# On symmetric and Hermitian rank distance codes 

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#### Abstract

Let $\mathcal{M}$ denote the set $\mathcal{S}_{n, q}$ of $n \times n$ symmetric matrices with entries in $\mathrm{GF}(q)$ or the set $\mathcal{H}_{n, q^{2}}$ of $n \times n$ Hermitian matrices whose elements are in $\operatorname{GF}\left(q^{2}\right)$. Then $\mathcal{M}$ equipped with the rank distance $d_{r}$ is a metric space. We investigate $d$-codes in $\left(\mathcal{M}, d_{r}\right)$ and construct $d$-codes whose sizes are larger than the corresponding additive bounds. In the Hermitian case, we show the existence of an $n$-code of $\mathcal{M}, n$ even and $n / 2$ odd, of size $\left(3 q^{n}-q^{n / 2}\right) / 2$, and of a 2 -code of size $q^{6}+q(q-1)\left(q^{4}+q^{2}+1\right) / 2$, for $n=3$. In the symmetric case, if $n$ is odd or if $n$ and $q$ are both even, we provide better upper bound on the size of a $2-$ code. In the case when $n=3$ and $q>2$, a 2 -code of size $q^{4}+q^{3}+1$ is exhibited. This provides the first infinite family of 2 -codes of symmetric matrices whose size is larger than the largest possible additive 2 -code and an answer to a question posed in [25, Section 7], see also [23, p. 176].


Keywords: symmetric rank distance codes; Hermitian rank distance codes; symplectic polar spaces; Hermitian polar spaces; Segre variety.

## 1 Introduction

Let $q$ be a power of a prime and let $\mathrm{GF}(q)$ be the finite field with $q$ elements. Denote by $\mathcal{S}_{n, q}$ the set of $n \times n$ symmetric matrices with entries in $\operatorname{GF}(q)$ and by $\mathcal{H}_{n, q^{2}}$ the set of $n \times n$ Hermitian matrices whose elements are in $\operatorname{GF}\left(q^{2}\right)$. Let $\mathcal{M}$ be $\mathcal{S}_{n, q}$ or $\mathcal{H}_{n, q^{2}}$. For two matrices $A, B \in \mathcal{M}$, define their rank distance to be

$$
d_{r}(A, B)=\operatorname{rk}(A-B) .
$$

Thus $d_{r}$ is a metric on $\mathcal{M}$ and $\left(\mathcal{M}, d_{r}\right)$ is a metric space. A rank metric code $\mathcal{C}$ is a non-empty subset of $\left(\mathcal{M}, d_{r}\right)$. The minimum distance of $\mathcal{C}$ is

$$
d_{r}(\mathcal{C})=\min \left\{d_{r}\left(c_{1}, c_{2}\right) \mid c_{1}, c_{2} \in \mathcal{C}, c_{1} \neq c_{2}\right\} .
$$

We will refer to a code in $\left(\mathcal{M}, d_{r}\right)$ with minimum distance $d$ as a $d$-code. A $d$-code is said to be maximal if it maximal with respect to set theoretic inclusion, whereas it is called maximum

[^0]if it has the largest possible size. If a $d$-code $\mathcal{C} \subset \mathcal{M}$ forms a subgroup of $(\mathcal{M},+)$, then $\mathcal{C}$ is called additive. Upper bounds on the size of a $d$-code of $\mathcal{M}$ were provided in [20, Corollary 7], [21, Lemma 3.5, Proposition 3.7], [24, Proposition 3.4] and [22, Theorems 1 and 2]. In the case when $\mathcal{C}$ is additive much better bounds can be obtained. Indeed in [21, Lemmas 3.5 and 3.6], [23. Theorem 4.3], the author proved that the largest additive $d$-codes of $\mathcal{S}_{n, q}$ have size at most either $q^{n(n-d+2) / 2}$ or $q^{(n+1)(n-d+1) / 2}$, according as $n-d$ is even or odd, respectively, whereas the size of the largest additive $d$-codes of $\mathcal{H}_{n, q^{2}}$ cannot exceed $q^{n(n-d+1)}$, see [22, Theorem 1]. Moreover there exist additive $d$-codes whose sizes meet the upper bounds for all possible value of $n$ and $d$, except when $\mathcal{M}=\mathcal{H}_{n, q^{2}}, n, d$ are both even and $3<d<n$, see [20, Theorems 12 and 16], [21, Theorem 4.4], [23, Theorem 5.3], [22, Theorems 4 and 5], [24, Theorem 6.1], [6], [7], [8]. If $d$ is odd, a $d$-code attaining the corresponding additive bound is maximum. This is not always true if $d$ is even. However not much is known about $d$-codes whose size is larger than the corresponding additive bound. In the Hermitian case, if $n$ is even, there is an $n$-code of $\mathcal{H}_{n, q^{2}}$ of size $q^{n}+1$ [22, Theorem 6], [11, Theorem 18]. In the symmetric case only sporadic examples of non-additive $d$-codes that are larger than the largest possible additive $d$-code are known [24, Tables 2 and 9].

Let $\mathcal{W}(2 n-1, q)$ be a non-degenerate symplectic polar space and $\mathcal{H}\left(2 n-1, q^{2}\right)$ be a nondegenerate Hermitian polar space. Let $\Pi_{1}$ be a generator of $\mathcal{W}(2 n-1, q)$ and let $\Lambda_{1}$ be a generator of $\mathcal{H}\left(2 n-1, q^{2}\right)$. It is known that there exists a bijection $\tau$ between the matrices of $\mathcal{S}_{n, q}$ or $\mathcal{H}_{n, q^{2}}$ and the generators of $\mathcal{W}(2 n-1, q)$ or $\mathcal{H}\left(2 n-1, q^{2}\right)$ disjoint from $\Pi_{1}$ or $\Lambda_{1}$, respectively, see [2, Proposition 9.5.10], [11]. Here an upper bound on the maximum number of generators of $\mathcal{W}(2 n-1, q)$ or $\mathcal{H}\left(2 n-1, q^{2}\right)$ pairwise intersecting in at most an $(n-3)$-dimensional projective space is derived, see Theorems 5.2 and 5.4. As a by product, by means of $\tau$, the following upper bound on the size of a 2 -code $\mathcal{C}$ of $\mathcal{S}_{n, q}$ is obtained:

$$
|\mathcal{C}| \leq \begin{cases}\sum_{j=0}^{\frac{n-1}{2}} \frac{q^{j}\left(q^{n-2 j+1}+1\right)}{q^{n-j+1}+1} \prod_{i=1}^{j} \frac{q^{2(n-i+1)}-1}{\left(q^{i}-1\right)\left(q^{i-1}+1\right)} & \text { if } n \text { is odd }  \tag{1.1}\\ \prod_{i=2}^{n}\left(q^{i}+1\right) & \text { if } n \text { is even }\end{cases}
$$

Since the previous known upper bound for the size of a 2 -code of $\mathcal{S}_{n, q}$ was $q^{n(n-1) / 2+1}\left(q^{n-1}+\right.$ 1)/ $(q+1)$ for $q$ odd [21, Proposition 3.7], and $q^{n(n+1) / 2}-q^{n}+1$ for $q$ even [24, Proposition 3.4], it follows that (1.1) provides better upper bounds if $n$ is odd or if $n$ and $q$ are both even.

By using $\tau$, it can be seen that an $n$-code $\mathcal{C}$ of $\mathcal{S}_{n, q}$ or $\mathcal{H}_{n, q^{2}}$ exists if and only if there exists a partial spread of $\mathcal{W}(2 n-1, q)$ or $\mathcal{H}\left(2 n-1, q^{2}\right)$ of size $|\mathcal{C}|+1$, see Lemmas 3.1] and 4.1, [11]. It is well-known that the points of $\mathcal{W}(2 n-1, q)$ can be partitioned into $q^{n}+1$ pairwise disjoint generators of $\mathcal{W}(2 n-1, q)$, that is, $\mathcal{W}(2 n-1, q)$ admits a spread. On the other hand, $\mathcal{H}\left(2 n-1, q^{2}\right)$ has no spread. If $n$ is odd, an upper bound for the largest partial spreads of $\mathcal{H}\left(2 n-1, q^{2}\right)$ is $q^{n}+1$ [29] and there are examples of partial spreads of that size [18]. If $n$ is even the situation is less clear: upper bounds can be found in [15], as for lower bounds there is a partial spread of $\mathcal{H}\left(2 n-1, q^{2}\right)$ of size $\left(3 q^{2}-q\right) / 2+1$ for $n=2, q>13$, [1, p. 32] and of size $q^{n}+2$ for $n \geq 4$ [11]. Here, generalizing the partial spread of $\mathcal{H}\left(3, q^{2}\right)$, we show the existence of a partial spread of
$\mathcal{H}\left(2 n-1, q^{2}\right)$, in the case when $n$ is even and $n / 2$ is odd, of size $\left(3 q^{n}-q^{n / 2}\right) / 2+1$ (cf. Theorem 6.4) and hence, if $n$ is even and $n / 2$ is odd, of an $n$-code of $\mathcal{H}_{n, q^{2}}$ of size $\left(3 q^{n}-q^{n / 2}\right) / 2$.

In the remaining part of the paper, we focus on the case $n=3$. First, a further improvement on the size of a 2 -code $\mathcal{C}$ of $\mathcal{S}_{3, q}$ is obtained (cf. Corollary 6.7):

$$
|\mathcal{C}| \leq \frac{q\left(q^{2}-1\right)\left(q^{2}+q+1\right)}{2}+1
$$

Then we construct 2 -codes of $\mathcal{S}_{3, q}$ and $\mathcal{H}_{3, q^{2}}$ of size $q^{4}+q^{3}+1$ and $q^{6}+q(q-1)\left(q^{4}+q^{2}+1\right) / 2$, respectively. This provides the first infinite family of 2 -codes of $\mathcal{S}_{3, q}$ whose size is larger than the largest possible additive 2 -code and an answer to a question posed in [25, Section 7], see also [23, p. 176].

## 2 Preliminaries

### 2.1 Projective and polar spaces

Let $\mathrm{PG}(r-1, q)$ be the projective space of projective dimension $r-1$ over $\mathrm{GF}(q)$ equipped with homogeneous projective coordinates $X_{1}, \ldots, X_{r}$. We will use the term $n$-space of $\operatorname{PG}(r-1, q)$ to denote an $n$-dimensional projective subspace of $\mathrm{PG}(r-1, q)$. We shall find it helpful to represent projectivities of $\mathrm{PG}(r-1, q)$ by invertible $r \times r$ matrices over $\mathrm{GF}(q)$ and to consider the points of $\mathrm{PG}(r-1, q)$ as column vectors, with matrices acting on the left. Let $U_{i}$ be the points having 1 in the $i$-th position and 0 elsewhere. Furthermore, we denote by $0_{n}$ and $I_{n}$ the $n \times n$ zero matrix and identity matrix, respectively; if $M$ is an $n \times n$ matrix over $\operatorname{GF}(q)$, we denote by $L(M)$ the ( $n-1$ )-space of $\mathrm{PG}(2 n-1, q)$ whose underlying vector space is the vector space spanned by the rows of the $n \times 2 n$ matrix $\left(\begin{array}{ll}I_{n} & M\end{array}\right)$; we also use the notation $L(M)=\left\langle\left(\begin{array}{ll}I_{n} & M\end{array}\right)\right\rangle$. If $m$ divides $r$, an $(m-1)$-spread of $\mathrm{PG}(r-1, q)$ is a set of pairwise disjoint $(m-1)$-spaces of $\mathrm{PG}(r-1, q)$ which partition the point set of $\operatorname{PG}(r-1, q)$.

A finite classical polar space $\mathbf{P}$ arises from a vector space of finite dimension over a finite field equipped with a non-degenerate reflexive sesquilinear form. In this paper we will be mainly concerned with symplectic polar spaces and Hermitian polar spaces. A projective subspace of maximal dimension contained in $\mathbf{P}$ is called a generator of $\mathbf{P}$. For further details on finite classical polar spaces we refer the readers to [13]. A partial spread $\mathbf{S}$ of $\mathbf{P}$ is a set of pairwise disjoint generators. A partial spread $\mathbf{S}$ of $\mathbf{P}$ is called a spread of $\mathbf{P}$ if $\mathbf{S}$ partitions the point set of $\mathbf{P}$.

### 2.1.1 Segre varieties

Consider the map defined by

$$
\xi: \mathrm{PG}(1, q) \times \mathrm{PG}(2, q) \longrightarrow \mathrm{PG}(5, q)
$$

taking a pair of points $x=\left(x_{1}, x_{2}\right)$ of $\mathrm{PG}(1, q), y=\left(y_{1}, y_{2}, y_{3}\right)$ of $\mathrm{PG}(2, q)$ to their product $\left(x_{1} y_{1}, \ldots, x_{2} y_{3}\right)$. This is a special case of a wider class of maps called Segre maps [13]. The image of $\xi$ is an algebraic variety called the Segre variety and denoted by $\Sigma_{1,2}$. The Segre variety
$\Sigma_{1,2}$ has two rulings, say $\mathcal{R}_{1}$ and $\mathcal{R}_{2}$, containing $q^{2}+q+1$ lines and $q+1$ planes, respectively, satisfying the following properties: two subspaces in the same ruling are disjoint, elements of different ruling intersect in exactly one point and each point of $\Sigma_{1,2}$ is contained in exactly one member of each ruling.

Notice that the set $\mathcal{R}_{1}$ consists of all the lines of $\operatorname{PG}(5, q)$ incident with three distinct members of $\mathcal{R}_{2}$ and, from [13, Theorem 25.6.1], three mutually disjoint planes of $\operatorname{PG}(5, q)$ define a unique Segre variety $\Sigma_{1,2}$. A line of $\operatorname{PG}(5, q)$ shares $0,1,2$ or $q+1$ points with $\Sigma_{1,2}$. Also, the automorphism group of $\Sigma_{1,2}$ in $\operatorname{PGL}(6, q)$ is a group isomorphic to $\operatorname{PGL}(2, q) \times \operatorname{PGL}(3, q)$ [13, Theorem 25.5.13]. For more details on Segre varieties, see [13].

### 2.2 Graphs

Recall some definitions and results from [3, 10]. Suppose $\Gamma$ is a (simple, undirected) graph having $\mathcal{V}$ as set of vertices. The adjacency matrix $A$ of $\Gamma$ is a symmetric real matrix whose rows and columns are indexed by $1, \ldots,|\mathcal{V}|$. The eigenvalues of $\Gamma$ are those of its adjacency matrix $A$. A graph $\Gamma$ is called regular of valency $k$ or $k$-regular when every vertex has precisely $k$ neighbors. If $\Gamma$ is regular of valency $k$, then $A \mathbf{1}=k \mathbf{1}$, where $\mathbf{1}$ denotes the all one column vector. Hence $k$ is an eigenvalue of $\Gamma$ and for every eigenvalue $\lambda$ of $\Gamma$, we have that $|\lambda| \leq k$. Furthermore the multiplicity of $k$ equals the number of connected components of $\Gamma$.

Let $\Gamma$ be a $k$-regular graph and let $\left\{\mathcal{V}_{1}, \ldots, \mathcal{V}_{m}\right\}$ be a partition of $\mathcal{V}$. Let $A$ be partitioned according to $\left\{\mathcal{V}_{1}, \ldots, \mathcal{V}_{m}\right\}$, that is,

$$
A=\left(\begin{array}{ccc}
A_{1,1} & \ldots & A_{1, m} \\
\vdots & & \vdots \\
A_{m, 1} & \ldots & A_{m, m}
\end{array}\right)
$$

such that $A_{i, i}$ is a square matrix for all $1 \leq i \leq m$. The quotient matrix $B$ is the $m \times m$ matrix with entries the average row sum of the blocks of $A$. More precisely,

$$
B=\left(b_{i, j}\right), b_{i, j}=\frac{1}{v_{i}} \mathbf{1}^{t} A_{i, j} \mathbf{1},
$$

where $v_{i}$ is the number of rows of $A_{i, j}$. If the row sum of each block $A_{i, j}$ is constant then the partition is called equitable or regular and we have $A_{i, j} \mathbf{1}=b_{i, j} \mathbf{1}$ for $1 \leq i, j \leq m$. One important class of equitable partitions arises from automorphisms of $\Gamma$, indeed the orbits of any group of automorphisms of $\Gamma$ form an equitable partition. The following result is well known and useful.

Lemma 2.1 (Lemma 2.3.1, 3], Theorem 9.4.1, [10]). Let $B$ be the quotient matrix of an equitable partition. If $\lambda$ is an eigenvalue of $B$, then $\lambda$ is an eigenvalue of $A$.

Let $\Gamma$ be a vertex-transitive graph and let $B$ be the quotient matrix of an equitable partition arising from the orbits of some subgroup of $\operatorname{Aut}(\Gamma)$. If $\left|\mathcal{V}_{i}\right|=1$ for some $i$, then every eigenvalue of $A$ is an eigenvalue of $B$.

A coclique of $\Gamma$ is a set of pairwise nonadjacent vertices. The independence number $\alpha(\Gamma)$ is the size of the largest coclique of $\Gamma$. Let $\lambda_{1} \geq \cdots \geq \lambda_{|\mathcal{V}|}$ be the eigenvalues of $\Gamma$. The following
results are due to Cvetković [5] and Hoffman, respectively; see also [3, Theorem 3.5.1, Theorem 3.5.2].

Lemma 2.2. $\alpha(\Gamma) \leq \min \left\{\left|\left\{i \mid \lambda_{i} \geq 0\right\}\right|,\left|\left\{i \mid \lambda_{i} \leq 0\right\}\right|\right\}$.
Lemma 2.3. $\alpha(\Gamma) \leq-\frac{|\mathcal{V}| \lambda|\mathcal{V}|}{k-\lambda|\mathcal{V}|}$.

## 3 Symmetric matrices and symplectic polar spaces

Let $\mathcal{W}(2 n-1, q)$ be the non-degenerate symplectic polar space of $\operatorname{PG}(2 n-1, q)$ associated with the following alternating bilinear form

$$
\left(X_{1}, \ldots, X_{2 n}\right)\left(\begin{array}{cc}
0_{n} & I_{n} \\
-I_{n} & 0_{n}
\end{array}\right)\left(\begin{array}{c}
Y_{1} \\
\vdots \\
Y_{2 n}
\end{array}\right)
$$

Let $\perp$ denote the symplectic polarity of $\operatorname{PG}(2 n-1, q)$ defining $\mathcal{W}(2 n-1, q)$ and let $\operatorname{PSp}(2 n, q)=$ $\operatorname{Sp}(2 n, q) /\langle-I\rangle$, where $\operatorname{Sp}(2 n, q)$ is the group of isometries of the alternating bilinear form previously defined. Hence $\operatorname{PSp}(2 n, q)$ consists of projectivities of $\operatorname{PG}(2 n-1, q)$ fixing $\mathcal{W}(2 n-1, q)$. It acts transitively on the generators of $\mathcal{W}(2 n-1, q)$. Denote by $\Pi_{1}$ the $(n-1)$-space of $\operatorname{PG}(2 n-1, q)$ spanned by $U_{n+1}, \ldots, U_{2 n}$. Then $\Pi_{1}$ is a generator of $\mathcal{W}(2 n-1, q)$. Let $G$ be the stabilizer of $\Pi_{1}$ in $\operatorname{PSp}(2 n, q)$. Then it is readily seen that an element of $G$ is represented by the matrix

$$
\left(\begin{array}{cc}
T^{-t} & 0_{n}  \tag{3.1}\\
S_{0} T^{-t} & T
\end{array}\right)
$$

where $T \in \mathrm{GL}(n, q)$ and $S_{0} \in \mathcal{S}_{n, q}$. Hence $G \simeq \mathcal{S}_{n, q} \rtimes\left(\mathrm{GL}(n, q) /\left\langle-I_{2 n}\right\rangle\right)$ has order

$$
\frac{q^{n(n+1) / 2} \prod_{i=0}^{n-1}\left(q^{n}-q^{i}\right)}{\operatorname{gcd}(2, q-1)}
$$

and it acts transitively on the set $\mathcal{G}$ of generators of $\mathcal{W}(2 n-1, q)$ disjoint from $\Pi_{1}$.
Define an action of $\mathcal{S}_{n, q} \rtimes \mathrm{GL}(n, q)$ on $\mathcal{S}_{n, q}$ as follows

$$
\left(\left(S_{0}, T\right), S\right) \in\left(\mathcal{S}_{n, q} \rtimes \mathrm{GL}(n, q)\right) \times \mathcal{S}_{n, q} \longmapsto T S T^{t}+S_{0} \in \mathcal{S}_{n, q} .
$$

Its orbitals are the relations of an association scheme, the so called association scheme of symmetric matrices [14, 31]. The following result enlightens a correspondence between $\mathcal{S}_{n, q}$ and $\mathcal{G}$, see also [2, Proposition 9.5.10].

Lemma 3.1. There is a bijection between $\mathcal{S}_{n, q}$ and $\mathcal{G}$ such that $\mathcal{S}_{n, q} \rtimes \operatorname{GL}(n, q)$ acts on $\mathcal{S}_{n, q}$ as $G$ acts on $\mathcal{G}$. In particular, a d-code of $\mathcal{S}_{n, q}$ corresponds to a set of generators of $\mathcal{W}(2 n-1, q)$ disjoint from $\Pi_{1}$ pairwise intersecting in at most an $(n-d-1)$-space, and conversely.

Proof. Let $S \in \mathcal{S}_{n, q}$. Since the rank of the matrix $\left(\begin{array}{cc}I_{n} & S \\ 0_{n} & I_{n}\end{array}\right)$ is $2 n$, it follows that $L(S)$ is disjoint from $\Pi_{1}$. The map $S \mapsto L(S)$ is injective. Moreover $\left|\mathcal{S}_{n, q}\right|=|\mathcal{G}|$ and $L(S)$ is a generator of $\mathcal{W}(2 n-1, q)$, indeed

$$
\left(\begin{array}{ll}
I_{n} & S
\end{array}\right)\left(\begin{array}{cc}
0_{n} & I_{n} \\
-I_{n} & 0_{n}
\end{array}\right)\binom{I_{n}}{S}=0 .
$$

Finally, let $g \in G$ be represented by the matrix (3.1). Then $L(S)^{g}=\left\langle\left(T^{-1} \quad S T^{t}+T^{-1} S_{0}\right)\right\rangle=$ $\left\langle\left(I_{n} \quad T S T^{t}+S_{0}\right)\right\rangle=L\left(T S T^{t}+S_{0}\right)$. This completes the proof of the first part of the statement. Let $\mathcal{C}$ be a $d$-code of $\mathcal{S}_{n, q}$ and let $S_{1}$ and $S_{2}$ be two different elements of $\mathcal{C}$. Since $\operatorname{rk}\left(S_{1}-S_{2}\right) \geq d$, it follows that $L\left(S_{1}\right) \cap L\left(S_{2}\right)$ is at most an $(n-d-1)$-space.

Let $\Pi_{2}=L\left(0_{n}\right)$. The previous lemma implies that the number of orbits of $G_{\Pi_{2}}$ on $\mathcal{G}$ equals the number of relations of the association scheme on symmetric matrices. Since $|\mathcal{G}|=q^{n(n+1) / 2}$ and $G$ acts transitively on $\mathcal{G}$, it follows that the stabilizer of $\Pi_{2}$ in $G$, namely $G_{\Pi_{2}}$, has order

$$
\frac{\prod_{i=0}^{n-1}\left(q^{n}-q^{i}\right)}{\operatorname{gcd}(2, q-1)}
$$

More precisely an element of $G_{\Pi_{2}}$ is represented by the matrix

$$
\left(\begin{array}{cc}
T^{-t} & 0_{n}  \tag{3.2}\\
0_{n} & T
\end{array}\right)
$$

where $T \in \operatorname{GL}(n, q)$. The group $G_{\Pi_{2}}$ acts transitively on points and hyperplanes of both $\Pi_{1}$ and $\Pi_{2}$. The action of the group $G_{\Pi_{2}}$ on points of $\mathcal{W}(2 n-1, q)$ has been studied in [16, p. 347]. For the sake of completeness a direct proof is given below.

Lemma 3.2. The orbits of $G_{\Pi_{2}}$ on points of $\mathcal{W}(2 n-1, q) \backslash\left(\Pi_{1} \cup \Pi_{2}\right)$ are

- $\mathcal{P}_{0}$ of size $\left(q^{n}-1\right)\left(q^{n-1}-1\right) /(q-1)$,
- $\mathcal{P}_{1}$ of size $q^{n-1}\left(q^{n}-1\right)$,
if $q$ is even and
- $\mathcal{P}_{0}$ of size $\left(q^{n}-1\right)\left(q^{n-1}-1\right) /(q-1)$,
- $\mathcal{P}_{1}, \mathcal{P}_{2}$, both of size $q^{n-1}\left(q^{n}-1\right) / 2$,
if $q$ is odd.
Proof. Let $g$ be the projectivity of $G_{\Pi_{2}}$ associated with the matrix (3.2), for some $T \in \mathrm{GL}(n, q)$. Then $g$ stabilizes $U_{n+1}$ if and only if the first column of $T$ is $(z, 0, \ldots, 0)^{t}$, for some $z \in \mathrm{GF}(q) \backslash\{0\}$, which is equivalent to the requirement that the first row of $T^{-t}$ is $\left(z^{-1}, 0, \ldots, 0\right)$, for some $z \in \mathrm{GF}(q) \backslash\{0\}$. It follows that $\operatorname{Stab}_{G_{\Pi_{2}}}\left(U_{n+1}\right)$ has two orbits on points of $\Pi_{2}$, namely $U_{n+1}^{\perp} \cap \Pi_{2}$ and $\Pi_{2} \backslash U_{n+1}^{\perp}$. Hence $G_{\Pi_{2}}$ permutes in a single orbit the lines of $\mathcal{W}(2 n-1, q)$ meeting both $\Pi_{1}$, $\Pi_{2}$ in a point. Similarly the lines not of $\mathcal{W}(2 n-1, q)$ meeting both $\Pi_{1}, \Pi_{2}$ in a point form a unique $G_{\Pi_{2}}$-orbit.

Let $P=U_{2}+U_{n+1}$. A projectivity of $G_{\Pi_{2}}$ stabilizing $P$ has to fix both $U_{2}$ and $U_{n+1}$. Straightforward calculations show that a member of $G_{\Pi_{2}}$ fixes $P$ if and only if it is associated with the matrix (3.2), where

$$
T=\left(\begin{array}{ccccc}
x & * & * & \ldots & * \\
0 & x^{-1} & 0 & \ldots & 0 \\
0 & * & & & \\
\vdots & \vdots & & T^{\prime} & \\
0 & * & & &
\end{array}\right)
$$

for some $T^{\prime} \in \mathrm{GL}(n-2, q)$ and $x \in \operatorname{GF}(q) \backslash\{0\}$. Therefore $\left|\operatorname{Stab}_{G_{\Pi_{2}}}(P)\right|=(q-1)\left(q^{n-1}-\right.$ 1) $\left(q^{n-2}-1\right)|\mathrm{GL}(n-2, q)| / g c d(2, q-1)$. Hence $\left|P^{G_{\Pi_{2}}}\right|=\left(q^{n}-1\right)\left(q^{n-1}-1\right) /(q-1)$, which equals the number of points of $\mathcal{W}(2 n-1, q) \backslash\left(\Pi_{1} \cup \Pi_{2}\right)$ that lie on the lines of $\mathcal{W}(2 n-1, q)$ meeting both $\Pi_{1}, \Pi_{2}$ in one point.

Let $z \in \operatorname{GF}(q) \backslash\{0\}$ and let $P=U_{1}+z U_{n+1}$. As before, a projectivity of $G_{\Pi_{2}}$ stabilizing $P_{z}$ has to fix both $U_{1}$ and $U_{n+1}$. Straightforward calculations show that the projectivity $g \in G_{\Pi_{2}}$ fixes the line $U_{1} U_{n+1}$ if and only if it is associated with the matrix (3.2), where

$$
T=\left(\begin{array}{cccc}
y & 0 & \ldots & 0 \\
0 & & & \\
\vdots & & T^{\prime \prime} & \\
0 & & &
\end{array}\right)
$$

for some $T^{\prime \prime} \in \mathrm{GL}(n-1, q)$ and $y \in \mathrm{GF}(q) \backslash\{0\}$. Moreover $g$ stabilizes $P_{z}$ if and only if $y= \pm 1$. In this case we have that $\left|\operatorname{Stab}_{G_{\Pi_{2}}}(P)\right|=|\mathrm{GL}(n-1, q)|$. Hence if $q$ is even, $\left|P_{z}^{G_{\Pi_{2}}}\right|=q^{n-1}\left(q^{n}-1\right)$, which equals the number of points of $\mathcal{W}(2 n-1, q) \backslash\left(\Pi_{1} \cup \Pi_{2}\right)$ that lie on the lines not belonging to $\mathcal{W}(2 n-1, q)$ and meeting both $\Pi_{1}, \Pi_{2}$ in one point. If $q$ is odd, then $\left|P_{z}^{G_{\Pi_{2}}}\right|=q^{n-1}\left(q^{n}-1\right) / 2$. Representatives for these two orbits are $P_{z_{1}}$ and $P_{z_{2}}$, where $z_{1}$ is a non-zero square in $\operatorname{GF}(q)$ and $z_{2}$ is a non-square of $\mathrm{GF}(q)$. Indeed there is no element of $G_{\Pi_{2}}$ sending $P_{z_{1}}$ to $P_{z_{2}}$. To see this fact assume on the contrary that there is a projectivity of $G_{\Pi_{2}}$ mapping $P_{z_{1}}$ to $P_{z_{2}}$. Then it has to fix the line $U_{1} U_{n+1}$. On the other hand such a projectivity sends the point $P_{z_{1}}$ to $P_{y^{2} z_{1}}$. Hence $z_{2}=y^{2} z_{1}$, a contradiction.

Remark 3.3. Note that if $q$ is odd and $\ell$ is a line such that $\ell \cap \Pi_{2}=P_{2}=\left(x_{1}, \ldots, x_{n}, 0, \ldots, 0\right)$, $\ell \cap \Pi_{1}=P_{1}=\left(0, \ldots, 0, y_{n+1}, \ldots, y_{2 n}\right)$ and $\ell$ is not a line of $\mathcal{W}(2 n-1, q)$, then the point $P=P_{1}+z P_{2} \in \ell$ belongs to $\mathcal{P}_{1}$ or to $\mathcal{P}_{2}$, according as $z\left(x_{1}, \ldots, x_{n}\right)\left(y_{n+1}, \ldots, y_{2 n}\right)^{t}$ is a nonzero square or a non-square in $\operatorname{GF}(q)$. Therefore $\left|\ell \cap \mathcal{P}_{1}\right|=\left|\ell \cap \mathcal{P}_{2}\right|=(q-1) / 2$. Moreover, it can be checked that there are projectivities of $\operatorname{Stab}_{\operatorname{PSp}(2 n-1, q)}\left(\left\{\Pi_{1}, \Pi_{2}\right\}\right) \backslash G_{\Pi_{2}}$ interchanging the two orbits and hence $\operatorname{Stab}_{\operatorname{PSp}(2 n-1, q)}\left(\left\{\Pi_{1}, \Pi_{2}\right\}\right)$ acts transitively on points of $\mathcal{W}(2 n-1, q) \backslash\left(\Pi_{1} \cup \Pi_{2}\right)$.

From Lemma 3.1, $\Pi_{3}$ is a generator of $\mathcal{W}(2 n-1, q)$ disjoint from both $\Pi_{1}$ and $\Pi_{2}$ if and only if $\Pi_{3}=L(A)$, where $A$ is an invertible matrix of $\mathcal{S}_{n, q}$. Hence there are

$$
q^{\frac{n(n+1)}{2}}-\left\lceil\frac{n}{2}\right\rceil^{2} \prod_{i=1}^{\left\lceil\frac{n}{2}\right\rceil}\left(q^{2 i-1}-1\right)
$$

generators of $\mathcal{W}(2 n-1, q)$ disjoint from both $\Pi_{1}$ and $\Pi_{2}$, see [19, Theorem 2], [17, Corollary 19]. The following result is well known. For the convenience of the reader a direct proof is provided.

Lemma 3.4 (Theorem 21, [17). Let $\Pi_{3}$ be a generator of $\mathcal{W}(2 n-1, q)$ disjoint from $\Pi_{1}$ and $\Pi_{2}$. The points $P \in \Pi_{3}$ such that there exists a line of $\mathcal{W}(2 n-1, q)$ through $P$ intersecting $\Pi_{1}$ and $\Pi_{2}$ are the absolute points of a non-degenerate polarity which is

- pseudo-symplectic if $q$ is even and $n$ is odd,
- orthogonal if $q$ and $n$ are odd,
- symplectic or pseudo-symplectic if $q$ and $n$ are even,
- elliptic orthogonal or hyperbolic orthogonal if $q$ is odd and $n$ is even.

Proof. Let $\Pi_{3}=L(A)$ be a generator of $\mathcal{W}(2 n-1, q)$ such that $\left|\Pi_{1} \cap \Pi_{3}\right|=\left|\Pi_{2} \cap \Pi_{3}\right|=0$. We show that there is a non-degenerate polarity $\rho$ of $\Pi_{3}$ associated with the matrix $A$. Observe that the ( $n-1$ )-space $\Pi_{3}$ has equations:

$$
\left(\begin{array}{c}
X_{n+1} \\
\vdots \\
X_{2 n}
\end{array}\right)=A\left(\begin{array}{c}
X_{1} \\
\vdots \\
X_{n}
\end{array}\right)
$$

Hence the point $P$ belongs to $\Pi_{3}$ if and only if $P=\left(x_{1}, \ldots, x_{n}, 0, \ldots, 0\right)+\left(0, \ldots, 0, x_{1}, \ldots, x_{n}\right)\left(0_{n} \quad A\right)^{t}$ and $P$ lies on the line $\ell$ joining the points $\left(0, \ldots, 0, x_{1}, \ldots, x_{n}\right)\left(0_{n} A\right)^{t} \in \Pi_{1}$ and $\left(x_{1}, \ldots, x_{n}, 0, \ldots, 0\right) \in$ $\Pi_{2}$. Thus $\ell^{\perp}$ is represented by the equations: $x_{1} X_{n+1}+\ldots+x_{n} X_{2 n}=\left(x_{1}, \ldots, x_{n}\right) A\left(X_{1}, \ldots, X_{n}\right)^{t}=$ 0 and a point $P^{\prime}=\left(y_{1}, \ldots, y_{n}, 0, \ldots, 0\right)+\left(0, \ldots, 0, y_{1}, \ldots, y_{n}\right)\left(0_{n} \quad A\right)^{t} \in \Pi_{3}$ belongs to $\ell^{\perp}$ if and only if

$$
\left(y_{1}, \ldots, y_{n}\right) A\left(\begin{array}{c}
x_{1} \\
\vdots \\
x_{n}
\end{array}\right)=0
$$

This concludes the proof.
Lemma 3.5. The group $G_{\Pi_{2}}$ has the following orbits on generators of $\mathcal{W}(2 n-1, q)$ disjoint from both $\Pi_{1}$ and $\Pi_{2}$ :

- one orbit if $q$ is even and $n$ is odd,
- two equally sized orbits if $q$ and $n$ are odd,
- two orbits having size

$$
q^{\frac{n(n-2)}{4}} \prod_{i=1}^{\frac{n}{2}}\left(q^{2 i-1}-1\right) \quad \text { and } \quad q^{\frac{n(n-2)}{4}}\left(q^{n}-1\right) \prod_{i=1}^{\frac{n}{2}}\left(q^{2 i-1}-1\right)
$$

if $q$ and $n$ are even,

- two orbits having size

$$
\frac{q^{\frac{n^{2}}{4}}\left(q^{\frac{n}{2}}+1\right) \prod_{i=1}^{\frac{n}{2}}\left(q^{2 i-1}-1\right)}{2} \quad \text { and } \quad \frac{q^{\frac{n^{2}}{4}}\left(q^{\frac{n}{2}}-1\right) \prod_{i=1}^{\frac{n}{2}}\left(q^{2 i-1}-1\right)}{2}
$$

if $q$ is odd and $n$ is even.

Proof. Let $\Pi_{3}=L(A)$ be a generator of $\mathcal{W}(2 n-1, q)$ such that $\left|\Pi_{1} \cap \Pi_{3}\right|=\left|\Pi_{2} \cap \Pi_{3}\right|=0$ and let $g$ be the projectivity of $G_{\Pi_{2}}$ associated with the matrix (3.2), for some $T \in \mathrm{GL}(n, q)$. From the proof of Lemma 3.2, $\Pi_{3}^{g}=L\left(T A T^{t}\right)$. Therefore $g$ stabilizes $\Pi_{3}$ if and only if $T A T^{t}=A$. It follows that [12, Appendix I]
$\left|\operatorname{Stab}_{G_{\Pi_{2}}}\left(\Pi_{3}\right)\right|= \begin{cases}q^{\frac{(n-1)^{2}}{4}} \prod_{i=1}^{\frac{n-1}{2}}\left(q^{2 i}-1\right) & \text { for } n \text { odd, } \\ q^{\frac{n^{2}}{4}} \prod_{i=1}^{2}\left(q^{2 i}-1\right) & \text { for } q, n \text { even, } a_{i j}=a_{j i}, a_{i i}=0, \\ q^{\frac{n^{2}}{4}} \prod_{i=1}^{\frac{n-2}{2}}\left(q^{2 i}-1\right) & \text { for } q, n \text { even, } a_{i j}=a_{j i}, a_{i i} \neq 0, \text { for some } i, \\ q^{\frac{n(n-2)}{4}}\left(q^{\frac{n}{2}}-1\right) \prod_{i=1}^{\frac{n-2}{2}}\left(q^{2 i}-1\right) & \text { for } q \operatorname{odd}, n \text { even, } \operatorname{det}(A) \text { square of } \operatorname{GF}(q) \backslash\{0\}, \\ q^{\frac{n(n-2)}{4}}\left(q^{\frac{n}{2}}+1\right) \prod_{i=1}^{\frac{n-2}{2}}\left(q^{2 i}-1\right) & \text { for } q \operatorname{odd}, n \text { even, } \operatorname{det}(A) \text { non-square of } \operatorname{GF}(q),\end{cases}$
where $A=\left(a_{i j}\right)$. The result follows.
Remark 3.6. We remark that if $q$ and $n$ are odd, then the generator $\Pi_{3}=L(A), A \in \mathcal{S}_{n, q}$, $\operatorname{rk}(A)=n$, belongs to the first or the second $G_{\Pi_{2}}$-orbit on generators of $\mathcal{W}(2 n-1, q)$ skew to $\Pi_{1}, \Pi_{2}$, according as $\operatorname{det}(A)$ is a square or a non-square in $\operatorname{GF}(q)$. It can be easily seen that there are projectivities of $\operatorname{Stab}_{\mathrm{PSp}(2 n-1, q)}\left(\left\{\Pi_{1}, \Pi_{2}\right\}\right) \backslash G_{\Pi_{2}}$ interchanging the two orbits. Hence $S_{t a b_{\operatorname{PSp}(2 n-1, q)}}\left(\left\{\Pi_{1}, \Pi_{2}\right\}\right)$ acts transitively on generators of $\mathcal{W}(2 n-1, q)$ disjoint from $\Pi_{1}$ and $\Pi_{2}$.

## $3.1 \mathcal{W}(5, q)$

Set $n=3$. Let $\mathcal{W}(5, q)$ be the symplectic polar space of $\operatorname{PG}(5, q)$ as described above. Recall that $\mathcal{G}$ is the set of $q^{6}$ planes of $\mathcal{W}(5, q)$ that are disjoint from $\Pi_{1}$, the group $G$ is the stabilizer of $\Pi_{1}$ in $\operatorname{PSp}(6, q), \Pi_{2}=L\left(0_{3}\right)$ and $G_{\Pi_{2}}$ is the stabilizer of $\Pi_{2}$ in $G$.

Following Lemma 3.2, let $\mathcal{P}_{0}$ be the set of points $R$ of $\mathcal{W}(5, q) \backslash\left(\Pi_{1} \cup \Pi_{2}\right)$ such that the line through $R$ intersecting $\Pi_{1}$ and $\Pi_{2}$ is a line of $\mathcal{W}(5, q)$ and let $\mathcal{P}$ be its complement in $\mathcal{W}(5, q) \backslash\left(\Pi_{1} \cup \Pi_{2}\right)$. Note that $\mathcal{P}$ coincides with $\mathcal{P}_{1}$ or $\mathcal{P}_{1} \cup \mathcal{P}_{2}$, according as $q$ is even or odd. Then $\left|\mathcal{P}_{0}\right|=\left(q^{2}-1\right)\left(q^{2}+q+1\right)$ and $|\mathcal{P}|=q^{5}-q^{2}$. Let $\ell$ be a line of $\mathcal{W}(5, q)$ disjoint from $\Pi_{1} \cup \Pi_{2}$. The hyperplane $\left\langle\Pi_{2}, \ell\right\rangle$ meets $\Pi_{1}$ in a line, say $r_{\ell}$, and the three-space $\left\langle\ell, r_{\ell}\right\rangle$ meets $\Pi_{2}$ in a line, say $t_{\ell}$. Hence the line $\ell$ defines a unique three-space $T_{\ell}=\left\langle r_{\ell}, t_{\ell}\right\rangle$ meeting both $\Pi_{1}, \Pi_{2}$ in a line.

Lemma 3.7. Let $\ell$ be a line of $\mathcal{W}(5, q)$ disjoint from $\Pi_{1} \cup \Pi_{2}$, then $\left|\ell \cap \mathcal{P}_{0}\right|$ belongs to $\{1, q+1\}$, if $q$ is even, and to $\{0,1,2\}$, if $q$ is odd.

Proof. There are two possibilities, either $T_{\ell}^{\perp}$ is a line of $\mathcal{W}(5, q)$ or it is not. In the former case, among the $q+1$ lines meeting $\ell, r_{\ell}, t_{\ell}$ in one point, there is exactly one line of $\mathcal{W}(5, q)$. If the latter case occurs, then $T_{\ell} \cap \mathcal{W}(5, q)$ is a $\mathcal{W}(3, q)$ and the regulus $\mathcal{R}$ determined by $\ell, r_{\ell}, t_{\ell}$ consists of lines of $\mathcal{W}(3, q)$. Thus its opposite regulus consists of either 1 or $q+1$ lines of $\mathcal{W}(3, q)$ if $q$ is even and of 0 or 2 lines of $\mathcal{W}(3, q)$ if $q$ is odd.

Let us partition the set of lines of $\mathcal{W}(5, q)$ disjoint from $\Pi_{1} \cup \Pi_{2}$. Let

$$
\left.\begin{array}{l}
\mathcal{L}_{0}=\left\{\ell \text { line of } \mathcal{W}(5, q):\left|\ell \cap\left(\Pi_{1} \cup \Pi_{2}\right)\right|=0,\left|\ell \cap \mathcal{P}_{0}\right|=q+1\right\}, \\
\mathcal{L}_{1}=\left\{\ell \text { line of } \mathcal{W}(5, q):\left|\ell \cap\left(\Pi_{1} \cup \Pi_{2}\right)\right|=0, T_{\ell}^{\perp} \text { is a line of } \mathcal{W}(5, q)\right\}, \\
\mathcal{L}_{2}=\{\ell \text { line of } \mathcal{W}(5, q)
\end{array}:\left|\ell \cap\left(\Pi_{1} \cup \Pi_{2}\right)\right|=0,\left|\ell \cap \mathcal{P}_{0}\right|=1, T_{\ell}^{\perp} \text { is not a line of } \mathcal{W}(5, q)\right\}, ~ l
$$

if $q$ is even, or

$$
\left.\begin{array}{l}
\mathcal{L}_{0}=\left\{\ell \text { line of } \mathcal{W}(5, q):\left|\ell \cap\left(\Pi_{1} \cup \Pi_{2}\right)\right|=0,\left|\ell \cap \mathcal{P}_{0}\right|=0, T_{\ell}^{\perp} \text { is not a line of } \mathcal{W}(5, q)\right\}, \\
\mathcal{L}_{1}=\left\{\ell \text { line of } \mathcal{W}(5, q):\left|\ell \cap\left(\Pi_{1} \cup \Pi_{2}\right)\right|=0, T_{\ell}^{\perp} \text { is a line of } \mathcal{W}(5, q)\right\}, \\
\mathcal{L}_{2}=\{\ell \text { line of } \mathcal{W}(5, q)
\end{array}:\left|\ell \cap\left(\Pi_{1} \cup \Pi_{2}\right)\right|=0,\left|\ell \cap \mathcal{P}_{0}\right|=2, T_{\ell}^{\perp} \text { is not a line of } \mathcal{W}(5, q)\right\}, ~ l
$$

if $q$ is odd. Note that in both cases if $\ell \in \mathcal{L}_{1}$, then $\left|\ell \cap \mathcal{P}_{0}\right|=1$, whereas if $q$ is even and $\ell \in \mathcal{L}_{0}$, then $T_{\ell}^{\perp}$ is not a line of $\mathcal{W}(5, q)$.

Lemma 3.8. If $q$ is even, then

$$
\left|\mathcal{L}_{0}\right|=q^{2}\left(q^{3}-1\right), \quad\left|\mathcal{L}_{1}\right|=q\left(q^{2}-1\right)\left(q^{3}-1\right), \quad\left|\mathcal{L}_{2}\right|=q^{2}\left(q^{2}-1\right)\left(q^{3}-1\right) .
$$

If $q$ is odd, then

$$
\left|\mathcal{L}_{0}\right|=\frac{q^{3}(q-1)\left(q^{3}-1\right)}{2}, \quad\left|\mathcal{L}_{1}\right|=q\left(q^{2}-1\right)\left(q^{3}-1\right), \quad\left|\mathcal{L}_{2}\right|=\frac{q^{3}(q+1)\left(q^{3}-1\right)}{2} .
$$

Proof. The line $T_{\ell}^{\perp}$ meets both $\Pi_{1}, \Pi_{2}$ in one point. If $T_{\ell}^{\perp}$ is a line of $\mathcal{W}(5, q)$, then $T_{\ell} \cap \mathcal{W}(5, q)$ consists of $q+1$ generators of $\mathcal{W}(5, q)$ through $T_{\ell}^{+}$and hence there are $q(q-1)^{2}$ lines of $\mathcal{W}(5, q)$ contained in $T_{\ell}$ and disjoint from $\Pi_{1} \cup \Pi_{2}$. Since there are $(q+1)\left(q^{2}+q+1\right)$ lines of $\mathcal{W}(5, q)$ meeting both $\Pi_{1}, \Pi_{2}$ in one point, we get $\left|\mathcal{L}_{1}\right|=q\left(q^{2}-1\right)\left(q^{3}-1\right)$. If $T_{\ell}^{\perp}$ is not a line of $\mathcal{W}(5, q)$, then $T_{\ell} \cap \mathcal{W}(5, q)$ is a non-degenerate symplectic polar space $\mathcal{W}(3, q)$ and the regulus $\mathcal{R}$ determined by $\ell, r_{\ell}, t_{\ell}$ is a regulus of $\mathcal{W}(3, q)$. The point line dual of $\mathcal{W}(3, q)$ is a parabolic quadric $\mathcal{Q}(4, q)$ and the lines $r_{\ell}$ and $t_{\ell}$ correspond to two points $R, T$ such that the line $R T$ meets $\mathcal{Q}(4, q)$ only in $R$ and $T$. Moreover, the regulus $\mathcal{R}$ corresponds to a conic $C$ of $\mathcal{Q}(4, q)$, where $R, T \in C$.

Assume that $q$ is even, then $\ell$ belongs either to $\mathcal{L}_{0}$ or to $\mathcal{L}_{2}$, according as the opposite regulus of $\mathcal{R}$ has $q+1$ or one line of $\mathcal{W}(3, q)$. In this case the parabolic quadric $\mathcal{Q}(4, q)$ has a nucleus, say $N$. Moreover, the opposite regulus of $\mathcal{R}$ has $q+1$ or one line of $\mathcal{W}(3, q)$ according as $N$ belongs to the plane $\langle C\rangle$ or does not. Therefore, in $\mathcal{W}(3, q), \ell$ can be chosen in $q-1$ ways such that it belongs to $\mathcal{L}_{0}$ and in $\left(q^{2}-1\right)(q-1)$ ways such that it belongs to $\mathcal{L}_{2}$. Since there are $q^{2}\left(q^{2}+q+1\right)$ lines not of $\mathcal{W}(5, q)$ meeting both $\Pi_{1}, \Pi_{2}$ in one point, we get $\left|\mathcal{L}_{0}\right|=q^{2}\left(q^{3}-1\right)$ and $\left|\mathcal{L}_{2}\right|=q^{2}\left(q^{2}-1\right)\left(q^{3}-1\right)$.

If $q$ is odd, then $\ell$ belongs either to $\mathcal{L}_{0}$ or to $\mathcal{L}_{2}$, according the opposite regulus of $\mathcal{R}$ has 0 or 2 lines of $\mathcal{W}(3, q)$. In this case the opposite regulus of $\mathcal{R}$ has 0 or 2 lines of $\mathcal{W}(3, q)$ according as the polar of $\langle C\rangle$ with respect to the orthogonal polarity of $\mathcal{Q}(4, q)$ is a line external or secant to $\mathcal{Q}(4, q)$. Therefore, in $\mathcal{W}(3, q), \ell$ can be chosen in $q(q-1)^{2} / 2$ ways such that it
belongs to $\mathcal{L}_{0}$ and in $\left(q^{3}-q\right) / 2$ ways such that it belongs to $\mathcal{L}_{2}$. Since there are $q^{2}\left(q^{2}+q+1\right)$ lines not of $\mathcal{W}(5, q)$ meeting both $\Pi_{1}, \Pi_{2}$ in one point, we get $\left|\mathcal{L}_{0}\right|=q^{3}(q-1)\left(q^{3}-1\right) / 2$ and $\left|\mathcal{L}_{2}\right|=q^{3}(q+1)\left(q^{3}-1\right) / 2$.

Lemma 3.9. Let $\Pi_{3}$ be a plane of $\mathcal{W}(5, q)$ skew to $\Pi_{1}$ and $\Pi_{2}$. The $q+1$ planes of the Segre variety $\Sigma_{1,2}$ of $\mathrm{PG}(5, q)$ determined by $\Pi_{1}, \Pi_{2}, \Pi_{3}$ are generators of $\mathcal{W}(5, q)$.

Proof. Let $A$ be an invertible matrix of $\mathcal{S}_{3, q}$ and consider the symplectic Segre variety $\Sigma_{1,2}$ determined by the planes $\Pi_{1}, \Pi_{2}$ and $L(A)$. Direct computations show that the remaining $q-2$ planes of $\Sigma_{1,2}$ are the planes $L(\lambda A)$, where $\lambda \in \operatorname{GF}(q) \backslash\{0,1\}$.

We will refer to a Segre variety $\Sigma_{1,2}$ of $\operatorname{PG}(5, q)$ whose $q+1$ planes are generators of $\mathcal{W}(5, q)$ as a symplectic Segre variety of $\mathcal{W}(5, q)$. As a consequence of Lemma 3.4, the following corollary arises.

Corollary 3.10. Let $\Sigma_{1,2}$ be a symplectic Segre variety containing $\Pi_{1}, \Pi_{2}$ and let $\Pi_{3}$ be a plane of $\Sigma_{1,2}, \Pi_{3} \neq \Pi_{1}, \Pi_{3} \neq \Pi_{2}$. Then $\Pi_{3}$ contains one line of $\mathcal{L}_{0}, q+1$ lines of $\mathcal{L}_{1}$ and $q^{2}-1$ lines of $\mathcal{L}_{2}$, if $q$ is even, and $q(q-1) / 2$ lines of $\mathcal{L}_{0}, q+1$ lines of $\mathcal{L}_{1}$ and $q(q+1) / 2$ lines of $\mathcal{L}_{2}$, if $q$ is odd.

Proof. From Lemma 3.4, there is a non-degenerate polarity $\rho$ of $\Pi_{3}$. Note that if $\ell$ is a line of $\Pi_{3}$, then $\ell^{\rho}=T_{\ell}^{\perp} \cap \Pi_{3}$. If $q$ is even, $\rho$ is a pseudo-polarity and the unique line of $\Pi_{3}$ belonging to $\mathcal{L}_{0}$ is the line $\ell_{0}$ consisting of its absolute points. The other lines of $\Pi_{3}$ belong to $\mathcal{L}_{1}$ if they pass through $\ell_{0}^{\rho}$ and to $\mathcal{L}_{2}$ otherwise. If $q$ is odd, $\rho$ is an orthogonal polarity and its absolute points form a conic, say $C$. A line of $\Pi_{3}$ belongs either to $\mathcal{L}_{1}$, or to $\mathcal{L}_{0}$ or to $\mathcal{L}_{2}$ according as it is tangent, external or secant to $\mathcal{C}$, respectively.

For a point $P \in \Pi_{2}$, let $\Sigma_{P}$ denote a 3 -space contained in $P^{\perp}$ and not containing $P$. When restricted to $\Sigma_{P}$, the polarity $\perp$ defines a non-degenerate symplectic polar space of $\Sigma_{P}$, say $\mathcal{W}_{P}$. Moreover $r_{P}=\Sigma_{P} \cap \Pi_{1}$ and $t_{P}=\Sigma_{P} \cap \Pi_{2}$ are lines of $\mathcal{W}_{P}$. In what follows we investigate the action of the group $G_{\Pi_{2}}$ on $\mathcal{G}$.

Lemma 3.11. The group $G_{\Pi_{2}}$ has the following orbits on $\mathcal{G}$ :

- the plane $\Pi_{2}$;
- $\mathcal{G}_{1}$ of size $q^{3}-1$ consisting of the planes of $\mathcal{G}$ meeting $\Pi_{2}$ in a line,
- $\mathcal{G}_{2}$ of size $q^{3}-1$ consisting of the planes of $\mathcal{G}$ meeting $\Pi_{2}$ in a point and no plane of $\mathcal{G}_{1}$ in a line,
- $\mathcal{G}_{3}$ of size $\left(q^{2}-1\right)\left(q^{3}-1\right)$ consisting of the planes of $\mathcal{G}$ meeting $\Pi_{2}$ in a point and $q$ planes of $\mathcal{G}_{1}$ in a line,
- $\mathcal{G}_{4}$ of size $q^{2}\left(q^{3}-1\right)(q-1)$ consisting of the planes of $\mathcal{G}$ disjoint from $\Pi_{2}$,
if $q$ is even and
- the plane $\Pi_{2}$;
- $\mathcal{G}_{1}$ of size $\left(q^{3}-1\right) / 2$ consisting of the planes of $\mathcal{G}$ meeting $\Pi_{2}$ in a line and having $q^{2}$ points of $\mathcal{P}_{1}$,
- $\mathcal{G}_{2}$ of size $\left(q^{3}-1\right) / 2$ consisting of the planes of $\mathcal{G}$ meeting $\Pi_{2}$ in a line and having $q^{2}$ points of $\mathcal{P}_{2}$,
- $\mathcal{G}_{3}$ of size $q(q-1)\left(q^{3}-1\right) / 2$ consisting of the planes of $\mathcal{G}$ meeting $\Pi_{2}$ in a point and $q+1$ planes of $\mathcal{G}_{1} \cup \mathcal{G}_{2}$ in a line,
- $\mathcal{G}_{4}$ of size $q(q+1)\left(q^{3}-1\right) / 2$ consisting of the planes of $\mathcal{G}$ meeting $\Pi_{2}$ in a point and $q-1$ planes of $\mathcal{G}_{1} \cup \mathcal{G}_{2}$ in a line,
- two orbits, say $\mathcal{G}_{5}$ and $\mathcal{G}_{6}$, both of size $q^{2}\left(q^{3}-1\right)(q-1) / 2$, consisting of planes of $\mathcal{G}$ disjoint from $\Pi_{2}$,
if $q$ is odd.
Proof. There are $q^{3}-1$ members of $\mathcal{G}$ intersecting $\Pi_{2}$ in a line. The number of generators of $\mathcal{W}(5, q)$ through $P$ disjoint from $\Pi_{1}$ and intersecting $\Pi_{2}$ exactly in $P$ equals the number of lines of $\mathcal{W}_{P}$ disjoint from both $r_{P}$ and $t_{P}$, and they are $q^{2}(q-1)$. As the point $P$ varies on $\Pi_{2}$ we get $q^{2}\left(q^{3}-1\right)$ generators of $\mathcal{W}(5, q)$ disjoint from $\Pi_{1}$ and intersecting $\Pi_{2}$ at exactly one point. Hence there are $q^{2}(q-1)\left(q^{3}-1\right)$ generators of $\mathcal{W}(5, q)$ disjoint from both $\Pi_{1}$ and $\Pi_{2}$. Alternatively, if $A \in \mathcal{S}_{3, q}$, then $L(A) \cap \Pi_{2}$ is a $(3-\operatorname{rk}(A)-1)$-space of $\Pi_{1}$.

Let $\pi$ be a plane of $\mathcal{G}$ intersecting $\Pi_{2}$ in a line. Since $G_{\Pi_{2}}$ is transitive on lines of $\Pi_{2}$, we may assume without loss of generality that $\pi \cap \Pi_{2}$ is the line $U_{2} U_{3}$. Then $\pi: X_{5}=X_{6}=0, X_{4}=z X_{1}$, for some $z \in \mathrm{GF}(q) \backslash\{0\}$. Let $g$ be the projectivity of $G_{\Pi_{2}}$ associated with the matrix (3.2), for some $T \in \operatorname{GL}(3, q)$. Then $g$ stabilizes $\pi$ if and only if

$$
T=\left(\begin{array}{ccc} 
\pm 1 & * & * \\
0 & T^{\prime} & \\
0 & &
\end{array}\right)
$$

where $T^{\prime} \in \mathrm{GL}(2, q)$. Hence $\left|\operatorname{Stab}_{G_{\Pi_{2}}}(\pi)\right|=\operatorname{gcd}(2, q-1) q^{2}|\mathrm{GL}(2, q)| / g c d(2, q-1)$ and $\left|\pi^{G_{\Pi_{2}}}\right|$ equals $q^{3}-1$ if $q$ is even or $\left(q^{3}-1\right) / 2$ if $q$ is odd. In the even characteristic case the planes of $\mathcal{G}$ meeting $\Pi_{2}$ in a line are permuted in a unique orbit, say $\mathcal{G}_{1}$. In the odd characteristic case, there are two orbits, say $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$; it can be seen that representatives for $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ are $\pi_{1}: X_{5}=X_{6}=X_{4}-z_{1} X_{1}=0$ and $\pi_{2}: X_{5}=X_{6}=X_{4}-z_{2} X_{1}=0$, respectively, where $z_{1}$ is a non-zero square in $\operatorname{GF}(q)$ and $z_{2}$ is a non-square of $\operatorname{GF}(q)$. Moreover $\operatorname{Stab}_{G_{\Pi_{2}}}(\pi)$ acts transitively on the $q^{2}$ points of $\pi \backslash \Pi_{2}$.

Let $\pi$ be a plane of $\mathcal{G}$ and intersecting $\Pi_{2}$ in the point $P$ and let $\ell$ be the line of $\mathcal{W}_{P}$ obtained by intersecting $\pi$ with $\Sigma_{P}$. Let $\mathcal{R}$ be the regulus determined by $r_{P}, t_{P}, \ell$ and $\mathcal{R}^{o}$ be its opposite regulus.

Assume that $q$ is even. There are $q-1$ possibilities for the line $\ell$ such that the regulus $\mathcal{R}^{o}$ contains $q+1$ lines of $\mathcal{W}_{P}$, and $\left(q^{2}-1\right)(q-1)$ possibilities for $\ell$ such that $\mathcal{R}^{o}$ contains exactly one
line of $\mathcal{W}_{P}$. Varying $P$ in $\Pi_{2}$ we get two sets, namely $\mathcal{G}_{2}$ and $\mathcal{G}_{3}$, of size $q^{3}-1$ and $\left(q^{2}-1\right)\left(q^{3}-1\right)$, respectively. Observe that there are 0 or $q$ planes of $\mathcal{G}_{1}$ meeting $\pi$ in a line, according as $\pi$ belongs to $\mathcal{G}_{2}$ or $\mathcal{G}_{3}$. We claim that $\mathcal{G}_{2}$ and $\mathcal{G}_{3}$ are two $G_{\Pi_{2}}$-orbits. Let $\pi$ be the plane with equations $X_{2}+X_{6}=X_{3}+X_{5}=X_{4}=0$. Direct computations show that $\pi \in \mathcal{G}_{2}$. The projectivity $g$ of $G_{\Pi_{2}}$ associated with the matrix (3.2), $T \in \operatorname{GL}(3, q)$, stabilizes $\pi$ if and only if

$$
T=\left(\begin{array}{ccc}
x & 0 & 0 \\
* & T^{\prime} & \\
* & &
\end{array}\right),
$$

where $x \in \mathrm{GF}(q) \backslash\{0\}$ and $T^{\prime} \in \mathrm{SL}(2, q)$. Hence $\left|\operatorname{Stab}_{G_{\Pi_{2}}}(\pi)\right|=q^{2}(q-1)|\operatorname{SL}(2, q)|$ and $\left|\pi^{G_{\Pi_{2}}}\right|=$ $q^{3}-1=\left|\mathcal{G}_{2}\right|$. Let $\pi$ be the plane having equations $X_{2}+X_{5}=X_{3}+X_{6}=X_{4}=0$. Direct computations show that $\pi \in \mathcal{G}_{3}$. The projectivity $g$ of $G_{\Pi_{2}}$ associated with the matrix (3.2), $T \in \mathrm{GL}(3, q)$, stabilizes $\pi$ if and only if

$$
T=\left(\begin{array}{ccc}
x & 0 & 0 \\
* & y & y^{\prime} \\
* & y^{\prime} & y
\end{array}\right)
$$

where $x, y, y^{\prime} \in \operatorname{GF}(q), x \neq 0$ and $y^{2}+y^{\prime 2}=1$. Hence $\left|\operatorname{Stab}_{G_{\Pi_{2}}}(\pi)\right|=q^{3}(q-1)$ and $\left|\pi^{G_{\Pi_{2}}}\right|=$ $\left(q^{2}-1\right)\left(q^{3}-1\right)=\left|\mathcal{G}_{3}\right|$.

Assume that $q$ is odd. There are $q(q-1)^{2} / 2$ possibilities for the line $\ell$ such that the regulus $\mathcal{R}^{o}$ contains no line of $\mathcal{W}_{P}$, and $q\left(q^{2}-1\right) / 2$ possibilities for $\ell$ such that $\mathcal{R}^{o}$ contains exactly two lines of $\mathcal{W}_{P}$. Varying $P$ in $\Pi_{2}$ we get two sets, say $\mathcal{G}_{3}$ and $\mathcal{G}_{4}$ of size $q(q-1)\left(q^{3}-1\right) / 2$ and $q(q+1)\left(q^{3}-1\right) / 2$ respectively. Observe that there are $q+1$ or $q-1$ planes of $\mathcal{G}_{1} \cup \mathcal{G}_{2}$ meeting $\pi$ in a line, according as $\pi$ belongs to $\mathcal{G}_{3}$ or $\mathcal{G}_{4}$. Again we want to show that $\mathcal{G}_{3}$ and $\mathcal{G}_{4}$ are two $G_{\Pi_{2}}$-orbits. Let $\alpha$ be a fixed non-square in $\operatorname{GF}(q)$ and let $\pi$ be the plane with equations $X_{5}-\alpha^{2} X_{2}=X_{6}+\alpha X_{3}=X_{4}=0$. Direct computations show that $\pi \in \mathcal{G}_{3}$. The projectivity $g$ of $G_{\Pi_{2}}$ associated with the matrix (3.2), $T \in \mathrm{GL}(3, q)$, stabilizes $\pi$ if and only if

$$
T=\left(\begin{array}{ccc}
x & 0 & 0 \\
* & y & -\alpha y^{\prime} \\
* & y^{\prime} & -y
\end{array}\right) \text { or } T=\left(\begin{array}{ccc}
x & 0 & 0 \\
* & y & \alpha y^{\prime} \\
* & y^{\prime} & y
\end{array}\right)
$$

where $x, y, y^{\prime} \in \operatorname{GF}(q), x \neq 0$ and $y^{2}-\alpha y^{\prime 2}=1$. Note that there are $q+1$ couple $\left(y, y^{\prime}\right) \in$ $\operatorname{GF}(q) \times \operatorname{GF}(q)$ such that $y^{2}-\alpha y^{\prime 2}=1$. Therefore $\left|\operatorname{Stab}_{G_{\Pi_{2}}}(\pi)\right|=q^{2}(q-1)(q+1)$ and $\left|\pi^{G_{\Pi_{2}}}\right|=$ $q(q-1)\left(q^{3}-1\right) / 2=\left|\mathcal{G}_{3}\right|$. Let $\pi$ be the plane having equations $X_{2}-X_{6}=X_{3}-X_{5}=X_{4}=0$. Direct computations show that $\pi \in \mathcal{G}_{4}$. The projectivity $g$ of $G_{\Pi_{2}}$ associated with the matrix (3.2), $T \in \mathrm{GL}(3, q)$, stabilizes $\pi$ if and only if

$$
T=\left(\begin{array}{ccc}
x & 0 & 0 \\
* & y & 0 \\
* & 0 & y^{-1}
\end{array}\right) \text { or } T=\left(\begin{array}{ccc}
x & 0 & 0 \\
* & 0 & y \\
* & y^{-1} & 0
\end{array}\right)
$$

where $x, y \in \operatorname{GF}(q) \backslash\{0\}$. In this case $\left|\operatorname{Stab}_{G_{\Pi_{2}}}(\pi)\right|=q^{2}(q-1)^{2}$ and $\left|\pi^{G_{\Pi_{2}}}\right|=q(q+1)\left(q^{3}-1\right) / 2=$ $\left|\mathcal{G}_{4}\right|$.

From Lemma 3.5, the group $G_{\Pi_{2}}$ has one or two orbits on generators of $\mathcal{W}(5, q)$ skew to $\Pi_{1}$ and $\Pi_{2}$.

Lemma 3.12. Assume that $q$ is even. Let $\Pi \in \mathcal{G}_{2}$ and $\Pi^{\prime} \in \mathcal{G}_{3}$. Then the number of planes of $\mathcal{G}_{2}$ meeting $\Pi$ or $\Pi^{\prime}$ in a line is zero or one, whereas the number of planes of $\mathcal{G}_{3}$ meeting $\Pi$ or $\Pi^{\prime}$ in a line equals $q^{2}-1$ or $q^{2}-q-2$.

Proof. Let $P=\Pi \cap \Pi_{2}, P^{\prime}=\Pi^{\prime} \cap \Pi_{2}, \ell=\Pi \cap \Sigma_{P}, \ell^{\prime}=\Pi^{\prime} \cap \Sigma_{P^{\prime}}, \mathcal{R}$ be the regulus determined by $r_{P}, t_{P}, \ell$ and $\mathcal{R}^{\prime}$ be the regulus determined by $r_{P}^{\prime}, t_{P}^{\prime}, \ell^{\prime}$. From the proof of Lemma 3.11, the opposite regulus of $\mathcal{R}$, say $\mathcal{R}^{o}$, consists of lines of $\mathcal{W}_{P}$, whereas the opposite regulus of $\mathcal{R}^{\prime}$, say $\mathcal{R}^{\prime o}$, has exactly one line of $\mathcal{W}_{P}^{\prime}$.

If a plane $\gamma \in \mathcal{G}_{2} \cup \mathcal{G}_{3}$ intersects $\Pi$ in a line, then $\gamma=\langle P, s\rangle$, where $s$ is a line of $\mathcal{W}_{P}$ intersecting $\ell$ and skew to $r_{P}$ and $t_{P}$. Moreover, $\gamma$ belongs to $\mathcal{G}_{2}$ or $\mathcal{G}_{3}$ according as there are $q+1$ or one line of $\mathcal{W}_{P}$ meeting $s, r_{P}, t_{P}$. Since the number of lines of $\mathcal{W}_{P}$ intersecting $\ell$ at a point and skew to $r_{P}$ and $t_{P}$ equals $q^{2}-1$ and, if $s$ is one of these lines, there is exactly one line of $\mathcal{W}_{P}$ meeting $s, r_{P}, t_{P}$, the statement holds true in this case.

Similarly, if a plane $\gamma \in \mathcal{G}_{2} \cup \mathcal{G}_{3}$ intersects $\Pi^{\prime}$ in a line, then $\gamma=\left\langle P^{\prime}, s\right\rangle$, where $s$ is a line of $\mathcal{W}_{P^{\prime}}$ intersecting $\ell^{\prime}$ and skew to $r_{P}^{\prime}$ and $t_{P}^{\prime}$. Moreover, $\gamma$ belongs to $\mathcal{G}_{2}$ or $\mathcal{G}_{3}$ according as there are $q+1$ or one line of $\mathcal{W}_{P^{\prime}}$ meeting $s, r_{P}^{\prime}$, $t_{P}^{\prime}$. The point line dual of $\mathcal{W}_{P^{\prime}}$ is a parabolic quadric $\mathcal{Q}(4, q)$, the lines $r_{P}^{\prime}, t_{P}^{\prime}$ and $\ell^{\prime}$ correspond to three points, say $R, T, L$, such that the line $R T$ meets $\mathcal{Q}(4, q)$ only in $R$ and $T$. Moreover, the regulus $\mathcal{R}^{\prime}$ corresponds to a conic $C$ of $\mathcal{Q}(4, q)$, where $R, T, L \in C$. Let $N$ be the nucleus of $\mathcal{Q}(4, q)$. Then $N$ does not belong to the plane $\langle C\rangle$ and the points $R, T, N$ span a plane meeting $\mathcal{Q}(4, q)$ in a conic, say $C^{\prime}$. Hence there is a plane $\gamma \in \mathcal{G}_{2}$ meeting $\Pi^{\prime}$ in a line if and only if there is a point $U$ of $C^{\prime}$ such that the line $U L$ is a line of $\mathcal{Q}(4, q)$. There exists only one such a point: the intersection point between the three-space containing the lines of $\mathcal{Q}(4, q)$ through $L$ and the conic $C^{\prime}$. Analogously, there is a plane $\gamma \in \mathcal{G}_{3}$ meeting $\Pi^{\prime}$ in a line if and only if there is a point $U \in \mathcal{Q}(4, q)$ not belonging to $C^{\prime}$ such that the line $U L$ is a line of $\mathcal{Q}(4, q)$ and the lines $U R$ and $U T$ are not lines of $\mathcal{Q}(4, q)$. There exist exactly $q^{2}-q-2$ points having these properties.

Lemma 3.13. Assume that $q$ is odd. Let $\Pi \in \mathcal{G}_{5}$ and $\Pi^{\prime} \in \mathcal{G}_{6}$, then either $\left|\Pi \cap \mathcal{P}_{1}\right|=\left|\Pi^{\prime} \cap \mathcal{P}_{2}\right|=$ $q(q-1) / 2$ and $\left|\Pi \cap \mathcal{P}_{2}\right|=\left|\Pi^{\prime} \cap \mathcal{P}_{1}\right|=q(q+1) / 2$ or $\left|\Pi \cap \mathcal{P}_{1}\right|=\left|\Pi^{\prime} \cap \mathcal{P}_{2}\right|=q(q+1) / 2$ and $\left|\Pi \cap \mathcal{P}_{2}\right|=\left|\Pi^{\prime} \cap \mathcal{P}_{1}\right|=q(q-1) / 2$. Moreover through a line of $\mathcal{L}_{0} \cup \mathcal{L}_{2}$, there pass $(q-1) / 2$ planes of $\mathcal{G}_{5}$ and $(q-1) / 2$ planes of $\mathcal{G}_{6}$, whereas the $q$ generators passing through a line of $\mathcal{L}_{1}$ and skew to $\Pi_{1}, \Pi_{2}$ are planes either of $\mathcal{G}_{5}$ or of $\mathcal{G}_{6}$.

Proof. Let $A$ be an invertible matrix of $\mathcal{S}_{3, q}$ and consider the symplectic Segre variety $\Sigma_{1,2}$ determined by the $\Pi_{1}, \Pi_{2}$ and $L(A)$. The planes of $\Sigma_{1,2}$ distinct from $\Pi_{1}$ and $\Pi_{2}$ are $L(\lambda A)$, where $\lambda \in \operatorname{GF}(q) \backslash\{0\}$. Taking into account Remark 3.6, we have that $(q-1) / 2$ members of $\Sigma_{1,2}$ belong to $\mathcal{G}_{5}$ and $(q-1) / 2$ members of $\Sigma_{1,2}$ belong to $\mathcal{G}_{6}$. Moreover, taking into account Remark 3.3, if $\lambda$ is a non-zero square of $\mathrm{GF}(q)$, then the point of $L(A)$ given by $(x, y, z, 0,0,0)+$ $(0,0,0, x, y, z)\left(0_{3} \quad A\right)^{t}$ belongs to $\mathcal{P}_{1}$ if and only if the point of $L(\lambda A)$ given by $(x, y, z, 0,0,0)+$ $(0,0,0, x, y, z) \lambda\left(0_{3} \quad A\right)^{t}$ belongs to $\mathcal{P}_{1}$, whereas if $\lambda$ is a non-square of $\operatorname{GF}(q)$, then the point of $L(A)$ given by $(x, y, z, 0,0,0)+(0,0,0, x, y, z)\left(0_{3} \quad A\right)^{t}$ belongs to $\mathcal{P}_{1}$ if and only if the point of $L(\lambda A)$ given by $(x, y, z, 0,0,0)+(0,0,0, x, y, z) \lambda\left(0_{3} \quad A\right)^{t}$ belongs to $\mathcal{P}_{2}$. Let $\Pi=L(\lambda A) \in \mathcal{G}_{5}$,
$\Pi^{\prime}=L\left(\lambda^{\prime} A\right) \in \mathcal{G}_{6}$. From Lemma 3.4, there is a non-degenerate conic $C$ (resp. $C^{\prime}$ ) of $\Pi$ (resp. $\left.\Pi^{\prime}\right)$. Observe that exactly one of the two following possibilities occurs: either $\Pi \cap \mathcal{P}_{1}$ are the points of $\Pi$ internal to $C, \Pi \cap \mathcal{P}_{2}$ are the points of $\Pi$ external to $C, \Pi^{\prime} \cap \mathcal{P}_{2}$ are the points of $\Pi^{\prime}$ internal to $C^{\prime}, \Pi^{\prime} \cap \mathcal{P}_{1}$ are the points of $\Pi^{\prime}$ external to $C^{\prime}$, or $\Pi \cap \mathcal{P}_{1}$ are the points of $\Pi$ external to $C, \Pi \cap \mathcal{P}_{2}$ are the points of $\Pi$ internal to $C, \Pi^{\prime} \cap \mathcal{P}_{2}$ are the points of $\Pi^{\prime}$ external to $C^{\prime}, \Pi^{\prime} \cap \mathcal{P}_{1}$ are the points of $\Pi^{\prime}$ internal to $C^{\prime}$.

Let $\ell$ be a line of $\mathcal{W}(5, q)$ and let $\Pi_{3}$ be a generator of $\mathcal{W}(5, q)$ skew to $\Pi_{1}, \Pi_{2}$ such that $\ell \subset \Pi_{3}$. Denote by $\rho$ the non-degenerate polarity of $\Pi_{3}$ arising from Lemma 3.4 and let $C_{3}$ be the corresponding non-degenerate conic. If $\ell \in \mathcal{L}_{0} \cup \mathcal{L}_{2}$, then $T_{\ell}^{\perp}$ is a line meeting both $\Pi_{1}, \Pi_{2}$ in a point and it is not a line of $\mathcal{W}(5, q)$. Hence, by Remark 3.3, $T_{\ell}^{\perp}$ contains $(q-1) / 2$ points of $\mathcal{P}_{1}$ and $(q-1) / 2$ points of $\mathcal{P}_{2}$. Since $\ell^{\rho}$ is the point $\Pi_{3} \cap T_{\ell}^{+}$, we have that through $\ell$ there pass $(q-1) / 2$ planes of $\mathcal{G}_{5}$ and $(q-1) / 2$ planes of $\mathcal{G}_{6}$. If $\ell \in \mathcal{L}_{1}$, then $T_{\ell}^{\perp}$ is a line of $\mathcal{W}(5, q)$ meeting both $\Pi_{1}, \Pi_{2}$ in a point. In this case $\ell \cap C_{3}$ consists of one point, say $Q$. The $q$ points of $\ell$ distinct from $Q$ are external to $C_{3}$ and hence they all lie in a unique point-orbit, that is either $\mathcal{P}_{1}$ or $\mathcal{P}_{2}$. Therefore the $q$ generators of $\mathcal{W}(5, q)$ passing through $\ell$ and skew to $\Pi_{1}, \Pi_{2}$ are such that the external points of their corresponding conics are all points belonging to the same point orbit, that is either $\mathcal{P}_{1}$ or $\mathcal{P}_{2}$. Hence these $q$ generators lie in the same $G_{\Pi_{2}}$-orbit which contains $\Pi_{3}$.

## 4 Hermitian matrices and Hermitian polar spaces

Let $\omega \in \mathrm{GF}\left(q^{2}\right) \backslash\{0\}$ such that $\omega^{q}=-\omega$ and let $\mathcal{H}\left(2 n-1, q^{2}\right)$ be the non-degenerate Hermitian polar space of $\operatorname{PG}\left(2 n-1, q^{2}\right)$ associated with the following Hermitian form

$$
\left(X_{1}, \ldots, X_{2 n}\right)\left(\begin{array}{cc}
0_{n} & \omega I_{n} \\
\omega^{q} I_{n} & 0_{n}
\end{array}\right)\left(\begin{array}{c}
Y_{1}^{q} \\
\vdots \\
Y_{2 n}^{q}
\end{array}\right)
$$

Let $\operatorname{PGU}\left(2 n, q^{2}\right)$ be the group of projectivities of $\operatorname{PG}\left(2 n-1, q^{2}\right)$ stabilizing $\mathcal{H}\left(2 n-1, q^{2}\right)$. Denote by $\Lambda_{1}$ the $(n-1)$-space of $\operatorname{PG}\left(2 n-1, q^{2}\right)$ spanned by $U_{n+1}, \ldots, U_{2 n}$. Then $\Lambda_{1}$ is a generator of $\mathcal{H}\left(2 n-1, q^{2}\right)$. Denote by $\bar{G}$ the stabilizer of $\Lambda_{1}$ in $\operatorname{PGU}\left(2 n, q^{2}\right)$. In this case an element of $\bar{G}$ is represented by the matrix

$$
\left(\begin{array}{cc}
T^{-t} & 0_{n}  \tag{4.1}\\
H_{0}^{q} T^{-t} & T^{q}
\end{array}\right)
$$

where $T \in \operatorname{GL}\left(n, q^{2}\right)$ and $H_{0} \in \mathcal{H}_{n, q^{2}}$. Hence $\bar{G} \simeq \mathcal{H}_{n, q^{2}} \rtimes\left(\operatorname{GL}\left(n, q^{2}\right) /\left\langle a I_{2 n}\right\rangle\right)$, where $a^{q+1}=1$, and

$$
|\bar{G}|=\frac{q^{n^{2}} \prod_{i=0}^{n-1}\left(q^{2 n}-q^{2 i}\right)}{q+1}
$$

Define an action of $\mathcal{H}_{n, q^{2}} \rtimes \operatorname{GL}\left(n, q^{2}\right)$ on $\mathcal{H}_{n, q^{2}}$ as follows

$$
\left(\left(H_{0}, T\right), H\right) \in\left(\mathcal{H}_{n, q^{2}} \rtimes \mathrm{GL}\left(n, q^{2}\right)\right) \times \mathcal{H}_{n, q^{2}} \longmapsto T H\left(T^{q}\right)^{t}+H_{0} \in \mathcal{H}_{n, q^{2}} .
$$

Its orbitals are the relations of an association scheme, the so called association scheme of Hermitian matrices [30, 23]. As in the symmetric case there is a correspondence between $\mathcal{H}_{n, q^{2}}$ and
the set $\overline{\mathcal{G}}$ of generators of $\mathcal{H}\left(2 n-1, q^{2}\right)$ disjoint from $\Lambda_{1}$ (see also [2, Proposition 9.5.10]). The proof is similar to that of the symmetric case and hence it is omitted.

Lemma 4.1. There is a bijection between $\mathcal{H}_{n, q^{2}}$ and $\overline{\mathcal{G}}$ such that $\mathcal{H}_{n, q^{2}} \rtimes \operatorname{GL}\left(n, q^{2}\right)$ acts on $\mathcal{H}_{n, q^{2}}$ as $\bar{G}$ acts on $\overline{\mathcal{G}}$. In particular, a d-code of $\mathcal{H}_{n, q^{2}}$ corresponds to a set of generators of $\mathcal{H}\left(2 n-1, q^{2}\right)$ disjoint from $\Lambda_{1}$ pairwise intersecting in at most an $(n-d-1)$-space, and conversely.

As before, if $\Lambda_{2}=L\left(0_{n}\right)$, the previous lemma implies that the number of orbits of $\bar{G}_{\Lambda_{2}}$ on $\overline{\mathcal{G}}$ equals the number of relations of the association scheme on Hermitian matrices.

Lemma 4.2 ( 27$]$ ). Let $\Pi_{3}$ be a generator of $\mathcal{H}\left(2 n-1, q^{2}\right)$ disjoint from $\Pi_{1}$ and $\Pi_{2}$. The points $P \in \Pi_{3}$ such that there exists a line of $\mathcal{H}\left(2 n-1, q^{2}\right)$ through $P$ intersecting $\Pi_{1}$ and $\Pi_{2}$ are the absolute points of a non-degenerate unitary polarity of $\Pi_{3}$.

## $5{ }^{2}$-codes of $\mathcal{S}_{n, q}$ or $\mathcal{H}_{n, q^{2}}$

Let $\Gamma_{\mathcal{W}}$ or $\Gamma_{\mathcal{H}}$ be the graph whose vertices are the generators of $\mathcal{W}(2 n-1, q)$ or $\mathcal{H}\left(2 n-1, q^{2}\right)$ and two vertices are adjacent whenever they meet in an $(n-2)$-space. Then $\Gamma_{\mathcal{W}}$ or $\Gamma_{\mathcal{H}}$ is a distance regular graph having diameter $n$, see [2, Section 9.4]. A coclique of $\Gamma_{\mathcal{W}}$ or $\Gamma_{\mathcal{H}}$ is a set of generators of $\mathcal{W}(2 n-1, q)$ or $\mathcal{H}\left(2 n-1, q^{2}\right)$ pairwise intersecting in at most an $(n-3)$-space.

Lemma 5.1 (Theorem 9.4.3, [2]). The eigenvalues $\theta_{j}, 0 \leq j \leq n$, of $\Gamma_{\mathcal{W}}$ are:

$$
\theta_{j}=\frac{q^{j}\left(q^{n-2 j+1}-1\right)}{q-1}-1, \text { with multiplicity } f_{j}=\frac{q^{j}\left(q^{n-2 j+1}+1\right)}{q^{n-j+1}+1} \prod_{i=1}^{j} \frac{q^{2(n-i+1)}-1}{\left(q^{i}-1\right)\left(q^{i-1}+1\right)} .
$$

The eigenvalues $\lambda_{j}, 0 \leq j \leq n$, of $\Gamma_{\mathcal{H}}$ are:
$\lambda_{j}=\frac{q^{2 j}\left(q^{2 n-4 j+1}-1\right)}{q^{2}-1}-\frac{1}{q+1}$, with multiplicity $g_{j}=\frac{q^{2 j}\left(q^{2 n-4 j+1}+1\right)}{q^{2 n-2 j+1}+1} \prod_{i=1}^{j} \frac{\left(q^{2 n-2 i+2}-1\right)\left(q^{2 n-2 i+1}+1\right)}{\left(q^{2 i}-1\right)\left(q^{2 i-1}+1\right)}$.
The eigenvalue $\theta_{j}$ is positive or negative, according as $0 \leq j \leq\left\lceil\frac{n-1}{2}\right\rceil$ or $\left\lceil\frac{n-1}{2}\right\rceil+1 \leq j \leq n$, respectively, and

$$
\operatorname{deg}\left(f_{j}\right)= \begin{cases}n j+j(n-2 j+1) & \text { if } 0 \leq j \leq\left\lceil\frac{n-1}{2}\right\rceil \\ n j+(j-1)(n-2 j+1) & \text { if }\left\lceil\frac{n-1}{2}\right\rceil+1 \leq j \leq n\end{cases}
$$

Moreover $\operatorname{deg}\left(f_{i}\right)<\operatorname{deg}\left(f_{j}\right)$, if $0 \leq i<j \leq\left\lceil\frac{n-1}{2}\right\rceil$, and $\operatorname{deg}\left(f_{i}\right)>\operatorname{deg}\left(f_{j}\right)$, if $\left\lceil\frac{n-1}{2}\right\rceil+1 \leq i<$ $j \leq n$. From the Cvetković bound (Lemma [2.2), it follows that $\alpha\left(\Gamma_{\mathcal{W}}\right) \leq \sum_{j=0}^{\frac{n-1}{2}} f_{j}$, if $n$ is odd. Note that the Hoffman bound gives a better upper bound for $\alpha\left(\Gamma_{\mathcal{W}}\right)$ than the Cvetković bound in the case when $n$ is even. Hence, the following result arises.

Theorem 5.2. Let $\mathcal{X}$ be a set of generators of $\mathcal{W}(2 n-1, q)$ pairwise intersecting in at most an $(n-3)$-space. Then

$$
|\mathcal{X}| \leq \begin{cases}\sum_{j=0}^{\frac{n-1}{2}} \frac{q^{j}\left(q^{n-2 j+1}+1\right)}{q^{n-j+1}+1} \prod_{i=1}^{j} \frac{q^{2(n-i+1)}-1}{\left(q^{i}-1\right)\left(q^{i-1}+1\right)} & \text { if } n \text { is odd } \\ \prod_{i=2}^{n}\left(q^{i}+1\right) & \text { if } n \text { is even } .\end{cases}
$$

From Lemma 3.1, the size of the largest 2 -codes of $\mathcal{S}_{n, q}$ coincides with the maximum number of generators of $\mathcal{W}(2 n-1, q)$ disjoint from $\Pi_{1}$ such that they pairwise intersect in at most an $(n-3)$-space. Hence we have the following corollary.

Corollary 5.3. Let $\mathcal{C}$ be a 2 -code of $\mathcal{S}_{n, q}$, then

$$
|\mathcal{C}| \leq \begin{cases}\sum_{j=0}^{\frac{n-1}{2}} \frac{q^{j}\left(q^{n-2 j+1}+1\right)}{q^{n-j+1}+1} \prod_{i=1}^{j} \frac{q^{2(n-i+1)}-1}{\left(q^{i}-1\right)\left(q^{i-1}+1\right)} & \text { if } n \text { is odd } \\ \prod_{i=2}^{n}\left(q^{i}+1\right) & \text { if } n \text { is even } .\end{cases}
$$

The term of highest degree in Corollary 5.3 is $q^{n(n+1) / 2-1} / 2$ if $n$ is odd and $q^{n(n+1) / 2-1}$ if $n$ is even. The previous known upper bound for the size of a 2 -code of $\mathcal{S}_{n, q}$ was $q^{n(n-1) / 2+1}\left(q^{n-1}+\right.$ 1)/ $(q+1)$ for $q$ odd [21, Proposition 3.7], and $q^{n(n+1) / 2}-q^{n}+1$ for $q$ even [24, Proposition 3.4]. Therefore Corollary 5.3 provides better upper bounds if $n$ is odd or if $n$ and $q$ are both even. Regarding 2-codes of $\mathcal{S}_{3, q}$, a further improvement will be obtained in Section 6,

In the Hermitian case, $\lambda_{j}$ is positive or negative, according as $0 \leq j \leq\left\lceil\frac{n-1}{2}\right\rceil$ or $\left\lceil\frac{n-1}{2}\right\rceil+1 \leq$ $j \leq n$, respectively, and

$$
\operatorname{deg}\left(g_{j}\right)= \begin{cases}4 j(n-j) & \text { if } 0 \leq j \leq\left\lceil\frac{n-1}{2}\right\rceil \\ 4 j(n-j)-(2 n-4 j+1) & \text { if }\left\lceil\frac{n-1}{2}\right\rceil+1 \leq j \leq n\end{cases}
$$

Moreover $\operatorname{deg}\left(g_{i}\right)<\operatorname{deg}\left(g_{j}\right)$, if $0 \leq i<j \leq\left\lceil\frac{n-1}{2}\right\rceil$, and $\operatorname{deg}\left(g_{i}\right)>\operatorname{deg}\left(g_{j}\right)$, if $\left\lceil\frac{n-1}{2}\right\rceil+1 \leq$ $i<j \leq n$. From the Cvetković bound, it follows that $\alpha\left(\Gamma_{\mathcal{H}}\right) \leq \sum_{j=0}^{\frac{n-1}{2}} g_{j}$, if $n$ is odd and $\alpha\left(\Gamma_{\mathcal{H}}\right) \leq \sum_{\frac{n}{2}+1}^{n} g_{j}$, if $n$ is even.

Theorem 5.4. Let $\mathcal{X}$ be a set of generators of $\mathcal{H}\left(2 n-1, q^{2}\right)$ pairwise intersecting in at most an ( $n-3$ )-space. Then

$$
|\mathcal{X}| \leq \begin{cases}\sum_{j=0}^{\frac{n-1}{2}} \frac{q^{2 j}\left(q^{2 n-4 j+1}+1\right)}{q^{2 n-2 j+1}+1} \prod_{i=1}^{j} \frac{\left(q^{2 n-2 i+2}-1\right)\left(q^{2 n-2 i+1}+1\right)}{\left(q^{2 i}-1\right)\left(q^{2 i-1}+1\right)} & \text { if } n \text { is odd } \\ \sum_{\frac{n}{2}+1}^{n} \frac{q^{2}\left(q^{2 n-4 j-1}+1\right)}{q^{2 n-2 j+1}+1} \prod_{i=1}^{j} \frac{\left(q^{2 n-2 i+2}-1\right)\left(q^{2 n-2 i+1}+1\right)}{\left(q^{2 i}-1\right)\left(q^{2 i-1}+1\right)} & \text { if } n \text { is even. } .\end{cases}
$$

From Lemma 4.1, the size of the largest 2 -codes of $\mathcal{H}_{n, q^{2}}$ coincides with the maximum number of members of $\overline{\mathcal{G}}$, the set of generators of $\mathcal{H}\left(2 n-1, q^{2}\right)$ disjoint from $\Lambda_{1}$, such that they pairwise intersect in at most an $(n-3)$-space. Let $\Gamma_{\mathcal{H}}^{\prime}$ be the induced subgraph of $\Gamma_{\mathcal{H}}$ on $\overline{\mathcal{G}}$. The graph $\Gamma_{\mathcal{W}}^{\prime}$ is also known as the last subconstituent of $\Gamma_{\mathcal{W}}$ or the Hermitian forms graph, see [2, Proposition 9.5.10]. In particular $\Gamma_{\mathcal{W}}^{\prime}$ is a distance-regular graph of diameter $n$. The eigenvalues of $\Gamma_{\mathcal{H}}^{\prime}$ are [2, Corollary 8.4.4]:

$$
\frac{(-q)^{2 n-j}-1}{q+1}, \text { with multiplicity } \prod_{i=1}^{j} \frac{(-q)^{n+1-i}-1}{(-q)^{i}-1} \prod_{i=0}^{j-1}\left(-(-q)^{n}-(-q)^{i}\right)
$$

respectively, with $0 \leq j \leq n$.
In this case the Hoffman bound gives a better upper bound for $\alpha\left(\Gamma_{\mathcal{H}}^{\prime}\right)$ than the Cvetković bound, that is $q^{(n-1)^{2}}\left(q^{2 n-1}+1\right) /(q+1)$. However this was already known [22, Theorem 2].

## 6 Large non-additive rank distance codes

In the case when $\mathcal{C}$ is additive much better bounds can be provided. Indeed in [23, Theorem 4.3], [22. Theorem 1] the author proved that the largest additive $d$-codes of $\mathcal{S}_{n, q}$ have size at most either $q^{n(n-d+2) / 2}$ or $q^{(n+1)(n-d+1) / 2}$, according as $n-d$ is even or odd, respectively, whereas the size of the largest additive $d$-codes of $\mathcal{H}_{n, q^{2}}$ cannot exceed $q^{n(n-d+1)}$. As far as regard codes whose size is larger than the additive bound not much is known. In the Hermitian case, if $n$ is even, there is an $n$-code of $\mathcal{H}_{n, q^{2}}$ of size $q^{n}+1$ [22, Theorem 6], [11, Theorem 18]. Observe that from Lemma 3.1 and Lemma 4.1, an $n$-code $\mathcal{C}$ of $\mathcal{S}_{n, q}$ or $\mathcal{H}_{n, q^{2}}$ exists if and only if there exists a partial spread of $\mathcal{W}(2 n-1, q)$ or $\mathcal{H}\left(2 n-1, q^{2}\right)$ of size $|\mathcal{C}|+1$. It is well-known that the points of $\mathcal{W}(2 n-1, q)$ can be partitioned into $q^{n}+1$ pairwise disjoint generators of $\mathcal{W}(2 n-1, q)$, that is, $\mathcal{W}(2 n-1, q)$ admits a spread. On the other hand, $\mathcal{H}\left(2 n-1, q^{2}\right)$ has no spread. If $n$ is odd, an upper bound for the largest partial spreads of $\mathcal{H}\left(2 n-1, q^{2}\right)$ is $q^{n}+1$ [29] and there are examples of partial spreads of that size [18]. If $n$ is even the situation is less clear: upper bounds can be found in [15], as for lower bounds there is a partial spread of $\mathcal{H}\left(2 n-1, q^{2}\right)$ of size $\left(3 q^{2}-q\right) / 2+1$ for $n=2, q>13$, [1, p. 32] and of size $q^{n}+2$ for $n \geq 4$ [11. Here, generalizing the partial spread of $\mathcal{H}\left(3, q^{2}\right)$, we show the existence of a partial spread of $\mathcal{H}\left(2 n-1, q^{2}\right)$, in the case when $n$ is even and $n / 2$ is odd, of size $\left(3 q^{n}-q^{n / 2}\right) / 2+1$, see Theorem 6.4. Hence the following result holds true.
Theorem 6.1. If $n$ is even and $n / 2$ is odd, then there exists an $n$-code of $\mathcal{H}_{n, q^{2}}$ of size $\frac{3 q^{n}-q^{n / 2}}{2}$.
For small values of $d, q$ and $n$, in [24] there are several $d$-codes of $\mathcal{S}_{n, q}$ and $\mathcal{H}_{n, q^{2}}$ whose sizes are larger than the corresponding additive bounds, namely a 2 -code of $\mathcal{S}_{3,2}$ of size 22 , a 2 -code of $\mathcal{S}_{3,3}$ of size 135 , a 2 -code of $\mathcal{S}_{3,4}$ of size 428 , a 2 -code of $\mathcal{S}_{3,5}$ of size 934 , a 2 -code of $\mathcal{S}_{3,7}$ of size 3100 , a 2 -code of $\mathcal{S}_{4,2}$ of size 320 , a 4 -code of $\mathcal{S}_{5,2}$ of size 96 , a 2 -code of $\mathcal{H}_{3,4}$ of size 120 and a 4 -code of $\mathcal{H}_{4,4}$ of size 37 . Besides these few examples, no $d$-codes whose sizes are larger than the largest possible additive $d$-codes are known.

In the remaining part of this section we focus on the case $n=3$. From Corollary 5.3, a 2 -code $\mathcal{C}$ of $\mathcal{S}_{3, q}$ has size at most

$$
\frac{q\left(q^{2}+1\right)\left(q^{2}+q+1\right)}{2}+1
$$

First, we improve on the upper bound of the size of $\mathcal{C}$. Then we construct 2 -codes of $\mathcal{S}_{3, q}$ and $\mathcal{H}_{3, q^{2}}$ that are larger than the largest possible additive 2 -codes. This provides an answer to a question posed in [25, Section 7], see also [23, p. 176]. The main results are summarized in the following theorem.

Theorem 6.2. Let $\mathcal{C}$ be a maximum 2 -code of $\mathcal{S}_{3, q}, q>2$, then

$$
q^{4}+q^{3}+1 \leq|\mathcal{C}| \leq \frac{q\left(q^{2}-1\right)\left(q^{2}+q+1\right)}{2}+1 .
$$

Let $\mathcal{C}$ be a maximum 2-code of $\mathcal{H}_{3, q^{2}}$, then

$$
q^{6}+\frac{q(q-1)\left(q^{4}+q^{2}+1\right)}{2} \leq|\mathcal{C}| \leq q^{4}\left(q^{4}-q^{3}+q^{2}-q+1\right)
$$

### 6.1 Partial spread of $\mathcal{H}\left(8 m-5, q^{2}\right)$

Let us consider the projective line $\operatorname{PG}\left(1, q^{4 m-2}\right)$ whose underlying vector space is $V\left(2, q^{4 m-2}\right)$, and let $\mathcal{H}\left(1, q^{4 m-2}\right)$ be a non-degenerate Hermitian polar space of $\operatorname{PG}\left(1, q^{4 m-2}\right)$ associated with $h$, where $h$ is a sesquilinear form on $V\left(2, q^{4 m-2}\right)$. The vector space $V\left(2, q^{4 m-2}\right)$ can be regarded as a $(4 m-2)$-dimensional vector space over $\operatorname{GF}\left(q^{2}\right)$, say $\bar{V}$. More precisely

$$
\bar{V}=\left\{\left(x, x^{q^{2}}, \ldots, x^{q^{4 m-4}}, y, y^{q^{2}}, \ldots, y^{q^{4 m-4}}\right) \mid(x, y) \in V\left(2, q^{4 m-2}\right)\right\} .
$$

Let $\mathrm{PG}\left(4 m-3, q^{2}\right)$ be the projective space whose underlying vector space is $\bar{V}$ and let

$$
T r_{q^{4 m-2} \mid q^{2}}: x \in \mathrm{GF}\left(q^{4 m-2}\right) \longmapsto \sum_{i=0}^{2 m-2} x^{q^{2 i}} \in \mathrm{GF}\left(q^{2}\right)
$$

denote the usual trace function. Note that

$$
\bar{h}=T r_{q^{4 m-2 \mid q^{2}}} \circ h: \bar{V} \times \bar{V} \longrightarrow \operatorname{GF}\left(q^{2}\right)
$$

is a non-degenerate sesquilinear form on $\bar{V}$ and hence there is a non-degenerate polar space $\mathcal{H}\left(4 m-3, q^{2}\right)$ of $\operatorname{PG}\left(4 m-3, q^{2}\right)$ associated with $\bar{h}$. See 9 for more details. Let $\rho$ be the unitary polarity of $\mathrm{PG}\left(4 m-3, q^{2}\right)$ defining $\mathcal{H}\left(4 m-3, q^{2}\right)$.

Lemma 6.3. There exists a $(2 m-2)-$ spread $\mathbf{S}$ of $\mathrm{PG}\left(4 m-3, q^{2}\right)$, such that $q^{2 m-1}+1$ members of $\mathbf{S}$ are generators of $\mathcal{H}\left(4 m-3, q^{2}\right)$ and the remaining $q^{4 m-2}-q^{2 m-1}$ are such that they occur in $\left(q^{4 m-2}-q^{2 m-1}\right) / 2$ pairs of type $\left\{\Delta, \Delta^{\rho}\right\}$, where $\left|\Delta \cap \Delta^{\rho}\right|=0$.

Proof. With the notation introduced above, if $W$ is a vector subspace of $V\left(2, q^{4 m-2}\right)$ of dimension one, then

$$
\left\{\left(x, x^{q^{2}}, \ldots, x^{q^{4 m-4}}, y, y^{q^{2}}, \ldots, y^{q^{4 m-4}}\right) \mid(x, y) \in W\right\}
$$

is a $(2 m-1)$-dimensional vector subspace of $\bar{V}$. Hence a point of $\operatorname{PG}\left(1, q^{4 m-2}\right)$ is sent to a $(2 m-2)$-space of $\mathrm{PG}\left(4 m-3, q^{2}\right)$ and two distinct $(2 m-2)$-spaces of $\mathrm{PG}\left(4 m-3, q^{2}\right)$ so obtained are pairwise skew. Let $\mathbf{S}$ be the set of $(2 m-2)$-spaces of $\mathrm{PG}\left(4 m-3, q^{2}\right)$ constructed in this way. Then $\mathbf{S}$ is a $(2 m-2)$-spread of $\mathrm{PG}\left(4 m-3, q^{2}\right)$ and $|\mathbf{S}|=q^{4 m-2}+1$.

Note that $\mathcal{H}\left(1, q^{4 m-2}\right)$ consists of $q^{2 m-1}+1$ points. The polarity of $\operatorname{PG}\left(1, q^{4 m-2}\right)$ defining $\mathcal{H}\left(1, q^{4 m-2}\right)$ fixes each of these $q^{2 m-1}+1$ points and interchanges in pairs the remaining $q^{4 m-2}-$ $q^{2 m-1}$ points of $\mathrm{PG}\left(1, q^{4 m-2}\right)$. Therefore an element $\Delta$ of $\mathbf{S}$ is a generator of $\mathcal{H}\left(4 m-3, q^{2}\right)$ if $\Delta$ corresponds to a point of $\mathcal{H}\left(1, q^{4 m-2}\right)$; otherwise $\left|\Delta \cap \Delta^{\rho}\right|=0$.

Let $\mathcal{H}\left(8 m-5, q^{2}\right)$ be a non-degenerate Hermitian polar space of $\mathrm{PG}\left(8 m-5, q^{2}\right)$ and let $\perp$ be the unitary polarity of $\mathrm{PG}\left(8 m-5, q^{2}\right)$ defining $\mathcal{H}\left(8 m-5, q^{2}\right)$.

Theorem 6.4. $\mathcal{H}\left(8 m-5, q^{2}\right)$ has a partial spread of size $\frac{3 q^{4 m-2}-q^{2 m-1}}{2}+1$.
Proof. Let $\Pi_{1}, \Pi_{2}, \Pi_{3}$ be three pairwise disjoint generators of $\mathcal{H}\left(8 m-5, q^{2}\right)$. Then $\Pi_{i} \simeq \operatorname{PG}(4 m-$ $3, q^{2}$ ). Moreover, there is a non-degenerate unitary polarity $\rho_{i}$ of $\Pi_{i}$, see Lemma 4.2, Let $\mathcal{H}_{i}$ be the

Hermitian polar space of $\Pi_{i}$, defined by $\rho_{i}, i=1,2,3$. From Lemma 6.3, there exists a $(2 m-2)-$ spread $\mathbf{S}$ of $\Pi_{1}$, such that $q^{2 m-1}+1$ members of $\mathbf{S}$ are generators of $\mathcal{H}_{1}$ and the remaining $q^{4 m-2}-q^{2 m-1}$ are such that they occur in $\left(q^{4 m-2}-q^{2 m-1}\right) / 2$ pairs of type $\left\{\Delta, \Delta^{\rho_{1}}\right\}$, where $\left|\Delta \cap \Delta^{\rho_{1}}\right|=0$. For an element $\Delta_{1}$ of $\mathbf{S}$, let $\Delta_{2}=\left\langle\Pi_{3}, \Delta_{1}\right\rangle \cap \Pi_{2}$ and $\Delta_{3}=\left\langle\Pi_{2}, \Delta_{1}\right\rangle \cap \Pi_{3}$. Then


If $\Delta_{1}$ is a generator of $\mathcal{H}_{1}$, then $\Delta_{i}^{\rho_{i}}=\Delta_{i}, i=1,2,3$, and $\left\langle\Delta_{1}, \Delta_{2}\right\rangle$ is a generator of $\mathcal{H}\left(8 m-5, q^{2}\right)$. Varying $\Delta_{1}$ among the $q^{2 m-1}+1$ members of $\mathbf{S}$ that are generators of $\mathcal{H}_{1}$, one obtains a set $Z_{1}$ of $q^{2 m-1}+1$ generators of $\mathcal{H}\left(8 m-5, q^{2}\right)$ that are pairwise disjoint.

If $\Delta_{1}$ is such that $\left|\Delta_{1} \cap \Delta_{1}^{\rho_{1}}\right|=0$, then consider the following generators of $\mathcal{H}\left(8 m-5, q^{2}\right)$ :

$$
\left\{\left\langle\Delta_{1}, \Delta_{2}^{\rho_{2}}\right\rangle,\left\langle\Delta_{2}^{\rho_{2}}, \Delta_{3}\right\rangle,\left\langle\Delta_{3}, \Delta_{1}^{\rho_{1}}\right\rangle,\left\langle\Delta_{1}^{\rho_{1}}, \Delta_{2}\right\rangle,\left\langle\Delta_{2}, \Delta_{3}^{\rho_{3}}\right\rangle,\left\langle\Delta_{3}^{\rho_{3}}, \Delta_{1}\right\rangle\right\} .
$$

Among these six generators, we can always choose three of them such that they are pairwise disjoint. For instance

$$
\left\langle\Delta_{1}, \Delta_{2}^{\rho_{2}}\right\rangle,\left\langle\Delta_{3}, \Delta_{1}^{\rho_{1}}\right\rangle,\left\langle\Delta_{2}, \Delta_{3}^{\rho_{3}}\right\rangle
$$

are pairwise skew since any two of them span the whole $\mathrm{PG}\left(8 m-5, q^{2}\right)$. Repeating this process for each of the $\left(q^{4 m-2}-q^{2 m-1}\right) / 2$ couples $\left\{\Delta_{1}, \Delta_{1}^{\rho_{1}}\right\}$ such that $\left|\Delta_{1} \cap \Delta_{1}^{\rho_{1}}\right|=0$, a set $Z_{2}$ of $3\left(q^{4 m-2}-q^{2 m-1}\right) / 2$ pairwise disjoint generators of $\mathcal{H}\left(8 m-5, q^{2}\right)$ is obtained. Again two members of $Z_{1} \cup Z_{2}$ span the whole $\operatorname{PG}\left(8 m-5, q^{2}\right)$. Therefore $Z_{1} \cup Z_{2}$ is a partial spread of $\mathcal{H}\left(8 m-5, q^{2}\right)$ of size $3\left(q^{4 m-2}-q^{2 m-1}\right) / 2+q^{2 m-1}+1=\left(3 q^{4 m-2}-q^{2 m-1}\right) / 2+1$.

### 6.2 2-codes of $\mathcal{S}_{3, q}$

Let $\perp$ be the symplectic polarity of $\operatorname{PG}(5, q)$ defining $\mathcal{W}(5, q)$. Recall that $\mathcal{G}$ is the set of $q^{6}$ planes of $\mathcal{W}(5, q)$ that are disjoint from $\Pi_{1}$, the group $G$ is the stabilizer of $\Pi_{1}$ in $\operatorname{PSp}(6, q), \Pi_{2}=L\left(0_{3}\right)$, and $G_{\Pi_{2}}$ is the stabilizer of $\Pi_{2}$ in $G$. For a point $P$ in $\Pi_{2}$, let $\Sigma_{P}$ denote a 3-space contained in $P^{\perp}$ and not containing $P$. When restricted to $\Sigma_{P}$, the polarity $\perp$ defines a non-degenerate symplectic polar space of $\Sigma_{P}$, say $\mathcal{W}_{P}$. Moreover $r_{P}=\Sigma_{P} \cap \Pi_{1}$ and $t_{P}=\Sigma_{P} \cap \Pi_{2}$ are lines of $\mathcal{W}_{P}$.

### 6.2.1 The upper bound

The graph $\Gamma_{\mathcal{W}}$ has valency $q\left(q^{2}+q+1\right)$. Let $\Gamma_{\mathcal{W}}^{\prime}$ be the induced subgraph of $\Gamma_{\mathcal{W}}$ on $\mathcal{G}$. The graph $\Gamma_{\mathcal{W}}^{\prime}$ is also known as the second subconstituent of $\Gamma_{\mathcal{W}}$, see [4]. Then $\Gamma_{\mathcal{W}}^{\prime}$ is connected, has valency $q^{3}-1$ and it is vertex-transitive, since $G \leq \operatorname{Aut}\left(\Gamma_{\mathcal{W}}^{\prime}\right)$ is transitive on $\mathcal{G}$. A 2 -code of $\mathcal{S}_{3, q}$ is a coclique of $\Gamma_{\mathcal{W}}^{\prime}$. We want to apply the Cvetković bound (Lemma 2.2) to the graph $\Gamma_{\mathcal{W}}^{\prime}$. In order to do that we need to compute the spectrum of $\Gamma_{\mathcal{W}}^{\prime}$. Consider the equitable partition arising from the action of the group $\mathcal{G}_{\Pi_{2}}$ on $\mathcal{G}$. Then, according to Lemma 3.11, the set $\mathcal{G}$ is partitioned into $\left\{\left\{\Pi_{2}\right\}, \mathcal{G}_{1}, \mathcal{G}_{2}, \mathcal{G}_{3}, \mathcal{G}_{4}\right\}$ if $q$ is even or into $\left\{\left\{\Pi_{2}\right\}, \mathcal{G}_{1}, \mathcal{G}_{2}, \mathcal{G}_{3}, \mathcal{G}_{4}, \mathcal{G}_{5}, \mathcal{G}_{6}\right\}$ if $q$ is odd. Let $B=\left(b_{i j}\right)$ denote the quotient matrix of this equitable partition. In other words, $b_{i j}$ is the number of planes of $\mathcal{G}_{j}$ intersecting a given plane of $\mathcal{G}_{i}$ in a line.

Lemma 6.5. If $q$ is even, then

$$
B=\left(\begin{array}{ccccc}
0 & q^{3}-1 & 0 & 0 & 0 \\
1 & q-2 & 0 & q^{3}-q & 0 \\
0 & 0 & 0 & q^{2}-1 & q^{2}(q-1) \\
0 & q & 1 & q^{2}-q-2 & q^{2}(q-1) \\
0 & 0 & 1 & q^{2}-1 & q^{3}-q^{2}-1
\end{array}\right)
$$

If $q$ is odd, then

$$
B=\left(\begin{array}{ccccccc}
0 & \frac{q^{3}-1}{2} & \frac{q^{3}-1}{2} & 0 & 0 & 0 & 0 \\
1 & \frac{q-3}{2} & \frac{q-1}{2} & \frac{q^{3}-q}{2} & \frac{q^{3}-q}{2} & 0 & 0 \\
1 & \frac{q-1}{2} & \frac{q-3}{2} & \frac{q^{3}-q}{2} & \frac{q^{3}-q}{2} & 0 & 0 \\
0 & \frac{q+1}{2} & \frac{q+1}{2} & \frac{(q-3)(q+1)}{2} & \frac{q^{2}-1}{2} & \frac{q^{2}(q-1)}{2} & \frac{q^{2}(q-1)}{2} \\
0 & \frac{q-1}{2} & \frac{q-1}{2} & \frac{(q-1)^{2}}{2} & \frac{q^{2}-1}{2} & \frac{q^{2}(q-1)}{2} & \frac{q^{2}(q-1)}{2} \\
0 & 0 & 0 & \frac{q(q-1)}{2} & \frac{q(q+1)}{2} & \frac{q^{2}(q-1)}{2}-1 & \frac{q^{2}(q-1)}{2} \\
0 & 0 & 0 & \frac{q(q-1)}{2} & \frac{q(q+1)}{2} & \frac{q^{2}(q-1)}{2} & \frac{q^{2}(q-1)}{2}-1
\end{array}\right) .
$$

Proof. Let $q$ be even. Every plane of $\mathcal{G}_{1}$ meets $\Pi_{2}$ in a line, no plane of $\cup_{i=2}^{4} \mathcal{G}_{i}$ meets $\Pi_{2}$ in a line and no plane of $\mathcal{G}_{1}$ meets a plane of $\mathcal{G}_{4}$ in a line. Hence $b_{12}=q^{3}-1, b_{21}=1$, $b_{25}=b_{52}=b_{1 j}=b_{i 1}=0, j \neq 2, i \neq 2$.

Let $\sigma \in \mathcal{G}_{1}$ and let $\gamma \in \cup_{i=1}^{3} \mathcal{G}_{i}$ such that $\gamma \cap \sigma$ is a line. If $\gamma \in \mathcal{G}_{1}$, then $\sigma \cap \Pi_{2}=\gamma \cap \Pi_{2}$. Hence $b_{22}=q-2$. From Lemma 3.11, $\gamma \notin \mathcal{G}_{2}$ and hence $b_{23}=0$. If $\gamma \in \mathcal{G}_{3}$, let $P$ be the point $\gamma \cap \Pi_{2}$. Then $P \in \sigma \cap \Pi_{2}$. Let $\ell=\sigma \cap \Sigma_{P}$ and $s=\gamma \cap \Sigma_{P}$. Then $\left|\ell \cap t_{P}\right|=1$ and $\left|\ell \cap r_{P}\right|=0$. On the other hand the line $s$ is skew to $r_{P}$ and $t_{P}$ and meets $\ell$ in a point. Since $s$ can be chosen in $q^{2}-q$ ways, we have that there are $q^{2}-q$ planes of $\mathcal{G}_{3}$ in $P^{\perp}$ meeting $\sigma$ in a line. Varying $P$ in $\sigma \cap \Pi_{2}$, we obtain $b_{24}=q^{3}-q$.

Let $\sigma \in \mathcal{G}_{2} \cup \mathcal{G}_{3}, P=\sigma \cap \Pi_{2}$, and let $\gamma \in \cup_{i=1}^{4} \mathcal{G}_{i}$ such that $\gamma \cap \sigma$ is a line. Note that necessarily $P \in \gamma$. By Lemma 3.11, there is no plane of $\mathcal{G}_{2}$ meeting a plane of $\mathcal{G}_{1}$ in a line and if $\sigma \in \mathcal{G}_{3}$, then there are $q$ planes of $\mathcal{G}_{1}$ meeting $\sigma$ in a line. Hence $b_{32}=0$ and $b_{42}=q$. From Lemma 3.12, it follows that $b_{33}=0, b_{34}=q^{2}-1, b_{43}=1$ and $b_{44}=q^{2}-q-2$. Through a line of $\sigma$ not containing $P$, there pass exactly $q-1$ planes of $\mathcal{G}_{4}$. Therefore $b_{45}=q^{2}(q-1)$.

Let $\sigma \in \mathcal{G}_{4}$ and let $\gamma \in \cup_{i=2}^{4} \mathcal{G}_{i}$ such that $\gamma \cap \sigma$ is a line. If $\gamma \in \mathcal{G}_{2}$ or $\mathcal{G}_{3}$, then $\sigma \cap \gamma \in \mathcal{L}_{0}$ or $\mathcal{L}_{2}$. Also, through a line of $\mathcal{L}_{0}$ or $\mathcal{L}_{2}$ there pass one plane of $\mathcal{G}_{2}$ or $\mathcal{G}_{3}$ and $q-1$ planes of $\mathcal{G}_{4}$, whereas through a line of $\mathcal{L}_{1}$ there are $q$ planes of $\mathcal{G}_{4}$. Since $\sigma$ contains one line of $\mathcal{L}_{0}, q+1$ lines of $\mathcal{L}_{1}$ and $q^{2}-1$ lines of $\mathcal{L}_{2}$, we have $b_{53}=1, b_{54}=q^{2}-1$ and $b_{55}=q^{3}-q^{2}-1$.

Let $q$ be odd. Every plane of $\mathcal{G}_{1} \cup \mathcal{G}_{2}$ meets $\Pi_{2}$ in a line, no plane of $\cup_{i=3}^{6} \mathcal{G}_{i}$ meets $\Pi_{2}$ in a line and no plane of $\mathcal{G}_{1} \cup \mathcal{G}_{2}$ meets a plane of $\mathcal{G}_{5} \cup \mathcal{G}_{6}$ in a line. Hence $b_{12}=b_{13}=\left(q^{3}-1\right) / 2$, $b_{21}=b_{31}=1, b_{26}=b_{27}=b_{36}=b_{37}=b_{62}=b_{72}=b_{63}=b_{73}=b_{1 j}=b_{i 1}=0, j \neq 2,3, i \neq 2,3$.

Let $\sigma \in \mathcal{G}_{1} \cup \mathcal{G}_{2}$ and let $\gamma \in \cup_{i=1}^{4} \mathcal{G}_{i}$ such that $\gamma \cap \sigma$ is a line. If $\gamma \in \mathcal{G}_{1} \cup \mathcal{G}_{2}$, then $\sigma \cap \Pi_{2}=\gamma \cap \Pi_{2}$. Hence $b_{22}=b_{33}=(q-3) / 2$ and $b_{23}=b_{32}=(q-1) / 2$. If $\gamma \in \mathcal{G}_{3} \cup \mathcal{G}_{4}$, let $P$ be the point $\gamma \cap \Pi_{2}$. Then $P \in \sigma \cap \Pi_{2}$. The lines $r_{P}, t_{P}, s=\Sigma_{P} \cap \gamma$ are three pairwise disjoint lines of $\mathcal{W}_{P}$. Let $\mathcal{R}$ be the regulus consisting of the lines of $\mathcal{W}_{P}$ intersecting both $r_{P}$ and $t_{P}$. The points covered by the lines of $\mathcal{R}$ form a hyperbolic quadric $\mathcal{Q}^{+}(3, q)$. The line $s$ is external or secant to $\mathcal{Q}^{+}(3, q)$
according as $\gamma \in \mathcal{G}_{3}$ or $\mathcal{G}_{4}$. Moreover the line $\ell=\sigma \cap \Sigma_{P}$ is tangent to $\mathcal{Q}^{+}(3, q)$ at the point $\ell \cap t_{P}$. For a point $L \in \ell \backslash t_{P}$, the plane $L^{\perp} \cap \Sigma_{P}$ meets $\mathcal{Q}^{+}(3, q)$ in a non-degenerate conic $C$; in this plane, through the point $L$ there are $(q-1) / 2$ lines of $\mathcal{W}_{P}$ external to $C$ and skew to both $r_{P}, t_{P}$ and $(q-1) / 2$ lines of $\mathcal{W}_{P}$ secant to $C$ and skew to both $r_{P}, t_{P}$. By varying the point $P$ over the line $\sigma \cap \Pi_{2}$, we get $b_{24}=b_{25}=b_{34}=b_{35}=\left(q^{3}-q\right) / 2$.

Let $\sigma \in \mathcal{G}_{3} \cup \mathcal{G}_{4}, P=\sigma \cap \Pi_{2}$, and let $\gamma \in \cup_{i=1}^{6} \mathcal{G}_{i}$ such that $\gamma \cap \sigma$ is a line. Note that necessarily $P \in \gamma$. As before, let $\mathcal{R}$ be the regulus consisting of the lines of $\mathcal{W}_{P}$ intersecting both $r_{P}$ and $t_{P}$ and denote by $\mathcal{Q}^{+}(3, q)$ the corresponding hyperbolic quadric. Let $\ell$ be the line $\sigma \cap \Sigma_{P}$ and $s=\gamma \cap \Sigma_{P}$. The line $\ell$ is skew to $r_{P}, t_{P}$ and it is external or secant to $\mathcal{Q}^{+}(3, q)$ according as $\sigma \in \mathcal{G}_{3}$ or $\mathcal{G}_{4}$. If $\gamma \in \mathcal{G}_{1} \cup \mathcal{G}_{2}$, then $s$ is a line of $\mathcal{W}_{P}$ meeting both $t_{P}$ and $\ell$, and it is disjoint from $r_{P}$. Also $s \cap \ell$ belongs to $\mathcal{P}_{1}$ or $\mathcal{P}_{2}$, according as $\gamma \in \mathcal{G}_{1}$ or $\mathcal{G}_{2}$, respectively. From the proof of Lemma 3.13, we have that $\left|\ell \cap \mathcal{P}_{1}\right|=\left|\ell \cap \mathcal{P}_{2}\right|=(q-1) / 2$ if $\ell$ is secant and $\left|\ell \cap \mathcal{P}_{1}\right|=\left|\ell \cap \mathcal{P}_{2}\right|=(q+1) / 2$ if $\ell$ is external. Hence $b_{42}=b_{43}=(q+1) / 2$ and $b_{52}=b_{53}=(q-1) / 2$. If $\gamma \in \mathcal{G}_{3} \cup \mathcal{G}_{4}$, then $s$ is a line of $\mathcal{W}_{P}$ intersecting $\ell$ and disjoint from $r_{P}$ and $t_{P}$. Also $\left|s \cap \mathcal{Q}^{+}(3, q)\right|$ equals 0 or 2 (i.e., $\ell$ belongs to $\mathcal{L}_{0}$ or $\mathcal{L}_{2}$, respectively), according as $\gamma \in \mathcal{G}_{3}$ or $\mathcal{G}_{4}$. If $\ell$ is external, through a point of $\ell$, there are $(q-1) / 2$ lines of $\mathcal{W}_{P}$ secant to $\mathcal{Q}^{+}(3, q)$ and skew to $r_{P}$ and $t_{P}$ and $(q-3) / 2$ lines of $\mathcal{W}_{P}$ external to $\mathcal{Q}^{+}(3, q)$ distinct from $\ell$ and skew to $r_{P}$ and $t_{P}$. Hence $b_{44}=(q-3)(q+1) / 2$ and $b_{45}=\left(q^{2}-1\right) / 2$. If $\ell$ is secant, through a point of $\ell$ not on $\mathcal{Q}^{+}(3, q)$, there are $(q-3) / 2$ lines of $\mathcal{W}_{P}$ secant to $\mathcal{Q}^{+}(3, q)$ distinct from $\ell$ and skew to $r_{P}$ and $t_{P}$ and $(q-1) / 2$ lines of $\mathcal{W}_{P}$ external to $\mathcal{Q}^{+}(3, q)$ and skew to $r_{P}$ and $t_{P}$; through a point of $\ell \cap \mathcal{Q}^{+}(3, q)$, there are $q-1$ lines of $\mathcal{W}_{P}$ secant to $\mathcal{Q}^{+}(3, q)$ distinct from $\ell$ and skew to $r_{P}$ and $t_{P}$. Hence $b_{54}=(q-1)^{2} / 2$ and $b_{55}=\left(q^{2}-1\right) / 2$. A line of $\sigma$ not containing $P$ belongs to $\mathcal{L}_{0}$ or $\mathcal{L}_{2}$ and, by Lemma 3.13, through such a line there pass exactly $(q-1) / 2$ planes of $\mathcal{G}_{5}$ and $(q-1) / 2$ planes of $\mathcal{G}_{6}$. Therefore $b_{46}=b_{47}=b_{56}=b_{57}=q^{2}(q-1) / 2$.

Let $\sigma \in \mathcal{G}_{5} \cup \mathcal{G}_{6}$ and let $\gamma \in \cup_{i=3}^{6} \mathcal{G}_{i}$ such that $\gamma \cap \sigma$ is a line. If $\gamma \in \mathcal{G}_{3}$ or $\mathcal{G}_{4}$, then $\sigma \cap \gamma \in \mathcal{L}_{0}$ or $\mathcal{L}_{2}$, respectively. Also, through a line of $\mathcal{L}_{0}$ or $\mathcal{L}_{2}$ there pass one plane of $\mathcal{G}_{3}$ or $\mathcal{G}_{4},(q-1) / 2$ planes of $\mathcal{G}_{5}$ and $(q-1) / 2$ planes of $\mathcal{G}_{6}$, see Lemma 3.13. Moreover, by Lemma 3.13, through a line of $\mathcal{L}_{1}$ there are $q$ planes disjoint from $\Pi_{1}$ and $\Pi_{2}$ and they belong either to $\mathcal{G}_{5}$ or to $\mathcal{G}_{6}$ according as $\sigma \in \mathcal{G}_{5}$ or $\sigma \in \mathcal{G}_{6}$, respectively. Since, by Corollary 3.10, $\sigma$ contains $q(q-1) / 2$ lines of $\mathcal{L}_{0}, q+1$ lines of $\mathcal{L}_{1}$ and $q(q+1) / 2$ lines of $\mathcal{L}_{2}$, it follows that $b_{64}=b_{74}=q(q-1) / 2$, $b_{65}=b_{75}=q(q+1) / 2, b_{66}=b_{77}=q^{2}(q-1) / 2-1$ and $b_{67}=b_{76}=q^{2}(q-1) / 2$.

Theorem 6.6. The spectrum of the graph $\Gamma_{\mathcal{W}}^{\prime}$ is

$$
\left(q^{3}-1\right)^{1},\left(q^{2}-1\right)^{\frac{q(q+1)\left(q^{3}-1\right)}{2}},(-1)^{\left(q^{3}-q^{2}+1\right)\left(q^{3}-1\right)},\left(-q^{2}-1\right)^{\frac{q(q-1)\left(q^{3}-1\right)}{2}} .
$$

Proof. The matrix $B$ described in Lemma 6.5 has four distinct eigenvalues, three of them are simple: $q^{3}-1, q^{2}-1$ and $-q^{2}-1$, whereas the multiplicity of the eigenvalue -1 is two or four, according as $q$ is even or odd. From Lemma [2.1, the graph $\Gamma_{\mathcal{W}}^{\prime}$ has four distinct eigenvalues: $q^{3}-1, q^{2}-1,-1,-q^{2}-1$ with multiplicities $m_{0}, m_{1}, m_{2}, m_{3}$, respectively. Note that $m_{0}=1$, since $\Gamma_{\mathcal{W}}^{\prime}$ is connected. Moreover, the following equations have to be satisfied (see for instance
[28, p. 142]):

$$
\begin{aligned}
1+m_{1}+m_{2}+m_{3} & =q^{6}, \\
q^{3}-1+\left(q^{2}-1\right) m_{1}-m_{2}-\left(q^{2}+1\right) m_{3} & =0, \\
\left(q^{3}-1\right)^{2}+\left(q^{2}-1\right)^{2} m_{1}+m_{2}+\left(q^{2}+1\right)^{2} m_{3} & =q^{6}\left(q^{3}-1\right) .
\end{aligned}
$$

It follows that $m_{1}=q(q+1)\left(q^{3}-1\right) / 2, m_{2}=\left(q^{3}-1\right)\left(q^{3}-q^{2}+1\right), m_{3}=q(q-1)\left(q^{3}-1\right) / 2$.
By applying the Cvetković bound (Lemma [2.2), we get $\alpha\left(\Gamma_{\mathcal{W}}^{\prime}\right) \leq \frac{q\left(q^{2}-1\right)\left(q^{2}+q+1\right)}{2}+1$.
Corollary 6.7. Let $\mathcal{C}$ be a 2 -code of $\mathcal{S}_{3, q}$, then $|\mathcal{C}| \leq \frac{q\left(q^{2}-1\right)\left(q^{2}+q+1\right)}{2}+1$.
Problem 6.8. We obtained a better upper bound for a 2 -code of $\mathcal{S}_{n, q}$ in the case $n=3$, by applying the Cvetković bound to the graph $\Gamma_{\mathcal{W}}^{\prime}$, the last subconstituent of $\Gamma_{\mathcal{W}}$. Determine whether or not this holds true for $n>3$.

### 6.2.2 The lower bound

Here we provide the first infinite family of 2 -codes of $\mathcal{S}_{3, q}$ whose size is larger than the largest possible additive 2-code.

Construction 6.9. Let $\mathcal{F}_{P}$ be a line-spread of $\mathcal{W}_{P}$ containing $r_{P}$ and $t_{P}$ and let $\mathcal{X}_{P}$ be the set of $q^{2}-1$ generators of $\mathcal{W}(5, q)$ passing through $P$ and meeting $\Sigma_{P}$ in a line of $\mathcal{F}_{P} \backslash\left\{r_{P}, t_{P}\right\}$. Define the set $\mathcal{X}$ as follows

$$
\bigcup_{P \in \Pi_{2}} \mathcal{X}_{P} \cup\left\{\Pi_{2}\right\} .
$$

Theorem 6.10. The set $\mathcal{X}$ consists of $(q+1)\left(q^{3}-1\right)+1$ planes of $\mathcal{W}(5, q)$ disjoint from $\Pi_{1}$ and pairwise intersecting in at most one point.

Proof. By construction every member of $\mathcal{X}$ distinct from $\Pi_{2}$ meets $\Pi_{2}$ in exactly one point. Let $\sigma_{1}, \sigma_{2} \in \mathcal{X} \backslash\left\{\Pi_{2}\right\}$. If $\sigma_{1} \cap \Pi_{2}=\sigma_{2} \cap \Pi_{2}$, then $\left|\sigma_{1} \cap \sigma_{2}\right|=1$ and there is nothing to prove. Hence let $P_{i}=\sigma_{i} \cap \Pi_{2}, i=1,2$, with $P_{1} \neq P_{2}$. Assume by contradiction that $\sigma_{1} \cap \sigma_{2}$ is a line, say $\ell$. If $\ell \cap \Pi_{2}$ is a point, say $R$, then $R \in \sigma_{1} \cap \sigma_{2}$. If $R \neq P_{1}$ then the line $R P_{1}$ would be contained in $\sigma_{1} \cap \Pi_{2}$, contradicting the fact that $\left|\sigma_{1} \cap \Pi_{2}\right|=1$. Similarly if $R \neq P_{2}$, then $R P_{2} \subseteq \sigma_{2} \cap \Pi_{2}$, a contradiction. Therefore $R=P_{1}=P_{2}$, contradicting the fact that $P_{1} \neq P_{2}$. Hence $\left|\ell \cap \Pi_{2}\right|=0$ and both $\sigma_{1}, \sigma_{2}$ are contained in $\ell^{\perp}$. However in this case the line $P_{1} P_{2}$ is a line of $\mathcal{W}(5, q)$ contained in $\ell^{\perp}$ and disjoint from $\ell$; a contradiction.

Corollary 6.11. There exists a 2 -code of $\mathcal{S}_{3, q}$ of size $\left(q^{2}-1\right)\left(q^{2}+q+1\right)+1$.
From Corollary 6.7 a 2 -code of $\mathcal{S}_{3,2}$ has at most 22 elements and hence the 2 -code of $\mathcal{S}_{3,2}$ obtained from Construction 6.9 is maximal; an alternative proof of its maximality will be exhibited (Corollary 6.15). Moreover, from [24], this code is the unique largest 2-code of $\mathcal{S}_{3,2}$ of size 22. Our next step is to show that, if $q>2$, the 2 -code of $\mathcal{S}_{3, q}$ provided in Construction 6.9 can be further enlarged. In order to do that some preliminary results are required.

Lemma 6.12. Let $\sigma \notin \mathcal{X}$ be a plane of $\mathcal{W}(5, q)$ disjoint from $\Pi_{1}$ and meeting $\Pi_{2}$ in at least one point. Then there exists a plane of $\mathcal{X}$ meeting $\sigma$ in a line.

Proof. Let $\sigma \notin \mathcal{X}$ be a plane disjoint from $\Pi_{1}$ and meeting $\Pi_{2}$ in one point, say $P$. Then $\sigma$ meets $\Sigma_{P}$ in a line, say $s$, and there are $q+1$ lines of $\mathcal{F}_{P} \backslash\left\{r_{P}, t_{P}\right\}$ meeting $s$ in one point. It follows that there are $q+1$ planes of $\mathcal{X}_{P}$ meeting $\sigma$ in a line.

Lemma 6.13. Through a point $R$ of $\mathcal{W}(5, q) \backslash\left(\Pi_{1} \cup \Pi_{2}\right)$ there pass either $q$ or $q+1$ planes of $\mathcal{X}$, according as the line through $R$ intersecting $\Pi_{1}$ and $\Pi_{2}$ is a line of $\mathcal{W}(5, q)$ or it is not.

Proof. Let $R$ be a point of $\mathcal{W}(5, q) \backslash\left(\Pi_{1} \cup \Pi_{2}\right)$, let $\ell_{R}$ be the unique line through $R$ intersecting both $\Pi_{1}, \Pi_{2}$ and let $R_{i}=\ell_{R} \cap \Pi_{i}, i=1,2$. Let $s$ denote the line $R^{\perp} \cap \Pi_{2}$. If a plane of $\mathcal{X}_{P}$ contains the point $R$, then $P \in s$. On the other hand for a fixed $P \in s$ there is at most one plane of $\mathcal{X}_{P}$ containing $R$. Hence there are at most $q+1$ planes of $\mathcal{X}$ through $R$. If $P \in s$ and $P \neq R_{2}$, then both $R_{2}$ and $R$ are in $P^{\perp}$. Hence $\ell_{R} \subseteq P^{\perp}$, the line $r_{P}=P^{\perp} \cap \Pi_{1}$ contains $R_{1}$ and $\left\langle P, r_{P}\right\rangle \cap \ell_{R}=\left\{R_{1}\right\}$. Therefore $R \notin\left\langle P, r_{P}\right\rangle$ and there exists a plane of $\mathcal{X}_{P}$ containing $R$. On the other hand, if $P=R_{2}$, then $R \in\left\langle P, r_{P}\right\rangle$. Finally note that $R_{2} \in s$ if and only if $\ell_{R}$ is a line of $\mathcal{W}(5, q)$.

Let $\mathcal{L}$ be the set of lines of $\mathcal{W}(5, q)$ disjoint from $\Pi_{1} \cup \Pi_{2}$ contained in a plane of $\mathcal{X}$. Then $|\mathcal{L}|=q^{2}(q+1)\left(q^{3}-1\right)$.

Lemma 6.14. If $q$ is even, then $\mathcal{L} \subseteq \mathcal{L}_{2}$. If $q$ is odd, then $\left|\mathcal{L} \cap \mathcal{L}_{0}\right|=\left|\mathcal{L} \cap \mathcal{L}_{2}\right|$.
Proof. Let $\ell$ be a line of $\mathcal{L}$. Thus there is a point $P \in \Pi_{2}$ and a plane $\sigma$ of $\mathcal{X}_{P}$ containing $\ell$. The three-space $T_{\ell}$ is contained in $P^{\perp}$ and does not contain $P$, otherwise $\left|\sigma \cap \Pi_{1}\right| \neq 0$. Hence $T_{\ell} \cap \mathcal{W}(5, q)$ is a non-degenerate symplectic polar space $\mathcal{W}(3, q)$ and $\left|\mathcal{L} \cap \mathcal{L}_{1}\right|=0$. Note that $\mathcal{D}=\left\{T_{\ell} \cap \gamma: \gamma \in \mathcal{X}_{P}\right\} \cup\left\{r_{\ell}, t_{\ell}\right\}$ is a line-spread of $\mathcal{W}(3, q)$. In the point-line dual of $\mathcal{W}(3, q)$, the line-spread $\mathcal{D}$ is an ovoid $\mathcal{O}$ of the parabolic quadric $\mathcal{Q}(4, q)$ and the regulus determined by $r_{\ell}, t_{\ell}, \ell$, would correspond to three points $P_{1}, P_{2}, P_{3}$ of a conic $C$ of $\mathcal{Q}(4, q)$ such that $P_{1}, P_{2}, P_{3} \in \mathcal{O}$.

Assume that $q$ is even. Then the parabolic quadric $\mathcal{Q}(4, q)$ has a nucleus $N$. If $\ell$ were in $\mathcal{L}_{0}$, then $N \in\langle C\rangle$. Consider a three-space $Z$ of the ambient projective space of $\mathcal{Q}(4, q)$ such that $N \notin Z$. By projecting points and lines of $\mathcal{Q}(4, q)$ from $N$ to $Z$, we obtain the points and lines of a non-degenerate symplectic polar space $\mathcal{W}$ of $Z$. In particular $C^{\prime}=\{N P \cap Z: P \in C\}$ is a line of $Z$ and $\mathcal{O}^{\prime}=\{N P \cap Z: P \in \mathcal{O}\}$ is an ovoid of $\mathcal{W}$. Then we would have $\left|C^{\prime} \cap \mathcal{O}^{\prime}\right| \geq 3$, a contradiction, see [26].

Assume that $q$ is odd. The line $\ell$ belongs to $\mathcal{L}_{2}$ if and only if the line polar to the plane $\left\langle P_{1}, P_{2}, P_{3}\right\rangle$ with respect to the polarity of $\mathcal{Q}(4, q)$ is secant to $\mathcal{Q}(4, q)$. Let $A$ be the conic obtained by intersecting $\mathcal{Q}(4, q)$ with the plane polar to the line $\left\langle P_{1}, P_{2}\right\rangle$ with respect to the orthogonal polarity of $\operatorname{PG}(4, q)$ associated with $\mathcal{Q}(4, q)$. Let us count the triple $\left(R, S, P_{3}\right)$, where $R, S \in A, R \neq S, P_{3} \in \mathcal{O} \backslash\left\{P_{1}, P_{2}\right\}$ and both $R P_{3}, S P_{3}$ are lines of $\mathcal{Q}(4, q)$. The point $R$ can be chosen in $q+1$ ways and for a fixed $R$, the point $P_{3}$ can be chosen in $q-1$ ways. Finally once $R$ and $P_{3}$ are fixed, the point $S$ is uniquely determined. Hence there are $q^{2}-1$ such triples. It
turns out that there are $\left(q^{2}-1\right) / 2$ points $P_{3} \in \mathcal{O} \backslash\left\{P_{1}, P_{2}\right\}$ such that the line polar to the plane $\left\langle P_{1}, P_{2}, P_{3}\right\rangle$ with respect to the polarity of $\mathcal{Q}(4, q)$ is secant to $\mathcal{Q}(4, q)$ and $\left(q^{2}-1\right) / 2$ points $P_{3} \in \mathcal{O} \backslash\left\{P_{1}, P_{2}\right\}$ such that the line polar to the plane $\left\langle P_{1}, P_{2}, P_{3}\right\rangle$ with respect to the polarity of $\mathcal{Q}(4, q)$ is external to $\mathcal{Q}(4, q)$.

Corollary 6.15. The 2 -code of $\mathcal{S}_{3,2}$ obtained from Construction 6.9 is maximal.
Proof. From Lemma 6.12, if there exists a plane $\sigma$ disjoint from $\Pi_{1}$ such that it meets every plane of $\mathcal{X}$ in at most one point, then $\sigma$ must be disjoint from $\Pi_{2}$. From Lemma 3.4, $\sigma$ contains exactly one line of $\mathcal{L}_{0}, 3$ lines of $\mathcal{L}_{1}$ and 3 lines of $\mathcal{L}_{2}$. Since $\left|\mathcal{L}_{2}\right|=|\mathcal{L}|$, the result follows.

Let $\Pi_{3}$ be a generator of $\mathcal{W}(5, q)$ disjoint from both $\Pi_{1}$ and $\Pi_{2}$. Let us denote by $\Pi_{i}, 1 \leq i \leq$ $q+1$, the $q+1$ planes of the unique symplectic Segre variety of $\mathcal{W}(5, q)$ containing $\Pi_{1}, \Pi_{2}, \Pi_{3}$. In what follows we want to prove that it is possible to construct $\mathcal{X}$ in such a way that the $q-1$ planes $\Pi_{i}, 3 \leq i \leq q+1$ can be added to it.

### 6.2.2.1 The even characteristic case

Assume that $q>2$ is even. Since the planes $\Pi_{1}, \ldots, \Pi_{q+1}$ are pairwise disjoint generators of $\mathcal{W}(5, q)$, from Lemma 3.4, there is a non-degenerate pseudo-polarity $\rho_{i}$ of $\Pi_{i}$. The set of absolute points of $\rho_{i}$ are those of a line $v_{i}$ of $\Pi_{i}$. Let $V_{i}=v_{i}^{\rho_{i}}$. Note that the unique line of $\mathcal{L}_{0}$ contained in $\Pi_{i}$ is $v_{i}, 3 \leq i \leq q+1$, while the $q+1$ lines of $\mathcal{L}_{1}$ contained in $\Pi_{i}$ are those through $V_{i}$, $3 \leq i \leq q+1$.

Let $Q$ be a point of $\Pi_{2}$ not on $v_{2}$ and distinct from $V_{2}$. Let $\Sigma_{Q}$ be a 3 -space contained in $Q^{\perp}$ and not containing $Q$. In particular we choose $\Sigma_{Q}$ spanned by the lines $Q^{\perp} \cap \Pi_{1}$ and $Q^{\rho_{2}}$. Note that $\Sigma_{Q} \cap \Pi_{i}=Q^{\perp} \cap \Pi_{i}$. Indeed, if $s$ is the unique line (not of $\mathcal{W}(5, q)$ ) containing $Q$ and meeting each of the planes $\Pi_{i}, 1 \leq i \leq q+1$, in one point, then $\left(s \cap \Pi_{i}\right)^{\rho_{i}}=\Sigma_{Q} \cap \Pi_{i}$. When restricted on $\Sigma_{Q}$, the polarity $\perp$ defines a non-degenerate symplectic polar space of $\Sigma_{Q}$, say $\mathcal{W}_{Q}$. As before, let $r_{Q}=\Sigma_{Q} \cap \Pi_{1}$ and $Q^{\rho_{2}}=t_{Q}=\Sigma_{Q} \cap \Pi_{2}$. Let $\mathcal{R}_{Q}$ be the set of $q+1$ lines of $\mathcal{W}_{Q}$ defined as follows

$$
\left\{\Sigma_{Q} \cap \Pi_{i}: 1 \leq i \leq q+1\right\}
$$

Then $\mathcal{R}_{Q}$ is a regulus of $\mathcal{W}_{Q}$ containing both $r_{Q}$ and $t_{Q}$; the opposite regulus of $\mathcal{R}_{Q}$ contains exactly one line of $\mathcal{W}_{Q}$ which is the line consisting of the points $v_{i} \cap\left(s \cap \Pi_{i}\right)^{\rho_{i}}$.

Lemma 6.16. There exists a Desarguesian line-spread of $\mathcal{W}_{P}$ having in common with $\mathcal{R}_{Q}$ exactly the lines $r_{P}$ and $t_{P}$.

Proof. Let $\mathcal{Q}(4, q)$ be the point line dual of $\mathcal{W}_{Q}$ and let $N$ be the nucleus of $\mathcal{Q}(4, q)$. The regulus $\mathcal{R}_{Q}$ corresponds to a conic $C$ of $\mathcal{Q}(4, q)$ such that $N \notin\langle C\rangle$ and the lines $r_{P}$ and $t_{P}$ correspond to two points of $C$, say $R$ and $T$. The result follows, since there are $q^{2} / 2-q$ elliptic quadrics of $\mathcal{Q}(4, q)$ meeting $C$ exactly in the points $R, T$.

For any point $Q$ of $\Pi_{2}$ different from $V_{2}$ and not on $v_{2}$, let $\mathcal{F}_{Q}$ be a Desarguesian line-spread of $\mathcal{W}_{Q}$ having in common with $\mathcal{R}_{Q}$ exactly the lines $r_{Q}$ and $t_{Q}$ and let $\mathcal{Y}_{Q}$ be the set of $q^{2}-1$
generators of $\mathcal{W}(5, q)$ passing through $Q$ and meeting $\Sigma_{Q}$ in a line of $\mathcal{F}_{Q} \backslash\left\{r_{Q}, t_{Q}\right\}$. For any point $P \in v_{2} \cup\left\{V_{2}\right\}$, let $\mathcal{X}_{P}$ be a set of $q^{2}-1$ generators of $\mathcal{W}(5, q)$ passing through $P$ defined as in Construction 6.9,

Define the set $\overline{\mathcal{X}}$ as follows

$$
\left(\bigcup_{P \in v_{2} \cup\left\{V_{2}\right\}} \mathcal{X}_{P}\right) \cup\left(\bigcup_{Q \in \Pi_{2} \backslash\left(v_{2} \cup\left\{V_{2}\right\}\right)} \mathcal{Y}_{Q}\right) \cup\left(\bigcup_{i=2}^{q+1} \Pi_{i}\right) .
$$

Theorem 6.17. The set $\overline{\mathcal{X}}$ consists of $q^{4}+q^{3}+1$ planes of $\mathcal{W}(5, q)$ disjoint from $\Pi_{1}$ and pairwise intersecting in at most one point.

Proof. It is enough to show that a plane $\sigma$ of

$$
\left(\bigcup_{P \in v_{2} \cup\left\{V_{2}\right\}} \mathcal{X}_{P}\right) \cup\left(\bigcup_{Q \in \Pi_{2} \backslash\left(v_{2} \cup\left\{V_{2}\right\}\right)} \mathcal{Y}_{Q}\right)
$$

meets $\Pi_{i}, 3 \leq i \leq q+1$, in at most one point. If $\sigma$ intersects $\Pi_{i}$ in a line, say $\ell$, from Lemma6.14, $\ell \in \mathcal{L} \subseteq \mathcal{L}_{2}$. Hence $\ell \neq v_{i}$ and $V_{i} \notin \ell$. Let $s$ be the unique line through the point $\ell^{\rho_{i}}$ meeting both $\Pi_{1}$ and $\Pi_{2}$ in one point. Let $Q=s \cap \Pi_{2}$. Then $Q$ coincides with $\ell^{\perp} \cap \Pi_{2}$ and $Q \notin v_{2} \cup\left\{V_{2}\right\}$. Moreover $\sigma=\langle Q, \ell\rangle \in \mathcal{Y}_{Q}$. But in this case the Desarguesian line-spread $\mathcal{F}_{Q}$ would have the three lines $\ell, r_{Q}, t_{Q}$ in common with $\mathcal{R}_{Q}$, contradicting the fact that $\left|\mathcal{F}_{Q} \cap \mathcal{R}_{Q}\right|=2$.

### 6.2.2.2 The odd characteristic case

Assume that $q$ is odd. Since the planes $\Pi_{1}, \ldots, \Pi_{q+1}$ are pairwise disjoint generators of $\mathcal{W}(5, q)$, from Lemma 3.4, there is a non-degenerate orthogonal polarity $\rho_{i}$ of $\Pi_{i}$. The set of absolute points of $\rho_{i}$ are those of a conic $\alpha_{i}$ of $\Pi_{i}$. Note that a line $\ell$ of $\Pi_{i}$ belongs to $\mathcal{L}_{j}$, according as $\left|\ell \cap \alpha_{i}\right|=j, 0 \leq j \leq 2,3 \leq i \leq q+1$.

Let $Q$ be a point of $\Pi_{2}$ not on $\alpha_{2}$. Let $\Sigma_{Q}$ be a 3 -space contained in $Q^{\perp}$ and not containing $Q$. In particular we choose $\Sigma_{Q}$ spanned by the lines $Q^{\perp} \cap \Pi_{1}$ and $Q^{\rho_{2}}$. Note that $\Sigma_{Q} \cap \Pi_{i}=Q^{\perp} \cap \Pi_{i}$. Indeed, if $s$ is the unique line containing $Q$ and meeting each of the planes $\Pi_{i}, 1 \leq i \leq q+1$, in one point, then $\left(s \cap \Pi_{i}\right)^{\rho_{i}}=\Sigma_{Q} \cap \Pi_{i}$. When restricted on $\Sigma_{Q}$, the polarity $\perp$ defines a nondegenerate symplectic polar space of $\Sigma_{Q}$, say $\mathcal{W}_{Q}$. As before, let $r_{Q}=\Sigma_{Q} \cap \Pi_{1}$ and $t_{Q}=\Sigma_{Q} \cap \Pi_{2}$. Let $\mathcal{R}_{Q}$ be the set of $q+1$ lines of $\mathcal{W}_{Q}$ defined as follows

$$
\left\{\Sigma_{Q} \cap \Pi_{i}: 1 \leq i \leq q+1\right\} .
$$

Then $\mathcal{R}_{Q}$ is a regulus of $\mathcal{W}_{Q}$ containing both $r_{Q}$ and $t_{Q}$ and the opposite regulus of $\mathcal{R}_{Q}$ contains exactly 0 or 2 lines of $\mathcal{W}_{Q}$. The proof of the next result is left to the reader.

Lemma 6.18. There exists a Desarguesian line-spread of $\mathcal{W}_{P}$ having in common with $\mathcal{R}_{Q}$ exactly the lines $r_{P}$ and $t_{P}$.

For any point $Q$ of $\Pi_{2}$ not on $\alpha_{2}$, let $\mathcal{F}_{Q}$ be a Desarguesian line-spread of $\mathcal{W}_{Q}$ having in common with $\mathcal{R}_{Q}$ exactly the lines $r_{Q}$ and $t_{Q}$ and let $\mathcal{Y}_{Q}$ be the set of $q^{2}-1$ generators of
$\mathcal{W}(5, q)$ passing through $Q$ and meeting $\Sigma_{Q}$ in a line of $\mathcal{F}_{Q} \backslash\left\{r_{Q}, t_{Q}\right\}$. For any point $P \in \alpha_{2}$, let $\mathcal{X}_{P}$ be a set of $q^{2}-1$ generators of $\mathcal{W}(5, q)$ passing through $P$ defined as in Construction 6.9,

Define the set $\overline{\mathcal{X}}$ as follows

$$
\left(\bigcup_{P \in \alpha_{2}} \mathcal{X}_{P}\right) \cup\left(\bigcup_{Q \in \Pi_{2} \backslash \alpha_{2}} \mathcal{Y}_{Q}\right) \cup\left(\bigcup_{i=2}^{q+1} \Pi_{i}\right)
$$

A proof similar to that given in the even characteristic case yields the following result.
Theorem 6.19. The set $\overline{\mathcal{X}}$ consists of $q^{4}+q^{3}+1$ planes of $\mathcal{W}(5, q)$ disjoint from $\Pi_{1}$ and pairwise intersecting in at most one point.

### 6.3 2-codes of $\mathcal{H}_{3, q^{2}}$

Let $\bar{\perp}$ be the Hermitian polarity of $\operatorname{PG}\left(5, q^{2}\right)$ defining $\mathcal{H}\left(5, q^{2}\right)$. Recall that $\overline{\mathcal{G}}$ is the set of $q^{9}$ planes of $\mathcal{H}\left(5, q^{2}\right)$ that are disjoint from $\Lambda_{1}$, the group $\bar{G}$ is the stabilizer of $\Lambda_{1}$ in $\operatorname{PGU}\left(6, q^{2}\right)$, $\Lambda_{2}=L\left(0_{3}\right)$, and $\bar{G}_{\Lambda_{2}}$ is the stabilizer of $\Lambda_{2}$ in $\bar{G}$. For a point $P$ in $\Lambda_{2}$, let $\bar{\Sigma}_{P}$ denote a 3 -space contained in $P^{\bar{\perp}}$ and not containing $P$. When restricted to $\bar{\Sigma}_{P}$, the polarity $\bar{\perp}$ defines a nondegenerate Hermitian polar space of $\bar{\Sigma}_{P}$, say $\mathcal{H}_{P}$. Moreover $\bar{r}_{P}=\bar{\Sigma}_{P} \cap \Lambda_{1}$ and $\bar{t}_{P}=\bar{\Sigma}_{P} \cap \Lambda_{2}$ are lines of $\mathcal{H}_{P}$.

Construction 6.20. Let $\overline{\mathcal{F}}_{P}$ be a partial spread of $\mathcal{H}_{P}$ containing $\bar{r}_{P}$ and $\bar{t}_{P}$ and let $\mathcal{Y}_{P}$ be the set of $\left|\overline{\mathcal{F}}_{P}\right|-2$ generators of $\mathcal{H}\left(5, q^{2}\right)$ passing through $P$ and meeting $\bar{\Sigma}_{P}$ in a line of $\overline{\mathcal{F}}_{P} \backslash\left\{\bar{r}_{P}, \bar{t}_{P}\right\}$. Define the set $\mathcal{Y}$ as follows

$$
\bigcup_{P \in \Lambda_{2}} \mathcal{Y}_{P} \cup\left\{\Lambda_{2}\right\} .
$$

A proof similar to that given in the symmetric case gives:
Theorem 6.21. The set $\mathcal{Y}$ consists of $\left(q^{4}+q^{2}+1\right)(|\mathcal{F}|-2)$ planes of $\mathcal{H}\left(5, q^{2}\right)$ disjoint from $\Lambda_{1}$ and pairwise intersecting in at most one point.

By selecting $\mathcal{F}$ as a partial spread of $\mathcal{H}\left(3, q^{2}\right)$ of size $\left(3 q^{2}-q+2\right) / 2$, see [1, p. 32], the following arises.

Corollary 6.22. There exists a 2 -code of $\mathcal{H}_{3, q^{2}}$ of size $q^{6}+\frac{q(q-1)\left(q^{4}+q^{2}+1\right)}{2}$.
Acknowledgments. This work was supported by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA- INdAM).

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    Mathematics Subject Classification (2010): Primary 51E20; Secondary 05C70; 05A05

