Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Combinatorial Theory, Series B 96 (2006) 445–454
Journal of Combinatorial Theory Series B
www.elsevier.com/locate/jctb

A characterization of cocircuit graphs of uniform oriented matroids

Juan José Montellano-Ballesteros, Ricardo Strausz

Instituto de Matemáticas, UNAM, Ciudad Universitaria, México D.F., 04510, Mexico

Received 6 September 2001
Available online 8 November 2005

Abstract

The *cocircuit graph* of an oriented matroid is the 1-skeleton of the cellular decomposition induced by the Topological Representation Theorem due to Folkman and Lawrence (1978) [J. Folkman, J. Lawrence, Oriented matroids, J. Combin. Theory Ser. B 25 (1978) 199–236]. In this paper we exhibit a characterization of such graphs (for the uniform case) via their natural embedding into $\mathcal{Q}_n^k$—the 1-skeleton of the n-cube's k-skeleton's dual complex. The main theorem reads, basically, as follows: *A graph $\mathcal{G}$ is the cocircuit graph of a d-dimensional uniform oriented matroid on n elements if and only if its order is $2\binom{n}{d+2}$, and it can be embedded antipodally and "metrically" into $\mathcal{Q}_n^{n-d-2}$.*

Keywords: Oriented matroid; Cocircuit graph; Graph embedding; Antipodal graph

1. Introduction

The *cocircuit graph* of an oriented matroid (see [1,3]) is the 1-skeleton of the cellular decomposition of the sphere induced by the pseudospheres that realize the oriented matroid via the Topological Representation Theorem of Folkman and Lawrence (1978) [4]. Since the cocircuit graph's vertices are labelled with signed vectors, they can be identified with some faces of the n-cube, where n is the number of elements of the oriented matroid. If the oriented matroid is uniform, all such faces have the same dimension, say k. Then the family of those faces induces a subcomplex of the k-skeleton of the n-cube (known as the Radon complex, see [6,7]), and the 1-skeleton of its dual complex is the cocircuit graph.

E-mail addresses: juancho@math.unam.mx (J.J. Montellano-Ballesteros), strausz@math.unam.mx (R. Strausz).

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2005.09.008

The 1-skeleton of the n-cube's k-skeleton's dual complex will be denoted by $\mathcal{Q}_n^k$. So, if $\mathcal{G}$ denotes the cocircuit graph of a uniform oriented matroid, there exists a natural embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$, for some n and k, and such an embedding is clearly antipodal (see also [5] for more on antipodal graphs and embeddings into the n-cube). The embedding is said to be *crabbed* if for all $X, Y \in V(\mathcal{G})$ such that $X \neq \pm Y$, there exists a *crab path* $P = (X = Z^0, Z^1, \dots, Z^m = Y)$ in $\mathcal{G}$; i.e., a path such that for all $i = 1, \dots, n$ and for each $\ell = 0, \dots, m-1$, if $(Z^\ell)_i = 0$ then $(Z^{\ell+1})_i \in \{0, X_i, Y_i\}$. In this paper we prove the following theorem:

Theorem 1. *A graph $\mathcal{G}$ is the circuit graph of a d-dimensional uniform oriented matroid of order $n > d+2$ if and only if the order of $\mathcal{G}$ is $2\binom{n}{d+2}$, and there exists an antipodal crabbed embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^{n-d-2}$ such that for every pair $X, Y \in V(\mathcal{G})$, if X and Y have the same support then $X = \pm Y$.*

This paper is organized as follows: In Section 2 some preliminary definitions are given; in particular, the k-dual of the n-cube is formally introduced. Section 3 is devoted to Theorem 1. Finally, in Section 4 the proof of Lemma 5, which describes the graph distance of $\mathcal{Q}_n^k$, is presented. This lemma allows us to prove an equivalent result to Theorem 1 (Corollary 6) whose statement is slightly closer to a formulation by graph properties than that of Theorem 1.

2. Preliminaries

2.1. Oriented matroids

Let E be any set with n elements and denote the set of (signed) vectors with n entries in $\{-, 0, +\}$ as $\{-, 0, +\}^E$. Given a signed vector $X = (X_e)_{e \in E}$, the set $X^\pm = \{e \in E\colon\ X_e \neq 0\}$ is called the *support* of X. The *zero set* of X is the complement of its support, $X^0 = E \setminus X^\pm = \{e \in E\colon\ X_e = 0\}$. Its *positive* and *negative* sets are $X^+ = \{e \in E\colon\ X_e = +\}$ and $X^- = \{e \in E\colon\ X_e = -\}$, respectively. The *opposite* $-X$ is defined by $(-X)_e = -(X_e)$. Additionally, in the family of signed vectors $\{-, 0, +\}^E$ a partial order $\leqslant$ can be defined as

$$X \leqslant Y \quad \Leftrightarrow \quad X^+ \subseteq Y^+ \quad \text{and} \quad X^- \subseteq Y^-.$$

Actually, this poset is the face lattice of the n-crosspolytope and dual of the n-cube.

Given two signed vectors X and Y, the *separator* of X and Y is the set

$$S(X, Y) = \{e \in E\colon\ X_e = -Y_e \neq 0\}.$$

Two signed vectors X, Y with the same support size ($|X^\pm| = |Y^\pm| < n$) will be said to be *adjacent* if there exist $i, j \in E$ such that $X_k = Y_k$ for all $k \notin \{i, j\}$, $X_i = 0 \neq Y_i$ and $Y_j = 0 \neq X_j$.

This notion of adjacency defines a graph G_n whose vertex set is the family of all signed vectors. It naturally leads to the notion of *moving a zero* from one place to another (nonzero place) which is a step of a walk in the graph. Therefore, the distance in G_n from one vector to another is the minimum number of moves of zeros needed to reach the destination vector. This motivates the following definition: the *traversen* of two signed vectors X, Y is

$$T(X, Y) = \{e \in E\colon\ X_e = 0 \neq Y_e \text{ or } Y_e = 0 \neq X_e\}.$$

Remark 2. X and Y are adjacent in G_n if and only if $S(X, Y) = \emptyset$ and $|T(X, Y)| = 2$.

An *oriented matroid* $\mathcal{M} = (E, \mathcal{C})$ of order $n = |E|$ is a set of signed vectors,

$$\mathcal{C} \subseteq \{-, 0, +\}^E,$$

with the following properties (cf. [2, p. 103]):

(C1) $\mathbf{0} \notin \mathcal{C}$,
(C2) $X \in \mathcal{C} \Rightarrow -X \in \mathcal{C}$,
(C3) $X, Y \in \mathcal{C}$ and $X^{\pm} \subseteq Y^{\pm} \Rightarrow X = \pm Y$,
(C4) $X, Y \in \mathcal{C}$, $X \neq \pm Y$ and $X_e = -Y_e \neq 0 \Rightarrow$ there exists $Z \in \mathcal{C}$ such that $Z^+ \subseteq X^+ \cup Y^+$, $Z^- \subseteq X^- \cup Y^-$ and $Z_e = 0$.

The elements of $\mathcal{C}$ are known as the *circuits* of the oriented matroid. We say that the oriented matroid is *d-dimensional* if the maximum size of a subset $X \subseteq E$ not containing the support of a circuit (the size of a maximum *independent* set) is $d + 1$. The oriented matroid is said to be *uniform* if all its circuits have support size $d + 2$.

Given an oriented matroid $\mathcal{M} = (E, \mathcal{C})$, its set of *vectors*, $\mathcal{V} = \mathcal{V}(\mathcal{M})$, is the minimal superset of $\mathcal{C}$ closed under composition ($X, Y \in \mathcal{V} \Rightarrow X \circ Y \in \mathcal{V}$), where *composition* is defined as

$$(X \circ Y)_e = \begin{cases} X_e, & \text{if } X_e \neq 0, \\ Y_e, & \text{otherwise.} \end{cases}$$

Finally, in the proof of Theorem 1, we will also use the following (cf. [2, Lemma 4.1.8]). If $\mathcal{V}$ denotes the set of vectors of an oriented matroid $\mathcal{M}$ on E and $A \subset E$, then the set of vectors of the *deletion* $\mathcal{M} \setminus A$ equals

$$\mathcal{V} \setminus A = \{X_{E \setminus A}\colon\ X \in \mathcal{V}\}$$

and the set of vectors of the *contraction* $\mathcal{M}/A$ equals

$$\mathcal{V}/A = \big\{X_{E \setminus A}\colon\ X \in \mathcal{V} \text{ and } A \subset X^0\big\},$$

where $X_{E \setminus A} \in \{-, 0, +\}^{E \setminus A}$ denotes the restriction of X to the ground set $E \setminus A$.

2.2. *The k-dual of the n-cube and the circuit graph of a uniform oriented matroid*

Associated with every oriented matroid $\mathcal{M} = (E, \mathcal{C})$ is a graph $\mathcal{G} = \mathcal{G}(\mathcal{M})$ whose vertices are the circuits of the matroid. Two of those vertices $X, Y \in \mathcal{C}$ are adjacent if $X \circ Y = Y \circ X$, and $Z \in \{X, Y\}$ for every circuit Z with $Z \leqslant X \circ Y$. This graph is what we call the *circuit graph* of the oriented matroid—in the literature [1,3] this graph is studied via the dual oriented matroid so it is better known as the *cocircuit graph*. It is well known that the cocircuit graph of an oriented matroid is the 1-skeleton of the cell decomposition of the sphere induced by the pseudospheres that realize the oriented matroid via the Topological Representation Theorem of Folkman and Lawrence [4]. Therefore, if $\mathcal{M} = (E, \mathcal{C})$ is a uniform oriented matroid, then two (co)circuits X, Y are adjacent in $\mathcal{G} = \mathcal{G}(\mathcal{M})$ if and only if $|S(X, Y)| = 0$ and $|T(X, Y)| = 2$. That is to say, in the topological representation, a vertex is the intersection of a fixed number of pseudospheres, say d, and each line is the intersection of $d - 1$ pseudospheres. So, while moving from vertex X to vertex Y, we traverse an edge (representing the vector $X \circ Y$) "losing" one zero, and when we reach the destination vertex Y, we "gain" another zero—recall the notion of "moving a zero."

Let $\mathcal{Q}_n^k$ denote the *k-dual of $\mathcal{Q}_n$* defined as follows ($k > 0$): the vertices of $\mathcal{Q}_n^k$ are the k-subcubes of $\mathcal{Q}_n$, two of which are adjacent if their respective subcubes intersect in a $(k - 1)$-subcube. Observe that $\mathcal{Q}_n^k$ is a graph of order $2^{n-k}\binom{n}{k}$ whose vertex set $V(\mathcal{Q}_n^k)$ can be identified

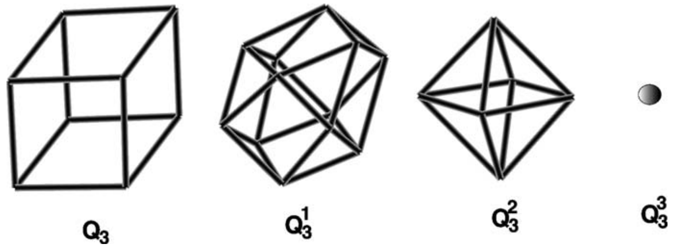

Fig. 1. The three k-duals of $\mathcal{Q}_3$.

with the set of signed vectors $\{X \in \{-, 0, +\}^n\colon\ |X^0| = k\}$ such that a pair $\{X, Y\}$ represents an edge of $\mathcal{Q}_n^k$ if and only if $|S(X, Y)| = 0$ and $|T(X, Y)| = 2$.

A graph *embedding* $G \hookrightarrow H$ is an injective function $f : V(G) \to V(H)$ of its vertices that sends edges to edges. In such a case we will identify the vertices of the domain with those of its image, and we will refer to the vertices of the domain with the name of their respective images. In particular, if a graph is embedded in $\mathcal{Q}_n^k$, each vertex of the graph will be denoted by the signed vector of its image. As usual, an embedding is said to be *isometric* if the graph distance of the domain is preserved by its image. An embedding $G \hookrightarrow \mathcal{Q}_n^k$ is said to be *antipodal* if it is closed by the antipodal automorphism of $\mathcal{Q}_n^k$; that is, if $X \in V(G)$ then $-X \in V(G)$.

From the definitions, it follows that $G_n = \bigcup_{k=1}^{n} \mathcal{Q}_n^k$, i.e., each $\mathcal{Q}_n^k$ is a connected component of G_n (Fig. 1). Clearly there is a natural embedding of the circuit graph of a uniform oriented matroid into $\mathcal{Q}_n^k$ for some n and k. Therefore, G_n is the *universal graph* for the circuit graphs that arise from uniform oriented matroids of order n.

3. Main theorem

In order to isolate the problem, let us summarise some basic observations in the following

Lemma 3.

(I) *If $\mathcal{G}$ is the circuit graph of a d-dimensional uniform oriented matroid of order $n > d + 2$ then there exists an antipodal embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^{n-d-2}$ such that*

$$X, Y \in V(\mathcal{G}) \quad \text{and} \quad X^{\pm} = Y^{\pm} \quad \Rightarrow \quad X = \pm Y. \tag{$*$}$$

(II) *If $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ is an antipodal embedding that satisfies* $(*)$, *then $V(\mathcal{G})$ fulfils the axioms* (C1), (C2) *and* (C3) *of an oriented matroid.*

Proof. (I) The existence of the embedding is clear. The embedding must by antipodal because of axiom (C2), and axiom (C3) implies $(*)$.

(II) The embedding and the antipodality of the embedding implies that the vertices of $\mathcal{G}$ fulfil (C1) and (C2). Since all vertices have the same support size, it follows from condition $(*)$ that axiom (C3) holds. □

Considering Lemma 3, we can see that, in order to find a characterization of the circuit graphs of uniform oriented matroids in terms of an embedding in $\mathcal{Q}_n^{n-d-2}$, we need to focus our attention on translating the weak elimination axiom (C4) into some additional properties of the embedding different from antipodality and condition $(*)$. As we will see, Theorem 1 implies that a sufficient

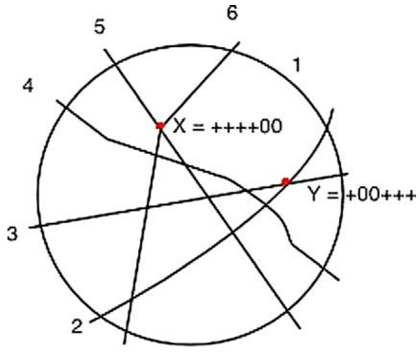

Fig. 2. A non-isometric embedding.

condition for a graph to be a circuit graph of a uniform oriented matroid is that the embedding is isometric (cf. [5]). However, it is not a necessary condition, as Fig. 2 illustrates. In it, the vertices X and Y are non-antipodal (in fact we are depicting only the projective half of the oriented matroid); $d_{\mathcal{Q}_n^{n-d-2}}(X,Y) = |S(X,Y)| + \frac{1}{2}|T(X,Y)| = 2$ but $d_{\mathcal{G}}(X,Y) = 3$.

The problem with isometric embeddings is that they require for every pair $X \neq \pm Y \in V(\mathcal{G})$ that there is an XY-path $P \subset \mathcal{G}$, such that for every $Z \in V(P)$ it follows that $d_{\mathcal{Q}_n^k}(Z,Y) < d_{\mathcal{Q}_n^k}(X,Y)$, and therefore (see Lemma 5 below)

$$\left|S(Z,Y)\right| < \left|S(X,Y)\right| \quad \text{or} \quad \left|T(Z,Y)\right| < \left|T(X,Y)\right|.$$

Furthermore, if $P = (X = Z^0, Z^1, \ldots, Z^m = Y)$ is a geodesic path, then for every $\ell < m$,

$$d_{\mathcal{Q}_n^{n-d-2}}\left(X, Z^\ell\right) < d_{\mathcal{Q}_n^{n-d-2}}\left(X, Z^{\ell+1}\right) \quad \text{and} \quad d_{\mathcal{Q}_n^{n-d-2}}\left(Y, Z^{\ell+1}\right) < d_{\mathcal{Q}_n^{n-d-2}}\left(Y, Z^\ell\right).$$

In other words, from the point of view of $\mathcal{Q}_n^{n-d-2}$, at every step along P, its elements move away from X and get closer to Y. But oriented matroids are more flexible than this.

Let us introduce a weaker (hence more general) concept of "metric" embeddings. An embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ is said to be a *crabbed embedding* if for every $i = 1, \ldots, n$ and every pair of vertices $X \neq \pm Y \in V(\mathcal{G})$ there exists an XY-path $P = (X = Z^0, Z^1, \ldots, Z^m = Y)$ in $\mathcal{G}$ such that for every pair of successive vertices $Z^\ell, Z^{\ell+1}$ of P, it happens that $(Z^{\ell+1})_i \in \{X_i, Y_i\}$ if $(Z^\ell)_i = (Z^{\ell+1})_j = 0 \neq (Z^\ell)_j$ (i.e., if we are moving a zero from i to j). In a certain way, every step takes "the right direction" in $\mathcal{Q}_n^k$. Then, for every $Z \in V(P)$ we have that

$$S(Z,Y) \subseteq S(X,Y) \quad \text{and} \quad \left|T(Z,Y)\right| \leqslant \left|T(X,Y)\right|,$$

and, as a consequence, $d_{\mathcal{Q}_n^k}(Z,Y) \leqslant d_{\mathcal{Q}_n^k}(X,Y)$. The XY-path P will be called a *crab path*. Clearly, an isometric embedding is crabbed.

Lemma 4. *Let $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ be an embedding. If $P \subset \mathcal{G}$ is a crab path from X to Y then for every $e \in S(X,Y)$ there exists a $Z \in V(P)$ such that $Z_e = 0$, $Z^- \subseteq X^- \cup Y^-$ and $Z^+ \subseteq X^+ \cup Y^+$.*

Proof. Changing an element of the separator $S(X,Y)$ from one sign to the other, while walking in an XY-path into $\mathcal{Q}_n^k$, requires to move the sign to zero and, after that, to the other sign. Thus in every XY-path and for every element in the separator, there exists a vertex Z with a zero in that position. To see that such a vertex satisfies the extra sign conditions, let $P = (X = Z^0, Z^1, \ldots, Z^{m-1}, Z^m = Y)$ be a crab XY-path. Now, let us suppose that there exists $s < m$

such that $(Z^s)^+ \nsubseteq X^+ \cup Y^+$. Let $i \in (Z^s)^+ \setminus (X^+ \cup Y^+)$ and $\ell = \min\{r\colon (Z^r)_i = (Z^s)_i = +\}$. Since $X_i \neq (Z^\ell)_i$, we have $\ell > 0$. By Remark 2, it follows that $(Z^{\ell-1})_i = 0$. But then, the step $(Z^{\ell-1}, Z^\ell)$ is not allowed in a crab XY-path, and this is a contradiction. □

Proof of Theorem 1. The sufficiency of Theorem 1 follows from the order of $\mathcal{G}$ and Lemmas 3 and 4. That is to say, given a graph $\mathcal{G}$ with the properties stated in Theorem 1, it follows that $\mathcal{G}$ is the circuit graph of a d-dimensional uniform oriented matroid of order $n > d + 2$.

For the necessity of Theorem 1, let $\mathcal{G} \hookrightarrow \mathcal{Q}_n^{n-d-2}$ be the natural embedding of the circuit graph $\mathcal{G}$ of an uniform oriented matroid $\mathcal{M} = (E, \mathcal{C})$. By Lemma 3, it remains to show that this embedding is crabbed. To do this, we will consider three cases. For $X \neq \pm Y \in V(\mathcal{G})$:

Case 1: $S(X, Y) = \emptyset = X^0 \cap Y^0$. In this case the composition $\tau = X \circ Y$ is a tope of the dual oriented matroid. In the topological representation this tope is the ball that results from intersecting a number of closed semispaces (cf. [2, Proposition 4.3.6]). We may suppose (via reorientation)—and with a little abuse of notation—that

$$\tau = \bigcap H_i^+ = + + \cdots +,$$

where $H_i^+ = \{V \in \mathcal{V}(\mathcal{M})\colon V_i \in \{0, +\}\}$. Since the boundary of such a ball $\partial\tau$ is connected and contains both X and Y, it contains a geodesic XY-path $P \subset \partial\tau$. As we walk along this path from X to Y, the signs of the elements of the path are those of the tope. As a consequence, it follows that P is a crab path.

For the remaining two cases, we will proceed by induction. Clearly if $|E| = 3$ then the embedding is crabbed. Now, let us suppose that for every oriented matroid of order less than n such an embedding is crabbed, and let $|E| = n$.

Case 2: $X^0 \cap Y^0 \neq \emptyset$. Let $j \in X^0 \cap Y^0$. Consider the matroid $\mathcal{M}' = \mathcal{M}/j$ and denote by $\mathcal{G}'$ its circuit graph. Also let X', Y' denote the contraction of the circuits X and Y. Clearly X' and Y' are two non-antipodal vertices. By the induction hypothesis, there exists a crab path $P' = (X' = Z^0, Z^1, \ldots, Z^m = Y')$ in $\mathcal{Q}_n^{n-d-3}$. Now, if we restore the jth coordinate to each vector Z^ℓ—i.e., if we consider the vectors $\hat{Z}^\ell \in \mathcal{C}$ for which $(\hat{Z}^\ell)_k = (Z^\ell)_k$ (for all $k \neq j$)—each vector receives a zero in that coordinate. Therefore, if Z^ℓ and $Z^{\ell+1}$ are adjacent in $\mathcal{G}'$ then $\hat{Z}^\ell$ and $\hat{Z}^{\ell+1}$ are adjacent in $\mathcal{G}$. So, the path $P = (X = \hat{Z}^0, \hat{Z}^1, \ldots, \hat{Z}^m = Y)$ is an XY-path in $\mathcal{Q}_n^{n-d-2}$ and since P' is a crab path, it is not difficult to see that P is also a crab path.

Case 3: $X^0 \cap Y^0 = \emptyset$ and $S(X, Y) \neq \emptyset$. Let $k \in S(X, Y)$. Consider the matroid $\mathcal{M}' = \mathcal{M} \setminus k$ and denote by $\mathcal{G}'$ its circuit graph. Also let X', Y' denote the restriction of the circuits X and Y. Once again, X' and Y' are two non-antipodal vertices. By the induction hypothesis, there exists a crab path $P' = (X' = Z^0, Z^1, \ldots, Z^m = Y')$. Restore the kth coordinate to each vector Z^ℓ, and consider the succession $(X = \hat{Z}^0, \hat{Z}^1, \ldots, \hat{Z}^m = Y)$ in $\mathcal{G}$. If $\hat{Z}^\ell$ and $\hat{Z}^{\ell+1}$ are not adjacent in $\mathcal{G}$, since Z^ℓ and $Z^{\ell+1}$ are adjacent in $\mathcal{G}'$ (i.e., $S(Z^\ell, Z^{\ell+1}) = \emptyset$ and $|T(Z^\ell, Z^{\ell+1})| = 2$), we must have that

$$S(\hat{Z}^\ell, \hat{Z}^{\ell+1}) = \{k\} \quad \text{and} \quad \left|T(\hat{Z}^\ell, \hat{Z}^{\ell+1})\right| = 2.$$

Let $T(\hat{Z}^\ell, \hat{Z}^{\ell+1}) = \{r, s\}$. Now, due to the weak elimination axiom (C4), there exists a circuit $W \in \mathcal{C}$ such that $k \in W^0$, $W^+ \subseteq (\hat{Z}^\ell)^+ \cup (\hat{Z}^{\ell+1})^+$ and $W^- \subseteq (\hat{Z}^\ell)^- \cup (\hat{Z}^{\ell+1})^-$. Observe that for every $i \notin \{r, s, k\}$, $W_i = (\hat{Z}^\ell)_i = (\hat{Z}^{\ell+1})_i$ and so, without loss of generality, suppose $0 = (\hat{Z}^\ell)_r \neq W_r = (\hat{Z}^{\ell+1})_r$ and $0 = (\hat{Z}^{\ell+1})_s \neq W_s = (\hat{Z}^\ell)_s$.

From here it is not difficult to see the following three facts:

(i) W is adjacent to both $\hat{Z}^\ell$ and $\hat{Z}^{\ell+1}$, and therefore $(\hat{Z}^\ell, W, \hat{Z}^{\ell+1})$ is a path in $\mathcal{G}$.
(ii) If $(\hat{Z}^\ell)_i = W_j = 0 \neq (\hat{Z}^\ell)_j$, then $j = k$, $i \neq s$ and as a result $W_i = (Z^{\ell+1})_i \in \{X_i, Y_i\}$.
(iii) If $W_i = (Z^{\ell+1})_j = 0 \neq W_j$, then $j = s$ and $i \neq r$. Since $k \in S(X, Y)$, if $i = k$ then $(Z^{\ell+1})_i \in \{X_i, Y_i\}$; otherwise $W_i = (Z^\ell)_i$ and $W_j = (Z^\ell)_j$. Accordingly, we have that $(Z^\ell)_i = (Z^{\ell+1})_j = 0 \neq (Z^\ell)_j$ which, on its turn, implies that $(Z^{\ell+1})_i \in \{X_i, Y_i\}$ since P' is a crab path.

In this way we can construct an XY-path in $\mathcal{G}$ from the succession $(X = \hat{Z}^0, \hat{Z}^1, \ldots, \hat{Z}^m = Y)$ which is a crab path since P' is a crab path. □

Let us present the following

Lemma 5. *The graph distance in $\mathcal{Q}_n^k$ $(k > 0)$ is, for $X \neq Y$*

$$d_{\mathcal{Q}_n^k}(X, Y) = \begin{cases} |S(X, Y)| + 1, & \text{if } X^\pm = Y^\pm, \\ |S(X, Y)| + \frac{1}{2}|T(X, Y)|, & \text{otherwise.} \end{cases}$$

The proof of this lemma is postponed to Section 4. From Lemma 5 it is not difficult to see:

Corollary 6. *A graph $\mathcal{G}$ is the circuit graph of a d-dimensional uniform oriented matroid of order $n > d + 2$ if and only if its order is $2\binom{n}{d+2}$ and there exists an antipodal crabbed embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^{n-d-2}$ such that for every pair $X \neq \pm Y \in V(\mathcal{G})$,*

$$d_{\mathcal{Q}_n^{n-d-2}}(X, Y) = \left|S(X, Y)\right| + \frac{1}{2}\left|T(X, Y)\right|.$$

Corollary 6 leads to a new axiomatization of uniform oriented matroids. However, the hypothesis of uniformity cannot be dropped without adding a new ingredient, because otherwise, the circuit graph of a non-uniform oriented matroid may not be embeddable in $\mathcal{Q}_n^k$. We believe that there should be a notion of distance related to the first barycentric subdivision of the n-cube that leads to a similar theorem for the general (non-uniform) case.

4. The distance lemma

Proof of Lemma 5. Let $X, Y \in V(\mathcal{Q}_n^k)$. First of all, we exhibit an XY-path with the desired length—this will show that the distance in $\mathcal{Q}_n^k$ is at most that in the statement of the lemma. There are four cases:

Case 1: $S(X, Y) = \emptyset$ and $T(X, Y) = \emptyset$. This condition is equivalent to $X = Y$.

Case 2: $S(X, Y) = \emptyset$ and $T(X, Y) \neq \emptyset$. Let $T_0(X, Y) = \{i \in E\colon\ X_i = 0 \neq Y_i\}$ (analogously $T_0(Y, X) = \{i \in E\colon\ Y_i = 0 \neq X_i\}$). Clearly $T(X, Y) = T_0(X, Y) \cup T_0(Y, X)$ and, since they have the same support size, $|T_0(X, Y)| = |T_0(Y, X)|$. Let us assign an arbitrary (but fixed) linear order to both previously defined sets: $T_0(X,Y) = (\tau_1, \ldots, \tau_{|T_0(X,Y)|})$ and $T_0(Y,X) = (\pi_1, \ldots, \pi_{|T_0(Y,X)|})$. Now, let $\{Z^1, Z^2, \ldots, Z^{\frac{1}{2}|T(X,Y)|}\}$ be defined as follows:

$$\left(Z^m\right)_i = \begin{cases} Y_i, & \text{if } i \in \{\tau_1, \ldots, \tau_m, \pi_1, \ldots, \pi_m\}, \\ X_i, & \text{otherwise.} \end{cases}$$

Observe that

$$S\left(X, Z^1\right) = S\left(Z^1, Z^2\right) = \cdots = S\left(Z^{(1/2)|T(X,Y)|-1}, Y\right) = \emptyset,$$
$$\left|T\left(X, Z^1\right)\right| = \left|T\left(Z^1, Z^2\right)\right| = \cdots = \left|T\left(Z^{(1/2)|T(X,Y)|-1}, Y\right)\right| = 2,$$

and $Z^{(1/2)|T(X,Y)|} = Y$. Therefore, by Remark 2, $(X, Z^1, Z^2, \ldots, Z^{(1/2)|T(X,Y)|} = Y)$ is an XY-path and its length is $\frac{1}{2}|T(X,Y)|$.

Case 3: $S(X,Y) \neq \emptyset$ and $T(X,Y) \neq \emptyset$. Let $\tau \in \{i \in E\colon X_i = 0 \neq Y_i\}$. Assign an arbitrary (but fixed) linear order to the separator $S(X,Y) = (\sigma_1, \ldots, \sigma_{|S(X,Y)|})$. Let $\{Z^1, Z^2, \ldots, Z^{|S(X,Y)|}\}$ be defined as follows:

$$\left(Z^m\right)_i = \begin{cases} Y_i, & \text{if } i \in \{\tau, \sigma_1, \ldots, \sigma_{m-1}\}, \\ 0, & \text{if } i = \sigma_m, \\ X_i, & \text{otherwise.} \end{cases}$$

Observe that,

$$S\left(X, Z^1\right) = S\left(Z^1, Z^2\right) = \cdots = S\left(Z^{|S(X,Y)|-1}, Z^{|S(X,Y)|}\right) = \emptyset \quad \text{and}$$
$$\left|T\left(X, Z^1\right)\right| = \left|T\left(Z^1, Z^2\right)\right| = \cdots = \left|T\left(Z^{|S(X,Y)|-1}, Z^{|S(X,Y)|}\right)\right| = 2.$$

Furthermore, $S(Z^{|S(X,Y)|}, Y) = \emptyset$ and

$$\left|T\left(Z^{|S(X,Y)|}, Y\right)\right| = \left|T(X,Y) \setminus \{\tau\} \cup \{\sigma_{S(X,Y)}\}\right| = \left|T(X,Y)\right|.$$

Now, let us construct a $Z^{|S(X,Y)|}Y$-path as in the previous case. Both paths combined become an XY-path of the desired length.

Case 4: $S(X,Y) \neq \emptyset$ and $T(X,Y) = \emptyset$. Let $\sigma \in S(X,Y)$ and $i_0 \in X^0 = Y^0$ be arbitrary (but fixed). Let Z^1 be defined as follows

$$\left(Z^1\right)_i = \begin{cases} 0, & \text{if } i = \sigma, \\ +, & \text{if } i = i_0, \\ X_i, & \text{otherwise.} \end{cases}$$

Observe that $S(Z^1, Y) = S(X,Y) \setminus \{\sigma\}$ and $T(Z^1, Y) = \{\sigma, i_0\}$. Therefore the previous cases apply.

To finish the proof, we have to show that the distance in $\mathcal{Q}_n^k$ is at least as stated in the lemma. We do it by induction.

Let $d : V(\mathcal{Q}_n^k) \times V(\mathcal{Q}_n^k) \to \mathbb{N}$ be the function

$$d(X,Y) = \begin{cases} |S(X,Y)| + 1, & \text{if } X^\pm = Y^\pm \text{ and } X \neq Y, \\ |S(X,Y)| + \frac{1}{2}|T(X,Y)|, & \text{otherwise.} \end{cases} \tag{$**$}$$

By Remark 2, it follows that $d(X,Y) = 1$ if and only if $d_{\mathcal{Q}_n^k}(X,Y) = 1$. Let us suppose that for every X, Y and for every $m < m_0$ ($m_0 \geqslant 2$), we have that $d(X,Y) = m$ if and only if

$d_{Q_n^k}(X,Y)=m$. Let $(X, Z^1, \ldots, Z^{m_0}=Y)$ be a geodesic XY-path (of length $d_{Q_n^k}(X,Y)$). Since we want to prove that $d(X,Y)\leqslant m_0$, suppose that $d(X,Y)>m_0$.

Since the path is geodesic, it follows that $d_{Q_n^k}(X,Y)=d_{Q_n^k}(X,Z^1)+d_{Q_n^k}(Z^1,Y)$, which by hypothesis implies that $m_0=1+d(Z^1,Y)$. Hence $d(X,Y)>1+d(Z^1,Y)$.

Let

$$\delta_{XY}=\begin{cases}1, & \text{if } X^{\pm}=Y^{\pm} \text{ and } X\neq Y,\\ 0, & \text{otherwise.}\end{cases}$$

Then we can denote $d(X,Y)=|S(X,Y)|+\frac{1}{2}|T(X,Y)|+\delta_{XY}$ including in one equation both cases of $(**)$. Recall that $X^{\pm}=Y^{\pm}$ if and only if $T(X,Y)=\emptyset$.

With this notation we have that

$$\bigl|S(X,Y)\bigr|+\frac{1}{2}\bigl|T(X,Y)\bigr|+\delta_{XY}>1+\bigl|S\bigl(Z^1,Y\bigr)\bigr|+\frac{1}{2}\bigl|T\bigl(Z^1,Y\bigr)\bigr|+\delta_{Z^1Y}.$$

Since X is adjacent to Z^1, there exist $i,j\in E$ such that for every $\ell\notin\{i,j\}$; $X_\ell=(Z^1)_\ell$, $X_i=0\neq(Z^1)_i$ and $X_j\neq 0=(Z^1)_j$. Then $S(X,Y)$ and $S(Z^1,Y)$, and respectively $T(X,Y)$ and $T(Z^1,Y)$, differ only in the ith and jth coordinates. This motivates the following notation: Given $i,j\in E$, let $S_{ij}(X,Y)=\{i,j\}\cap S(X,Y)$ and $T_{ij}(X,Y)=\{i,j\}\cap T(X,Y)$. Thus,

$$\bigl|S_{ij}(X,Y)\bigr|+\frac{1}{2}\bigl|T_{ij}(X,Y)\bigr|+\delta_{XY}>1+\bigl|S_{ij}\bigl(Z^1,Y\bigr)\bigr|+\frac{1}{2}\bigl|T_{ij}\bigl(Z^1,Y\bigr)\bigr|+\delta_{Z^1Y}.$$

We consider two cases:

Case 1: $T_{ij}(X,Y)=\emptyset$. Since $X_i=0\neq(Z^1)_i$ and $X_j\neq 0=(Z^1)_j$, we have $Y_i=0$ and $Y_j\neq 0$. Therefore, $\{i,j\}\subset T_{ij}(Z^1,Y)$ and $i\notin S_{ij}(X,Y)$. But

$$2\geqslant\bigl|S_{ij}(X,Y)\bigr|+\frac{1}{2}\bigl|T_{ij}(X,Y)\bigr|+\delta_{XY}>1+\bigl|S_{ij}\bigl(Z^1,Y\bigr)\bigr|+\frac{1}{2}\bigl|T_{ij}\bigl(Z^1,Y\bigr)\bigr|+\delta_{Z^1Y}\geqslant 2,$$

an obvious contradiction.

Case 2: $T_{ij}(X,Y)\neq\emptyset$. Clearly, in this case, $\delta_{XY}=0$. Then

$$\bigl|S_{ij}(X,Y)\bigr|+\frac{1}{2}\bigl|T_{ij}(X,Y)\bigr|>1+\bigl|S_{ij}\bigl(Z^1,Y\bigr)\bigr|+\frac{1}{2}\bigl|T_{ij}\bigl(Z^1,Y\bigr)\bigr|.$$

Since $X_i=0$ we have $i\notin S_{ij}(X,Y)$, which implies $j\in S_{ij}(X,Y)$. Therefore $j\in T_{ij}(Z^1,Y)$, which implies that

$$1+\frac{1}{2}\geqslant\bigl|S_{ij}(X,Y)\bigr|+\frac{1}{2}\bigl|T_{ij}(X,Y)\bigr|>1+\bigl|S_{ij}\bigl(Z^1,Y\bigr)\bigr|+\frac{1}{2}\bigl|T_{ij}\bigl(Z^1,Y\bigr)\bigr|\geqslant 1+\frac{1}{2},$$

a new contradiction.

This concludes the proof. □

Acknowledgments

The authors express their gratitude to the referees for their insightful comments and remarks, which helped to improve the quality of this paper.

References

[1] E. Babson, L. Finschi, K. Fukuda, Cocircuit graphs and efficient orientation reconstruction in oriented matroids, European J. Combin. 22(5) (2001) 587–600.
[2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G.M. Ziegler, Oriented Matroids, Encyclopedia Math. Appl., vol. 43, Cambridge Univ. Press, Cambridge, UK, 1993.
[3] L. Finschi, K. Fukuda, Generation of oriented matroids—A graph theoretical approach, Discrete Comput. Geom. 27 (2002) 117–136.
[4] J. Folkman, J. Lawrence, Oriented matroids, J. Combin. Theory Ser. B 25 (1978) 199–236.
[5] K. Fukuda, K. Handa, Antipodal graphs and oriented matroids, Discrete Math. 111 (1993) 245–256.
[6] J.J. Montellano-Ballesteros, R. Strausz, Counting polytopes via the Radon complex, J. Combin. Theory Ser. A 106 (2004) 109–121.
[7] R. Strausz, Separoides: el Complejo de Radon, MSc thesis, Universidad Nacional Autónoma de México, México, http://www.math.unam.mx/strausz/Papers/MScThesis.pdf, 2001.