# Graphs with bounded tree-width and large odd-girth are almost bipartite 

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#### Abstract

We prove that for every $k$ and every $\varepsilon>0$, there exists $g$ such that every graph with tree-width at most $k$ and odd-girth at least $g$ has circular chromatic number at most $2+\varepsilon$.


## 1 Introduction

It has been a challenging problem to prove the existence of graphs of arbitrary high girth and chromatic number [2]. On the other hand, graphs with large girth that avoid a fixed minor are known to have low chromatic number (in particular, this applies to graphs embedded on a fixed surface). More precisely, as Thomassen observed [8], a graph that avoids a fixed minor and has large girth is 2-degenerate, and hence 3-colorable. Further, Galluccio, Goddyn and Hell [3] proved the following theorem, which essentially states that graphs with large girth that avoid a fixed minor are almost bipartite.

[^0]Theorem 1 (Galluccio, Goddyn and Hell, 2001). For every graph $H$ and every $\varepsilon>0$, there exists an integer $g$ such that the circular chromatic number of every $H$-minor free graph of girth at least $g$ is at most $2+\varepsilon$.

A natural way to weaken the girth-condition is to require the graphs to have high odd-girth (the odd-girth is the length of a shortest odd cycle). However, Young [9] constructed 4-chromatic projective graphs with arbitrary high odd-girth. Thus, the high odd-girth requirement is not sufficient to ensure 3-colorability, even for graphs embedded on a fixed surface. Klostermeyer and Zhang [4], though, proved that the circular chromatic number of every planar graph of sufficiently high odd-girth is arbitrarily close to 2 . In particular, the same is true for $K_{4}$-minor free graphs, i.e. graphs with tree-width at most 2. We prove that the conclusion is still true for any class of graphs of bounded tree-width, which answers a question of Pan and Zhu [6, Question 6.5] also appearing as Question 8.12 in the survey by Zhu [10].

Theorem 2. For every $k$ and every $\varepsilon>0$, there exists $g$ such that every graph with tree-width at most $k$ and odd-girth at least $g$ has circular chromatic number at most $2+\varepsilon$.

Motivated by tree-width duality, Nešetřil and Zhu [5] proved the following theorem.

Theorem 3 (Nešetřil and Zhu, 1996). For every $k$ and every $\varepsilon>0$, there exists $g$ such that every graph $G$ with tree-width at most $k$ and homomorphic to a graph $H$ with girth at least $g$ has circular chromatic number at most $2+\varepsilon$.

To see that Theorem 2 implies Theorem 3, observe that if $G$ has an odd cycle of length $g$, then $H$ has an odd cycle of length at most $g$.

## 2 Notation

A $(p, q)$-coloring of a graph is a coloring of the vertices with colors from the set $\{0, \ldots, p-1\}$ such that the colors of any two adjacent vertices $u$ and $v$ satisfy $q \leq|c(u)-c(v)| \leq p-q$. The circular chromatic number $\chi_{c}(G)$ of a graph $G$ is the infimum (and it can be shown to be the minimum) of the ratios $p / q$ such that $G$ has a $(p, q)$-coloring. For every finite graph $G$, it holds that $\chi(G)=\left\lceil\chi_{c}(G)\right\rceil$ and there is $(p, q)$-coloring of $G$ for every $p$ and $q$ with $p / q \geq \chi_{c}(G)$. In particular, the circular chromatic number of $G$ is at most $2+1 / k$ if and only if $G$ is homomorphic to a cycle of length $2 k+1$. The
reader is referred to the surveys by Zhu [10, 11] for more information about circular colorings.

A p-precoloring is a coloring $\varphi$ of a subset $A$ of vertices of a graph $G$ with colors from $\{0, \ldots, p-1\}$, and its extension is a coloring of the whole graph $G$ that coincides with $\varphi$ on $A$. The following lemma can be seen as a corollary of a theorem of Albertson and West [1, Theorem 1], and it is the only tool we use from this area.

Lemma 4. For every $p$ and $q$ with $2<p / q$, there exists $d$ such that any p-precoloring of vertices with mutual distances at least d of a bipartite graph $H$ extends to a $(p, q)$-coloring of $H$.

A $k$-tree is a graph obtained from a complete graph of order $k+1$ by adding vertices of degree $k$ whose neighborhood is a clique. The tree-width of a graph $G$ is the smallest $k$ such that $G$ is a subgraph of a $k$-tree. Graphs with tree-width at most $k$ are also called partial $k$-trees.

A rooted partial $k$-tree is a partial $k$-tree $G$ with $k+1$ distinguished vertices $v_{1}, \ldots, v_{k+1}$ such that there exists a $k$-tree $G^{\prime}$ that is a supergraph of $G$ and the vertices $v_{1}, \ldots, v_{k+1}$ form a clique in $G^{\prime}$. We also say that the partial $k$ tree is rooted at $v_{1}, \ldots, v_{k+1}$. If $G$ is a partial $k$-tree rooted at $v_{1}, \ldots, v_{k+1}$ and $G^{\prime}$ is a partial $k$-tree rooted at $v_{1}^{\prime}, \ldots, v_{k+1}^{\prime}$, then the graph $G \oplus G^{\prime}$ obtained by identifying $v_{i}$ and $v_{i}^{\prime}$ is again a rooted partial $k$-tree (identify the cliques in the corresponding $k$-trees).

Fix $p$ and $q$. If $G$ is a rooted partial $k$-tree, then $\mathcal{F}(G)$ is the set of all $p$-precolorings of the $k+1$ distinguished vertices of $G$ that can be extended to a $(p, q)$-coloring of $G$.

The next lemma is a standard application of results in the area of graphs of bounded tree-width [7].

Lemma 5. Let $k$ and $N$ be positive integers such that $N \geq k+1$. If $G$ is a partial $k$-tree with at least $3 N$ vertices, then there exist partial rooted $k$-trees $G_{1}$ and $G_{2}$ such that $G$ is isomorphic to $G_{1} \oplus G_{2}$ and $G_{1}$ has at least $N+1$ and at most $2 N$ vertices.

If $G$ is a partial $k$-tree rooted at $v_{1}, \ldots, v_{k+1}$, then its type is a $(k+1) \times$ $(k+1)$ matrix $M$ such that $M_{i j}$ is the length of the shortest path between the vertices $v_{i}$ and $v_{j}$. If there is no such path, $M_{i j}$ is equal to $\infty$. Any matrix $M$ that is a type of a partial rooted $k$-tree satisfies the triangle inequality (setting $\infty+x=\infty$ for any $x$ ). A symmetric matrix $M$ whose entries are nonnegative integers and $\infty$ (and zeroes only on the main diagonal) that satisfies the triangle inequality is a type. A type is bipartite if $M_{i j}+M_{j k}+M_{i k} \equiv 0$ $\bmod 2$ for any three finite entries $M_{i j}, M_{j k}$ and $M_{i k}$. Two bipartite types $M$
and $M^{\prime}$ are compatible if $M_{i j}$ and $M_{i j}^{\prime}$ have the same parity whenever both of them are finite. We define a binary relation on bipartite types as follows: $M \preceq M^{\prime}$ if and only if $M$ and $M^{\prime}$ are compatible and $M_{i j} \leq M_{i j}^{\prime}$ for every $i$ and $j$. Note that the relation $\preceq$ is a partial order.

We finish this section with the following lemma. Its straightforward proof is included to help us in familiarizing with the just introduced notation.

Lemma 6. Let $G^{1}$ and $G^{2}$ be two bipartite rooted partial $k$-trees with types $M^{1}$ and $M^{2}$ such that there exists a bipartite type $M^{0}$ with $M^{0} \preceq M^{1}$ and $M^{0} \preceq M^{2}$. Then the types $M^{1}$ and $M^{2}$ are compatible, $G^{1} \oplus G^{2}$ is a bipartite rooted partial $k$-tree and its type $M$ satisfies $M^{0} \preceq M$.

Proof. The types $M^{1}$ and $M^{2}$ are compatible: if both $M_{i j}^{1}$ and $M_{i j}^{2}$ are finite, then $M_{i j}^{0}$ is finite and has the same parity as $M_{i j}^{1}$ and $M_{i j}^{2}$. Hence, the entries $M_{i j}^{1}$ and $M_{i j}^{2}$ have the same parity.

Let $M$ be the type of $G^{1} \oplus G^{2}$. Note that it does not hold in general that $M_{i j}=\min \left\{M_{i j}^{1}, M_{i j}^{2}\right\}$. We show that $M^{0} \preceq M$ which will also imply that $G^{1} \oplus G^{2}$ is bipartite since $M^{0}$ is a bipartite type. Consider a shortest path $P$ between two distinguished vertices $v_{i}$ and $v_{i^{\prime}}$ and split $P$ into paths $P_{1}, \ldots, P_{\ell}$ delimited by distinguished vertices on $P$. Note that $\ell \leq k$ since $P$ is a path. Let $j_{0}=i$ and let $j_{i}$ be the index of the end-vertex of $P_{i}$ for $i \in\{1, \ldots, \ell\}$. In particular, $j_{\ell}=i^{\prime}$. Each of the paths $P_{1}, \ldots, P_{\ell}$ is fully contained in $G^{1}$ or in $G^{2}$ (possibly in both if it is a single edge). Since $M^{0} \preceq M^{1}$ and $M^{0} \preceq M^{2}$, the length of $P_{i}$ is at least $M_{j_{i-1} j_{i}}^{0}$, and it has the same parity as $M_{j_{i-1} j_{i}}^{0}$. Since $M^{0}$ is a bipartite type (among others, it satisfies the triangle inequality), the length of $P$, which is $M_{i i^{\prime}}$, has the same parity as $M_{j_{0} j_{e}}^{0}=M_{i i^{\prime}}^{0}$ and is at least $M_{i i^{\prime}}^{0}$. This implies that $M^{0} \preceq M$.

## 3 The Main Lemma

In this section, we prove a lemma which forms the core of our argument. To this end, we first prove another lemma that asserts that for every $k, p$ and $q$, the set of types of all bipartite rooted partial $k$-trees forbidding a fixed set of $p$-precolorings from extending (and maybe some other precolorings, too) has always a maximal element. We formulate the lemma slightly differently to facilitate its application.

Lemma 7. For every $k, p$ and $q$, there exists a finite number of (bipartite) types $M^{1}, \ldots, M^{m}$ such that for any bipartite rooted partial $k$-tree $G$ with type $M$, there exists a bipartite rooted partial $k$-tree $G^{\prime}$ with type $M^{i}$ for some $i \in\{1, \ldots, m\}$ such that $\mathcal{F}\left(G^{\prime}\right) \subseteq \mathcal{F}(G)$ and $M \preceq M^{i}$.

Proof. Let $d \geq 2$ be the constant from Lemma 4 applied for $p$ and $q$. Let $M^{1}, \ldots, M^{m}$ be all bipartite types with entries from the set $\left\{1, \ldots, D^{(k+1)^{2}}\right\} \cup$ $\{\infty\}$ where $D=4 d$. Thus, $m$ is finite and does not exceed $\left(D^{(k+1)^{2}}+\right.$ 1) ${ }^{k(k+1) / 2}$.

Let $G$ be a bipartite rooted partial $k$-tree with type $M$. If $M$ is one of the types $M^{1}, \ldots, M^{m}$, then there is nothing to prove (just choose $i$ such that $M=M^{i}$ ). Otherwise, one of its entries is finite and exceeds $D^{(k+1)^{2}}$.

For $i \in\left\{1, \ldots,(k+1)^{2}\right\}$, let $J^{i}$ be the set of all positive integers between $D^{i-1}$ and $D^{i}-1$ (inclusively). Let $i_{0}$ be the smallest integer such that no entry of $M$ is contained in $J^{i_{0}}$. Since $M$ has at most $k(k+1) / 2$ different entries, such an index $i_{0}$ exists. Note that if $i_{0}=1$, then Lemma 4 implies that $\mathcal{F}(G)$ contains all possible $p$-precolorings, and the sought graph $G^{\prime}$ is the bipartite rooted partial $k$-tree composed of $k+1$ isolated vertices, with the all- $\infty$ type.

Two vertices $v_{i}$ and $v_{j}$ at which $G$ is rooted are close if $M_{i j}$ is at most $D^{i_{0}-1}$. The relation $\approx$ of being close is an equivalence relation on $v_{1}, \ldots, v_{k+1}$. Indeed, it is reflexive and symmetric by the definition, and we show now that it is transitive. Suppose that $M_{i j}$ and $M_{j k}$ are both at most $D^{i_{0}-1}$. Then, the distance between $v_{i}$ and $v_{k}$ is at most $M_{i j}+M_{j k} \leq 2 D^{i_{0}-1}-2 \leq D^{i_{0}}-1$ since $D \geq 2$. Consequently, by the choice of $i_{0}$, the distance between $v_{i}$ and $v_{k}$ is at most $D^{i_{0}-1}-1$ and thus $v_{i} \approx v_{k}$.

Let $C_{1}, \ldots, C_{\ell}$ be the equivalence classes of the relation $\approx$. Note that $C_{1}, \ldots, C_{\ell}$ is a finer partition than that given by the equivalence relation of being connected.

Since $G$ is bipartite, we can partition its vertices into two color classes, say red and blue. For every $i \in\{1, \ldots, \ell\}$, contract the closed neighborhood of a vertex $v$ if $v$ is a blue vertex and its distance from any vertex of $C_{i}$ is at least $D^{i_{0}-1}$ and keep doing so as long as such a vertex exists. Observe that the resulting graph is uniquely defined. After discarding the components that do not contain the vertices of $C_{i}$, we obtain a bipartite partial $k$-tree $G_{i}$ rooted at the vertices of $C_{i}$ : it is bipartite as we have always contracted closed neighborhoods of vertices of the same color (blue) to a single (red) vertex, and its tree-width is at most $k$ since the tree-width is preserved by contractions. Moreover, the distance between any two vertices of $C_{i}$ has not decreased since any path between them through any of the newly arising vertices has length at least $2 D^{i_{0}-1}-2 \geq D^{i_{0}-1}$.

Now, let $G^{\prime}$ be the bipartite rooted partial $k$-tree obtained by taking the disjoint union of $G_{1}, \ldots, G_{\ell}$. The type $M^{\prime}$ of $G^{\prime}$ can be obtained from the type of $G$ : set $M_{i j}^{\prime}$ to be $M_{i j}$ if the vertices $v_{i}$ and $v_{j}$ are close, and $\infty$ otherwise. Thus, $M^{\prime}$ is one of the types $M^{1}, \ldots, M^{m}$ and $M \preceq M^{\prime}$. It remains to show that $\mathcal{F}\left(G^{\prime}\right) \subseteq \mathcal{F}(G)$.

Let $c \in \mathcal{F}\left(G^{\prime}\right)$ be a $p$-precoloring that extends to $G^{\prime}$, and recall that $D \geq 4$. For $i \in\{1, \ldots, \ell\}$, let $A_{i}$ be the set of all red vertices at distance at most $D^{i_{0}-1}$ and all blue vertices at distance at most $D^{i_{0}-1}-1$ from $C_{i}$, and let $R_{i}$ be the set of all red vertices at distance $D^{i_{0}-1}-1$ or $D^{i_{0}-1}$ from $C_{i}$. Set $B_{i}=A_{i} \backslash R_{i}$ ( $B_{i}$ is the "interior" of $A_{i}$ and $R_{i}$ its "boundary"). The extension of $c$ to $G_{i}$ naturally defines a coloring of all vertices of $A_{i}: G_{i}$ is the subgraph of $G$ induced by $A_{i}$ with some red vertices of $R_{i}$ identified (two vertices of $R_{i}$ are identified if and only if they are in the same component of the graph $G-B_{i}$ ).

Let $H$ be the following auxiliary graph obtained from $G$ : remove the vertices of $B=B_{1} \cup \cdots \cup B_{\ell}$ and, for $i \in\{1, \ldots, \ell\}$, identify every pair of vertices of $R_{i}$ that are in the same component of $G-B$. Let $R$ be the set of vertices of $H$ corresponding to some vertices of $R_{1} \cup \cdots \cup R_{\ell}$. Precolor the vertices of $R$ with the colors given by the colorings of $G_{i}$ (note that two vertices of $R_{i}$ in the same component of $G-B_{i}$ are also in the same component of $G-B$, so this is well-defined). The graph $H$ is bipartite as only red vertices have been identified. The distance between any two precolored vertices is at least $d$ : consider two precolored vertices $r$ and $r^{\prime}$ at distance at most $d-1$. Let $i$ and $i^{\prime}$ be such that $r \in R_{i}$ and $r^{\prime} \in R_{i^{\prime}}$. If $i=i^{\prime}$, then $r$ and $r^{\prime}$ are in the same component of $G-B$ and thus $r=r^{\prime}$. If $i \neq i^{\prime}$ then by the definition of $R_{i}$ and $R_{i^{\prime}}$, the vertex $r$ is in $G$ at distance at most $D^{i_{0}-1}$ from some vertex $v$ of $C_{i}$ and $r^{\prime}$ is at distance at most $D^{i_{0}-1}$ from some vertex $v^{\prime}$ of $C_{i^{\prime}}$. So, the distance between $v$ and $v^{\prime}$ is at most $2 D^{i_{0}-1}+d<D^{i_{0}}-1$. Since $M$ has no entry from $J^{i_{0}}$, the vertices $v$ and $v^{\prime}$ must be close and thus $i=i^{\prime}$, a contradiction.

Since the distance between any two precolored vertices is at least $d$, the precoloring extends to $H$ by Lemma 4 and in a natural way it defines a coloring of $G$. We conclude that every $p$-precoloring that extends to $G^{\prime}$ also extends to $G$ and thus $\mathcal{F}\left(G^{\prime}\right) \subseteq \mathcal{F}(G)$.

We now prove our main lemma, which basically states that there is only a finite number of bipartite rooted partial $k$-trees that can appear in a minimal non- $(p, q)$-colorable graph with tree-width $k$ and a given odd girth.

Lemma 8. For every $k, p$ and $q$, there exist a finite number $m$ and bipartite rooted partial $k$-trees $G^{1}, \ldots, G^{m}$ with types $M^{1}, \ldots, M^{m}$ such that for any bipartite rooted partial $k$-tree $G$ with type $M$ there exists $i$ such that $\mathcal{F}\left(G^{i}\right) \subseteq$ $\mathcal{F}(G)$ and $M \preceq M^{i}$.

Proof. Let $M^{1}, \ldots, M^{m}$ be the types from Lemma 7. We define the graph $G^{i}$ as follows: for every $p$-precoloring $c$ that does not extend to a bipartite partial rooted $k$-tree with type $M^{i}$, fix any partial rooted $k$-tree $G_{c}^{i}$ with
type $M^{i}$ such that $c$ does not extend to $G_{c}^{i}$. Set $G^{i}=\bigoplus_{c} G_{c}^{i}$, where $c$ runs over all such $p$-precolorings. If the above sum of partial $k$-trees is non-empty, then the type $M$ of $G^{i}$ is $M^{i}$. Indeed, $M \preceq M^{i}$ by the definition of $G^{i}$, and Lemma 6 implies that $M^{i} \preceq M$. If all the $p$-precolorings of the $k+1$ vertices in the root extend to each partial $k$-tree of type $M^{i}$, then let $G^{i}$ be the graph consisting of $k+1$ isolated vertices. This happens in particular for the all- $\infty$ type.

Let us verify the statement of the lemma. Let $G$ be a bipartite rooted partial $k$-tree and let $M$ be the type of $G$. If $\mathcal{F}(G)$ is composed of all $p$ precolorings, the sought graph $G^{i}$ is the one composed of $k+1$ isolated vertices. Hence, we assume that $\mathcal{F}(G)$ does not contain all $p$-precolorings, i.e., there are $p$-precolorings that do not extend to $G$. By Lemma 7 , there exists a bipartite rooted partial $k$-tree $G^{\prime}$ with type $M^{\prime}$ such that $M \preceq M^{\prime}=M^{i}$ for some $i$ and $\mathcal{F}\left(G^{\prime}\right) \subseteq \mathcal{F}(G)$. For every $p$-precoloring $c$ that does not extend to $G^{\prime}$ (and there exists at least one such $p$-precoloring $c$ ), some graph $G_{c}^{i}$ has been glued into $G^{i}$. Hence, $\mathcal{F}\left(G^{i}\right) \subseteq \mathcal{F}\left(G^{\prime}\right) \subseteq \mathcal{F}(G)$. Since the type of $G^{i}$ is $M^{i}$, the conclusion of the lemma follows.

## 4 Proof of Theorem [2]

We are now ready to prove Theorem 2, which is recalled below.
Theorem 2. For every $k$ and every $\varepsilon>0$, there exists $g$ such that every graph with tree-width at most $k$ and odd-girth at least $g$ has circular chromatic number at most $2+\varepsilon$.

Proof. Fix $p$ and $q$ such that $2<p / q \leq 2+\varepsilon$. Let $G^{1}, \ldots, G^{m}$ be the bipartite partial $k$-trees from Lemma 8 applied for $k, p$ and $q$. Set $N$ to be the largest order of the graphs $G^{i}$ and set $g$ to be $3 N$. We assert that each partial $k$-tree with odd-girth $g$ has circular chromatic number at most $p / q$. Assume that this is not the case and let $G$ be a counterexample with the fewest vertices.

The graph $G$ has at least $3 N$ vertices (otherwise, it has no odd cycles and thus it is bipartite). By Lemma 5 , $G$ is isomorphic to $G_{1} \oplus G_{2}$, where $G_{1}$ and $G_{2}$ are rooted partial $k$-trees and the number of vertices of $G_{1}$ is between $N+1$ and $2 N$. By the choice of $g$, the graph $G_{1}$ has no odd cycle and thus it is a bipartite rooted partial $k$-tree. By Lemma 图, there exists $i$ such that $\mathcal{F}\left(G^{i}\right) \subseteq \mathcal{F}\left(G_{1}\right)$ and $M_{1} \preceq M^{i}$ where $M_{1}$ is the type of $G_{1}$ and $M^{i}$ is the type of $G^{i}$. Let $G^{\prime}$ be the partial $k$-tree $G^{i} \oplus G_{2}$.

First, $G^{\prime}$ has fewer vertices than $G$ since the number of vertices of $G^{i}$ is at most $N$ and the number of vertices of $G_{1}$ is at least $N+1$. Second, $G^{\prime}$ has no $(p, q)$-coloring: if it had a $(p, q)$-coloring, then the corresponding
$p$-precoloring of the $k+1$ vertices shared by $G^{i}$ and $G_{2}$ would extend to $G_{1}$ since $\mathcal{F}\left(G^{i}\right) \subseteq \mathcal{F}\left(G_{1}\right)$ and thus $G$ would have a $(p, q)$-coloring, too. Finally, $G^{\prime}$ has no odd cycle of length at most $g$ : if it had such a cycle, replace any path between vertices $v_{j}$ and $v_{j^{\prime}}$ of the root of $G^{i}$ with a path of at most the same length between them in $G_{1}$ (recall that $M_{1} \preceq M^{i}$ ). If such paths for different pairs of $v_{j}$ and $v_{j^{\prime}}$ on the considered odd cycle intersect, take their symmetric difference. In this way, we obtain an Eulerian subgraph of $G=G_{1} \oplus G_{2}$ with an odd number of edges such that the number of its edges does not exceed $g$. Consequently, this Eulerian subgraph has an odd cycle of length at most $g$, which violates the assumption on the odd-girth of $G$. We conclude that $G^{\prime}$ is a counterexample with less vertices than $G$, a contradiction.

We end by pointing out that the approach used yields an upper bound of $3(k+1) \cdot 2^{2^{p^{k+1}}\left((4 d)^{(k+1)^{2}}+1\right)^{k^{2}}}$ for the smallest $g$ such that all graphs with tree-width at most $k$ and odd-girth at least $g$ have circular chromatic number at most $p / q$, whenever $p / q>2$. More precisely, the value of $N$ cannot exceed
 where $C$ is a set of $p$-precolorings of the root and $M$ is a type such that there is a bipartite rooted partial $k$-tree of type $M$ to which no coloring of $C$ extends. Let $n_{P}$ be the size of a smallest such partial $k$-tree. We obtain a sequence of at most $2^{p^{k+1}} \times\left((4 d)^{(k+1)^{2}}+1\right)^{k^{2}}$ integers. The announced bound follows from the following fact: if the sequence is sorted in increasing order, then each term is at most twice the previous one.

Indeed, consider the tree-decomposition of the partial $k$-tree $G_{P}$ chosen for the pair $P$. If the bag containing the root has a single child, then we delete a vertex of the root, and set a vertex in the single child to be part of the root. We obtain a partial $k$-tree to which some $p$-precolorings of $C$ do not extend. Thus, $n_{P} \leq 1+n_{P^{\prime}}$ for some pair $P^{\prime}$ and $n_{P^{\prime}}<n_{P}$. If the bag containing the root has more than one child, then $G_{P}$ can be obtained by identifying the roots of two smaller partial $k$-trees $G$ and $G^{\prime}$. By the minimality of $G_{P}$, the orders of $G$ and $G^{\prime}$ are $n_{P_{1}}$ and $n_{P_{2}}$ for two pairs $P_{1}$ and $P_{2}$ such that $n_{P_{i}}<n_{P}$ for $i \in\{1,2\}$. This yields the stated fact, which in turn implies the given bound, since the smallest element of the sequence is $k+1$.

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