

BRÄNDÉN'S (p, q) -EULERIAN POLYNOMIALS, ANDRÉ PERMUTATIONS AND CONTINUED FRACTIONS

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ABSTRACT. In 2008 Brändén proved a (p, q) -analogue of the γ -expansion formula for Eulerian polynomials and conjectured the divisibility of the γ -coefficient $\gamma_{n,k}(p, q)$ by $(p+q)^k$. As a follow-up, in 2012 Shin and Zeng showed that the fraction $\gamma_{n,k}(p, q)/(p+q)^k$ is a polynomial in $\mathbb{N}[p, q]$. The aim of this paper is to give a combinatorial interpretation of the latter polynomial in terms of André permutations, a class of objects first defined and studied by Foata, Schützenberger and Strehl in the 1970s. It turns out that our result provides an answer to a recent open problem of Han, which was the impetus of this paper.

1. INTRODUCTION

The Euler number E_n , namely the coefficient of $x^n/n!$ in the expansion of $\sec(x) + \tan(x)$, is well studied and has many combinatorial interpretations and different refinements; see [FS73, Vi81, St09, GSZ, SZ10, SZ12, JV14, MPP]. It was André [An79] who first proved that E_n is the number of alternating permutations $a_1 \dots a_n$ of $1 \dots n$, i.e., $a_1 > a_2 < \dots$. Among the many remarkable identities for the Euler numbers there is the less known J-type continued fraction

$$\sum_{n=0}^{\infty} E_{n+1} x^n = \frac{1}{1 - x - \frac{x^2}{1 - 2x - \frac{3x^2}{1 - 3x - \frac{6x^2}{1 - 4x - \frac{10x^2}{1 - \dots}}}}}. \quad (1.1)$$

This formula does not appear in Flajolet's classic [Fl80] and its connection with the work of Stieltjes [St90] was unveiled only in 2018, see [So18]. More recently, Han [Ha19] considered a q -version of (1.1) and asked for a combinatorial interpretation for the corresponding q -Euler numbers $E_n(q)$ (see (1.3) below). Motivated by Han's question, we shall study the more general polynomials $D_n(p, q, t)$ defined by the following continued fraction, which is

a (p, q) -analogue of (1.1):

$$\sum_{n=0}^{\infty} D_{n+1}(p, q, t) x^n = \frac{1}{1 - x - \frac{\binom{2}{2}_{p,q} t x^2}{1 - [2]_{p,q} x - \frac{\binom{3}{2}_{p,q} t x^2}{1 - [3]_{p,q} x - \frac{\binom{4}{2}_{p,q} t x^2}{1 - [4]_{p,q} x - \frac{\binom{5}{2}_{p,q} t x^2}{1 - \dots}}}}}, \quad (1.2)$$

where the (p, q) -analogue of n is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = \sum_{i+j=n-1} p^i q^j \quad (n \in \mathbb{N})$$

and the (p, q) -analogue of the binomial coefficient $\binom{n}{k}$ is defined by

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q} \cdots [n - k + 1]_{p,q}}{[1]_{p,q} \cdots [k]_{p,q}} \quad (0 \leq k \leq n).$$

Comparing (1.1) and (1.2) yields that

$$D_n(1, 1, 1) = E_n \quad (n \geq 1).$$

The q -Euler number $E_n(q)$ of Han [Ha19] can be expressed as

$$E_n(q) := D_n(1, q, 1) = D_n(q, 1, 1) \quad (n \geq 1). \quad (1.3)$$

The first few values of $D_n(p, q, t)$ are $D_1(p, q, t) = D_2(p, q, t) = 1$, and

$$\begin{aligned} D_3(p, q, t) &= 1 + t, & D_4(p, q, t) &= 1 + (p + q + 2)t, \\ D_5(p, q, t) &= 1 + ((p + q)^2 + 2(p + q) + 3)t + (p^2 + pq + q^2 + 1)t^2. \end{aligned} \quad (1.4)$$

It turns out that the polynomials $D_n(p, q, t)$ are related to the γ -coefficients of Brändén's (p, q) -analogue of Eulerian polynomials [Br08]. In this paper we shall interpret $D_n(p, q, t)$ in terms of *André permutations*, which were introduced and studied by Foata, Schützenberger and Strehl [FS73, FS74, FS76] in the 1970s. There are three ingredients in our proof: the connection of these polynomials with the γ -coefficients of Brändén's (p, q) -analogue of Eulerian polynomials [Br08], Shin-Zeng's continued fraction expansion of the γ -coefficients of generalized Eulerian polynomials [SZ12] and a new action on the permutations without double descents.

Recall that any polynomial $h(t) = \sum_{i=0}^n h_i t^i$ satisfying $h_i = h_{n-i}$ can be expressed uniquely in the form $\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_i t^i (1+t)^{n-2i}$. The coefficients γ_i are called the γ -coefficients of $h(t)$. If the γ -coefficients γ_i are all nonnegative, then $h(t)$ is said to be γ -positive. The unimodality of the sequence (h_0, \dots, h_n) is a direct consequence of γ -positivity. Let \mathfrak{S}_n be the set of permutations of $[n] := \{1, \dots, n\}$. For a permutation $\sigma := \sigma_1 \sigma_2 \dots \sigma_n$ of $[n]$, the *descent number* $\text{des } \sigma$ is the number of descent positions, i.e. $i < n$ such that $\sigma_i > \sigma_{i+1}$, and the *excedance number* $\text{exc } \sigma$ is the number of excedance positions, i.e. $i \in [n]$ such

$n \setminus k$	0	1	2	3	$n \setminus k$	0	1	2	3	$E_n = \sum_k d_{n,k}$
1	1				1	1				1
2	1				2	1				1
3	1	2			3	1	1			2
4	1	8			4	1	4			5
5	1	22	16		5	1	11	4		16
6	1	52	136		6	1	26	34		61
7	1	114	720	272	7	1	57	180	34	272

FIGURE 1. The first values of $\gamma_{n,k}$ (left), $d_{n,k}$ and E_n for $0 \leq 2k < n \leq 7$.

that $\sigma_i > i$. Thanks to the work of MacMahon [Mac15] and Riordan [Ri51] we can define the Eulerian polynomials $A_n(t)$ by

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma}.$$

The following γ -decompositions for $A_n(t)$ are well-known [FS73, Section 4].

Theorem 1 (Foata and Schützenberger).

$$A_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k} \quad (1.5)$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} 2^k d_{n,k} t^k (1+t)^{n-1-2k}, \quad (1.6)$$

where $\gamma_{n,k} = 2^k d_{n,k}$ and $d_{n,k}$ are positive integers satisfying the recurrence

$$\begin{aligned} d_{1,0} &= 1 \quad \text{and for } n \geq 2, k \geq 0, \\ d_{n,k} &= (k+1)d_{n-1,k} + (n-2k)d_{n-1,k-1}. \end{aligned} \quad (1.7)$$

Moreover, the sum $\sum_k d_{n,k}$ is precisely the Euler number E_n , see Figure 1.

In the last two decades although various refinements of (1.5) have been given in combinatorics and geometry (see [PRW, Pe15, At18, SW20]), similar extension of (1.6) does not seem to be known. In this paper we will provide two refinements of (1.6) (see (1.12) and (1.16)).

Definition 1. For a permutation $\sigma = \sigma_1 \dots \sigma_n$ of $[n]$ with $\sigma_0 = \sigma_{n+1} = 0$, the entry σ_i is

- a *peak* if $\sigma_{i-1} < \sigma_i$ and $\sigma_i > \sigma_{i+1}$;
- a *valley* if $\sigma_{i-1} > \sigma_i$ and $\sigma_i < \sigma_{i+1}$;
- a *double ascent* if $\sigma_{i-1} < \sigma_i$ and $\sigma_i < \sigma_{i+1}$;
- a *double descent* if $\sigma_{i-1} > \sigma_i$ and $\sigma_i > \sigma_{i+1}$.

Let $\text{pk } \sigma$ (resp. $\text{val } \sigma$, $\text{da } \sigma$, $\text{dd } \sigma$) denote the number of peaks (resp. valleys, double ascents, double descents) in σ . Note that $\text{des } \sigma = \text{val } \sigma + \text{dd } \sigma$ and $\text{pk } \sigma = \text{val } \sigma + 1$. Let $\mathcal{G}_{n,k} = \{\sigma \in \mathfrak{S}_n : \text{val } \sigma = k, \text{dd } \sigma = 0\}$. For example, the elements of $\mathcal{G}_{4,1}$ are

$$13\bar{2}4, \quad 14\bar{2}3, \quad \bar{2}134, \quad 23\bar{1}4, \quad 24\bar{1}3, \quad 3\bar{1}24, \quad 34\bar{1}2, \quad 4\bar{1}23.$$

Definition 2. For a permutation σ of $[n]$, let $\sigma_{[k]}$ be the *subword* of σ consisting of $1, \dots, k$ in the order they appear in σ . Then, the permutation σ is an André permutation if $\sigma_{[k]}$ has no double descents (and ends with an ascent) for all $1 \leq k \leq n$.

For example, the permutation $\sigma = 43512$ is not André since the subword 4312 of σ contains a double descent 3, while the permutation $\tau = 31245$ is André. Let \mathfrak{D}_n be the set of André permutations of $[n]$. For instance, the set \mathfrak{D}_4 consists of five permutations $1234, 1423, 3124, 3412$, and 4123 . Let $\mathfrak{D}_{n,k}$ be the set of André permutations of $[n]$ with k descents. For example, the elements of $\mathfrak{D}_{5,2}$ are

$$3\bar{1}5\bar{2}4, \quad 4\bar{1}5\bar{2}3, \quad 5\bar{1}4\bar{2}3, \quad 5\bar{3}4\bar{1}2.$$

Proposition 1 ([FS73, FS76]). *The coefficients $\gamma_{n,k}$ and $d_{n,k}$ equal the cardinalities of $\mathcal{G}_{n,k}$ and $\mathfrak{D}_{n,k}$, respectively.*

For $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, the statistic $(31-2)\sigma$ is the number of pairs (i, j) such that $2 \leq i < j \leq n$ and $\sigma_{i-1} > \sigma_j > \sigma_i$. Similarly, the statistic $(2-13)\sigma$ is the number of pairs (i, j) such that $1 \leq i < j \leq n-1$ and $\sigma_{j+1} > \sigma_i > \sigma_j$. In 2008 Brändén [Br08] defined a (p, q) -analogue of Eulerian polynomials and proved a (p, q) -analogue of (1.5). In this paper we shall use the following variant of Brändén's (p, q) -Eulerian polynomials in [SZ12]

$$A_n(p, q, t) := \sum_{\sigma \in \mathfrak{S}_n} p^{(2-13)\sigma} q^{(31-2)\sigma} t^{\text{des } \sigma}. \quad (1.8)$$

For $0 \leq k \leq (n-1)/2$ define the (p, q) -analogue of $\gamma_{n,k}$ and $d_{n,k}$ in (1.5) and (1.6) by

$$\gamma_{n,k}(p, q) = \sum_{\sigma \in \mathcal{G}_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma}, \quad (1.9)$$

$$d_{n,k}(p, q) = \sum_{\sigma \in \mathfrak{D}_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma - k}. \quad (1.10)$$

Our main results are the following two theorems.

Theorem 2. *We have*

$$A_n(p, q, t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(p, q) t^k (1+t)^{n-1-2k} \quad (1.11)$$

$$= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (p+q)^k d_{n,k}(p, q) t^k (1+t)^{n-1-2k}. \quad (1.12)$$

Remark 1. An equivalent γ -expansion of (1.11) was proved by Brändén [Br08] using the *modified Foata-Stehl* action. The divisibility of $\gamma_{n,k}(p, q)$ by $(p + q)^k$ was conjectured by Brändén (*op.cit.*) and proved by Shin and Zeng [SZ12] using the combinatorial theory of continued fractions.

Theorem 3. *We have*

$$D_n(p, q, t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} d_{n,k}(p, q) t^k \quad (1.13)$$

$$= \sum_{\sigma \in \mathfrak{D}_n} p^{(2-13)\sigma} q^{(31-2)\sigma - \text{des } \sigma} t^{\text{des } \sigma}. \quad (1.14)$$

Remark 2. It is not difficult to see that $(31-2)\sigma \geq \text{des } \sigma$ for any $\sigma \in \mathfrak{D}_n$, see (iii) of Proposition 3.

Example. We enumerate the André permutations in \mathfrak{D}_3 and \mathfrak{D}_4 with their number of patterns (2-13) and (31-2). The valleys are in boldface.

$\sigma \in \mathfrak{D}_3$	$(2-13)\sigma$	$(31-2)\sigma$	$\text{des } \sigma$
123	0	0	0
312	0	1	1

$\sigma \in \mathfrak{D}_4$	$(2-13)\sigma$	$(31-2)\sigma$	$\text{des } \sigma$
1234	0	0	0
14 23	0	1	1
3124	1	1	1
3412	0	1	1
4123	0	2	1

These are in accordance with (1.4).

Combining Theorem 2 with Theorem 1 in [SZ16], which is (1.15) below, we derive a q -analogue of (1.5) and (1.6).

Corollary 1. *We have*

$$\sum_{\sigma \in \mathfrak{S}_n} q^{(\text{inv} - \text{exc})\sigma} t^{\text{exc } \sigma} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(q^2, q) t^k (1+t)^{n-1-2k} \quad (1.15)$$

$$= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (1+q)^k d_{n,k}(q) t^k (1+t)^{n-1-2k}, \quad (1.16)$$

where

$$d_{n,k}(q) = \sum_{\sigma \in \mathfrak{D}_{n,k}} q^{2(2-13)\sigma + (31-2)\sigma}.$$

By (1.3) and Theorem 3 we derive two interpretations for Han's q -Euler numbers [Ha19].

Corollary 2. *We have*

$$E_n(q) = \sum_{\sigma \in \mathfrak{D}_n} q^{(2-13)\sigma} \quad (1.17)$$

$$= \sum_{\sigma \in \mathfrak{D}_n} q^{(31-2)\sigma - \text{des } \sigma}. \quad (1.18)$$

In Section 3 we shall give a triple sum formula for $D_n(1, q, t)$ (cf. Theorem 5) and also a simple sum formula for $D_n(1, -1, t)$ (cf. Theorem 6).

2. PROOF OF MAIN THEOREMS

2.1. x-factorization and MFS-action. We recall some definitions from [FS73, FS74]. A permutation w of a finite subset $\{a_1 < a_2 < \dots < a_n\}$ of \mathbb{N} is a word $w = x_1 \dots x_n$. The word u obtained by juxtaposing two words v and w in this order is written $u = vw$. The word v (resp. w) is the left (resp. right) factor of u . More generally, a factorization of length q ($q > 0$) of a word w is any sequence (w_1, w_2, \dots, w_q) of words (some of them possibly empty) such that the juxtaposition product $w_1 w_2 \dots w_q$ is equal to w . The following definition was given as a lemma in [FS74, Lemma 1].

Definition 3. Let $w = x_1 x_2 \dots x_n$ ($n > 0$) be a permutation and x be one of the letters x_i ($1 < i < n$). Then w has a unique factorization (w_1, w_2, x, w_4, w_5) of length 5, called its x -factorization, which is characterized by the three properties

- (i) w_1 is empty or its last letter is less than x ;
- (ii) w_2 (resp. w_4) is empty or all its letters are greater than x ;
- (iii) w_5 is empty or its first letter is less than x .

For instance, for $x = 4$ the x -factorization of $w = 76314582$ is given by $(7631, \epsilon, 4, 58, 2)$, where ϵ denotes the empty word. It is not difficult to check the following facts about x -factorization (see [FS76]).

Proposition 2. *Let (w_1, w_2, x, w_4, w_5) be the x -factorization of a permutation. Then*

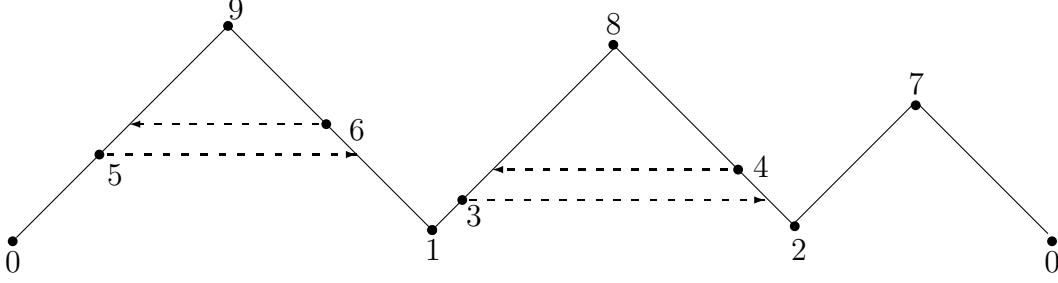
- x is a peak of σ iff $w_2 = \epsilon$ and $w_4 = \epsilon$,
- x is a valley of σ iff $w_2 \neq \epsilon$ and $w_4 \neq \epsilon$,
- x is a double ascent of σ iff $w_2 = \epsilon$ and $w_4 \neq \epsilon$,
- x is a double descent of σ iff $w_2 \neq \epsilon$ and $w_4 = \epsilon$.

We can characterize André permutations in terms of x -factorization [FS73].

Proposition 3. *A permutation $\sigma \in \mathfrak{S}_n$ is an André permutation if it is empty or satisfies the following:*

- (i) σ has no double descents,
- (ii) $n - 1$ is not a descent position, i.e. $\sigma_{n-1} < \sigma_n$,
- (iii) If σ_i is a valley of σ with σ_i -factorization $(w_1, w_2, \sigma_i, w_4, w_5)$, then $\min(w_2) > \min(w_4)$, i.e., the minimum letter of w_2 is larger than the minimum letter of w_4 .

Proof. For $\sigma \in \mathfrak{S}_n$ satisfying (i)–(iii), let $\tau := \sigma_{[k]} = \tau_1 \dots \tau_k$ for $1 \leq k < n$.

FIGURE 2. MFS-actions on $\sigma = 5\,9\,6\,1\,3\,8\,4\,2\,7$ with $\sigma(0) = \sigma(10) = 0$

- If $x := \tau_i$ is a double descent of τ , then x must be a valley in σ by (i); as all letters (if any) between τ_{i-1} and x in σ are larger than x , and all the letters between x and τ_{i+1} are larger than k , the x -factorization of σ is (w_1, w_2, x, w_4, w_5) with $\min(w_4) > k$ and $\min(w_2) \leq \tau_{i-1} \leq k$, so $\min(w_4) > \min(w_2)$ which is impossible by (iii).
- If τ_{k-1} is a descent in $\sigma_{[k]}$, then τ_k must be a valley in σ because all the letters after x in σ are larger than k . Let $x = \tau_k$, then the same argument as above leads to a contradiction.

Thus, for all $1 \leq k \leq n$, the restriction $\sigma_{[k]}$ is an André permutation. \square

We recall Brändén's modified Foata-Strehl action or MFS-action [Br08]. For $x \in [n]$, the MFS-action $\tilde{\varphi}_x$ on $\sigma \in \mathfrak{S}_n$ with x -factorization (w_1, w_2, x, w_4, w_5) is defined as follows:

$$\tilde{\varphi}_x(\sigma) = \begin{cases} w_1 w_4 x w_2 w_5 & \text{if } x \text{ is a double ascent or double descent,} \\ \sigma & \text{if } x \text{ is a valley or peak.} \end{cases}$$

Thus $\tilde{\varphi}_x$ is an involution acting on \mathfrak{S}_n and it is not hard to see that $\tilde{\varphi}_x$ and $\tilde{\varphi}_y$ commute for all $x, y \in [n]$. Hence for any subset $X \subseteq [n]$ we may define the action $\tilde{\varphi}_X$ on $\sigma \in \mathfrak{S}_n$ by

$$\tilde{\varphi}_X(\sigma) = \prod_{x \in X} \tilde{\varphi}_x(\sigma).$$

See Figure 2 for illustration, where exchanging w_2 and w_4 in the x -factorisation is equivalent to moving x from a double ascent to a double descent or vice versa.

For $\sigma \in \mathfrak{S}_n$ let $\text{Orb}(\sigma) = \{g(\sigma) : g \in \mathbb{Z}_2^n\}$ be the orbit of σ under the action MFS-action. Let $\tilde{\sigma}$ be the unique element in $\text{Orb}(\sigma)$ without double descents. The next theorem follows from the work of [Br08, FS76].

Theorem 4. *For any $\tilde{\sigma} \in \mathfrak{S}_n$ without double decent, we have*

$$\sum_{\sigma \in \text{Orb}(\tilde{\sigma})} p^{(2-13)\sigma} q^{(31-2)\sigma} t^{\text{des } \sigma} = p^{(2-13)\tilde{\sigma}} q^{(31-2)\tilde{\sigma}} t^{\text{des } \tilde{\sigma}} (1+t)^{n-1-2\text{des } \tilde{\sigma}}.$$

Proof. For $\sigma \in \mathfrak{S}_n$, consider a (2-13) triple $(\sigma_j, \sigma_i, \sigma_{i+1})$, where $1 \leq j < i \leq n$ and $\sigma_i < \sigma_j < \sigma_{i+1}$, where (σ_i, σ_{i+1}) is a pair of consecutive valley and peak, which means that there are no other peaks and valleys in between σ_i and σ_{i+1} . The number of such triples is invariant under the action since σ_i and σ_{i+1} cannot move and a_j can not move over the peak σ_{i+1} . A similar reasoning applies to (31-2). \square

Let $A_n(p, q, t, u, v, w)$ be the generalized Eulerian polynomials defined by

$$A_n(p, q, t, u, v, w) := \sum_{\sigma \in \mathfrak{S}_n} p^{(2-13)\sigma} q^{(31-2)\sigma} t^{\text{des}\sigma} u^{\text{da}\sigma} v^{\text{dd}\sigma} w^{\text{val}\sigma}. \quad (2.1)$$

As $\text{des} = \text{val} + \text{dd}$ we derive the following generalization of (1.11) from Theorem 4. This was first proved in [SZ12] by using combinatorial theory of continued fractions.

Corollary 3. *For the γ -coefficients $\gamma_{n,k}(p, q)$ in (1.9) we have*

$$A_n(p, q, t, u, v, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(p, q) (tw)^k (u + vt)^{n-1-2k}. \quad (2.2)$$

2.2. New action on permutations without double descent. Let \mathcal{G}_n be the set of permutations of $[n]$ without double descent. For any permutation $\sigma \in \mathcal{G}_n$ and $x \in [n]$ we shall identify σ with its x -factorization, i.e., $\sigma = (w_1, w_2, x, w_4, w_5) = w_1 w_2 x w_4 w_5$, and let $y_1 := \min(w_2)$, $y_2 := \min(w_4)$. A valley x of σ is said to be

- *good* (resp. *bad*) if $y_1 > y_2$ (resp. $y_1 < y_2$);
- of *type I* if $\min(y_1, y_2)$ is a peak or double ascent,
- of *type II* if $\min(y_1, y_2)$ is a valley.

We denote by $\text{Val}\sigma$ the set of valleys of σ .

Proposition 4. *Let $\sigma \in \mathcal{G}_n$ and $x \in \text{Val}\sigma$ with $y = \min(y_1, y_2)$.*

- (i) *If y is a peak, then $w_4 = y$ (resp. $w_2 = y$) if $y_1 > y_2$ (resp. $y_1 < y_2$).*
- (ii) *If y is a double ascent, then $w_4 = yw_4''$ (resp. $w_2 = yw_2''$) with $w_2'', w_4'' \neq \epsilon$ if $y_1 > y_2$ (resp. $y_1 < y_2$).*
- (iii) *If y is a valley, then $w_4 = w_4' y w_4''$ (resp. $w_2 = w_2' y w_2''$) with $w_2', w_2'', w_4', w_4'' \neq \epsilon$ if $y_1 > y_2$ (resp. $y_1 < y_2$).*

Proof. We assume that $y = y_2$.

- (i) If y_2 is a peak, then $w_4 = y_2$ for, otherwise, the word w_4 contains a letter next to y_2 and smaller than y_2 .
- (ii) If y_2 is a double ascent, then $w_4 = yw_4'$ with $w_4' \neq \epsilon$.
- (iii) If y_2 is a valley, then $w_4 = w_4' y w_4''$ with $w_4', w_4'' \neq \epsilon$.

The case $y = y_1$ can be proved similarly. \square

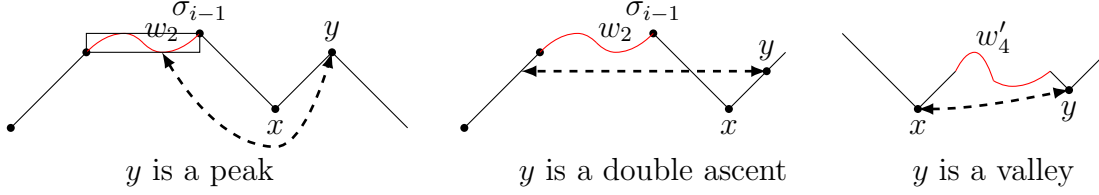
Definition 4. For $\sigma \in \mathcal{G}_n$ and each $x \in \text{Val}\sigma$ with $y = \min(y_1, y_2)$, we define its transform $\varphi(\sigma, x)$ as follows:

- (i) If y is a peak, then

$$\varphi(\sigma, x) = \begin{cases} (w_1, y, x, w_2, w_5) & \text{if } y = y_2, \\ (w_1, w_4, x, y, w_5) & \text{if } y = y_1. \end{cases}$$

- (ii) If y is a double ascent, then

$$\varphi(\sigma, x) = \begin{cases} (w_1, yw_2, x, w'', w_5) & \text{if } y = y_2 \text{ and } w_4 = yw'', \\ (w_1, w'', x, yw_4, w_5) & \text{if } y = y_1 \text{ and } w_2 = yw''. \end{cases}$$

FIGURE 3. The transform $\varphi(x, \sigma)$ according to the type of y

(iii) If y is a valley, then

$$\varphi(\sigma, x) = \begin{cases} (w_1, w_2 y w', x, w'', w_5) & \text{if } y = y_2 \text{ and } w_4 = w' y w'', \\ (w_1, w', x, w'' y w_4, w_5) & \text{if } y = y_1 \text{ and } w_2 = w' y w'' \end{cases}$$

with $w', w'' \neq \epsilon$.

Obviously this transformation switches y from left to right or right to left of x and $\varphi(\varphi(\sigma, x), x) = \sigma$. The three cases (i), (ii), (iii) of the transformation with $y = y_2$ are depicted in Figure 2.2.

We record the basic properties of this transformation in the following proposition.

Proposition 5. *If $\sigma \in \mathcal{G}_{n,k}$ and $x \in \text{Val } \sigma$, then $\varphi(\sigma, x) \in \mathcal{G}_{n,k}$ and*

$$\begin{aligned} (2-13) \varphi(\sigma, x) &= \begin{cases} (2-13) \sigma + 1 & \text{if } x \text{ is good} \\ (2-13) \sigma - 1 & \text{if } x \text{ is bad;} \end{cases} \\ (31-2) \varphi(\sigma, x) &= \begin{cases} (31-2) \sigma - 1 & \text{if } x \text{ is good} \\ (31-2) \sigma + 1 & \text{if } x \text{ is bad.} \end{cases} \end{aligned} \tag{2.3}$$

Proof. We assume that $y := \min(y_1, y_2) = y_2$. By definition 4, the descent number is unchanged after the transform, i.e., $\text{des } \sigma = \text{des } \varphi(\sigma, x)$. Next we verify that $\varphi(\sigma, x)$ has no double descent for the above three cases. For (i), the only possible double descent in $\varphi(\sigma, x) = (w_1, y, x, w_2, w_5)$ is $\text{last}(w_2)$, but this is impossible for, otherwise, the same letter $\text{last}(w_2)$ would be a double descent in σ . The case (ii) is clear. For case (iii), the only possible double descent in $\varphi(\sigma, x) = (w_1, w_2 y w'_4, x, w''_4, w_5)$ is $\text{last}(w'_4)$, but this is also impossible for, otherwise, the same letter $\text{last}(w'_4)$ would be a double descent in σ . Thus $\varphi(\sigma, x) \in \mathcal{G}_{n,k}$.

It remains to prove (2.3). For a permutation $\sigma \in \mathfrak{S}_n$, it is convenient to say that a (31-2) pattern $(\sigma_j, \sigma_{j+1}, \sigma_i)$ is created by σ_i , and a (2-13) pattern $(\sigma_i, \sigma_j, \sigma_{j+1})$ is created by σ_i . Let $x \in \text{Val } \sigma$ and (w_1, w_2, x, w_4, w_5) be the x -factorization of σ and $y := \min(w_4)$. We examine three cases corresponding to the above (i)-(iii), respectively.

- (i) If y is a peak, then $\tau = (w_1, y, x, w_2, w_5)$. Let $a \in [n]$.
 - If $a = y$ and creates the (31-2)-pattern $(\text{last}(w_2), x, y)$ in σ , then a creates the (2-13)-pattern $(y, x, \text{first}(w_2))$ in τ .
 - If $a = x$ and creates the (2-13)-pattern $(x, y, \text{first}(w_5))$ in σ , then a creates the same pattern $(x, \text{last}(w_2), \text{first}(w_5))$ in τ .

- If $a \neq x, y$ and creates the (2-13)-pattern $(a, \text{last}(w_1), \text{first}(w_2))$ in σ , then a creates the same pattern $(a, \text{last}(w_1), y)$ in τ .
- If $a \neq x, y$ and creates the (31-2)-pattern $(y, \text{first}(w_5), a)$ in σ , then a creates the same pattern $(\text{last}(w_2), \text{first}(w_5), a)$ in τ .

Hence, if $a \in [n] \setminus \{y\}$, the number of (31-2)-patterns (resp. (2-13)-patterns) created by a is unchanged under the action.

- (ii) If y is a double ascent, then $\tau = (w_1, yw_2, x, w'_4, w_5)$ with $w_4 = yw'_4$ and $w'_4 \neq \epsilon$. If y creates the (31-2)-pattern $(\text{last}(w_2), x, y)$ in σ , then y creates the (2-13)-pattern $(y, x, \text{first}(w_4))$ in τ . As in (i) we can verify that the number of patterns created by other entry $a \in [n]$ will not change.
- (iii) If y is a valley, then $\tau = (w_1, w_2yw'_4, x, w''_4, w_5)$ with $w_4 = w'yw'_4$ and $w'_4, w''_4 \neq \epsilon$. If y creates the (31-2)-pattern $(\text{last}(w_2), x, y)$ in σ , then y creates the (2-13)-pattern $(y, x, \text{first}(w'_4))$ in τ . Again as in case (i) for other entry $a \in [n]$, the number of (31-2) and (2-13) created by a will not change.

The proof is thus completed. \square

Next we define the transform $\varphi(\sigma, S)$ for any subset S of $\text{Val}(\sigma)$ with $\sigma \in \mathcal{G}_n$.

Definition 5. Let $\sigma \in \mathcal{G}_n$. For any $S \subseteq \text{Val} \sigma$, let $\{S_1, S_2\}$ be the partition of S such that

- (1) S_1 is the subset of S consisting of valleys of type I, say i_1, \dots, i_l ;
- (2) S_2 is the subset of S consisting of valleys of type II, say $j_k < \dots < j_2 < j_1$.

Define the transforms

$$\begin{aligned}\varphi(\sigma, S_1) &= \varphi(i_l, \dots, \varphi(i_2, \varphi(i_1, \sigma))), \\ \varphi(\sigma, S_2) &= \varphi(j_k, \dots, \varphi(j_2, \varphi(j_1, \sigma))), \\ \varphi(\sigma, S) &= \varphi(\varphi(\sigma, S_1), S_2).\end{aligned}$$

Remark 3. The image $\varphi(\sigma, S_1)$ is independent of the order of i_1, \dots, i_l while $\varphi(\sigma, S_2)$ is defined for the elements of S_2 in the decreasing order $j_1 > j_2 > \dots > j_l$.

In what follows, if $w = w_1 \dots w_n$ is a permutation of $[n]$ and u a subword of w , i.e., $u = w_i w_{i+1} \dots w_j$ ($1 \leq i \leq j \leq n$), we write $u \subseteq w$ and $u \in w$ if u is a single letter w_i .

Proposition 6. If $\sigma \in \mathfrak{D}_{n,k}$ and $S \subseteq \text{Val}(\sigma)$, then $\tau := \varphi(\sigma, S) \in \mathcal{G}_{n,k}$ is well defined and

$$S = \{x \in \text{Val}(\tau) \mid x \text{ is a bad guy}\}. \quad (2.4)$$

Proof. For $\sigma \in \mathcal{G}_{n,k}$ and $x_1, x_2 \in S$, let $(w_1, w_2, x_1, w_4, w_5)$ and $(w'_1, w'_2, x_2, w'_4, w'_5)$ be the x_1 -factorization and x_2 -factorization of σ , respectively. Assume that $x_1 < x_2$, hence $x_1 \in w'_1$ or $x_1 \in w'_5$. Thus, if $w_2 x_1 w_4 \cap w'_2 x_2 w'_4 \neq \emptyset$, then $w'_2 x_2 w'_4 \subseteq w_4$ or $w'_2 x_2 w'_4 \subseteq w_2$, namely, one of the following holds:

$$w'_2 x_2 w'_4 \subseteq w_1, \quad w'_2 x_2 w'_4 \subseteq w_2, \quad w'_2 x_2 w'_4 \subseteq w_4, \quad w'_2 x_2 w'_4 \subseteq w_5.$$

Thus when we apply φ to the elements of S_1 the image $\varphi(\sigma, S_1)$ is independent from the order of elements, and $\varphi(\sigma, S_2)$ is well defined if we apply φ to the elements of S_2 in decreasing order. As for each $\sigma \in \mathfrak{D}_{n,k}$ and any $x \in \text{Val}(\sigma)$ there holds $y_1 > y_2$, and for each element of S the application of φ switches y_2 from right to left of x , we have (2.4). \square

For any set S we denote by 2^S the set of all subsets of S . In what follows, for $\sigma \in \mathcal{G}_{n,k}$ we will identify $\text{Val}(\sigma)$ with $[k]$ under the map $a_i \mapsto i$ for $i \in [k]$ if $\text{Val}(\sigma)$ consists of $a_1 < a_2 < \dots < a_k$, and identify any subset $S \in \text{Val}(\sigma)$ with its image $S' \in 2^{[k]}$. Thus we will use $2^{[k]}$ instead of $2^{\text{Val}(\sigma)}$.

Proposition 7. *The map $\varphi : \mathfrak{D}_{n,k} \times 2^{[k]} \rightarrow \mathcal{G}_{n,k}$ is a bijection such that for $(\sigma, S) \in \mathfrak{D}_{n,k} \times 2^{[k]}$ we have*

$$\begin{aligned} (2-13) \sigma + |S| &= (2-13) \varphi(\sigma, S) \\ (31-2) \sigma - |S| &= (31-2) \varphi(\sigma, S). \end{aligned} \tag{2.5}$$

Proof. By Propositions 5 and 6, for $(\sigma, S) \in \mathfrak{D}_{n,k} \times 2^{[k]}$ the image $\tau := \varphi(\sigma, S)$ is an element in $\mathcal{G}_{n,k}$ and satisfies (2.4) and (2.5). To show that φ is bijective, we construct the reverse of φ . For any $\tau \in \mathcal{G}_{n,k}$, let $\text{Val}^{(1)}(\tau)$ and $\text{Val}^{(2)}(\tau)$ be the sets of valleys of τ of types I and II, respectively, and define

$$\begin{aligned} S_2(\tau) &:= \{x \in \text{Val}^{(2)}(\tau) \mid x \text{ is a bad guy}\}; \\ S_1(\tau) &:= \{x \in \text{Val}^{(1)}(\tau) \mid x \text{ is a bad guy}\}. \end{aligned}$$

Note that $\varphi(\sigma, \emptyset) = \sigma$ and $\sigma \in \mathfrak{D}_{n,k}$ if and only if $S_1(\sigma) \cup S_2(\sigma) = \emptyset$. We recover the pair $(\sigma, S) \in \mathfrak{D}_{n,k} \times 2^{[k]}$ such that $\varphi(\sigma, S) = \tau$ by applying the following algorithm:

- (i) Input $(\sigma, S) := (\tau, \emptyset)$.
- (ii) While $S_2(\sigma) := \{x \in \text{Val}^{(2)}(\sigma) \mid y_1 < y_2\} \neq \emptyset$ do $(\sigma, S) := (\varphi(\sigma, z), S \cup \{z\})$ with $z := \min S_2(\sigma)$.
- (iii) Let (σ, S) be the output of (ii) with $S_1(\sigma) := \{x \in \text{Val}^{(1)}(\sigma) \mid y_1 < y_2\}$, and $(\sigma, S) := (\varphi(\sigma, S_1(\sigma)), S \cup S_1(\sigma))$.

To see that the loop (ii) is finite we just need to verify (easy!) that z is a good guy in $\varphi(\sigma, z)$, which implies that if $z' = \min S_2(\varphi(\sigma, z))$ then $z' > z$. It is clear that $S = S_1(\tau) \cup S_2(\tau)$ and $\varphi(\sigma, S) = \tau$. \square

For the reader's convenience, we run the map $\varphi : \mathfrak{D}_{5,2} \times 2^{[2]} \rightarrow \mathcal{G}_{5,2}$ in Figure 4, and give one example for $\varphi : \mathfrak{D}_{5,2} \times 2^{[2]} \rightarrow \mathcal{G}_{5,2}$ and $\varphi^{-1} : \mathcal{G}_{13,5} \rightarrow \mathfrak{D}_{13,5} \times 2^{[5]}$, respectively.

(A) If $\sigma = 31524 \in \mathfrak{D}_{5,2}$ and $S = \{1, 2\}$, then

- for the valley 1, $\min(y_1, y_2) = 2$ is a valley, so 1 is a type II valley.
- for the valley 2, $\min(y_1, y_2) = 4$ is a peak, so 2 is a type I valley.

So, we should first deal with the valley 2, the 2-factorization is $(w_1, w_2, x, w_4, w_5) = (31, 5, 2, 4, \emptyset)$ according to case (i) of φ , we just exchange 4 and 5, and get 31425; then we apply φ to the valley 1 in 31425, the 1-factorization is $(w_1, w_2, x, w_4, w_5) = (\emptyset, 3, 1, 425, \emptyset)$, which is case (iii) of φ , we just exchange 1 and 2, and get $\varphi(\sigma, S) = 32415$.

(B) For $\tau = 11 \mathbf{2} 12 13 \mathbf{1} 6 \mathbf{4} 5 \mathbf{3} 8 \mathbf{9} \mathbf{7} 10 \in \mathcal{G}_{13,5}$.

- (i) First let $(\sigma, S) := (\tau, \emptyset)$. We have $S_2 = \{1, 3\}$ and $\min(S_2) = 1$, so $S = \{1\}$ and

$$\sigma := \varphi(\sigma, 1) = 11 \mathbf{1} 12 13 \mathbf{2} 6 \mathbf{4} 5 \mathbf{3} 8 \mathbf{9} \mathbf{7} 10.$$

$\sigma \in \mathfrak{D}_{5,2}$	$(2-13)$	$(31-2)$	$S \in 2^{\text{Val}\sigma}$	$\tau \in \mathcal{G}_{5,2}$	$(2-13)$	$(31-2)$
31524	2	2	\emptyset	31524	2	2
31524	2	2	$\{1\}$	32514	3	1
31524	2	2	$\{2\}$	31425	3	1
31524	2	2	$\{1, 2\}$	32415	4	0
41523	1	3	\emptyset	41523	1	3
41523	1	3	$\{1\}$	42513	2	2
41523	1	3	$\{2\}$	41325	2	2
41523	1	3	$\{1, 2\}$	42315	3	1
51423	0	4	\emptyset	51423	0	4
51423	0	4	$\{1\}$	52413	1	3
51423	0	4	$\{2\}$	51324	1	3
51423	0	4	$\{1, 2\}$	52314	2	2
53412	0	2	\emptyset	53412	0	2
53412	0	2	$\{1\}$	21534	1	1
53412	0	2	$\{3\}$	43512	1	1
53412	0	2	$\{1, 3\}$	21435	2	0

FIGURE 4. The bijection $\varphi : (\sigma, S) \mapsto \tau$ from $\mathfrak{D}_{5,2} \times 2^{[2]}$ to $\mathcal{G}_{5,2}$

As $S_2 = \{3\}$, we have $\min S_2 = 3$ and $S := \{1, 3\}$, hence

$$\sigma := \varphi(\sigma, 3) = 11 \mathbf{1} 12 13 \mathbf{2} 6 \mathbf{3} 5 \mathbf{4} 8 9 \mathbf{7} 10.$$

(ii) Now for σ , $S_2 = \emptyset$ and $S_1 = \{4, 7\}$, then $S := \{1, 3, 4, 7\}$ and

$$\sigma := \varphi(\sigma, S_1) = 11 \mathbf{1} 12 13 \mathbf{2} 6 \mathbf{3} 10 \mathbf{7} 8 9 \mathbf{4} 5 \in \mathfrak{D}_{13,5}.$$

We can check that $\varphi(\sigma, S) = \tau$.

2.3. Proof of Theorem 2. Clearly (1.11) is a special case of Corollary 3, and (1.12) is equivalent to

$$(p+q)^k \sum_{\sigma \in \mathfrak{D}_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma-k} = \sum_{\sigma \in \mathcal{G}_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma}. \quad (2.6)$$

As $(p+q)^k = \sum_{S \in 2^{[k]}} p^{|S|} q^{k-|S|}$ we can rewrite the above identity as

$$\sum_{(\sigma, S) \in \mathfrak{D}_{n,k} \times 2^{[k]}} p^{(2-13)\sigma + |S|} q^{(31-2)\sigma - |S|} = \sum_{\sigma \in \mathcal{G}_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma}.$$

The result follows from Proposition 7. □

2.4. Proof of Theorem 3. We shall use the J-type continued fraction as a formal power series defined by

$$\sum_{n=0}^{\infty} \mu_n t^n = \frac{1}{1 - b_0 t - \frac{\lambda_1 t^2}{1 - b_1 t - \frac{\lambda_2 t^2}{1 - \dots}}},$$

where (b_n) and (λ_{n+1}) ($n \geq 0$) are two sequences in a commutative ring. When $b_n = 0$ we obtain the S-type continued fraction:

$$\sum_{n=0}^{\infty} \mu_n t^n = \frac{1}{1 - \frac{\lambda_1 t}{1 - \frac{\lambda_2 t}{1 - \dots}}}.$$

Recall the following continued fraction expansion formula from [SZ12, (28)]:

$$\sum_{n \geq 1} A_n(p, q, t, u, v, w) x^{n-1} = \frac{1}{1 - (u + tv)[1]_{p,q}x - \frac{[1]_{p,q}[2]_{p,q}twx^2}{1 - (u + tv)[2]_{p,q}x - \frac{[2]_{p,q}[3]_{p,q}twx^2}{\dots}}} \quad (2.7)$$

with $b_n = (u + tv)[n + 1]_{p,q}$ and $\lambda_n = [n]_{p,q}[n + 1]_{p,q}tw$.

By Theorem 2 and substituting (t, u, v, w) with $(p + q, 0, 1, t)$ in (2.2), we obtain

$$A_n(p, q, p + q, 0, 1, t) = (p + q)^{n-1} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} d_{n,k}(p, q) t^k.$$

Thus, substituting (t, u, v, w) with $(p + q, 0, 1, t)$ in (2.7) and replacing x by $x/(p + q)$ we obtain the same continued fraction in (1.2). This proves (1.13). \square

3. TWO EXPLICIT FORMULAE

First we derive a formula for $D_n(1, q, t)$ from the work of Shin-Zeng [SZ12], Han-Mao-Zeng [HMZ] and Josuat-Vergès [JV11].

Theorem 5. *For $n \geq 1$ we have*

$$D_n(1, q, t) = \frac{1}{v(1 - q)} \left(\frac{1 + u}{(1 + uv)(1 - q^2)} \right)^{n-1} \times \sum_{k=0}^n (-1)^k \left(\sum_{j=0}^{n-k} v^j \binom{n}{j} \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \right) \cdot \left(\sum_{i=0}^k v^i q^{i(k+1-i)} \right),$$

where

$$u = \frac{1 + q^2 - 2(1 + q)t - (1 + q)\sqrt{(1 + q)^2 - 4(1 + q)t}}{2(q - t(1 + q))}, \quad (3.1)$$

$$v = \frac{(1 + q) - 2t - \sqrt{(1 + q)^2 - 4t(1 + q)}}{2t}. \quad (3.2)$$

Proof. By (2.2) we have

$$A_n(1, q, q, 1, 1, t(1 + q^{-1})) = (1 + q)^{n-1} D_n(1, q, t). \quad (3.3)$$

From Corollary 3.2 in [HMZ] we derive

$$A_n(1, q, q, 1, 1, t(1 + q^{-1})) = \left(\frac{1 + u}{1 + uv} \right)^{n-1} \sum_{\sigma \in \mathfrak{S}_n} v^{\text{des } \sigma} q^{(31-2)\sigma},$$

where u and v are given by (3.1) and (3.2). By Theorem 6.3 in [JV11], we have

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_n} y^{\text{des } \sigma} q^{(31-2)\sigma} &= \sum_{\sigma \in \mathfrak{S}_n} y^{\text{asc } \sigma} q^{(13-2)\sigma} \\ &= \frac{1}{y(1 - q)^n} \sum_{n=0}^n (-1)^k \left(\sum_{j=0}^{n-k} y^j \binom{n}{j} \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \right) \\ &\quad \times \left(\sum_{i=0}^k y^i q^{i(k+1-i)} \right). \end{aligned}$$

Putting the above three formulae together completes the proof. \square

A *Motzkin path* of length n is a sequence of points $\eta := (\eta_0, \dots, \eta_n)$ in the integer plane $\mathbb{Z} \times \mathbb{Z}$ such that

- $\eta_0 = (0, 0)$ and $\eta_n = (n, 0)$,
- $\eta_i - \eta_{i-1} \in \{(1, 0), (1, 1), (1, -1)\}$,
- $\eta_i := (x_i, y_i) \in \mathbb{N} \times \mathbb{N}$ for $i = 0, \dots, n$.

In other words, a Motzkin path of length n is a lattice path starting at $(0, 0)$, ending at $(n, 0)$, and never going below the x -axis, consisting of up-steps $\mathbf{U} = (1, 1)$, level-steps $\mathbf{L} = (1, 0)$, and down-steps $\mathbf{D} = (1, -1)$. Let \mathcal{MP}_n be the set of Motzkin paths of length n . Clearly we can identify Motzkin paths of length n with words w on $\{\mathbf{U}, \mathbf{L}, \mathbf{D}\}$ of length n such that all prefixes of w contain no more \mathbf{D} 's than \mathbf{U} 's and the number of \mathbf{D} 's equals the number of \mathbf{D} 's. The height of a step (η_i, η_{i+1}) is the coordinate of the starting point η_i . Given a Motzkin path $p \in \mathcal{MP}_n$ and two sequences (b_i) and (λ_i) of a commutative ring R , we weight each up-step by 1, and each level-step (resp. down-step) at height i by b_i (resp. λ_i) and define the weight $w(p)$ of p by the product of the weights of all its steps. The following result of Flajolet [Fl80] is our starting point.

Lemma 1 (Flajolet). *We have*

$$\sum_{n=0}^{\infty} \left(\sum_{p \in \mathcal{MP}_n} w(p) \right) t^n = \frac{1}{1 - b_0 t - \frac{\lambda_1 t^2}{1 - b_1 t - \frac{\lambda_2 t^2}{1 - b_2 t - \dots}}}.$$

A Motzkin path without level-steps is called a *Dyck path*, and a Motzkin path without level-steps at odd height is called an *André path*. We denote by $\mathcal{AP}_{n,k}$ the set of André paths of half-length n with k level-steps, and \mathcal{DP}_n the set of Dyck paths of half length n .

Lemma 2. *Let $b_i = 0$ ($i \geq 0$) and $\lambda_i = \lfloor \frac{i+1}{2} \rfloor$ ($i \geq 1$). Then*

$$n! = \sum_{p \in \mathcal{MP}_n} w(p).$$

In other words, the factorial $n!$ is the generating polynomial of \mathcal{DP}_n .

Proof. Recall the following formula of Euler:

$$\sum_{n \geq 0} n! x^n = \frac{1}{1 - \frac{1x}{1 - \frac{1x}{1 - \frac{2x}{1 - \frac{2x}{\dots}}}}} \quad (3.4)$$

with $\lambda_n = \lfloor (n+1)/2 \rfloor$. The result then follows from Lemma 1. \square

Remark 4. A bijective proof of Euler's formula (3.4) is known, see [PZ19, (4.9)].

Lemma 3. *Let $b_{2i} = 1$, $b_{2i+1} = 0$ ($i \geq 0$) and $\lambda_k = \lfloor \frac{k+1}{2} \rfloor t$ ($i \geq 1$). Then*

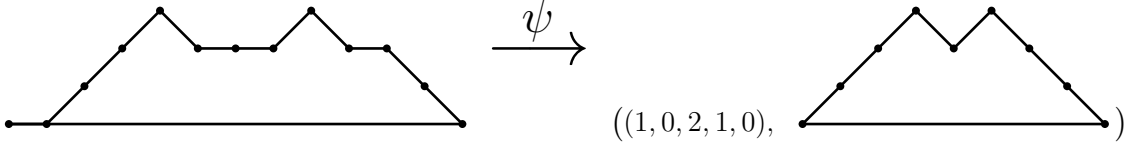
$$D_{n+1}(1, -1, t) = \sum_{p \in \mathcal{AP}_n} w(p).$$

In other words, the polynomial $D_{n+1}(1, -1, t)$ is the generating polynomial of André paths of length n .

Proof. When $(p, q) = (1, -1)$ formula(1.2) reduces to

$$\sum_{n=0}^{\infty} D_{n+1}(1, -1, t) x^n = \frac{1}{1 - x - \frac{tx^2}{1 - \frac{tx^2}{1 - x - \frac{2tx^2}{1 - \frac{2tx^2}{1 - x - \dots}}}}} \quad (3.5)$$

with coefficients $b_{2i} = 1$, $b_{2i+1} = 0$ and $\lambda_i = \lfloor \frac{i+1}{2} \rfloor t$. The result follows from Lemma 1. \square

FIGURE 5. The bijection $\psi : \mathcal{AP}_{12,4} \rightarrow \mathcal{Y}_{12,4} \times \mathcal{DP}_4$

Let

$$\mathcal{Y}_{n,k} := \{(y_1, \dots, y_{k+1}) \in \mathbb{N}^{k+1} : y_1 + \dots + y_{k+1} = n - 2k\}.$$

Example. An illustration of the bijection ψ is given in Figure 5.

Lemma 4. For $0 \leq k \leq \lfloor n/2 \rfloor$, there is an explicit bijection $\psi : \mathcal{AP}_{n,n-2k} \rightarrow \mathcal{Y}_{n,k} \times \mathcal{DP}_k$ such that if $\psi(u) = (y, p)$ with $u \in \mathcal{AP}_{n,n-2k}$ and $(y, p) \in \mathcal{Y}_{n,k} \times \mathcal{DP}_k$ then $w(u) = w(p)$, where the weight is associated to the sequences (b_i) and (λ_i) with $b_{2i} = 1$, $b_{2i+1} = 0$ ($i \geq 0$), and $\lambda_k = \lfloor \frac{k+1}{2} \rfloor t$ ($i \geq 1$).

Proof. Since an André path (word) on $\{\mathbf{U}, \mathbf{D}, \mathbf{L}\}$ has only level-steps at even height and starts from height 0, so the subword between two consecutive level-steps \mathbf{L} 's must be of even length and is a word on the alphabet $\{\mathbf{UU}, \mathbf{DD}, \mathbf{UD}, \mathbf{DU}\}$. Thus, any André word $u \in \mathcal{AP}_{n,n-2k}$ can be written in a unique way as follows:

$$u = \mathbf{L}^{y_1} w_1 \mathbf{L}^{y_2} w_2 \dots w_k \mathbf{L}^{y_{k+1}} \quad \text{with} \quad w_i \in \{\mathbf{UU}, \mathbf{DD}, \mathbf{UD}, \mathbf{DU}\}.$$

Let $y := (y_1, \dots, y_{k+1})$ and $p := w_1 \dots w_k$. As the path p is obtained by removing out all the level-steps \mathbf{L} 's from the André path u , each step in p keeps the same height in u , and $(y, p) \in \mathcal{Y}_{n,k} \times \mathcal{DP}_k$. Let $\psi(u) = (y, p)$. It is clear that this is the desired bijection. \square

Theorem 6. For $n \geq 1$ we have

$$D_n(1, -1, t) = \sum_{k=0}^{n-1} \binom{n-1-k}{k} k! t^k. \quad (3.6)$$

Remark 5. If $t = 1$, in view of (1.3), the above result is equivalent to

$$E_n(-1) = \sum_{k=0}^{n-1} \binom{n-1-k}{k} k!, \quad (3.7)$$

which was posted by P. Barry, see A122852 in OEIS [OEIS]. Han [Ha19, Theorem 7.1] gave a non-trivial (sic) proof of (3.7) by showing that both sides of (3.7) satisfy the same recurrence relation, which had been conjectured by R. J. Mathar. Our proof of (3.6) is combinatorial and insightful for the summation formula in (3.6).

Proof of Theorem 6. By Lemmas 3 and 4 we have

$$D_{n+1}(1, -1, t) = \sum_{k \geq 0} \sum_{(y,p) \in \mathcal{Y}_{n,k} \times \mathcal{DP}_k} w(p).$$

Since the cardinality of $\mathcal{Y}_{n,k}$ is $\binom{n-k}{k}$, and the generating polynomial of \mathcal{DP}_k is equal to $k!t^k$ by Lemma 2, summing over all $0 \leq k \leq \lfloor n/2 \rfloor$ we obtain Theorem 3.6. \square

Remark 6. It is a challenge to show directly that Theorem 6 is the limit case of Theorem 5 when $q \rightarrow -1$.

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