# A tight bound on the collection of edges in MSTs of induced subgraphs 

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Let $G=(V, E)$ be a complete $n$-vertex graph with distinct positive edge weights. We prove that for $k \in\{1,2, \ldots, n-1\}$, the set consisting of the edges of all minimum spanning trees (MSTs) over induced subgraphs of $G$ with $n-k+1$ vertices has at most $n k-\binom{k+1}{2}$ elements. This proves a conjecture of Goemans and Vondrak [1]. We also show that the result is a generalization of Mader's Theorem, which bounds the number of edges in any edge-minimal $k$-connected graph.

## 1 Introduction

Let $G=(V, E)$ be a complete $n$-vertex graph with distinct positive edge weights. For any set $X \subseteq V$, denote by $G[V \backslash X]$ the subgraph of $G$ induced by $V \backslash X$. We will also sometimes write this graph as $(V \backslash X, E)$, ignoring edges in $E$ incident on vertices in $X$. $\operatorname{MST}(G[V \backslash X])$ denotes the set of edges in the graph's minimum spanning tree. (The MST is unique due to the assumption that the edge weights are distinct.)

For $k \in\{1,2, \ldots, n-1\}$, define

$$
M_{k}(G)=\bigcup_{X \subseteq V,|X|=k-1} \operatorname{MST}(G[V \backslash X])
$$

Note that for $k=1$ we have $M_{1}(G)=\operatorname{MST}(G)$. In [1, Goemans and Vondrak considered the problem of finding a sparse set of edges which, with high probability, contain the MST of a random subgraph of $G$. In this context they proved an upper bound on $M_{k}(G)$, namely that $\left|M_{k}(G)\right|<\left(1+\frac{e}{2}\right) k n$, and they conjectured that one should be able to improve the bound to $\left|M_{k}(G)\right| \leq n k-\binom{k+1}{2}$. In this paper we prove this conjecture.

## Theorem 1

For any complete graph $G$ on $n$ vertices with distinct positive edge weights,

$$
\begin{equation*}
\left|M_{k}(G)\right| \leq n k-\binom{k+1}{2} . \tag{1}
\end{equation*}
$$

As Goemans and Vondrak recognized, the bound is tight: for any $n$ and $k$ it is easy to produce edge weights giving equality in (1). One way is to fix an arbitrary set $V^{\prime} \subseteq V$ with cardinality $k$, and partition the edges $E$ into three sets $E_{0}, E_{1}$ and $E_{2}$ where, for $i \in\{0,1,2\}, E_{i}$ contains all edges of $E$ having exactly $i$ endpoints in $V^{\prime}$. Assign arbitrary distinct positive weights to the edges in $E$ such that all weights on $E_{2}$ are smaller than those on $E_{1}$, which in turn are smaller than those on $E_{0}$. It can easily be verified that $M_{k}(G)=E_{2} \cup E_{1}$ and thus $\left|M_{k}(G)\right|=n k-\binom{k+1}{2}$.

Theorem $\mathrm{s}_{\mathrm{s}}$ assumption that $G$ is complete is not meaningfully restrictive. If $G$ is such that deletion of some $k-1$ vertices leaves it disconnected, then the notion of $M_{k}(G)$ does not make sense; otherwise, it does not matter if other edges of $G$ are simply very costly or are absent.

The bound of Theorem 1 applies equally if we consider the edge set of MSTs of induced subgraphs of size at most $n-k+1$ (rather than exactly that number). This is an immediate consequence of the following remark.

## Remark 2

For any complete graph $G$ on $n$ vertices with distinct positive edge weights, and $k \in\{1,2, \ldots, n-2\}, M_{k+1}(G) \supseteq M_{k}(G)$.

Proof. We will show that any edge $e$ in $M_{k}(G)$ is also in $M_{k+1}(G)$. By definition, $e \in M_{k}(G)$ means that there is some vertex set $X$ of cardinality $|X|=k-1$ for which $e \in \operatorname{MST}\left(G_{k}\right)$, where $G_{k}=G[V \backslash X]$.

Consider any leaf vertex $v$ of $\operatorname{MST}\left(G_{k}\right)$, with neighbor $u$. We claim that deleting $v$ from $G_{k}$ (call the resulting graph $G_{k+1}$ ) results in the same MST less the edge $\{u, v\}$, i.e., that $\operatorname{MST}\left(G_{k+1}\right)=\operatorname{MST}\left(G_{k}\right) \backslash\{\{u, v\}\}$. This follows from considering the progress of Kruskal's algorithm on the two graphs. Before edge $\{u, v\}$ is added to $\operatorname{MST}\left(G_{k}\right)$, the two processes progress identically: every edge added to $\operatorname{MST}\left(G_{k}\right)$ is also a cheapest edge for the smaller graph $G_{k+1}$. The edge $e$, added to $\operatorname{MST}\left(G_{k}\right)$, of course has no parallel in $G_{k+1}$. As further edges are considered in order of increasing cost, again, every edge added to $\operatorname{MST}\left(G_{k}\right)$ will also be added to $\operatorname{MST}\left(G_{k+1}\right)$, using the fact that none of these edges is incident on $v$.

Thus, if $v$ is not a vertex of $e$, then $e \in \operatorname{MST}\left(G_{k+1}\right)$. Since $\operatorname{MST}\left(G_{k}\right)$ has at least two leaves, it has at least one leaf $v$ not in $e$, unless $\operatorname{MST}\left(G_{k}\right)=e$, which is impossible since $G_{k}$ has at least 3 vertices.

## Outline of the paper

In Section 2 we define a " $k$-constructible" graph, and show that every graph $\left(V, M_{k}(G)\right)$ is $k$-constructible, and every $k$-constructible graph is a subgraph of some graph $\left(V, M_{k}(G)\right)$. This allows a simpler reformulation of Theorem 1 as Theorem 6, which also generalizes a theorem of Mader [3]. We prove Theorem 6] in Section 3.

## $2 k$-constructible graphs

We begin by recalling Menger's theorem for undirected graphs, which motivates our definition of $k$-constructible graphs. Two vertices in an undirected graph are called $k$-connected if there are $k$ (internally) vertex-disjoint paths connecting them.

## Theorem 3 (Menger's theorem)

Let $s, t$ be two vertices in an undirected graph $G=(V, E)$ such that $\{s, t\} \notin E$. Then $s$ and $t$ are $k$-connected in $G$ if and only if after deleting any $k-1$ vertices (distinct from $s$ and $t$ ), $s$ and $t$ are still connected.

Definition 4 ( $k$-constructible graph)
A graph $G=(V, E)$ is called $k$-constructible if there exists an ordering $O=\left\langle e_{1}, e_{2}, \ldots, e_{m}\right\rangle$ of the edges in $E$ such that for all $i \in\{1,2, \ldots, m\}$ the graph ( $\left.V,\left\{e_{1}, e_{2}, \ldots, e_{i-1}\right\}\right)$ contains at most $k-1$ vertex-disjoint paths between the two endpoints of $e_{i}$. We say that $O$ is a $k$-construction order for the graph $G$.

Note that 1-constructible graphs are forests, and edge-maximal 1-constructible graphs are spanning trees. We therefore have in particular that graphs of the form $M_{1}(G)$ (i.e., MSTs, recalling the $G$ is complete) are edge-maximal 1-constructible graphs. A slightly weaker statement is true for all $k$ : every graph $M_{k}(G)$ is $k$-constructible (Theorem 5II), and every $k$-constructible graph is a subgraph of some graph $M_{k}(G)$ (Theorem 5Fiii).
Note that a stronger statement, that the graphs of the form $M_{k}(G)$ are exactly the edgemaximal $k$-constructible graphs, is not true. To see this consider a cycle $C_{4}$ of length four. Assign weights $1, \ldots, 4$ to these four edges (in arbitrary order) and weights 5,6 to the remaining edges of the complete graph on four vertices. It is easily checked that $M_{2}(G)=C_{4}$. But $M_{2}(G)$ is not edge-maximal, as a diagonal to the cycle $C_{4}$ can be added without destroying 2-constructibility.

## Theorem 5

i) For every complete graph $G=(V, E)$ with distinct positive edge weights, $\left(V, M_{k}(G)\right)$ is $k$-constructible.
ii) Let $G=(V, E)$ be $k$-constructible. Then there exist distinct positive edge weights for the complete graph $\widetilde{G}=(V, \widetilde{E})$ such that $E \subseteq M_{k}(\widetilde{G})$.

Proof. Part (ili): Let $G=(V, E)$ be a complete graph on $n$ vertices with distinct positive edge weights. Let $\left\langle e_{1}, e_{2}, \ldots, e_{\binom{n}{2}}\right\rangle$ be the ordering of the edges in $E$ by increasing edge weights and $O=\left\langle e_{r_{1}}, e_{r_{2}}, \ldots, e_{r_{\left|M_{k}(G)\right|}}\right\rangle$ be the ordering of the edges in $M_{k}(G)$ by increasing edge weights. We will now show that $O$ is a $k$-construction order for $\left(V, M_{k}(G)\right)$. Let $i \in\left\{1,2, \ldots,\left|M_{k}(G)\right|\right\}$. As $e_{r_{i}} \in M_{k}(G)$ there exists a set $X \subseteq V$ with $|X|=k-1$ and $e_{r_{i}} \in \operatorname{MST}(G \backslash X)$, implying that the two endpoints of $e_{r_{i}}$ are not connected in the graph ( $V \backslash X,\left\{e_{1}, e_{2}, \ldots, e_{r_{i}-1}\right\}$ ). By Menger's theorem, this implies that there are at most $k-1$ vertex-disjoint paths between the two endpoints of $e_{r_{i}}$ in $\left(V,\left\{e_{1}, e_{2}, \ldots, e_{r_{i}-1}\right\}\right)$. This statement remains thus true for the subgraph $\left(V,\left\{e_{r_{1}}, e_{r_{2}}, \ldots, e_{r_{i-1}}\right\}\right)$. The ordering $O$ is thus a $k$-construction order for $\left(V, M_{k}(G)\right)$.

Part (iii): Conversely let $G=(V, E)$ be a $k$-constructible graph with $k$-construction order $O=\left\langle e_{1}, e_{2}, \ldots, e_{|E|}\right\rangle$. Let $(V, \widetilde{E})$ be the complete graph on $V$. We assign the following edge weights $\widetilde{w}$ to the edges in $\widetilde{E}$. We assign the weight 1 to $e_{1}, 2$ to $e_{2}$ and so on. The remaining edges $\widetilde{E} \backslash E$ get arbitrary distinct weights greater than $|E|$. In order to show that the graph $\widetilde{G}=(V, \widetilde{E}, \widetilde{w})$ satisfies $E \subseteq M_{k}(\widetilde{G})$ consider an arbitrary edge $e_{i} \in E$ and let $C \subseteq V$ with $|C|=k-1$ be a vertex set separating the two endpoints of $e_{i}$ in the graph $G_{i-1}=\left(V,\left\{e_{1}, e_{2}, \ldots, e_{i-1}\right\}\right)$. Applying Kruskal's algorithm to $\widetilde{G}[V \backslash C]$, the set of all edges considered before $e_{i}$ is contained in $E\left(G_{i-1}\right)$, leaving the endpoints of $e_{i}$ separated, so $e_{i}$ will be accepted: $e_{i} \in \operatorname{MST}(\widetilde{G}[V \backslash C]) \subseteq M_{k}(\widetilde{G})$.

We remark that the first part of the foregoing proof shows an efficient construction of $M_{k}(G)$ : follow a generalization of Kruskal's algorithm, considering edges in order of increasing weight, adding an edge if (prior to addition) its endpoints are at most ( $k-1$ )connected. Connectivity can be tested as a flow condition, so that the algorithm runs in polynomial time - far more efficient than the naive $\left.\Omega\binom{n}{k}\right)$ protocol suggested by the definition of $M_{k}(G)$. This again was already observed in [1].

By Theorem 5 the following theorem is equivalent to Theorem 1 .

## Theorem 6

For $k \geq 1$, every $k$-constructible graph $G=(V, E)$ with $n \geq k+1$ vertices satisfies

$$
\begin{equation*}
|E| \leq n k-\binom{k+1}{2} . \tag{2}
\end{equation*}
$$

Theorem 6 generalizes a result of Mader [3], based on results in [2], concerning " $k$ minimal" graphs (edge-minimal $k$-connected graphs). Every $k$-minimal graph is $k$ constructible, since every order of its edges is a $k$-construction order. The following theorem is thus a corollary of Theorem 6.

Theorem 7 (Mader's theorem)
Every $k$-minimal graph with $n$ vertices has at most $n k-\binom{k+1}{2}$ edges.
Note that Mader's theorem (Theorem[7) is weaker than Theorem6, because while every $k$-minimal graph is $k$-constructible, the converse is false: not every $k$-constructible graph
is $k$-minimal. An example with $k=2$ is a cycle $C_{4}$ with length four with an additional diagonal $e$. The vertex set remains 2 -connected even upon deletion of the edge $e$, so the graph is not 2 -minimal, but it is 2 -constructible (by any order where $e$ is not last).

## 3 Proof of the main theorem

In this section we prove Theorem 6, We fix $k$ and prove the theorem by induction on $n$. The theorem is trivially true for $n=k+1$, so assume that $n \geq k+2$ and that the theorem is true for all smaller values of $n$. We prove (2) for a $k$-constructible graph $G=(V, E)$ on $n$ vertices and $m$ edges which, without loss of generality, we may assume is edge-maximal (no edges may be added to $G$ leaving it $k$-constructible). Fix a $k$-construction order

$$
O=\left\langle e_{1}, e_{2}, \ldots, e_{m}\right\rangle
$$

of $G$ and (for any $i \leq m$ ) let $G_{i}=\left(V,\left\{e_{1}, e_{2}, \ldots, e_{i}\right\}\right)$. Also fix a set $C \subseteq V$ of size $|C|=k-1$ such that the two endpoints of $e_{m}$ lie in two different components $Q^{1}, Q^{2} \subseteq V$ of $G_{m-1}[V \backslash C]$ (the set $C$ exists by $k$-constructibility of $G$ and Menger's theorem). The edge maximality of $G$ implies that $Q^{1}, Q^{2}, C$ form a partition of $V$. Let $V^{1}=Q^{1} \cup C$ and $V^{2}=Q^{2} \cup C$. (If there were a third component $Q^{3}$ then, even after adding $e_{m}$, any $v_{1} \in Q^{1}$ and $v_{3} \in Q^{3}$ are at most $(k-1)$-connected and so the edge $\left\{v_{1}, v_{3}\right\}$ could be added, contradicting maximality.)

Our goal is to define two graphs $G^{1}=\left(V^{1}, E^{1}\right)$ and $G^{2}=\left(V^{2}, E^{2}\right)$ that satisfy the following property.

## Property 8

- $G^{1}$ and $G^{2}$ are both $k$-constructible.
- $E^{1}$ contains all edges of $G\left[V^{1}\right]$.
- $E^{2}$ contains all edges of $G\left[V^{2}\right]$.
- For every pair of vertices $c_{1}, c_{2} \in C$ not connected by an edge in $G$, there is an edge $\left\{c_{1}, c_{2}\right\}$ in either $E^{1}$ or in $E^{2}$ (but not both).

If we can find graphs $G^{1}$ and $G^{2}$ satisfying Property 8 , then the proof can be finished as follows. Note that we have the following equality:

$$
\left|E^{1}\right|+\left|E^{2}\right|=(m-1)+|G[C]|+\left(\binom{k-1}{2}-|G[C]|\right) .
$$

The term $m-1$ comes from the fact that $E^{1} \cup E^{2}$ covers all edges of $G$ except $e_{m}$, the term $|G[C]|$ represents the double counting of edges contained in $C$, and the last term counts the edges which are covered by $E^{1}$ and $E^{2}$ but not in $G$.

We therefore have

$$
m=1+\left|E^{1}\right|+\left|E^{2}\right|-\binom{k-1}{2} .
$$

Applying the inductive hypothesis on $G^{1}$ and $G^{2}$ (which by Property 8 are $k$-constructible) we get the desired result:

$$
\begin{aligned}
m & \leq 1+\left(\left|V^{1}\right| k-\binom{k+1}{2}\right)+\left(\left|V^{2}\right| k-\binom{k+1}{2}\right)-\binom{k-1}{2} \\
& \leq 1+(n+k-1) k-2\binom{k+1}{2}-\binom{k-1}{2} \\
& =n k-\binom{k+1}{2},
\end{aligned}
$$

where in the second inequality we have used $\left|V_{1}\right|+\left|V_{2}\right|=n+|C|=n+k-1$.
We will finally concentrate on finding $G^{1}=\left(V^{1}, E^{1}\right)$ and $G^{2}=\left(V^{2}, E^{2}\right)$ satisfying Property 8
Let $B=\binom{C}{2} \backslash E$ be the set of all anti-edges in $G[C]$. ( $\binom{C}{2}$ denotes the set of unordered pairs of elements of $C$.) For $\left\{c_{1}, c_{2}\right\} \in B$, let $\ell\left(c_{1}, c_{2}\right)$ be the smallest value of $i$ such that $c_{1}$ and $c_{2}$ are $k$-connected in $G_{i}$. (Considering $k$ vertex-disjoint paths between $c_{1}$ and $c_{2}$ in $G_{i}$, and noting that deletion of the single edge $e_{i}$ leaves them at least $k-1$ connected, it follows that $c_{1}$ and $c_{2}$ are precisely ( $k-1$ )-connected in $G_{i-1}$.) Define $B_{i}=\left\{\left\{c_{1}, c_{2}\right\}: \ell\left(c_{1}, c_{2}\right)=i\right\}$. Since by edge maximality of $G$ every pair $\left\{c_{1}, c_{2}\right\}$ is $k$-connected in $G_{m}=G$, it follows that $B_{1}, B_{2}, \ldots, B_{m}$ form a partition of $B$.
Our basic strategy to define the graphs $G^{1}$ and $G^{2}$ (and appropriate orderings of their edges which prove that they are $k$-constructible) is as follows. In a particular way, we will partition each $B_{i}$ as $B_{i}^{1} \cup B_{i}^{2}$, and determine orders $O_{i}^{1}$ and $O_{i}^{2}$ on their respective edges. Let $G^{1}$ be the graph constructed by the order

$$
\begin{equation*}
O^{1}=\left\langle e_{1}, O_{1}^{1}, e_{2}, O_{2}^{1}, \ldots, e_{m}, O_{m}^{1}\right\rangle \tag{3}
\end{equation*}
$$

where (recalling that $G^{1}$ has vertex set $V^{1}$ ) we ignore any edge $e_{i} \notin\binom{V^{1}}{2}$. (There is no issue with edges from $O_{i}^{1}$, as these belong to $\binom{C}{2} \subseteq\binom{V^{1}}{2}$.) Define $G^{2}$ symmetrically. We need to show that the graphs $G^{1}$ and $G^{2}$ satisfy Property 8 the central point will be to ensure that $O^{1}$ is a $k$-construction order for $G^{1}$, and $O^{2}$ for $G^{2}$. (By definition of the edges $B_{i}$, note that every edge $e \in O_{i}^{1}$ when added after $e_{i}$ in the order $O$ violates $k$-constructibility, but in the following we show how $O_{i}^{1}, O_{i}^{2}$ can be chosen such that it will not violate $k$-constructibility in $G^{1}$; likewise for edges $e \in O_{i}^{2}$ and $G^{2}$.)

To show that $O^{1}$ and $O^{2}$ are $k$-construction orders we need to check that, just before an edge is added, its endpoints are at most $(k-1)$-connected. To prove this, we distinguish between edges $e_{i} \in E$ and edges $e \in B$. We first dispense with the easier case of an edge $e_{i} \in E$. Proposition 9 shows that (for any orders $O_{i}$ of $B_{i}$ ) in the edge sequence $\left\langle e_{1}, O_{1}, \ldots, e_{m}, O_{m}\right\rangle$, every edge $e_{i}$ has endpoints which are at most $(k-1)$-connected upon its addition to the graph $\left(V,\left\{e_{1}, O_{1}, \ldots, e_{i-1}, O_{i-1}\right\}\right)$. It follows that the endpoints are also at most $(k-1)$-connected upon the edge's addition to $G^{1}$ (respectively, $G^{2}$ ), i.e., in the graph $\left(V^{1},\left\{e_{1}, O_{1}^{1}, \ldots, e_{i-1}, O_{i-1}^{1}\right\}\right)$, where as usual we disregard edges not in $\binom{V_{1}}{2}$.

## Proposition 9

Let $i \in\{1,2, \ldots, m\}$ and $v_{1}, v_{2} \in V$ such that $\left\{v_{1}, v_{2}\right\}$ is not an edge in $G_{i-1}$. If the maximum number of vertex-disjoint paths between $v_{1}$ and $v_{2}$ in $G_{i-1}$ is $r \leq k-1$, then the maximum number of vertex-disjoint paths between $v_{1}$ and $v_{2}$ in the graph $\left(V,\left\{e_{1}, e_{2}, \ldots, e_{i-1}\right\} \cup \bigcup_{l=1}^{i-1} B_{l}\right)$ is $r$, too.

Proof. For any $i, v_{1}, v_{2}$ as above, let $S \subseteq V,|S|=r$, be a set separating $v_{1}$ and $v_{2}$ in $G_{i-1}$. As $|S|=r<k, S$ cannot separate two $k$-connected vertices in $G_{i}$. This implies that any two vertices in $V \backslash S$ that are $k$-connected in $G_{i-1}$ lie in the same connected component of $G_{i-1}[V \backslash S]$. As every edge in $\bigcup_{l=1}^{i-1} B_{l}$ connects two vertices that are $k$-connected in $G_{i-1}$, adding the edges $\bigcup_{l=1}^{i-1} B_{l}$ to $G_{i-1}[V \backslash S]$ does not change the component structure of $G_{i-1}[V \backslash S]$. The set $S$ thus remains a separating set for $v_{1}$ and $v_{2}$ in the graph $\left(V,\left\{e_{1}, e_{2}, \ldots, e_{i-1}\right\} \cup \bigcup_{l=1}^{i-1} B_{l}\right)$, proving that $v_{1}$ and $v_{2}$ are at most $r$-connected in this graph.

Proposition 9

With Proposition 9 addressing edges $e_{i} \in E$, to ensure $k$-constructibility of $O^{1}$ and $O^{2}$, it suffices to choose for $j \in\{1,2\}$ and $i \in\{1,2, \ldots, m\}$ the orders $O_{i}^{j}$ in such a way that successively adding any edge $e \in O_{i}^{j}$ to the graph $G_{i}\left[V^{j}\right]$ connects two vertices which were at most ( $k-1$ )-connected.

Let $C_{i} \subseteq V$ with $\left|C_{i}\right|=k-1$ a set separating the endpoints of $e_{i}$ in the graph $G_{i-1}$. Let $U, W \subseteq V$ be the two components of $G_{i-1}\left[V \backslash C_{i}\right]$ containing the two endpoints of the edge $e_{i}$. We define $C^{U}=C \cap U, C^{W}=C \cap W$. Figure $\square_{1}$ illustrates these sets.


Figure 1: Sets defined to prove Propositions $10 \sqrt{12}$

The following proposition shows that the edges $B_{i}$ form a bipartite graph.

## Proposition 10

$$
B_{i} \subseteq C^{U} \times C^{W}
$$

Proof. Suppose by way of contradiction that $\exists e \in B_{i} \backslash\left(C^{U} \times C^{W}\right)$. Let

$$
O^{\prime}=\left\langle e_{1}, \ldots, e_{i-1}, e, e_{i}, \ldots, e_{m}\right\rangle,
$$

the edge order obtained by inserting $e$ immediately before $e_{i}$ in the original order $O=$ $\left\langle e_{1}, e_{2}, \ldots, e_{m}\right\rangle$. We will show that $O^{\prime}$ is a $k$-construction order, thus contradicting the edge maximality of $G$. For edges up to $e_{i-1}$ this is immediate from the fact that $O$ is a $k$-construction order. Proposition 9 shows that edges $e_{i+1}$ and later do not violate $k$ constructibility. (Literally, Proposition 9 applies to the order $\left\langle e_{1}, \ldots, e_{i}, e, e_{i+1}, \ldots, e_{m}\right\rangle$ rather than to $O^{\prime}$, but for edges $e_{i+1}$ and later the swap of $e_{i}$ and $e$ is irrelevant.) The edge $e$ itself does not violate $k$-constructibility, since by the definition of $B_{i}$ its two endpoints are at most $k-1$ connected in $G_{i-1}$. This leaves only edge $e_{i}$ to check, but since $e \notin U \times W, C_{i}$ remains a separating set with cardinality $k-1$ for the two endpoints of $e_{i}$ in the graph ( $V,\left\{e_{1}, e_{2}, \ldots, e_{i-1}, e\right\}$ ). Thus $O^{\prime}$ is a $k$-construction order, giving the desired contradiction.

We will now describe a method for constructing the orders $O_{i}^{1}, O_{i}^{2}$. Our approach is to define an order $L=\left\langle v_{1}, v_{2}, \ldots, v_{r}\right\rangle$ on (a subset of) the vertices of $C^{U} \cup C^{W}$ and to assign to every vertex $v \in C^{U} \cup C^{W}$ a label $\alpha(v) \in\{1,2\}$. The two orders $O_{i}^{1}, O_{i}^{2}$ are then defined as follows. We begin with $O_{i}^{1}, O_{i}^{2}=\emptyset$ and add all edges in $B_{i}$ which are incident to $v_{1}$ at the end of $O_{i}^{\alpha\left(v_{1}\right)}$ in any order. In the next step all edges of $B_{i}$ which are incident to $v_{2}$ and not already assigned to one of the orders $O_{i}^{1}, O_{i}^{2}$ are added at the end of $O_{i}^{\alpha\left(v_{2}\right)}$ in any order. This is repeated until all edges are assigned.

In what follows we show how to choose a vertex order $L$ and labels $\alpha$ so that $O^{1}$ and $O^{2}$ are $k$-construction orders. Just as $O^{1}$ and $O^{2}$ are built iteratively, so is $L$, starting with $L=\emptyset$.

For any $X \subseteq C^{U} \cup C^{W}$, we define $B_{i}(X)$ to be the set of edges in $B_{i}$ incident on vertices in $X$, i.e., $B_{i}(X)=\left\{e \in B_{i} \mid e \cap X \neq \emptyset\right\}$.

## Proposition 11

Let $j \in\{1,2\}$ and $X \subseteq C^{U} \cup C^{W}$. We then have that $\forall e \in B_{i} \backslash B_{i}(X)$ there are at most $\left|C_{i} \cap V^{j}\right|+|X|$ vertex-disjoint paths between the two endpoints of $e$ in the graph $\left(V,\left\{e_{1}, e_{2}, \ldots, e_{i}\right\} \cup B_{i}(X)\right)\left[V^{j}\right]$.

Proof. Observe that the set $\left(C_{i} \cap V^{j}\right) \cup X$ separates the two endpoints of the edge $e$ in the graph $\left(V,\left\{e_{1}, e_{2}, \ldots, e_{i}\right\} \cup B_{i}(X)\right)\left[V^{j}\right]$. As this set has cardinality $\left|C_{i} \cap V^{j}\right|+|X|$ the result follows by Menger's theorem.

Let $X^{1}$ be the set of vertices labeled 1 contained in the partially constructed $L$, and $X^{2}$ those labeled 2. If we can find a vertex $v \in\left(C^{U} \cup C^{W}\right) \backslash\left(X^{1} \cup X^{2}\right)$ where the number of "new" edges incident on $v$ satisfies

$$
\begin{equation*}
\left|B_{i}(v) \backslash\left(B_{i}\left(X^{1} \cup X^{2}\right)\right)\right| \leq k-1-\min \left\{\left|C_{i} \cap V^{1}\right|+\left|X^{1}\right|,\left|C_{i} \cap V^{2}\right|+\left|X^{2}\right|\right\} \tag{4}
\end{equation*}
$$

then by Proposition 11, adding $v$ at the end of the current order $L$ and labeling it $\arg \min _{j \in\{1,2\}}\left\{\left|C_{i} \cap V^{j}\right|+\left|X^{j}\right|\right\}$ does not violate $k$-constructibility of the orders $O^{1}$ and $O^{2}$.

The following proposition shows that, until the process is complete (until $B_{i}\left(X^{1} \cup X^{2}\right)=$ $B_{i}$ ), such a vertex $v$ can always be found.

## Proposition 12

Let $X^{1}, X^{2} \subset C^{U} \cup C^{W}$ be two disjoint sets. If $B_{i}\left(X^{1} \cup X^{2}\right) \subsetneq B_{i}$, then there exists a vertex $v \in\left(C^{U} \cup C^{W}\right) \backslash\left(X^{1} \cup X^{2}\right)$ that satisfies (4).

Proof. Note that $C^{U}, C^{W}$, and $C \cap C_{i}$ are disjoint and contained in $C$, so

$$
\begin{equation*}
\left|C^{U}\right|+\left|C^{W}\right|+\left|C \cap C_{i}\right| \leq|C|=k-1, \tag{5}
\end{equation*}
$$

where $|C|=k-1$ by definition. Also,

$$
\begin{equation*}
\left|V^{1} \cap C_{i}\right|+\left|V^{2} \cap C_{i}\right|-\left|C \cap C_{i}\right|=\left|C_{i}\right|=k-1 . \tag{6}
\end{equation*}
$$

¿From the fact that the right side of (5) is equal to $2(k-1)$ minus that of (6), we get

$$
\begin{equation*}
\left|C^{U}\right|+\left|C^{W}\right| \leq\left(k-1-\left|V^{1} \cap C_{i}\right|\right)+\left(k-1-\left|V^{2} \cap C_{i}\right|\right) . \tag{7}
\end{equation*}
$$

By disjointness of $C^{U}$ and $C^{W}$,

$$
\begin{align*}
& \left|C^{U} \backslash\left(X^{1} \cup X^{2}\right)\right|+\left|C^{W} \backslash\left(X^{1} \cup X^{2}\right)\right|  \tag{8}\\
& \quad=\left|C^{U}\right|+\left|C^{W}\right|-\left|X^{1}\right|-\left|X^{2}\right| \\
& \quad \leq\left(k-1-\left|V^{1} \cap C_{i}\right|-\left|X^{1}\right|\right)+\left(k-1-\left|V^{2} \cap C_{i}\right|-\left|X^{2}\right|\right), \tag{9}
\end{align*}
$$

using (77) in the last inequality. Thus, the smaller summand in (8) is at most the larger summand in (9), and without loss of generality we suppose that

$$
\begin{equation*}
\left|C^{U} \backslash\left(X^{1} \cup X^{2}\right)\right| \leq k-1-\left|V^{1} \cap C_{i}\right|-\left|X^{1}\right| . \tag{10}
\end{equation*}
$$

By the hypothesis $B_{i}\left(X^{1} \cup X^{2}\right) \subsetneq B_{i}$, there is an edge $e \in B_{i} \backslash B_{i}\left(X^{1} \cup X^{2}\right)$; by Proposition 10, $e=\{u, w\}$ with $u \in C^{U}$ and $w \in C^{W}$; and by definition of $B_{i}\left(X^{1} \cup X^{2}\right)$, $u, w \notin X^{1} \cup X^{2}$, i.e., $u \in C^{U} \backslash\left(X^{1} \cup X^{2}\right)$ and $w \in C^{W} \backslash\left(X^{1} \cup X^{2}\right)$. Then $v=w$ satisfies (4) because the new edges on $w$ must go to so-far-unused vertices in $C^{U}$ :

$$
\left|B_{i}(w) \backslash B_{i}\left(X^{1} \cup X^{2}\right)\right| \leq\left|C^{U} \backslash\left(X^{1} \cup X^{2}\right)\right|,
$$

whence (10) closes the argument.

Therefore there always exist two $k$-construction orders $O^{1}, O^{2}$ as desired, which completes the proof of Theorem 6.

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## References

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