A tight bound on the collection of edges in MSTs of induced subgraphs

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Let G = (V, E) be a complete *n*-vertex graph with distinct positive edge weights. We prove that for $k \in \{1, 2, ..., n-1\}$, the set consisting of the edges of all minimum spanning trees (MSTs) over induced subgraphs of Gwith n - k + 1 vertices has at most $nk - \binom{k+1}{2}$ elements. This proves a conjecture of Goemans and Vondrak [1]. We also show that the result is a generalization of Mader's Theorem, which bounds the number of edges in any edge-minimal k-connected graph.

1 Introduction

Let G = (V, E) be a complete *n*-vertex graph with distinct positive edge weights. For any set $X \subseteq V$, denote by $G[V \setminus X]$ the subgraph of G induced by $V \setminus X$. We will also sometimes write this graph as $(V \setminus X, E)$, ignoring edges in E incident on vertices in X. $MST(G[V \setminus X])$ denotes the set of edges in the graph's minimum spanning tree. (The MST is unique due to the assumption that the edge weights are distinct.)

For $k \in \{1, 2, ..., n-1\}$, define

$$M_k(G) = \bigcup_{X \subseteq V, |X|=k-1} MST(G[V \setminus X]) .$$

Note that for k = 1 we have $M_1(G) = \text{MST}(G)$. In [1], Goemans and Vondrak considered the problem of finding a sparse set of edges which, with high probability, contain the MST of a random subgraph of G. In this context they proved an upper bound on $M_k(G)$, namely that $|M_k(G)| < (1 + \frac{e}{2})kn$, and they conjectured that one should be able to improve the bound to $|M_k(G)| \le nk - {k+1 \choose 2}$. In this paper we prove this conjecture.

Theorem 1

For any complete graph G on n vertices with distinct positive edge weights,

$$|M_k(G)| \le nk - \binom{k+1}{2}.$$
 (1)

As Goemans and Vondrak recognized, the bound is tight: for any n and k it is easy to produce edge weights giving equality in (1). One way is to fix an arbitrary set $V' \subseteq V$ with cardinality k, and partition the edges E into three sets E_0 , E_1 and E_2 where, for $i \in \{0, 1, 2\}$, E_i contains all edges of E having exactly i endpoints in V'. Assign arbitrary distinct positive weights to the edges in E such that all weights on E_2 are smaller than those on E_1 , which in turn are smaller than those on E_0 . It can easily be verified that $M_k(G) = E_2 \cup E_1$ and thus $|M_k(G)| = nk - {k+1 \choose 2}$.

Theorem 1's assumption that G is complete is not meaningfully restrictive. If G is such that deletion of some k - 1 vertices leaves it disconnected, then the notion of $M_k(G)$ does not make sense; otherwise, it does not matter if other edges of G are simply very costly or are absent.

The bound of Theorem 1 applies equally if we consider the edge set of MSTs of induced subgraphs of size at most n - k + 1 (rather than exactly that number). This is an immediate consequence of the following remark.

Remark 2

For any complete graph G on n vertices with distinct positive edge weights, and $k \in \{1, 2, ..., n-2\}, M_{k+1}(G) \supseteq M_k(G).$

Proof. We will show that any edge e in $M_k(G)$ is also in $M_{k+1}(G)$. By definition, $e \in M_k(G)$ means that there is some vertex set X of cardinality |X| = k - 1 for which $e \in MST(G_k)$, where $G_k = G[V \setminus X]$.

Consider any leaf vertex v of $MST(G_k)$, with neighbor u. We claim that deleting v from G_k (call the resulting graph G_{k+1}) results in the same MST less the edge $\{u, v\}$, i.e., that $MST(G_{k+1}) = MST(G_k) \setminus \{\{u, v\}\}$. This follows from considering the progress of Kruskal's algorithm on the two graphs. Before edge $\{u, v\}$ is added to $MST(G_k)$, the two processes progress identically: every edge added to $MST(G_k)$ is also a cheapest edge for the smaller graph G_{k+1} . The edge e, added to $MST(G_k)$, of course has no parallel in G_{k+1} . As further edges are considered in order of increasing cost, again, every edge added to $MST(G_k)$ will also be added to $MST(G_{k+1})$, using the fact that none of these edges is incident on v.

Thus, if v is not a vertex of e, then $e \in MST(G_{k+1})$. Since $MST(G_k)$ has at least two leaves, it has at least one leaf v not in e, unless $MST(G_k) = e$, which is impossible since G_k has at least 3 vertices.

Outline of the paper

In Section 2 we define a "k-constructible" graph, and show that every graph $(V, M_k(G))$ is k-constructible, and every k-constructible graph is a subgraph of some graph $(V, M_k(G))$. This allows a simpler reformulation of Theorem 1 as Theorem 6, which also generalizes a theorem of Mader [3]. We prove Theorem 6 in Section 3.

2 k-constructible graphs

We begin by recalling Menger's theorem for undirected graphs, which motivates our definition of k-constructible graphs. Two vertices in an undirected graph are called k-connected if there are k (internally) vertex-disjoint paths connecting them.

Theorem 3 (Menger's theorem)

Let s,t be two vertices in an undirected graph G = (V, E) such that $\{s, t\} \notin E$. Then s and t are k-connected in G if and only if after deleting any k - 1 vertices (distinct from s and t), s and t are still connected.

Definition 4 (*k*-constructible graph)

A graph G = (V, E) is called k-constructible if there exists an ordering $O = \langle e_1, e_2, \ldots, e_m \rangle$ of the edges in E such that for all $i \in \{1, 2, \ldots, m\}$ the graph $(V, \{e_1, e_2, \ldots, e_{i-1}\})$ contains at most k - 1 vertex-disjoint paths between the two endpoints of e_i . We say that O is a k-construction order for the graph G.

Note that 1-constructible graphs are forests, and edge-maximal 1-constructible graphs are spanning trees. We therefore have in particular that graphs of the form $M_1(G)$ (i.e., MSTs, recalling the G is complete) are edge-maximal 1-constructible graphs. A slightly weaker statement is true for all k: every graph $M_k(G)$ is k-constructible (Theorem 5.i), and every k-constructible graph is a subgraph of some graph $M_k(G)$ (Theorem 5.ii).

Note that a stronger statement, that the graphs of the form $M_k(G)$ are exactly the edgemaximal k-constructible graphs, is not true. To see this consider a cycle C_4 of length four. Assign weights $1, \ldots, 4$ to these four edges (in arbitrary order) and weights 5,6 to the remaining edges of the complete graph on four vertices. It is easily checked that $M_2(G) = C_4$. But $M_2(G)$ is not edge-maximal, as a diagonal to the cycle C_4 can be added without destroying 2-constructibility.

Theorem 5

- i) For every complete graph G = (V, E) with distinct positive edge weights, $(V, M_k(G))$ is k-constructible.
- ii) Let G = (V, E) be k-constructible. Then there exist distinct positive edge weights for the complete graph $\widetilde{G} = (V, \widetilde{E})$ such that $E \subseteq M_k(\widetilde{G})$.

Proof. Part (i): Let G = (V, E) be a complete graph on n vertices with distinct positive edge weights. Let $\langle e_1, e_2, \ldots, e_{\binom{n}{2}} \rangle$ be the ordering of the edges in E by increasing edge weights and $O = \langle e_{r_1}, e_{r_2}, \ldots, e_{r_{|M_k(G)|}} \rangle$ be the ordering of the edges in $M_k(G)$ by increasing edge weights. We will now show that O is a k-construction order for $(V, M_k(G))$. Let $i \in \{1, 2, \ldots, |M_k(G)|\}$. As $e_{r_i} \in M_k(G)$ there exists a set $X \subseteq V$ with |X| = k - 1 and $e_{r_i} \in MST(G \setminus X)$, implying that the two endpoints of e_{r_i} are not connected in the graph $(V \setminus X, \{e_1, e_2, \ldots, e_{r_i-1}\})$. By Menger's theorem, this implies that there are at most k - 1 vertex-disjoint paths between the two endpoints of e_{r_i} in $(V, \{e_1, e_2, \ldots, e_{r_i-1}\})$. This statement remains thus true for the subgraph $(V, \{e_{r_1}, e_{r_2}, \ldots, e_{r_{i-1}}\})$. The ordering O is thus a k-construction order for $(V, M_k(G))$.

Part (ii): Conversely let G = (V, E) be a k-constructible graph with k-construction order $O = \langle e_1, e_2, \ldots, e_{|E|} \rangle$. Let (V, \widetilde{E}) be the complete graph on V. We assign the following edge weights \widetilde{w} to the edges in \widetilde{E} . We assign the weight 1 to e_1 , 2 to e_2 and so on. The remaining edges $\widetilde{E} \setminus E$ get arbitrary distinct weights greater than |E|. In order to show that the graph $\widetilde{G} = (V, \widetilde{E}, \widetilde{w})$ satisfies $E \subseteq M_k(\widetilde{G})$ consider an arbitrary edge $e_i \in E$ and let $C \subseteq V$ with |C| = k - 1 be a vertex set separating the two endpoints of e_i in the graph $G_{i-1} = (V, \{e_1, e_2, \ldots, e_{i-1}\})$. Applying Kruskal's algorithm to $\widetilde{G}[V \setminus C]$, the set of all edges considered before e_i is contained in $E(G_{i-1})$, leaving the endpoints of e_i separated, so e_i will be accepted: $e_i \in MST(\widetilde{G}[V \setminus C]) \subseteq M_k(\widetilde{G})$.

We remark that the first part of the foregoing proof shows an efficient construction of $M_k(G)$: follow a generalization of Kruskal's algorithm, considering edges in order of increasing weight, adding an edge if (prior to addition) its endpoints are at most (k-1)connected. Connectivity can be tested as a flow condition, so that the algorithm runs in polynomial time — far more efficient than the naive $\Omega\left(\binom{n}{k}\right)$ protocol suggested by the definition of $M_k(G)$. This again was already observed in [1].

By Theorem 5, the following theorem is equivalent to Theorem 1.

Theorem 6

For $k \ge 1$, every k-constructible graph G = (V, E) with $n \ge k + 1$ vertices satisfies

$$|E| \le nk - \binom{k+1}{2} \,. \tag{2}$$

Theorem 6 generalizes a result of Mader [3], based on results in [2], concerning "k-minimal" graphs (edge-minimal k-connected graphs). Every k-minimal graph is k-constructible, since every order of its edges is a k-construction order. The following theorem is thus a corollary of Theorem 6.

Theorem 7 (Mader's theorem)

Every k-minimal graph with n vertices has at most $nk - \binom{k+1}{2}$ edges.

Note that Mader's theorem (Theorem 7) is weaker than Theorem 6, because while every k-minimal graph is k-constructible, the converse is false: not every k-constructible graph

is k-minimal. An example with k = 2 is a cycle C_4 with length four with an additional diagonal e. The vertex set remains 2-connected even upon deletion of the edge e, so the graph is not 2-minimal, but it is 2-constructible (by any order where e is not last).

3 Proof of the main theorem

In this section we prove Theorem 6. We fix k and prove the theorem by induction on n. The theorem is trivially true for n = k+1, so assume that $n \ge k+2$ and that the theorem is true for all smaller values of n. We prove (2) for a k-constructible graph G = (V, E) on n vertices and m edges which, without loss of generality, we may assume is edge-maximal (no edges may be added to G leaving it k-constructible). Fix a k-construction order

$$O = \langle e_1, e_2, \dots, e_m \rangle$$

of G and (for any $i \leq m$) let $G_i = (V, \{e_1, e_2, \ldots, e_i\})$. Also fix a set $C \subseteq V$ of size |C| = k-1 such that the two endpoints of e_m lie in two different components $Q^1, Q^2 \subseteq V$ of $G_{m-1}[V \setminus C]$ (the set C exists by k-constructibility of G and Menger's theorem). The edge maximality of G implies that Q^1, Q^2, C form a partition of V. Let $V^1 = Q^1 \cup C$ and $V^2 = Q^2 \cup C$. (If there were a third component Q^3 then, even after adding e_m , any $v_1 \in Q^1$ and $v_3 \in Q^3$ are at most (k-1)-connected and so the edge $\{v_1, v_3\}$ could be added, contradicting maximality.)

Our goal is to define two graphs $G^1 = (V^1, E^1)$ and $G^2 = (V^2, E^2)$ that satisfy the following property.

Property 8

- G^1 and G^2 are both k-constructible.
- E^1 contains all edges of $G[V^1]$.
- E^2 contains all edges of $G[V^2]$.
- For every pair of vertices $c_1, c_2 \in C$ not connected by an edge in G, there is an edge $\{c_1, c_2\}$ in either E^1 or in E^2 (but not both).

If we can find graphs G^1 and G^2 satisfying Property 8, then the proof can be finished as follows. Note that we have the following equality:

$$|E^{1}| + |E^{2}| = (m-1) + |G[C]| + \left(\binom{k-1}{2} - |G[C]|\right)$$
.

The term m-1 comes from the fact that $E^1 \cup E^2$ covers all edges of G except e_m , the term |G[C]| represents the double counting of edges contained in C, and the last term counts the edges which are covered by E^1 and E^2 but not in G.

We therefore have

$$m = 1 + |E^1| + |E^2| - \binom{k-1}{2}$$

Applying the inductive hypothesis on G^1 and G^2 (which by Property 8 are k-constructible) we get the desired result:

$$m \leq 1 + \left(|V^1|k - {\binom{k+1}{2}} \right) + \left(|V^2|k - {\binom{k+1}{2}} \right) - {\binom{k-1}{2}}$$

$$\leq 1 + (n+k-1)k - 2{\binom{k+1}{2}} - {\binom{k-1}{2}}$$

$$= nk - {\binom{k+1}{2}},$$

where in the second inequality we have used $|V_1| + |V_2| = n + |C| = n + k - 1$.

We will finally concentrate on finding $G^1 = (V^1, E^1)$ and $G^2 = (V^2, E^2)$ satisfying Property 8.

Let $B = \binom{C}{2} \setminus E$ be the set of all anti-edges in G[C]. $\binom{C}{2}$ denotes the set of unordered pairs of elements of C.) For $\{c_1, c_2\} \in B$, let $\ell(c_1, c_2)$ be the smallest value of i such that c_1 and c_2 are k-connected in G_i . (Considering k vertex-disjoint paths between c_1 and c_2 in G_i , and noting that deletion of the single edge e_i leaves them at least k - 1connected, it follows that c_1 and c_2 are precisely (k - 1)-connected in G_{i-1} .) Define $B_i = \{\{c_1, c_2\} : \ell(c_1, c_2) = i\}$. Since by edge maximality of G every pair $\{c_1, c_2\}$ is k-connected in $G_m = G$, it follows that B_1, B_2, \ldots, B_m form a partition of B.

Our basic strategy to define the graphs G^1 and G^2 (and appropriate orderings of their edges which prove that they are k-constructible) is as follows. In a particular way, we will partition each B_i as $B_i^1 \cup B_i^2$, and determine orders O_i^1 and O_i^2 on their respective edges. Let G^1 be the graph constructed by the order

$$O^{1} = \langle e_{1}, O_{1}^{1}, e_{2}, O_{2}^{1}, \dots, e_{m}, O_{m}^{1} \rangle,$$
(3)

where (recalling that G^1 has vertex set V^1) we ignore any edge $e_i \notin {\binom{V^1}{2}}$. (There is no issue with edges from O_i^1 , as these belong to $\binom{C}{2} \subseteq \binom{V^1}{2}$.) Define G^2 symmetrically. We need to show that the graphs G^1 and G^2 satisfy Property 8; the central point will be to ensure that O^1 is a k-construction order for G^1 , and O^2 for G^2 . (By definition of the edges B_i , note that every edge $e \in O_i^1$ when added after e_i in the order O violates k-constructibility, but in the following we show how O_i^1, O_i^2 can be chosen such that it will not violate k-constructibility in G^1 ; likewise for edges $e \in O_i^2$ and G^2 .)

To show that O^1 and O^2 are k-construction orders we need to check that, just before an edge is added, its endpoints are at most (k-1)-connected. To prove this, we distinguish between edges $e_i \in E$ and edges $e \in B$. We first dispense with the easier case of an edge $e_i \in E$. Proposition 9 shows that (for any orders O_i of B_i) in the edge sequence $\langle e_1, O_1, \ldots, e_m, O_m \rangle$, every edge e_i has endpoints which are at most (k-1)-connected upon its addition to the graph $(V, \{e_1, O_1, \ldots, e_{i-1}, O_{i-1}\})$. It follows that the endpoints are also at most (k-1)-connected upon the edge's addition to G^1 (respectively, G^2), i.e., in the graph $(V^1, \{e_1, O_1^1, \ldots, e_{i-1}, O_{i-1}^1\})$, where as usual we disregard edges not in $\binom{V_1}{2}$.

Proposition 9

Let $i \in \{1, 2, ..., m\}$ and $v_1, v_2 \in V$ such that $\{v_1, v_2\}$ is not an edge in G_{i-1} . If the maximum number of vertex-disjoint paths between v_1 and v_2 in G_{i-1} is $r \leq k-1$, then the maximum number of vertex-disjoint paths between v_1 and v_2 in the graph $(V, \{e_1, e_2, ..., e_{i-1}\} \cup \bigcup_{l=1}^{i-1} B_l)$ is r, too.

Proof. For any i, v_1, v_2 as above, let $S \subseteq V$, |S| = r, be a set separating v_1 and v_2 in G_{i-1} . As |S| = r < k, S cannot separate two k-connected vertices in G_i . This implies that any two vertices in $V \setminus S$ that are k-connected in G_{i-1} lie in the same connected component of $G_{i-1}[V \setminus S]$. As every edge in $\bigcup_{l=1}^{i-1} B_l$ connects two vertices that are k-connected in G_{i-1} , adding the edges $\bigcup_{l=1}^{i-1} B_l$ to $G_{i-1}[V \setminus S]$ does not change the component structure of $G_{i-1}[V \setminus S]$. The set S thus remains a separating set for v_1 and v_2 in the graph $(V, \{e_1, e_2, \ldots, e_{i-1}\} \cup \bigcup_{l=1}^{i-1} B_l)$, proving that v_1 and v_2 are at most r-connected in this graph.

 \square Proposition 9

With Proposition 9 addressing edges $e_i \in E$, to ensure k-constructibility of O^1 and O^2 , it suffices to choose for $j \in \{1, 2\}$ and $i \in \{1, 2, ..., m\}$ the orders O_i^j in such a way that successively adding any edge $e \in O_i^j$ to the graph $G_i[V^j]$ connects two vertices which were at most (k-1)-connected.

Let $C_i \subseteq V$ with $|C_i| = k - 1$ a set separating the endpoints of e_i in the graph G_{i-1} . Let $U, W \subseteq V$ be the two components of $G_{i-1}[V \setminus C_i]$ containing the two endpoints of the edge e_i . We define $C^U = C \cap U$, $C^W = C \cap W$. Figure 1 illustrates these sets.

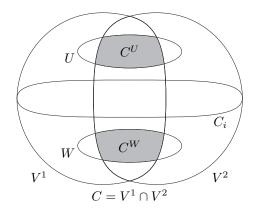


Figure 1: Sets defined to prove Propositions 10–12.

The following proposition shows that the edges B_i form a bipartite graph.

Proposition 10

$$B_i \subseteq C^U \times C^W$$

Proof. Suppose by way of contradiction that $\exists e \in B_i \setminus (C^U \times C^W)$. Let

$$O' = \langle e_1, \ldots, e_{i-1}, e, e_i, \ldots, e_m \rangle,$$

the edge order obtained by inserting e immediately before e_i in the original order $O = \langle e_1, e_2, \ldots, e_m \rangle$. We will show that O' is a k-construction order, thus contradicting the edge maximality of G. For edges up to e_{i-1} this is immediate from the fact that O is a k-construction order. Proposition 9 shows that edges e_{i+1} and later do not violate k-constructibility. (Literally, Proposition 9 applies to the order $\langle e_1, \ldots, e_i, e, e_{i+1}, \ldots, e_m \rangle$ rather than to O', but for edges e_{i+1} and later the swap of e_i and e is irrelevant.) The edge e itself does not violate k-constructibility, since by the definition of B_i its two endpoints are at most k - 1 connected in G_{i-1} . This leaves only edge e_i to check, but since $e \notin U \times W$, C_i remains a separating set with cardinality k-1 for the two endpoints of e_i in the graph $(V, \{e_1, e_2, \ldots, e_{i-1}, e\})$. Thus O' is a k-construction order, giving the desired contradiction.

We will now describe a method for constructing the orders O_i^1 , O_i^2 . Our approach is to define an order $L = \langle v_1, v_2, \ldots, v_r \rangle$ on (a subset of) the vertices of $C^U \cup C^W$ and to assign to every vertex $v \in C^U \cup C^W$ a label $\alpha(v) \in \{1, 2\}$. The two orders O_i^1 , O_i^2 are then defined as follows. We begin with $O_i^1, O_i^2 = \emptyset$ and add all edges in B_i which are incident to v_1 at the end of $O_i^{\alpha(v_1)}$ in any order. In the next step all edges of B_i which are incident to v_2 and not already assigned to one of the orders O_i^1, O_i^2 are added at the end of $O_i^{\alpha(v_2)}$ in any order. This is repeated until all edges are assigned.

In what follows we show how to choose a vertex order L and labels α so that O^1 and O^2 are k-construction orders. Just as O^1 and O^2 are built iteratively, so is L, starting with $L = \emptyset$.

For any $X \subseteq C^U \cup C^W$, we define $B_i(X)$ to be the set of edges in B_i incident on vertices in X, i.e., $B_i(X) = \{e \in B_i \mid e \cap X \neq \emptyset\}.$

Proposition 11

Let $j \in \{1,2\}$ and $X \subseteq C^U \cup C^W$. We then have that $\forall e \in B_i \setminus B_i(X)$ there are at most $|C_i \cap V^j| + |X|$ vertex-disjoint paths between the two endpoints of e in the graph $(V, \{e_1, e_2, \ldots, e_i\} \cup B_i(X))[V^j]$.

Proof. Observe that the set $(C_i \cap V^j) \cup X$ separates the two endpoints of the edge e in the graph $(V, \{e_1, e_2, \ldots, e_i\} \cup B_i(X))[V^j]$. As this set has cardinality $|C_i \cap V^j| + |X|$ the result follows by Menger's theorem.

Let X^1 be the set of vertices labeled 1 contained in the partially constructed L, and X^2 those labeled 2. If we can find a vertex $v \in (C^U \cup C^W) \setminus (X^1 \cup X^2)$ where the number of "new" edges incident on v satisfies

$$|B_i(v) \setminus (B_i(X^1 \cup X^2))| \le k - 1 - \min\{|C_i \cap V^1| + |X^1|, |C_i \cap V^2| + |X^2|\}$$
(4)

then by Proposition 11, adding v at the end of the current order L and labeling it $\arg\min_{j\in\{1,2\}}\{|C_i \cap V^j| + |X^j|\}$ does not violate k-constructibility of the orders O^1 and O^2 .

The following proposition shows that, until the process is complete (until $B_i(X^1 \cup X^2) = B_i$), such a vertex v can always be found.

Proposition 12

Let $X^1, X^2 \subset C^U \cup C^W$ be two disjoint sets. If $B_i(X^1 \cup X^2) \subsetneq B_i$, then there exists a vertex $v \in (C^U \cup C^W) \setminus (X^1 \cup X^2)$ that satisfies (4).

Proof. Note that C^U , C^W , and $C \cap C_i$ are disjoint and contained in C, so

$$|C^{U}| + |C^{W}| + |C \cap C_{i}| \le |C| = k - 1 , \qquad (5)$$

where |C| = k - 1 by definition. Also,

$$|V^{1} \cap C_{i}| + |V^{2} \cap C_{i}| - |C \cap C_{i}| = |C_{i}| = k - 1.$$
(6)

From the fact that the right side of (5) is equal to 2(k-1) minus that of (6), we get

$$|C^{U}| + |C^{W}| \le (k - 1 - |V^{1} \cap C_{i}|) + (k - 1 - |V^{2} \cap C_{i}|).$$
(7)

By disjointness of C^U and C^W ,

$$|C^U \setminus (X^1 \cup X^2)| + |C^W \setminus (X^1 \cup X^2)|$$
(8)

$$= |C^{U}| + |C^{W}| - |X^{1}| - |X^{2}|$$

$$\leq (k - 1 - |V^{1} \cap C_{i}| - |X^{1}|) + (k - 1 - |V^{2} \cap C_{i}| - |X^{2}|), \qquad (9)$$

using (7) in the last inequality. Thus, the smaller summand in (8) is at most the larger summand in (9), and without loss of generality we suppose that

$$|C^{U} \setminus (X^{1} \cup X^{2})| \le k - 1 - |V^{1} \cap C_{i}| - |X^{1}|.$$
(10)

By the hypothesis $B_i(X^1 \cup X^2) \subseteq B_i$, there is an edge $e \in B_i \setminus B_i(X^1 \cup X^2)$; by Proposition 10, $e = \{u, w\}$ with $u \in C^U$ and $w \in C^W$; and by definition of $B_i(X^1 \cup X^2)$, $u, w \notin X^1 \cup X^2$, i.e., $u \in C^U \setminus (X^1 \cup X^2)$ and $w \in C^W \setminus (X^1 \cup X^2)$. Then v = w satisfies (4) because the new edges on w must go to so-far-unused vertices in C^U :

$$|B_i(w) \setminus B_i(X^1 \cup X^2)| \le |C^U \setminus (X^1 \cup X^2)|,$$

whence (10) closes the argument.

Therefore there always exist two k-construction orders O^1, O^2 as desired, which completes the proof of Theorem 6.

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References

- Michel X. Goemans and Jan Vondrák. Covering minimum spanning trees of random subgraphs. *Random Structures and Algorithms*, 29(3):257–276, 2005.
- [2] R. Halin. A theorem on *n*-connected graphs. Journal on Combinatorial Theory, 7:150–154, 1969.
- [3] W. Mader. Minimale *n*-fach zusammenhängende Graphen. Journal für die Reine und Angewandte Mathematik, 249:201–207, 1971.