last Λ to get e'. We see that $e' \in \mathbf{I}_{n-1}^{(\Lambda)}(\geq,\neq,>)$ is of Type IV. Also $\mathrm{asc}(e) = \mathrm{asc}(e') + 1$.
- τ is of the form $\Lambda\, b_{\geq 2}\, \Lambda$. Then we delete one of the b's to get some $e' \in \mathbf{I}_{n-1}^{(\Lambda)}(\geq,\neq,>)$. This time e' is of Type III and $\mathrm{asc}(e) = \mathrm{asc}(e')$.

Case 4. If e is of Type IV, then $\tau(e) = \Lambda\, b$ for some $b < \Lambda$. We as well delete Λ from e to get e'. Notice that we also have $e' \in \mathbf{I}_{n-1}(>,\neq,\geq)$ with largest entry smaller than Λ. We have three subcases as follows.

- $b = e_n = e_{n-2}$. Then e' is of Type I and in this case $\mathrm{asc}(e) = \mathrm{asc}(e') + 1$.
- $b = e_n > e_{n-2}$. Then $b \geq \max\{e_1, e_2, \ldots, e_{n-2}\}$. Otherwise, there exists some e_i in $e_1e_2\cdots e_{n-3}$ such that $e_i > b = e_n$. Now the subsequence $e_ie_{n-2}e_n$ satisfies $e_i > e_n > e_{n-2}$, which is not allowed. If $b > \max\{e_1, e_2, \ldots, e_{n-2}\}$, then e' is of Type II and $\mathrm{asc}(e) = \mathrm{asc}(e')$; if $b = \max\{e_1, e_2, \ldots, e_{n-2}\}$, then $\tau(e_1e_2\cdots e_{n-2})$ must be of the form $b\, c_{\geq 1}$ where $c = e_{n-2} < b$ and hence e' is of Type III and $\mathrm{asc}(e) = \mathrm{asc}(e')$.
- $b = e_n < e_{n-2}$. Then $\tau(e_1e_2\cdots e_{n-2})$ must be of the form a. Otherwise, there exists some $e_i \geq e_{n-2}$ with $i < n-2$. Hence, the subsequence $e_ie_{n-2}e_n$ satisfies $e_i \geq e_{n-2} > e_n$ and thus satisfies (2.2). But this is not allowed. Thus, e' is of Type IV and $\mathrm{asc}(e) = \mathrm{asc}(e') + 1$.

The proofs of (b') and (d') are also similar to those for Lemma 3.1. $\square$

Finally, we define, for $1 \le i \le 4$,

$$g_i(n,\Lambda) := \sum_{e\in \mathbf{I}_{n,i}^{(\Lambda)}(\ge,\ne,>)} z^{n-1-\mathrm{asc}(e)}. \tag{3.6}$$

Then in view of (3.5),

$$g_i(n,\Lambda) = z^{n-1}\big[h_i(n,\Lambda)\big]_{z\mapsto z^{-1}}$$

and conversely,

$$h_i(n,\Lambda) = z^{n-1}\big[g_i(n,\Lambda)\big]_{z\mapsto z^{-1}}.$$

Thus, the initial values of the g_i's are

$$g_1(1,\Lambda) = g_3(1,\Lambda) = g_4(1,\Lambda) = 0 \quad \text{for all } \Lambda \ge 0,$$

$$g_2(1,\Lambda) = \begin{cases} 1 & \text{for } \Lambda = 0,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$g_3(2,\Lambda) = 0 \quad \text{for all } \Lambda \ge 0.$$

Also, the recurrences for the g_i's can be translated with no difficulty from those for the h_i's.

Lemma 3.4. *For $n \ge 2$, we have*

(a). *for $\Lambda \ge 0$,*

$$g_1(n,\Lambda) = zg_1(n-1,\Lambda) + zg_2(n-1,\Lambda) + zg_3(n-1,\Lambda) + zg_4(n-1,\Lambda);$$

(b). *for $\Lambda = 0$ and $\Lambda \ge n$,*

$$g_2(n,\Lambda) = 0,$$

and for $1 \le \Lambda \le n-1$,

$$g_2(n,\Lambda) = \sum_{0\le \Lambda' < \Lambda} \big(g_1(n-1,\Lambda') + g_2(n-1,\Lambda') + g_3(n-1,\Lambda') + g_4(n-1,\Lambda')\big);$$

(c). *for $\Lambda \ge 0$,*

$$g_3(n,\Lambda) = zg_3(n-1,\Lambda) + g_4(n-1,\Lambda);$$

(d). *for $\Lambda = 0$ and $\Lambda \ge n-1$,*

$$g_4(n,\Lambda) = 0,$$

and for $1 \le \Lambda \le n-2$,

$$g_4(n,\Lambda) = \sum_{0\le \Lambda' < \Lambda} \big(g_1(n-1,\Lambda') + zg_2(n-1,\Lambda') + zg_3(n-1,\Lambda') + g_4(n-1,\Lambda')\big).$$

Further,

(b'). *for $1 \le \Lambda \le n-1$,*

$$g_2(n,\Lambda) - g_2(n,\Lambda-1) = z^{-1}g_1(n,\Lambda-1);$$

(d'). *for $1 \le \Lambda \le n-2$,*

$$g_4(n,\Lambda) - g_4(n,\Lambda-1) = zg_2(n-1,\Lambda) + g_3(n,\Lambda-1).$$

Similar to Proposition 3.2, we have the following equalities.

Proposition 3.5. *For $n \geq 1$,*

$$\begin{cases} g_1(n,\Lambda) = 0, & \text{if } \Lambda > n-2, \\ g_2(n,\Lambda) = 0, & \text{if } \Lambda > n-1, \\ g_3(n,\Lambda) = 0, & \text{if } \Lambda > n-3, \\ g_4(n,\Lambda) = 0, & \text{if } \Lambda > n-2. \end{cases}$$

3.2. Generating functions. Let

$$\mathcal{F}_1(t) = \mathcal{F}_1(t;q) := \sum_{n\geq 2} \sum_{\Lambda=0}^{n-2} f_1(n,\Lambda) t^{n-2-\Lambda} q^n,$$

$$\mathcal{F}_2(t) = \mathcal{F}_2(t;q) := \sum_{n\geq 2} \sum_{\Lambda=0}^{n-1} f_2(n,\Lambda) t^{n-1-\Lambda} q^n,$$

$$\mathcal{F}_3(t) = \mathcal{F}_3(t;q) := \sum_{n\geq 3} \sum_{\Lambda=0}^{n-3} f_3(n,\Lambda) t^{n-3-\Lambda} q^n,$$

$$\mathcal{F}_4(t) = \mathcal{F}_4(t;q) := \sum_{n\geq 2} \sum_{\Lambda=0}^{n-2} f_4(n,\Lambda) t^{n-2-\Lambda} q^n.$$

It is easy to translate the recurrences of the f_i's in Lemma 3.1 to functional equations of $\mathcal{F}_1(t)$, $\mathcal{F}_2(t)$, $\mathcal{F}_3(t)$ and $\mathcal{F}_4(t)$.

Lemma 3.6. *We have*

$$\begin{cases} \mathcal{F}_1(t) - q^2 = tq\mathcal{F}_1(t) + q\mathcal{F}_2(t) + t^2 q\mathcal{F}_3(t) + tq\mathcal{F}_4(t), \\ t\mathcal{F}_2(t) - \big(\mathcal{F}_2(t) - \mathcal{F}_2(0)\big) = zt\mathcal{F}_1(t), \\ \mathcal{F}_3(t) = tq\mathcal{F}_3(t) + q\mathcal{F}_4(t), \\ t\mathcal{F}_4(t) - \big(\mathcal{F}_4(t) - \mathcal{F}_4(0)\big) = tq\mathcal{F}_2(t) + zt\mathcal{F}_3(t). \end{cases} \tag{3.7}$$

Proof. We show the first and second equations as instances. The proof of the third one is similar to the first and the proof of the fourth one resembles the second.

First, by Lemma 3.1(a) and Proposition 3.2, we have

$$\begin{aligned} \mathcal{F}_1(t) &= \sum_{n\geq 2} \sum_{\Lambda=0}^{n-2} f_1(n,\Lambda) t^{n-2-\Lambda} q^n \\ &= \sum_{n\geq 2} \sum_{\Lambda=0}^{n-2} \big(f_1(n-1,\Lambda) + f_2(n-1,\Lambda) + f_3(n-1,\Lambda) + f_4(n-1,\Lambda)\big) t^{n-2-\Lambda} q^n \\ &= \sum_{n\geq 1} \sum_{\Lambda=0}^{n-1} \big(f_1(n,\Lambda) + f_2(n,\Lambda) + f_3(n,\Lambda) + f_4(n,\Lambda)\big) t^{n-1-\Lambda} q^{n+1} \\ &= tq \sum_{n\geq 1} \sum_{\Lambda=0}^{n-2} f_1(n,\Lambda) t^{n-2-\Lambda} q^n + q \sum_{n\geq 1} \sum_{\Lambda=0}^{n-1} f_2(n,\Lambda) t^{n-1-\Lambda} q^n \end{aligned}$$

$$+ t^2 q \sum_{n\geq 1} \sum_{\Lambda=0}^{n-3} f_3(n,\Lambda) t^{n-3-\Lambda} q^n + tq \sum_{n\geq 1} \sum_{\Lambda=0}^{n-2} f_4(n,\Lambda) t^{n-2-\Lambda} q^n.$$

The first equation follows by recalling the initial values (3.2), (3.3) and (3.4).

For the second equation, we apply Lemma 3.1(b') and Proposition 3.2. Then

$$\begin{aligned}
\mathcal{F}_2(t) &= \sum_{n\geq 2} \sum_{\Lambda=0}^{n-1} f_2(n,\Lambda) t^{n-1-\Lambda} q^n \\
&= \sum_{n\geq 2} \sum_{\Lambda=1}^{n-1} f_2(n,\Lambda) t^{n-1-\Lambda} q^n \quad (\text{since } f_2(n,0)=0 \text{ for } n\geq 2 \text{ by Lemma 3.1(b)}) \\
&= \sum_{n\geq 2} \sum_{\Lambda=1}^{n-1} \big(z f_1(n,\Lambda-1) + f_2(n,\Lambda-1)\big) t^{n-1-\Lambda} q^n \\
&= \sum_{n\geq 2} \sum_{\Lambda=0}^{n-2} \big(z f_1(n,\Lambda) + f_2(n,\Lambda)\big) t^{n-2-\Lambda} q^n \\
&= z \sum_{n\geq 2} \sum_{\Lambda=0}^{n-2} f_1(n,\Lambda) t^{n-2-\Lambda} q^n + t^{-1} \sum_{n\geq 2} \sum_{\Lambda=0}^{n-2} f_2(n,\Lambda) t^{n-1-\Lambda} q^n \\
&= z\mathcal{F}_1(t) + t^{-1}\big(\mathcal{F}_2(t) - \mathcal{F}_2(0)\big),
\end{aligned}$$

which is essentially the second equation. □

We treat $\mathcal{F}_1(t)$, $\mathcal{F}_2(t)$, $\mathcal{F}_3(t)$ and $\mathcal{F}_4(t)$ as unknowns and solve the above system so that they are expressed in terms of $\mathcal{F}_2(0)$, $\mathcal{F}_4(0)$, z, q and t. In particular, we have the following expression for $\mathcal{F}_4(t)$.

Lemma 3.7. *We have*

$$K_f(t)\mathcal{F}_4(t) = (1-qt)P_f(t), \tag{3.8}$$

where

$$\begin{aligned}
P_f(t) = \mathcal{F}_4(0) &- \big(q\mathcal{F}_2(0) + \mathcal{F}_4(0) + q\mathcal{F}_4(0) - zq\mathcal{F}_4(0)\big)t \\
&+ \big(zq^3 + q^2\mathcal{F}_2(0) + q\mathcal{F}_4(0)\big)t^2
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
K_f(t) = 1 &- (2+2q-2zq)t + (1+4q-2zq+q^2-2zq^2+z^2q^2)t^2 \\
&- (2q+2q^2-zq^2)t^3 + q^2t^4.
\end{aligned} \tag{3.10}$$

Analogously, we define

$$\mathcal{G}_1(t) = \mathcal{G}_1(t;q) := \sum_{n\geq 2} \sum_{\Lambda=0}^{n-2} g_1(n,\Lambda) t^{n-2-\Lambda} q^n,$$

$$\mathcal{G}_2(t) = \mathcal{G}_2(t;q) := z \sum_{n\geq 2} \sum_{\Lambda=0}^{n-1} g_2(n,\Lambda) t^{n-1-\Lambda} q^n,$$

$$\mathcal{G}_3(t) = \mathcal{G}_3(t;q) := \sum_{n\geq 3} \sum_{\Lambda=0}^{n-3} g_3(n,\Lambda) t^{n-3-\Lambda} q^n,$$

$$\mathcal{G}_4(t)=\mathcal{G}_4(t;q):=\sum_{n\geq 2}\sum_{\Lambda=0}^{n-2} g_4(n,\Lambda)t^{n-2-\Lambda}q^n.$$

By the recurrences of the g_i's in Lemma 3.4, the following system holds true.

Lemma 3.8. *We have*

$$\begin{cases}\mathcal{G}_1(t)-zq^2=ztq\mathcal{G}_1(t)+q\mathcal{G}_2(t)+zt^2q\mathcal{G}_3(t)+ztq\mathcal{G}_4(t),\\ t\mathcal{G}_2(t)-\big(\mathcal{G}_2(t)-\mathcal{G}_2(0)\big)=t\mathcal{G}_1(t),\\ \mathcal{G}_3(t)=ztq\mathcal{G}_3(t)+q\mathcal{G}_4(t),\\ t\mathcal{G}_4(t)-\big(\mathcal{G}_4(t)-\mathcal{G}_4(0)\big)=tq\mathcal{G}_2(t)+t\mathcal{G}_3(t).\end{cases}\tag{3.11}$$

We may also solve the above system for $\mathcal{G}_1(t)$, $\mathcal{G}_2(t)$, $\mathcal{G}_3(t)$ and $\mathcal{G}_4(t)$. In particular, we have the following expression for $\mathcal{G}_4(t)$.

Lemma 3.9. *We have*

$$K_g(t)\mathcal{G}_4(t)=(1-zqt)P_g(t),\tag{3.12}$$

where

$$\begin{aligned}P_g(t)=\mathcal{G}_4(0)&-\big(q\mathcal{G}_2(0)+\mathcal{G}_4(0)-q\mathcal{G}_4(0)+zq\mathcal{G}_4(0)\big)t\\ &+\big(zq^3+zq^2\mathcal{G}_2(0)+zq\mathcal{G}_4(0)\big)t^2\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}K_g(t)=1&-(2-2q+2zq)t+(1-2q+4zq+q^2-2zq^2+z^2q^2)t^2\\ &-(2zq-zq^2+2z^2q^2)t^3+(z^2q^2-zq^3+z^2q^3)t^4.\end{aligned}\tag{3.14}$$

Remark 3.1. We could, of course, derive kernel equations for $\mathcal{F}_2(t)$ and $\mathcal{G}_2(t)$ instead of $\mathcal{F}_4(t)$ and $\mathcal{G}_4(t)$. But such changes will not make any essential difference after the application of the kernel method; we are still led to Theorem 4.1.

4. Proof of Theorem 1.1

The kernel method is a powerful tool to treat functional equations. This method originated as an exercise in the first volume of Donald Knuth's book "*The Art of Computer Programming*" [13, Exercise 4, §2.2.1, p. 243]. There are also collections of examples to illustrate this ingenious approach; see for instance [21].

The objective of this section is to apply the kernel method to establish the following surprising relations, one of which will lead to a proof of Theorem 1.1.

Theorem 4.1. *We have*

$$\begin{cases}f_1(n,n-2)=g_1(n,n-2) & \textit{for } n\geq 3,\\ f_2(n,n-1)=zg_2(n,n-1) & \textit{for } n\geq 2,\\ f_3(n,n-3)=g_3(n,n-3) & \textit{for } n\geq 3,\\ f_4(n,n-2)=g_4(n,n-2) & \textit{for } n\geq 2.\end{cases}$$

4.1. Roots of the kernel polynomials. Before applying the kernel method to $\mathcal{F}_4(t)$ and $\mathcal{G}_4(t)$, let us first investigate properties of the roots of the two kernel polynomials $K_f(t)$ and $K_g(t)$.

Lemma 4.2. *Let r_1, r_2, r_3 and r_4 be the four roots of the quartic polynomial $K_f(t)$. Then the quartic polynomial $K_g(t)$ has roots s_1, s_2, s_3 and s_4 such that for $1 \le i \le 4$,*

$$s_i = \frac{r_i}{1-(1-z)qr_i}. \tag{4.1}$$

Proof. We have

$$K_f(t) = q^2(t-r_1)(t-r_2)(t-r_3)(t-r_4).$$

Since $K_f(t)$ has constant term 1, we know that the quartic polynomial $K_f^\star(t) := t^4K_f(t^{-1})$ is monic. Further,

$$K_f^\star(t) = (t-r_1^{-1})(t-r_2^{-1})(t-r_3^{-1})(t-r_4^{-1}).$$

Similarly, if $K_g^\star(t) := t^4K_g(t^{-1})$, then

$$K_g^\star(t) = (t-s_1^{-1})(t-s_2^{-1})(t-s_3^{-1})(t-s_4^{-1}).$$

Therefore, to obtain the desired relations, it suffices to show

$$K_g^\star(t) = K_f^\star\big(t+(1-z)q\big),$$

which is easy to verify. □

For the sake of simplicity when utilizing the general formula for roots of quartic equations, we assume that $0<q<1$ and $z>0$. It can be computed that as $q \to 0^+$, $K_f(t)$ has four roots

$$\begin{aligned}
r_1 &= 1+\big(z+\sqrt{z}\big)q + O_z(q^2),\\
r_2 &= 1+\big(z-\sqrt{z}\big)q + O_z(q^2),\\
r_3 &= q^{-1}+\sqrt{z}q^{-1/2}-\frac{z}{2}+O_z(q^{1/2}),\\
r_4 &= q^{-1}-\sqrt{z}q^{-1/2}-\frac{z}{2}+O_z(q^{1/2}).
\end{aligned}$$

Let s_i be as in Lemma 4.2 so that they are roots of $K_g(t)$. Then

$$\begin{aligned}
s_1 &= 1+\big(1+\sqrt{z}\big)q + O_z(q^2),\\
s_2 &= 1+\big(1-\sqrt{z}\big)q + O_z(q^2),\\
s_3 &= \frac{1}{z}q^{-1}+\frac{1}{z^{3/2}}q^{-1/2}+\frac{2-3z}{2z^2}+O_z(q^{1/2}),\\
s_4 &= \frac{1}{z}q^{-1}-\frac{1}{z^{3/2}}q^{-1/2}+\frac{2-3z}{2z^2}+O_z(q^{1/2}).
\end{aligned}$$

4.2. Applying the kernel method. To apply the kernel method, we need to choose roots of $K_f(t)$ and $K_g(t)$ that can be expanded as a formal power series in q. So only r_1, r_2 and s_1, s_2 are admissible, respectively. Recall (3.8):

$$K_f(t)\mathcal{F}_4(t) = (1-qt)P_f(t).$$

We substitute the roots $t = r_1$ and r_2 into the above and arrive at $P_f(t) = 0$. Then recalling (3.9) yields the system

$$\begin{cases} 0 = \mathcal{F}_4(0) - \big(q\mathcal{F}_2(0) + \mathcal{F}_4(0) + q\mathcal{F}_4(0) - zq\mathcal{F}_4(0)\big)r_1 \\ \qquad + \big(zq^3 + q^2\mathcal{F}_2(0) + q\mathcal{F}_4(0)\big)r_1^2, \\ 0 = \mathcal{F}_4(0) - \big(q\mathcal{F}_2(0) + \mathcal{F}_4(0) + q\mathcal{F}_4(0) - zq\mathcal{F}_4(0)\big)r_2 \\ \qquad + \big(zq^3 + q^2\mathcal{F}_2(0) + q\mathcal{F}_4(0)\big)r_2^2. \end{cases}$$

Solving the above system for $\mathcal{F}_2(0)$ and $\mathcal{F}_4(0)$ gives

$$\begin{cases} \mathcal{F}_2(0) = \dfrac{zq^2\big(r_1 + r_2 - (1 + (1-z)q)r_1r_2\big)}{1 - q(r_1 + r_2) + (1-z)q^2r_1r_2}, \\ \mathcal{F}_4(0) = \dfrac{zq^3r_1r_2}{1 - q(r_1 + r_2) + (1-z)q^2r_1r_2}. \end{cases}$$

Likewise, we substitute the roots $t = s_1$ and s_2 into $P_g(t) = 0$ and use (3.13) to obtain a similar system, which leads to the solution

$$\begin{cases} \mathcal{G}_2(0) = \dfrac{zq^2\big(s_1 + s_2 - (1 - (1-z)q)s_1s_2\big)}{1 - zq(s_1 + s_2) - z(1-z)q^2s_1s_2}, \\ \mathcal{G}_4(0) = \dfrac{zq^3s_1s_2}{1 - zq(s_1 + s_2) - z(1-z)q^2s_1s_2}. \end{cases}$$

Finally, making use of the relations

$$\begin{cases} s_1 = \dfrac{r_1}{1 - (1-z)qr_1}, \\ s_2 = \dfrac{r_2}{1 - (1-z)qr_2}, \end{cases}$$

we find that

$$\begin{cases} \mathcal{F}_2(0) = \mathcal{G}_2(0), \\ \mathcal{F}_4(0) = \mathcal{G}_4(0). \end{cases}$$

Further, by (3.7) and (3.11), we have

$$\begin{cases} \mathcal{F}_1(0) = q\mathcal{F}_2(0) + q^2, \\ \mathcal{G}_1(0) = q\mathcal{G}_2(0) + zq^2, \end{cases}$$

and

$$\begin{cases} \mathcal{F}_3(0) = q\mathcal{F}_4(0), \\ \mathcal{G}_3(0) = q\mathcal{G}_4(0). \end{cases}$$

Theorem 4.1 therefore follows.

4.3. Proof of Theorem 1.1. Recalling the definition of the f_i's, we find that, for $n \geq 1$,

$$\begin{aligned} \sum_{e \in \mathbf{I}_n(>,\neq,\geq)} z^{\operatorname{asc}(e)} &= \sum_{\Lambda=0}^{n-1} \big(f_1(n,\Lambda) + f_2(n,\Lambda) + f_3(n,\Lambda) + f_4(n,\Lambda)\big) \\ &= z^{-1} f_2(n+1, n), \end{aligned}$$

where we make use of Lemma 3.1(b). Similarly, we have, for $n \geq 1$,

$$\sum_{e \in \mathbf{I}_n(\geq, \neq, >)} z^{n-1-\mathrm{asc}(e)} = \sum_{\Lambda=0}^{n-1} \big(g_1(n, \Lambda) + g_2(n, \Lambda) + g_3(n, \Lambda) + g_4(n, \Lambda) \big)$$
$$= g_2(n+1, n),$$

where Lemma 3.4(b) is utilized. By the second relation in Theorem 4.1, we find that for $n \geq 1$,

$$z^{-1} f_2(n+1, n) = g_2(n+1, n),$$

and therefore complete the proof of (1.2).

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(G. E. Andrews) Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

E-mail address: `gea1@psu.edu`

(S. Chern) Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

E-mail address: `shanechern@psu.edu; chenxiaohang92@gmail.com`