

# TWO-REGULAR SUBGRAPHS OF ODD-UNIFORM HYPERGRAPHS

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**ABSTRACT.** Let  $k \geq 3$  be an odd integer and let  $n$  be a sufficiently large integer. We prove that the maximum number of edges in an  $n$ -vertex  $k$ -uniform hypergraph containing no 2-regular subgraphs is  $\binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$ , and the equality holds if and only if  $H$  is a full  $k$ -star with center  $v$  together with a maximal matching omitting  $v$ . This verifies a conjecture of Mubayi and Verstraëte.

## 1. INTRODUCTION

Turán problems are central in extremal graph theory. In general, Turán-type problems question on the maximum number of edges of a (hyper)graph that does not contain certain subgraph(s). Their generalizations to hypergraphs appear to be extremally hard – for example, despite many existing works, the Turán density of tetrahedron (four triples on four vertices) is still unknown (see [8]).

Erdős [3] asked to determine the maximum size  $f_k(n)$  of an  $n$ -vertex  $k$ -uniform hypergraph without any generalized 4-cycles, i.e., four distinct edges  $A, B, C, D$  such that  $A \cup B = C \cup D$  and  $A \cap B = C \cap D = \emptyset$ . For  $k = 2$ , this reduces to a well-known problem of studying the Turán number for the 4-cycle. It is known that  $f_2(n) = (1+o(1))n^{3/2}$  [2, 4] and the exact value of  $f_2(n)$  for infinitely many  $n$  is obtained in [6]. For  $k \geq 3$ , Füredi [7] showed that  $\binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor \leq f_k(n) \leq \frac{7}{2}\binom{n}{k-1}$  and conjectured the following.<sup>1</sup>

**Conjecture 1.1.** For  $k \geq 4$  and  $n \in \mathbb{N}$ ,  $f_k(n) = \binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$ .

The lower bound is achieved by a full  $k$ -star together with a maximal matching omitting its center. Here a *full  $k$ -star* is a  $k$ -uniform  $n$ -vertex hypergraph which consists of all  $\binom{n-1}{k-1}$  sets of size  $k$  containing a given vertex  $v$ , and the given vertex  $v$  is called the *center* of the full  $k$ -star. The most recent result on  $f_k(n)$  is due to Pikhurko and Verstraëte [13], who showed that  $f_k(n) \leq \min\{1 + 2/\sqrt{k}, 7/4\}\binom{n}{k-1}$ , and  $f_3(n) \leq \frac{13}{9}\binom{n}{2}$ . This improves a result by Mubayi and Verstraëte [11]. In [9], the second author made a related conjecture about  $k$ -uniform hypergraphs containing no  $r$  pairs of disjoint sets with the same union when  $k$  is sufficiently bigger than  $r$ .

Since the generalized 4-cycles are 2-regular, i.e., each vertex has degree 2, one way to relax the original problem of Erdős is to consider the maximum size of  $n$ -vertex (hyper)graphs without any 2-regular sub(hyper)graphs (or more generally, without any  $r$ -regular subgraphs). In fact, the (relaxed) problem has its own interest even for graphs. Although it is trivial for  $r = 2$ , Pyber [14] proved that the largest number of edges in a graph with no  $r$ -regular subgraphs is  $O(n \log n)$  for any  $r \geq 2$ , and in [15], Pyber, Rödl and Szemerédi showed that there are graphs with no  $r$ -regular subgraphs having  $\Omega(n \log \log n)$  edges for any  $r \geq 3$ .

For non-uniform hypergraphs, it is easy to see that any hypergraph with no  $r$ -regular subgraphs has at most  $2^{n-1} + r - 1$  edges and Kostochka and the second author [10] showed that if  $n \geq \max\{425, r + 1\}$  then any  $n$ -vertex hypergraph with no  $r$ -regular subgraphs having the maximum number of edges must contain a vertex of degree  $2^{n-1}$ . For uniform hypergraphs, the

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<sup>1</sup>In fact, Füredi [7] found a slightly better lower bound for  $k = 3$ , namely,  $f_3(n) \geq \binom{n}{2}$  for  $n \equiv 1$  or  $5 \pmod{20}$ .

problem becomes more interesting. One natural candidate for the extremal example of  $k$ -uniform hypergraphs with no 2-regular subgraphs is the full  $k$ -star. Indeed, Mubayi and Verstraëte [12] proved the following.

**Theorem 1.2.** [12] *For every even integer  $k \geq 4$ , there exists  $n_k$  such that the following holds for all  $n \geq n_k$ . If  $H$  is an  $n$ -vertex  $k$ -uniform hypergraph with no 2-regular subgraphs, then  $|H| \leq \binom{n-1}{k-1}$ . Moreover, equality holds if and only if  $H$  is a full  $k$ -star.*

In [9] the second author generalized the arguments in [12] and showed similar results for  $k$ -uniform hypergraphs with no  $r$ -regular subgraphs when  $r \in \{3, 4\}$ . Moreover, for odd  $k$ , Mubayi and Verstraëte [12] conjectured that  $|H| \leq \binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$ , and the only extremal graph is the full  $k$ -star plus a matching omitting its center. In this paper, we prove this conjecture.

**Theorem 1.3.** *For every odd integer  $k \geq 3$ , there exists  $n_k$  such that the following holds for all  $n \geq n_k$ . If  $H$  is an  $n$ -vertex  $k$ -uniform hypergraph with no 2-regular subgraphs, then*

$$|H| \leq \binom{n-1}{k-1} + \left\lfloor \frac{n-1}{k} \right\rfloor.$$

*Moreover, equality holds if and only if  $H$  is a full  $k$ -star with center  $v$  together with a maximal matching omitting  $v$ .*

Theorem 1.2 [12] is proved via the stability approach introduced by Erdős and Simonovits [16], which has been widely used in extremal set theory. To prove Theorem 1.3, we also use the stability approach as well as some other ideas from [12]. One advantage when  $k$  is even is that there exist 2-regular  $k$ -uniform hypergraphs on  $3k/2$  vertices. In contrast, for odd  $k$ , the smallest 2-regular  $k$ -uniform hypergraphs have order  $2k$  and thus the analysis is more difficult (this is also the reason why more edges are allowed in the extremal graph for odd  $k$ , which makes the structure more complicated). In our proof, we use some new tricks to overcome this difficulty.

## 2. PRELIMINARIES

For a positive integer  $N$  we write  $[N]$  to denote the set  $\{1, \dots, N\}$ . We write  $V(H)$  for the set of vertices,  $E(H)$  for the set of edges in a hypergraph  $H$ . For a hypergraph  $H$ , we view  $H$  as a collection of edges, thus sometimes  $H$  refers to  $E(H)$ . We say that  $H$  is a  $k$ -uniform hypergraph or  $k$ -uniform family if every edge of  $H$  has size exactly  $k$ . Moreover, we always say *subgraph* instead of subhypergraph. For a hypergraph  $H$  and a set  $S \subseteq V(H)$ ,

$$N_H(S) := \{e \setminus S : e \in E(H), S \subseteq e\} \quad \text{and} \quad d_H(S) = |N_H(S)|.$$

We say a set  $S$  is an  $s$ -set if  $|S| = s$ . For a vertex  $x \in V(H)$ , we write  $N_H(x) := N_H(\{x\})$  and  $d_H(x) := d_H(\{x\})$ . We say  $\{S, S'\}$  is an *equipartition* of a set  $A$  if  $|S| = |S'|$ ,  $S \cap S' = \emptyset$  and  $S \cup S' = A$ .

In order to prove Theorem 1.3, we use the following two theorems proved in [12]. These theorems give a rough structure of near-extremal hypergraphs.

**Theorem 2.1.** [12] *For given  $\varepsilon > 0$  and  $k \in \mathbb{N}$ , there exists  $n_0 = n_0(k, \varepsilon)$  such that the following holds for all  $n \geq n_0$ . If  $H$  is an  $n$ -vertex  $k$ -uniform hypergraph with no 2-regular subgraphs, then*

$$|H| \leq (1 + \varepsilon) \binom{n-1}{k-1}.$$

**Theorem 2.2.** [12] *For given  $\varepsilon > 0$  and  $k \in \mathbb{N}$ , there exists  $n_1 = n_1(k, \varepsilon)$  such that the following holds for all  $n \geq n_1$ . If  $H$  is an  $n$ -vertex  $k$ -uniform hypergraph with no 2-regular subgraphs with  $|H| \geq \binom{n-1}{k-1}$ , then  $H$  contains a vertex  $v$  with  $d_H(v) \geq (1 - \varepsilon) \binom{n-1}{k-1}$ .*

We use the following result of Frankl [5, Theorem 10.3].

**Theorem 2.3.** *For integers  $t \geq 1$  and  $n \geq 2k$ , if an  $n$ -vertex  $k$ -uniform hypergraph  $H$  has more than  $t \binom{n-1}{k-1}$  edges, then  $H$  has a matching of size  $t + 1$ .*

We also use the following result of Balogh, Bohman and Mubayi [1]. If an intersecting  $k$ -uniform hypergraph is a subgraph of a full  $k$ -star, then it is called *trivial*, otherwise *non-trivial*. Moreover, we say that a  $k$ -uniform hypergraph  $H$  is covered by a set  $X \subseteq \binom{V(H)}{2}$  of pairs of vertices of  $H$  if for every hyperedge  $e$  of  $H$ , there is a pair  $\{x, y\} \in X$  such that  $\{x, y\} \subseteq e$ .

**Lemma 2.4.** [1] *Let  $H$  be a non-trivial intersecting  $k$ -uniform hypergraph. Then  $H$  can be covered by at most  $k^2 - k + 1$  pairs of vertices.*

### 3. PROOF OF THEOREM 1.3

Let  $k \geq 3$  be an odd integer. Let  $\varepsilon := \varepsilon(k) > 0$  be sufficiently small and let  $n(k, \varepsilon)$  be a sufficiently large integer. For  $n \geq n(k, \varepsilon)$ , let  $H$  be an  $n$ -vertex  $k$ -uniform hypergraph with no 2-regular subgraphs. By removing edges if necessary, we may assume that

$$|H| = \binom{n-1}{k-1} + \left\lfloor \frac{n-1}{k} \right\rfloor. \quad (3.1)$$

To prove Theorem 1.3, it is enough to show that  $H$  contains a full  $k$ -star, because a full  $k$ -star with two additional intersecting edges always gives a 2-regular subgraph. To derive a contradiction, we assume that  $H$  does not contain any full  $k$ -star. Since  $n$  is sufficiently large, Theorem 2.2 implies that there is a vertex  $v \in V(H)$  such that  $d_H(v) \geq \binom{n-1}{k-1} - \varepsilon^3 n^{k-1}$ . Let  $V' := V(H) \setminus \{v\}$ ,  $H^* := H[V']$  and  $\tilde{H} := \{e \setminus \{v\} : |e| = k, v \in e \notin H\}$ . Note that any  $(k-1)$ -set  $A \subseteq V'$  with  $A \notin \tilde{H}$  satisfies that  $A \cup \{v\} \in H$ . Let

$$x := |\tilde{H}| = \binom{n-1}{k-1} - d_H(v) \leq \varepsilon^3 n^{k-1}. \quad (3.2)$$

Since  $H$  does not contain a full  $k$ -star with center  $v$ , we have  $x \geq 1$ . Then (3.1) and (3.2) imply that

$$|H^*| = x + \left\lfloor \frac{n-1}{k} \right\rfloor. \quad (3.3)$$

Important idea for the proof is that a pair of two intersecting edges in  $H^*$  ensures  $\tilde{H}$  to contain more  $(k-1)$ -sets (Claim 3.1). Since  $x > 0$ , there exists a pair of intersecting edges in  $H^*$  and Claim 3.1 implies that value of  $x = |\tilde{H}|$  is larger. However, by (3.3), larger value of  $x$  guarantees more pairs of intersecting edges in  $H^*$  which again implies the value of  $x$  is larger. This circulation of logic gives a contradiction as  $x$  cannot be too large by (3.2).

To turn this idea into a mathematical proof, we need to prove some technical claims. Here, we give a brief outline of our proof. We start with some simple but useful claims (Subsection 3.1), and in particular, we show that  $x \geq n - 2k + 1$  (Claim 3.3). Thus together with (3.2), we may assume that there exists an integer  $2 \leq \ell \leq k-1$  such that

$$\varepsilon^3 n^{\ell-1} \leq x \leq \varepsilon^3 n^\ell.$$

We next find pairwise disjoint  $(\ell-1)$ -sets  $S_1, S_2, \dots, S_{2k} \subseteq V'$  such that  $d_{\tilde{H}}(S_i) \leq \binom{k}{\ell-1} x / \binom{n-1}{\ell-1}$  for  $i \in [2k]$ , which play an important role in the proof. Let  $\mathcal{T} := \bigcup_{i=1}^{2k} N_{\tilde{H}}(S_i)$  be a collection of  $(k-\ell)$ -sets in  $V'$ . Let  $H_1 := \{e \in H^* : \exists T \in \mathcal{T} \text{ such that } T \subseteq e\}$  and let  $H_0 := H^* \setminus H_1$ . Our goal is to show that  $|H_1| \leq \varepsilon x$  (Subsection 3.2) and  $|H_0| < (1-\varepsilon)x + \left\lfloor \frac{n-1}{k} \right\rfloor$  (Subsection 3.3), which together imply that  $|H^*| = |H_1| + |H_0| < x + \left\lfloor \frac{n-1}{k} \right\rfloor$ , contradicting (3.3). In fact, the technical parts are Subsections 3.2 and 3.3, in which the essential argument is some clever double counting also used in [12]. However, as mentioned in Section 1, our case is more complicated than that in [12], so we have to proceed a more careful analysis (including introducing  $\ell$  and  $\mathcal{T}$ ) and use some new tricks (e.g. analyzing the intersecting property of certain family and utilizing Lemma 2.4).

**3.1. Preparation.** First we prove the following easy claim.

**Claim 3.1.** *Assume we have  $e_1, e_2 \in H^*$ ,  $A \subseteq V'$  and  $\{S, S'\}$  such that*

- $A \cap (e_1 \cup e_2) = \emptyset$ ,
- $|A| = |e_1 \cap e_2| - 1$ ,
- $\{S, S'\}$  is an equipartition of  $e_1 \triangle e_2$ .

*Then either  $A \cup S \in \tilde{H}$  or  $A \cup S' \in \tilde{H}$ .*

*Proof.* If both  $(k-1)$ -sets  $A \cup S$  and  $A \cup S'$  are not in  $\tilde{H}$ , then

$$e_1, e_2, A \cup S \cup \{v\} \text{ and } A \cup S' \cup \{v\}$$

form a 2-regular subgraph of  $H$ , a contradiction.  $\square$

Now we prove the following two claims regarding lower bounds on  $x$ .

**Claim 3.2.** *Let  $t \in [k-1]$ . If  $H^*$  contains two edges  $e_1, e_2$  such that  $|e_1 \cap e_2| = t$ , then*

$$x \geq \frac{1}{2} \binom{2k-2t}{k-t} \binom{n-2k+t-1}{t-1}.$$

*Proof.* Suppose  $e_1, e_2 \in H^*$  such that  $|e_1 \cap e_2| = t$ . Consider a set  $A \in \binom{V' \setminus (e_1 \cup e_2)}{t-1}$  and an equipartition  $\{S, S'\}$  of  $e_1 \triangle e_2$ . For each  $A$  and  $\{S, S'\}$ , Claim 3.1 implies that  $A \cup S \in \tilde{H}$  or  $A \cup S' \in \tilde{H}$ . Moreover, distinct choices of  $(A, \{S, S'\})$  give us distinct  $(k-1)$ -sets in  $\tilde{H}$ .

Since there are  $\binom{n-2k+t-1}{t-1}$  distinct choices of  $A$  and  $\frac{1}{2} \binom{2k-2t}{k-t}$  distinct choices of  $\{S, S'\}$ , we have  $x = |\tilde{H}| \geq \frac{1}{2} \binom{2k-2t}{k-t} \binom{n-2k+t-1}{t-1}$ .  $\square$

**Claim 3.3.** *The hypergraph  $H^*$  contains two edges  $e_1, e_2$  such that  $|e_1 \cap e_2| \geq 2$ . Moreover,  $x \geq n - 2k + 1$ .*

*Proof.* Assume  $H^*$  does not contain such two edges. Then for any  $u \in V'$  and  $S, S' \in N_{H^*}(u)$ , we have  $S \cap S' = \emptyset$ . If there are two  $(k-1)$ -sets  $S, S' \in N_{H^*}(u)$  such that  $S, S' \notin \tilde{H}$ , then

$$S \cup \{u\}, S' \cup \{u\}, S \cup \{v\} \text{ and } S' \cup \{v\}$$

form a 2-regular subgraph of  $H$ , a contradiction. Thus for any  $u \in V'$ , we have  $|N_{H^*}(u) \cap \tilde{H}| \geq |N_{H^*}(u)| - 1$ . Moreover, by our assumption, we have  $N_{H^*}(u) \cap N_{H^*}(u') = \emptyset$  for any distinct  $u, u' \in V'$ . Thus

$$x = |\tilde{H}| \geq \sum_{u \in V'} |N_{H^*}(u) \cap \tilde{H}| \geq \sum_{u \in V'} (d_{H^*}(u) - 1) = k|H^*| - (n-1) \stackrel{(3.3)}{\geq} kx - (k-1).$$

Since  $k \geq 3$ , we get  $x \leq 1$ . However, the assumption that  $x \geq 1$  and (3.3) imply that there are two edges  $e_1, e_2 \in H^*$  with  $|e_1 \cap e_2| \geq 1$ . So by Claim 3.2, we have  $x \geq \frac{1}{2} \binom{2k-2}{k-1} \binom{n-2k}{0} \geq 3$ , a contradiction. Thus  $H^*$  contains two edges  $e_1, e_2$  with  $|e_1 \cap e_2| \geq 2$ . Hence Claim 3.2 implies that  $x \geq \frac{1}{2} \binom{2k-4}{k-2} \binom{n-2k+1}{1} \geq n - 2k + 1$ .  $\square$

By (3.2) and Claim 3.3, there exists an integer  $\ell$  such that

$$\varepsilon^3 n^{\ell-1} \leq x \leq \varepsilon^3 n^\ell \tag{3.4}$$

and  $2 \leq \ell \leq k-1$ . Throughout the rest of the paper,  $\ell$  denotes such integer satisfying (3.4).

The following claim finds  $2k$  pairwise disjoint  $(\ell-1)$ -sets which have low degree in  $\tilde{H}$ .

**Claim 3.4.** *There are pairwise disjoint  $(\ell-1)$ -sets  $S_1, S_2, \dots, S_{2k} \subseteq V'$  such that  $d_{\tilde{H}}(S_i) \leq \binom{k}{\ell-1} x / \binom{n-1}{\ell-1}$  for  $i \in [2k]$ .*

*Proof.* Let  $F := \{S \in \binom{V'}{\ell-1} : d_{\tilde{H}}(S) \leq \binom{k}{\ell-1}x / \binom{n-1}{\ell-1}\}$  and  $F' := \binom{V'}{\ell-1} \setminus F$ . So it suffices to find a matching of size  $2k$  in  $F$ . Then

$$\binom{k-1}{\ell-1}x = \sum_{S \in \binom{V'}{\ell-1}} d_{\tilde{H}}(S) \geq 0 \cdot |F| + \frac{\binom{k}{\ell-1}x}{\binom{n-1}{\ell-1}}|F'|.$$

So we have

$$|F'| \leq \frac{\binom{k-1}{\ell-1}}{\binom{k}{\ell-1}} \binom{n-1}{\ell-1} = \frac{k-\ell+1}{k} \binom{n-1}{\ell-1} \leq \frac{k-1}{k} \binom{n-1}{\ell-1},$$

as  $\ell \geq 2$ . Since  $|F| + |F'| = \binom{n-1}{\ell-1}$ , we have  $|F| \geq \frac{1}{k} \binom{n-1}{\ell-1} > 2k \binom{|V'|-1}{\ell-2}$ . Then by Theorem 2.3,  $F$  contains a matching  $\{S_1, \dots, S_{2k}\}$  of size  $2k$  as desired.  $\square$

Let  $S_1, \dots, S_{2k}$  be pairwise disjoint  $(\ell-1)$ -sets as in Claim 3.4. Let  $\mathcal{T} := \bigcup_{i=1}^{2k} N_{\tilde{H}}(S_i)$  be a collection of  $(k-\ell)$ -sets in  $V'$ . So we have

$$|\mathcal{T}| \leq \sum_{i=1}^{2k} |N_{\tilde{H}}(S_i)| \leq \frac{2k \binom{k}{\ell-1}x}{\binom{n-1}{\ell-1}} \leq 2k^{k+1}n^{1-\ell}x. \quad (3.5)$$

Note that for any  $(k-\ell)$ -set  $T \notin \mathcal{T}$ , we have  $T \cup S_i \cup \{v\} \in H$  if  $T \cap S_i = \emptyset$ . Let  $W = \bigcup_{T \in \mathcal{T}} T$ , then

$$|W| \leq (k-\ell)|\mathcal{T}| \stackrel{(3.5)}{\leq} 2k^{k+2}n^{1-\ell}x \stackrel{(3.4)}{\leq} \varepsilon^2 n. \quad (3.6)$$

Let  $H_1 := \{e \in H^* : \exists T \in \mathcal{T} \text{ such that } T \subset e\}$  and let  $H_0 := H^* \setminus H_1$ .

We finish this subsection with an essential claim that bounds the degrees of vertex sets of size at most  $\ell$  from above.

**Claim 3.5.** *Any  $\ell$ -set  $L \subseteq V'$  satisfies that  $|N_{H^*}(L) \setminus \mathcal{T}| \leq 1$ . Moreover, for any set  $B \subseteq V'$  with  $|B| = b \leq \ell$ , it satisfies*

$$d_{H_0}(B) \leq \binom{n-1-b}{\ell-b} \text{ and } d_{H^*}(B) \leq \binom{n-1-b}{\ell-b} (1 + |\mathcal{T}|).$$

*Proof.* Suppose that there exists an  $\ell$ -set  $L \subseteq V'$  such that  $|N_{H^*}(L) \setminus \mathcal{T}| \geq 2$ . Then there are two distinct  $(k-\ell)$ -sets  $E_1, E_2 \in N_{H^*}(L) \setminus \mathcal{T}$ . Since  $2 \leq \ell \leq k-1$ , there exists  $i \in [2k]$  such that  $S_i$  is disjoint from  $E_1 \cup E_2 \cup L$ . Also we choose a (possibly empty) set  $A \subseteq V' \setminus W$  such that  $|A| = |E_1 \cap E_2|$  and  $A \cap (E_1 \cup E_2 \cup L \cup S_i) = \emptyset$ . This choice is possible since

$$|V' \setminus (W \cup E_1 \cup E_2 \cup L \cup S_i)| \stackrel{(3.6)}{\geq} (n-1) - \varepsilon^2 n - 2k + \ell - (\ell-1) \geq k.$$

We claim that both  $S_i \cup A \cup (E_1 \setminus E_2)$  and  $S_i \cup A \cup (E_2 \setminus E_1)$  are not in  $\tilde{H}$ . Indeed, if  $A = \emptyset$ , then  $E_j \setminus E_{3-j} = E_j$  for  $j \in [2]$ . Since  $E_j \notin \mathcal{T} = \bigcup_{i=1}^{2k} N_{\tilde{H}}(S_i)$ , we obtain  $S_i \cup E_j = S_i \cup (E_j \setminus E_{3-j}) \notin \tilde{H}$  for  $j \in [2]$ . If  $A \neq \emptyset$ , then since  $A \cap W = \emptyset$ , we have for  $j \in [2]$ ,  $A \cup (E_j \setminus E_{3-j}) \not\subseteq W$ . So  $A \cup (E_j \setminus E_{3-j}) \notin \mathcal{T}$  and thus  $S_i \cup A \cup (E_j \setminus E_{3-j}) \notin \tilde{H}$  for  $j \in [2]$ . Thus

$$L \cup E_1, L \cup E_2, \{v\} \cup S_i \cup A \cup (E_1 \setminus E_2) \text{ and } \{v\} \cup S_i \cup A \cup (E_2 \setminus E_1)$$

form a 2-regular subgraph of  $H$ , a contradiction. Thus the first part of the claim holds.

For any  $B \subseteq V'$  with  $|B| = b \leq \ell$ ,

$$d_{H_0}(B) \leq \sum_{B \subseteq L, |L|=\ell} d_{H_0}(L) \leq \sum_{B \subseteq L, |L|=\ell} |N_{H^*}(L) \setminus \mathcal{T}| \leq \sum_{B \subseteq L, |L|=\ell} 1 = \binom{n-1-b}{\ell-b}.$$

Since  $d_{H^*}(L) \leq |N_{H^*}(L) \setminus \mathcal{T}| + |\mathcal{T}| \leq 1 + |\mathcal{T}|$  for any  $\ell$ -set  $L$ , we also have

$$d_{H^*}(B) \leq \sum_{B \subseteq L, |L|=\ell} d_{H^*}(L) \leq \sum_{B \subseteq L, |L|=\ell} (1 + |\mathcal{T}|) = \binom{n-1-b}{\ell-b} (1 + |\mathcal{T}|). \quad \square$$

In the next two subsections, we show that  $|H_1| \leq \varepsilon x$  (Subsection 3.2) and  $|H_0| < (1-\varepsilon)x + \lfloor \frac{n-1}{k} \rfloor$  (Subsection 3.3), which together imply that  $|H^*| = |H_1| + |H_0| < x + \lfloor \frac{n-1}{k} \rfloor$ . This contradicts (3.3) and thus completes the proof of Theorem 1.3.

**3.2. Size of  $H_1$ .** In this subsection, we show that  $|H_1| \leq \varepsilon x$ . We first consider the case  $\ell \leq k-2$ .

**Claim 3.6.** *If  $\ell \leq k-2$ , then  $|H_1| \leq \varepsilon x$ .*

*Proof.* We first claim that we may assume that  $|\mathcal{T}| > 0$ ,  $|H_1| \geq 3|\mathcal{T}|$  and  $\ell \geq (k+1)/2$ . Indeed, since  $|\mathcal{T}| = 0$  implies  $|H_1| = 0 \leq \varepsilon x$ , we may assume that  $|\mathcal{T}| > 0$ . If  $|H_1| < 3|\mathcal{T}|$ , then by (3.5),  $|H_1| \leq 6k^{k+1}n^{1-\ell}x \leq \varepsilon x$  because  $\ell \geq 2$  and  $n$  is sufficiently large. Thus we may assume that  $|H_1| \geq 3|\mathcal{T}|$ . Finally, since  $|H_1| \geq 3|\mathcal{T}| > |\mathcal{T}|$ , there is a  $(k-\ell)$ -set  $T \in \mathcal{T}$  which is a subset of two distinct edges  $e_1, e_2$  of  $H_1$ . Since  $|e_1 \cap e_2| \geq |T| \geq k-\ell$ , Claim 3.2 and (3.4) implies that

$$\frac{1}{2} \binom{2\ell}{\ell} \binom{n-k-\ell-1}{k-\ell-1} \leq x \stackrel{(3.4)}{\leq} \varepsilon^3 n^\ell.$$

Since  $n$  is sufficiently large and  $\varepsilon$  is small, this implies that  $\ell > k-\ell-1$ . Thus we have  $\ell \geq (k+1)/2$  since  $k$  is odd.

Let  $p$  be the number of tuples  $(T, \{e_1, e_2\}, f)$  with the following properties.

(P.1.1)  $T \in \mathcal{T}$ ,  $\{e_1, e_2\} \in \binom{H_1}{2}$  and  $f \in \tilde{H}$ ,

(P.1.2)  $T \subseteq e_1 \cap e_2$ ,

(P.1.3)  $f \cap (e_1 \cap e_2) = \emptyset$  and  $\{|f \cap e_1|, |f \cap e_2|\} = \{1, |e_2 \setminus e_1| - 1\}$ .

First we find a lower bound on  $p$ . Fix a  $(k-\ell)$ -set  $T$  in  $\mathcal{T}$  and a pair  $\{e_1, e_2\} \in P(T)$ , where  $P(T) := \{\{e_1, e_2\} \in \binom{H_1}{2} : T \subseteq e_1 \cap e_2\}$ . Let  $A$  be an arbitrary set of size  $|e_1 \cap e_2| - 1$  in  $V' \setminus (e_1 \cup e_2)$  and let  $\{S, S'\}$  be an equipartition of  $e_1 \triangle e_2$  such that  $|S \cap e_1| = 1$ . Then Claim 3.1 implies that one of  $A \cup S$  and  $A \cup S'$  belongs to  $\tilde{H}$  and it satisfies (P.1.3). Note that distinct choices of  $(A, \{S, S'\})$  give us distinct  $(k-1)$ -sets in  $\tilde{H}$ .

Note that  $|e_1 \cap e_2| \geq |T| = k-\ell$ . Since there are at least  $\binom{n-2k}{|e_1 \cap e_2| - 1} \geq \binom{n-2k}{k-\ell-1}$  distinct choices of  $A$  and at least one choice of equipartition  $\{S, S'\}$  with  $|S \cap e_1| = 1$ , we obtain

$$\begin{aligned} p &\geq \sum_{T \in \mathcal{T}} \sum_{\{e_1, e_2\} \in P(T)} \binom{n-2k}{k-\ell-1} = \binom{n-2k}{k-\ell-1} \sum_{T \in \mathcal{T}} \binom{d_{H_1}(T)}{2} \\ &\geq \binom{n-2k}{k-\ell-1} |\mathcal{T}| \left( \frac{1}{|\mathcal{T}|} \sum_{T \in \mathcal{T}} d_{H_1}(T) \right) \geq \binom{n-2k}{k-\ell-1} |\mathcal{T}| \left( \frac{|H_1|}{|\mathcal{T}|} \right). \end{aligned}$$

Note that we get the penultimate inequality from the convexity of the real function  $f(z) = \binom{z}{2} = z(z-1)/2$ . Since  $|H_1| \geq 3|\mathcal{T}|$ , we have that  $\binom{|H_1|}{2} \geq |H_1|^2/(3|\mathcal{T}|^2)$  and thus

$$p \geq \frac{1}{3} \binom{n-2k}{k-\ell-1} \frac{|H_1|^2}{|\mathcal{T}|}. \quad (3.7)$$

Now we find an upper bound of  $p$ . Clearly there are at most  $x = |\tilde{H}|$  choices for the  $(k-1)$ -set  $f$  and there are at most  $|\mathcal{T}|$  choices for  $T \in \mathcal{T}$ . For given  $f$ , we choose two disjoint subsets  $S_1, S_2 \subseteq f$  with  $|S_1| = 1$ . There are at most  $(k-1)2^{k-2}$  ways to choose  $S_1$  and  $S_2$ .

Assume that  $f, T, S_1$  and  $S_2$  are fixed, and we count the number of pairs of distinct edges  $e_1, e_2 \in H_1$  such that  $T \subseteq e_1 \cap e_2$ ,  $e_1 \cap f = S_1$ ,  $e_2 \cap f = S_2$ , and  $|e_2 \setminus e_1| - 1 = |e_2 \cap f| = |S_2|$ . We choose  $e_1 \in H_1$  with  $T \cup S_1 \subseteq e_1$ , and a set  $B \subseteq e_1 \setminus (T \cup S_1)$  with  $|B| = k - |T| - |S_2| - 1$ . By

Claim 3.5, there are  $d_{H_1}(T \cup S_1) \leq \binom{n}{2\ell-k-1}(|\mathcal{T}| + 1)$  ways to choose such an edge  $e_1$  and there are at most  $2^k$  ways to choose such a set  $B$ . Then we choose  $e_2 \in H_1$  such that  $T \cup B \cup S_2 \subseteq e_2$  and  $e_1 \cap e_2 = T \cup B$ . There are at most  $d_{H_1}(T \cup B \cup S_2) \leq 1$  way to choose such a set  $e_2$  by Claim 3.5. Thus for fixed  $f, T, S_1, S_2$ , the number of choices of  $e_1, e_2$  is at most  $2^k \binom{n}{2\ell-k-1}(|\mathcal{T}| + 1)$ . Thus we obtain

$$\begin{aligned} p &\leq \sum_{f \in \tilde{H}} \sum_{T \in \mathcal{T}} \sum_{S_1, S_2} 2^k \binom{n}{2\ell-k-1} (|\mathcal{T}| + 1) \\ &\leq x|\mathcal{T}|(k-1)2^{k-2} \cdot 2^k \binom{n}{2\ell-k-1} (|\mathcal{T}| + 1) \leq k2^{2k} \binom{n}{2\ell-k-1} |\mathcal{T}|^2 x. \end{aligned} \quad (3.8)$$

Note that the third sum is over  $S_1, S_2$  satisfying  $|S_1| = 1, S_1 \subseteq f, S_2 \subseteq f \setminus S_1$ . From (3.7) and (3.8), we get

$$|H_1|^2 \leq 3k2^{2k} \binom{n}{2\ell-k-1} \binom{n-2k}{k-\ell-1}^{-1} |\mathcal{T}|^3 x \leq k^{5k} n^{3\ell-2k} |\mathcal{T}|^3 x \stackrel{(3.5)}{\leq} k^{10k} n^{3-2k} x^4.$$

Thus, we get

$$|H_1| \leq k^{5k} n^{3/2-k} x^2 \stackrel{(3.4)}{\leq} k^{5k} \varepsilon^3 n^{\ell+3/2-k} x \leq \varepsilon x,$$

because  $\ell \leq k-2$ . □

Now assume that  $\ell = k-1$ . In this case  $\mathcal{T}$  is a collection of singletons, any vertex in  $W$  belongs to  $\mathcal{T}$  and  $|W| = |\mathcal{T}|$ . We partition  $H_1 = G_1 \cup G_2 \cup \dots \cup G_k$ , where  $G_i = \{e \in H_1 \mid |e \cap W| = i\}$  for each  $i \in [k]$ . Since  $\ell = k-1$ , the fact that  $\varepsilon$  is small and (3.5) imply

$$3|W|^{k-1} \leq 3(2k^{k+1}n^{2-k}x)^{k-1} \leq k^{2k^2}(xn^{1-k})^{k-2}x \stackrel{(3.4)}{\leq} k^{2k^2}(\varepsilon^3)^{k-2}x \leq \varepsilon x/k. \quad (3.9)$$

Now we show that  $|G_i| \leq \varepsilon x/k$  for all  $i \in [k]$  which together imply that  $|H_1| \leq \varepsilon x$ .

**Claim 3.7.**  $|G_k| \leq \varepsilon x/k$  and  $|G_{k-1}| \leq \varepsilon x/k$ .

*Proof.* First, since  $G_k$  does not contain any 2-regular subgraphs, by Theorem 2.1, there exists  $n_0 = n_0(k, 2)$  such that if  $|W| \geq n_0$  then  $|G_k| \leq 3\binom{|W|}{k-1}$ . If  $|W| \leq n_0$ , then by (3.4) we have  $|G_k| \leq n_0^k < \varepsilon x/k$  since  $n$  is large enough and  $x \geq n - 2k + 1$  by Claim 3.3. Otherwise  $|W| > n_0$ , then  $|G_k| \leq 3\binom{|W|}{k-1} \leq \varepsilon x/k$  by (3.9). Second, since  $\ell = k-1$  we have

$$|G_{k-1}| = \sum_{L \in \binom{W}{k-1}} |N_{H^*}(L) \setminus \mathcal{T}| \stackrel{\text{Claim 3.5}}{\leq} \sum_{L \in \binom{W}{k-1}} 1 \leq |W|^{k-1} \stackrel{(3.9)}{\leq} \varepsilon x/k. \quad \square$$

Now we estimate  $|G_i|$  for  $1 \leq i \leq k-2$ .

**Claim 3.8.**  $|G_i| \leq \varepsilon x/k$  for  $i \in [k-2]$ .

*Proof.* Assume  $|G_i| > \varepsilon x/k$  for some  $i \in [k-2]$ , then (3.9) implies that  $|G_i| \geq 3|W|^{k-1}$ . Let  $p_i$  be the number of the tuples  $(S, \{e_1, e_2\}, f)$  with the following properties.

(P.2.1)  $e_1, e_2 \in G_i$  and  $f \in \tilde{H}$ ,

(P.2.2)  $S \in \binom{W}{i}$  and  $S \subseteq e_1 \cap e_2$ ,

(P.2.3)  $f \cap (e_1 \cap e_2) = \emptyset$ , and  $\{|f \cap e_1|, |f \cap e_2|\} = \{1, |e_1 \setminus e_2| - 1\}$ .

Let  $P_i(S) := \{ \{e_1, e_2\} \in \binom{G_i}{2} : S \subseteq e_1 \cap e_2 \}$ . By convexity, we have

$$\sum_{S \in \binom{W}{i}} \sum_{\{e_1, e_2\} \in P_i(S)} 1 \geq \sum_{S \in \binom{W}{i}} \binom{d_{G_i}(S)}{2} \geq \binom{|W|}{i} \binom{|G_i|/\binom{|W|}{i}}{2} \geq \frac{|G_i|^2}{3\binom{|W|}{i}}, \quad (3.10)$$

where we used  $\sum_{S \in \binom{W}{i}} d_{G_i}(S) = |G_i|$  and  $|G_i| \geq 3|W|^{k-1} \geq 3\binom{W}{i}$ .

Consider a set  $S \subseteq W$  of size  $i$ , and a pair  $\{e_1, e_2\} \in P_i(S)$ . Let  $A$  be an arbitrary set of size  $|e_1 \cap e_2| - 1$  in  $V' \setminus (e_1 \cup e_2)$ , and let  $A_1, A_2$  be a partition of  $e_1 \triangle e_2$  such that  $|A_1| = 1$ . The number of ways to choose  $A$  is at least  $\binom{n-2k}{|e_1 \cap e_2| - 1} \geq \binom{n-2k}{i-1}$  and the number of ways to choose  $A_1, A_2$  is at least one. By Claim 3.1, at least one of  $A \cup A_1 \in \tilde{H}$  and  $A \cup A_2 \in \tilde{H}$  holds. Then either  $(S, \{e_1, e_2\}, A \cup A_1)$  or  $(S, \{e_1, e_2\}, A \cup A_2)$  satisfies (P.2.1)–(P.2.3). Since distinct choices of  $(S, \{e_1, e_2\}, A, \{A_1, A_2\})$  give us distinct tuples, we have

$$p_i \geq \sum_{S \in \binom{W}{i}} \sum_{\{e_1, e_2\} \in P_i(S)} \sum_A \sum_{A_1, A_2} 1 \geq \sum_{S \in \binom{W}{i}} \sum_{\{e_1, e_2\} \in P_i(S)} \binom{n-2k}{i-1} \stackrel{(3.10)}{\geq} \binom{n-2k}{i-1} \frac{|G_i|^2}{3\binom{|W|}{i}}. \quad (3.11)$$

Now we find an upper bound of  $p_i$ . Clearly there are at most  $x = |\tilde{H}|$  choices of  $f \in \tilde{H}$  and at most  $\binom{|W|}{i}$  choices of  $S \in \binom{W}{i}$  with  $S \cap f = \emptyset$ . For given  $f$  and  $S$ , we choose two disjoint sets  $A_1, A_2 \subseteq f$  with  $|A_1| = 1$ . There are at most  $(k-1)2^{k-2}$  ways to choose such  $A_1$  and  $A_2$ .

Assume  $f, S, A_1, A_2$  are given, and we count the number of pairs  $\{e_1, e_2\} \in P_i(S)$  such that  $e_1 \cap f = A_1$ ,  $e_2 \cap f = A_2$  and  $|e_2 \setminus e_1| - 1 = |e_2 \cap f| = |A_2|$ . We choose  $e_1 \in G_i$  such that  $S \cup A_1 \subseteq e_1$  and  $e_1 \setminus (S \cup A_1) \subseteq V' \setminus W$ , and the number of ways to choose such  $e_1$  is at most

$$\sum_{S \cup A_1 \subseteq L \in \binom{V'}{k-1}} |N_{H^*}(L) \setminus W| \stackrel{\text{Claim 3.5}}{\leq} \sum_{S \cup A_1 \subseteq L \in \binom{V'}{k-1}} 1 = \binom{n-i-2}{k-i-2}.$$

We also choose a set  $B \subseteq e_1 \setminus (S \cup A_1)$  with  $|B| = k - |S| - |A_2| - 1$ . There are at most  $2^k$  ways to choose such a set  $B$ . Then we choose  $e_2 \in G_i$  such that  $S \cup B \cup A_2 \subseteq e_2$ ,  $e_1 \cap e_2 = S \cup B$  and  $e_2 \setminus (S \cup B \cup A_2) \subseteq V' \setminus W$ , and the number of ways to choose such  $e_2$  is at most  $|N_{H^*}(S \cup B \cup A_2) \setminus W| \leq 1$  by Claim 3.5. Overall, for fixed  $f, S, A_1, A_2$ , the number of choices of  $e_1, e_2$  is at most  $2^k \binom{n-i-2}{k-i-2}$ . Thus we obtain

$$\begin{aligned} p_i &\leq \sum_{f \in \tilde{H}} \sum_{S \in \binom{W}{i}} \sum_{A_1, A_2} 2^k \binom{n}{k-i-2} \\ &\leq x \binom{|W|}{i} (k-1) 2^{k-2} \cdot 2^k \binom{n}{k-i-2} \leq k 2^{2k} x \binom{|W|}{i} \binom{n}{k-i-2}. \end{aligned} \quad (3.12)$$

Note that the third sum is over  $A_1, A_2$  satisfying  $|A_1| = 1, A_1 \subseteq f, A_2 \subseteq f \setminus A_1$ .

From (3.11) and (3.12) and the fact that  $|W| = |\mathcal{T}|$ , we obtain

$$\begin{aligned} |G_i|^2 &\leq 3k 2^{2k} \binom{|W|}{i}^2 \binom{n}{k-i-2} \binom{n-2k}{i-1}^{-1} x \\ &\stackrel{(3.5)}{\leq} k^{3k} (2k^{k+1} n^{2-k} x)^{2i} n^{k-2i-1} x \\ &\leq k^{10k^2} n^{-(2i-1)(k-1)} x^{2i+1} \\ &\stackrel{(3.4)}{\leq} k^{10k^2} n^{-(2i-1)(k-1)} (\varepsilon^3 n^{k-1})^{2i-1} x^2 \leq k^{10k^2} \varepsilon^3 x^2 < \varepsilon^2 x^2 / k^2. \end{aligned}$$

This contradicts that  $|G_i| > \varepsilon x / k$ . Thus the claim holds.  $\square$

**3.3. Size of  $H_0$ .** At last we show that  $|H_0| < (1 - \varepsilon)x + \lfloor \frac{n-1}{k} \rfloor$ . Assume to the contrary, that

$$|H_0| \geq (1 - \varepsilon)x + \left\lfloor \frac{n-1}{k} \right\rfloor. \quad (3.13)$$

For any  $u \in V'$ , let  $F_u := N_{H_0}(u) \setminus \tilde{H}$ . We first observe that  $F_u$  is an intersecting family.



**Claim 3.9.** *For any  $u \in V'$ ,  $F_u$  forms an intersecting family.*

*Proof.* If not, then there are two disjoint  $(k-1)$ -sets  $A, A' \in F_u = N_{H_0}(u) \setminus \tilde{H}$ . Since  $A, A' \notin \tilde{H}$ ,

$$A \cup \{u\}, A' \cup \{u\}, A \cup \{v\} \text{ and } A' \cup \{v\}$$

form a 2-regular subgraph of  $H$ , a contradiction.  $\square$

If  $f \in \tilde{H}$  belongs to  $N_{H_0}(u) \cap N_{H_0}(u')$  for distinct  $u, u' \in V'$ , then fix any  $L \in \binom{f}{\ell}$ , we have  $d_{H_0}(L) \geq 2$ , contradicting Claim 3.5. Thus  $f \in \tilde{H}$  belongs to  $N_{H_0}(u)$  for at most one  $u \in V'$ , which implies that

$$x = |\tilde{H}| \geq \sum_{u \in V'} |N_{H_0}(u) \cap \tilde{H}| = \sum_{u \in V'} (d_{H_0}(u) - |F_u|). \quad (3.14)$$

Note that this implies that  $\ell \geq 3$  and  $k \geq 5$ . In fact, if  $\ell = 2$ , then Claim 3.5 implies that  $d_{H_0}(\{u, u'\}) \leq 1$  for any two distinct vertices  $u, u' \in V'$ , i.e., any two edges in  $H_0$  share at most one vertex. Thus  $N_{H_0}(u)$  forms a matching. By Claim 3.9,  $|F_u| \leq 1$  as  $F_u$  is an intersecting subfamily of a matching. By (3.14),

$$x \geq \sum_{u \in V'} (d_{H_0}(u) - 1) = k|H_0| - (n - 1) \stackrel{(3.13)}{\geq} k(1 - \varepsilon)x - (k - 1).$$

Thus  $x \leq 1$ , contradicting (3.4). Thus  $\ell \geq 3$ . Since  $3 \leq \ell \leq k - 1$  by (3.4) and  $k$  is odd, we have  $k \geq 5$ .

Let

$$X := \{u \in V' : F_u \text{ is a trivial intersecting family}\}$$

and for  $u \in X$ , let  $p(u)$  be a vertex in  $V'$  such that every  $(k-1)$ -set in  $F_u$  contains  $p(u)$ . We claim that

$$\sum_{u \in X} |F_u| \geq (1 - \varepsilon)(k - 1)|H_0|. \quad (3.15)$$

We first show that for  $u \notin X$ ,  $|F_u| \leq k^2 n^{\ell-3}$ . Indeed, since  $u \notin X$ ,  $F_u$  is a non-trivial intersecting  $(k-1)$ -uniform family. By Lemma 2.4, there are pairs of vertices  $w_1 w'_1, \dots, w_t w'_t$  with  $t \leq (k-1)^2 - (k-1) + 1 \leq k^2$  which together cover all  $(k-1)$ -sets in  $F_u$ . Since  $\ell \geq 3$ , Claim 3.5 implies

$$|F_u| \leq \sum_{i=1}^t d_{H_0}(\{u, w_i, w'_i\}) \leq k^2 \binom{n-4}{\ell-3} \leq k^2 n^{\ell-3}.$$

Then note that  $\sum_{u \in V'} d_{H_0}(u) = k|H_0|$ . From (3.14), we get

$$\begin{aligned} x &\geq \sum_{u \in V'} (d_{H_0}(u) - |F_u|) \geq \sum_{u \in V'} d_{H_0}(u) - \sum_{u \in X} |F_u| - \sum_{u \in V' \setminus X} |F_u| \\ &\geq k|H_0| - \sum_{u \in X} |F_u| - \sum_{u \in V' \setminus X} k^2 n^{\ell-3} \geq k|H_0| - \sum_{u \in X} |F_u| - k^2 n^{\ell-2}. \end{aligned}$$

Since  $n$  is sufficiently large, (3.4) implies that  $k^2 n^{\ell-2} \leq \varepsilon^4 n^{\ell-1} \leq \varepsilon x$ . Thus we get

$$\sum_{u \in X} |F_u| \geq k|H_0| - x - \varepsilon x \stackrel{(3.13)}{\geq} k|H_0| - \frac{(1 + \varepsilon)|H_0|}{1 - \varepsilon} \geq (1 - \varepsilon)(k - 1)|H_0|$$

as  $k \geq 5$ . So (3.15) is proved.

For  $t \in [k-1]$ , let  $q_t$  be the number of the tuples  $(u, \{e_1, e_2\}, f)$  with the following properties.

- (Q1)<sub>t</sub>  $u \in X$ ,  $e_i \setminus \{u\} \in F_u$  for  $i \in [2]$  and  $|e_1 \cap e_2| = t$ ,
- (Q2)  $f \in \tilde{H}$ ,  $f \cap (e_1 \cap e_2) = \emptyset$ ,
- (Q3)  $\{|f \cap e_1|, |f \cap e_2|\} = \{1, |e_1 \setminus e_2| - 1\}$ .

For  $u \in X$  and  $t \in [k-1]$ , we let

$$F_u^t := \{\{e_1, e_2\} : e_i \setminus \{u\} \in F_u \text{ for } i \in [2], |e_1 \cap e_2| = t\},$$

and

$$P^t := \{(u, \{e_1, e_2\}) : u \in X, \{e_1, e_2\} \in F_u^t\}.$$

Note that  $F_u^1 = \emptyset$  for any  $u \in X$  since  $F_u$  is an intersecting family. Since  $u \in X$ , we have  $\{u, p(u)\} \subseteq e_1 \cap e_2$  for  $(u, \{e_1, e_2\}) \in P^t$ . By convexity of function  $f(z) = \binom{z}{2} = \frac{z(z-1)}{2}$ , we have

$$\begin{aligned} \sum_{t=2}^{k-1} |P^t| &= \sum_{u \in X} \binom{|F_u|}{2} \geq |X| \binom{\frac{1}{|X|} \sum_{u \in X} |F_u|}{2} \\ &\stackrel{(3.15)}{\geq} \frac{(1-\varepsilon)^2(k-1)^2|H_0|^2}{2n} - \frac{1}{2}(1-\varepsilon)(k-1)|H_0| \geq \frac{(1-2\varepsilon)(k-1)^2|H_0|^2}{2n}. \end{aligned} \quad (3.16)$$

Here, we get the last inequality since we have  $\varepsilon^2|H_0| \geq \varepsilon^2x \geq \varepsilon^5n^2 > 2n$  from (3.4), the fact that  $\ell \geq 3$  and  $n$  is large.

Now we find a lower bound of  $q_t$ . Note that  $q_1 = 0$  since  $F_u$  is an intersecting family for any  $u \in X$ . For  $2 \leq t \leq k-1$ , first fix a vertex  $u \in X$  and let  $\{e_1, e_2\} \in F_u^t$ . We choose a set  $A \subseteq V' \setminus (e_1 \cup e_2)$  of size  $t-1$ . We also choose an equipartition  $\{S, S'\}$  of  $e_1 \triangle e_2$  such that  $|S \cap e_1| = 1$ . The number of choices of such  $\{S, S'\}$  is  $(k-t)^2$ . Then Claim 3.1 implies that either  $A \cup S$  or  $A \cup S'$  belongs to  $\tilde{H}$  and it satisfies (Q3) as it plays the role of  $f$ . Note that for distinct choices of  $(A, \{S, S'\})$ , we get distinct  $(k-1)$ -sets  $f$  in  $\tilde{H}$ .

So for  $2 \leq t \leq k-1$ ,

$$q_t \geq \sum_{u \in X} \sum_{\{e_1, e_2\} \in F_u^t} (k-t)^2 \binom{n-2k}{t-1} = (k-t)^2 \binom{n-2k}{t-1} |P^t|.$$

Since  $k \geq 5$  and  $n$  is large,  $(k-t)^2 \binom{n-2k}{t-1} \geq n(n-2k)$  for  $t \geq 3$ . Thus we obtain

$$q_2 \geq (k-2)^2(n-2k)|P^2| \quad \text{and} \quad q_t \geq n(n-2k)|P^t| \quad \text{for } 3 \leq t \leq k-1. \quad (3.17)$$

Next we find an upper bound of  $q_t$ . Clearly there are at most  $x = |\tilde{H}|$  choices for  $f \in \tilde{H}$ . We choose two disjoint sets  $A_1, A_2 \subseteq f$  such that  $|A_1| = 1, |A_2| = k-t-1$ . The number of such choices is at most  $(k-1)\binom{k-2}{k-t-1}$ . Now we choose  $e_1$  in  $H_0$  containing  $A_1$ . The number of choices for  $e_1$  is at most  $|H_0|$ . Once  $e_1$  is chosen, we choose  $u \in X \cap (e_1 \setminus A_1)$  such that  $\{u, p(u)\} \subseteq e_1$ . There are at most  $k-1$  such choices for  $u$ . Now we choose a  $(t-2)$ -subset  $B \subseteq e_1 \setminus (A_1 \cup \{u, p(u)\})$ , and there are  $\binom{k-3}{t-2}$  ways to choose such  $B$ . For given  $A_2, B, u$ , we choose  $e_2$  such that  $A_2 \cup B \cup \{u, p(u)\} \subseteq e_2$ . Since  $|A_2 \cup B \cup \{u, p(u)\}| = k-1$ , it contains a subset  $L$  of size  $\ell$ , thus Claim 3.5 implies that  $d_{H_0}(A_2 \cup B \cup \{u, p(u)\}) \leq d_{H_0}(L) \leq 1$ . Thus the number of choices of  $e_2$  is at most 1. Thus we get

$$q_t \leq x(k-1) \binom{k-2}{k-t-1} |H_0| (k-1) \binom{k-3}{t-2}. \quad (3.18)$$

Thus we obtain

$$\begin{aligned} (k-2)^2 n(n-2k) \sum_{t=2}^{k-1} |P^t| &\stackrel{(3.17)}{\leq} nq_2 + (k-2)^2 \sum_{t=3}^{k-1} q_t \\ &\stackrel{(3.18)}{\leq} (k-1)^2(k-2)xn|H_0| + \sum_{t=3}^{k-1} (k-1)^4 \binom{k-2}{k-t-1} \binom{k-3}{t-2} x|H_0| \\ &\leq (1+\varepsilon)(k-1)^2(k-2)xn|H_0|. \end{aligned}$$

Note that we get the last inequality since  $n$  is sufficiently large. By (3.16), we get

$$\frac{1}{2}(1-2\varepsilon)(k-2)^2(k-1)^2|H_0|^2(n-2k) \leq (k-2)^2n(n-2k) \sum_{t=2}^{k-1} |P^t| \leq (1+\varepsilon)(k-1)^2(k-2)xn|H_0|.$$

Since  $|H_0| > 0$ , dividing both sides by  $(k-2)(k-1)^2|H_0|/2$  gives

$$(1-2\varepsilon)(k-2)|H_0|(n-2k) \leq 2(1+\varepsilon)xn.$$

Since we have  $(1-\varepsilon)x \leq |H_0|$  from (3.13),

$$(1-2\varepsilon)(k-2)(1-\varepsilon)x(n-2k) \leq 2(1+\varepsilon)xn.$$

Since  $x \geq 1$ , we get  $(1-3\varepsilon)(k-2)(n-2k) \leq 2(1+\varepsilon)n$ , which is a contradiction since  $k \geq 5$ ,  $\varepsilon$  is small and  $n$  is large enough. So (3.13) does not hold and we are done.

#### 4. CONCLUDING REMARKS

In our proof of Theorem 1.3, except the use of Theorems 2.1 and 2.2, we only use the assumption that  $H$  does not contain any 2-regular subgraphs on  $2k$  vertices. This motivates the following conjecture.

**Conjecture 4.1.** *For every integer  $k \geq 3$ , there exists  $n_k$  such that the following holds for all  $n \geq n_k$ . If  $H$  is an  $n$ -vertex  $k$ -uniform hypergraph with no 2-regular subgraphs on  $2k$  vertices, then*

$$|H| \leq \binom{n-1}{k-1} + \left\lfloor \frac{n-1}{k} \right\rfloor.$$

*Moreover, equality holds if and only if  $H$  is a full  $k$ -star with center  $v$  together with a maximal matching omitting  $v$ .*

For  $k \geq 4$ , Conjecture 1.1 implies Conjecture 4.1. Note that Conjecture 4.1 stands between Conjecture 1.1 and the result on forbidding 2-regular subgraphs. In some sense it is more close to Conjecture 1.1 – because only finitely many (independent of  $n$ ) configurations are forbidden (in contrast, by forbidding all 2-regular subgraphs, the number of instances forbidden is related to  $n$ ). By our proof, to show Conjecture 4.1 for odd integers  $k$ , it suffices to prove an asymptotical result and a stability result.

In this paper we focused on forbidding 2-regular subgraphs. It is natural to consider hypergraphs without  $r$ -regular subgraphs for  $r \geq 3$  (see Question 6.9 in [9]). We remark that Construction 6.8 in [9] gives a lower bound on the maximum number of edges in such a hypergraph.

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