

Chromatic index determined by fractional chromatic index

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Abstract

Given a graph G possibly with multiple edges but no loops, denote by Δ the maximum degree, μ the multiplicity, χ' the chromatic index and χ'_f the fractional chromatic index of G , respectively. It is known that $\Delta \leq \chi'_f \leq \chi' \leq \Delta + \mu$, where the upper bound is a classic result of Vizing. While deciding the exact value of χ' is a classic NP-complete problem, the computing of χ'_f is in polynomial time. In fact, it is shown that if $\chi'_f > \Delta$ then $\chi'_f = \max_{H \subseteq G} \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor}$, where the maximality is taken over all induced subgraphs H of G . Gupta (1967), Goldberg (1973), Andersen (1977), and Seymour (1979) conjectured that $\chi' = \lceil \chi'_f \rceil$ if $\chi' \geq \Delta + 2$, which is commonly referred as Goldberg's conjecture. It has been shown that Goldberg's conjecture is equivalent to the following conjecture of Jakobsen: For any positive integer m with $m \geq 3$, every graph G with $\chi' > \frac{m}{m-1}\Delta + \frac{m-3}{m-1}$ satisfies $\chi' = \lceil \chi'_f \rceil$. Jakobsen's conjecture has been verified for m up to 15 by various researchers in the last four decades. We use an extended form of a Tashkinov tree to show that it is true for $m \leq 23$. With the same technique, we show that if $\chi' \geq \Delta + \sqrt[3]{\Delta/2}$ then $\chi' = \lceil \chi'_f \rceil$. The previous best known result is for graphs with $\chi' > \Delta + \sqrt{\Delta/2}$ obtained by Scheide, and by Chen, Yu and Zang, independently. Moreover, we show that Goldberg's conjecture holds for graphs G with $\Delta \leq 23$ or $|V(G)| \leq 23$.

Keywords. Edge chromatic index; Fractional chromatic index; Critical graph; Tashkinov tree; Extended Tashkinov tree

1 Introduction

Graphs considered in this paper may contain multiple edges but no loops. Let G be a graph and $\Delta := \Delta(G)$ be the maximum degree of G . A (proper) k -edge-coloring φ of G is a mapping φ from $E(G)$ to $\{1, 2, \dots, k\}$ (whose elements are called colors) such that no two adjacent edges receive the

same color. The *chromatic index* $\chi' := \chi'(G)$ is the least integer k such that G has a k -edge-coloring. In graph edge-coloring, the central question is to determine the chromatic index χ' for graphs. We refer the book [17] of Stiebitz, Scheide, Toft and Favrholt and the elegant survey [12] of McDonald for literature on the recent progress of graph edge-colorings. Clearly, $\chi' \geq \Delta$. Conversely, Vizing showed that $\chi' \leq \Delta + \mu$, where $\mu := \mu(G)$ is the multiplicity of G . However, determining the exact value of χ' is a very difficult problem. Holyer [8] showed that the problem is NP-hard even restricted to simple cubic graphs. To estimate χ' , the notion of fractional chromatic index is introduced.

A *fractional edge coloring* of G is a non-negative weighting $w(\cdot)$ of the set $\mathcal{M}(G)$ of matchings in G such that, for every edge $e \in E(G)$, $\sum_{M \in \mathcal{M}: e \in M} w(M) = 1$. Clearly, such a weighting $w(\cdot)$ exists. The *fractional chromatic index* $\chi'_f := \chi'_f(G)$ is the minimum total weight $\sum_{M \in \mathcal{M}} w(M)$ over all fractional edge colorings of G . By definitions, we have $\chi' \geq \chi'_f \geq \Delta$. It follows from Edmonds' characterization of the matching polytope [3] that χ'_f can be computed in polynomial time and

$$\chi'_f = \max \left\{ \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} : H \subseteq G \text{ with } |V(H)| \geq 3 \right\} \text{ if } \chi'_f > \Delta.$$

It is not difficult to show that the above maximality can be restricted to induced subgraphs H with odd number of vertices. So, in the case of $\chi'_f > \Delta$, we have

$$\lceil \chi'_f \rceil = \max \left\{ \left\lceil \frac{2|E(H)|}{|V(H)| - 1} \right\rceil : \text{induced subgraphs } H \subseteq G \text{ with } |V(H)| \geq 3 \text{ and odd} \right\}.$$

A graph G is called *elementary* if $\chi' = \lceil \chi'_f \rceil$. Gupta (1967) [7], Goldberg (1973) [5], Andersen (1977) [1], and Seymour (1979) [15] independently made the following conjecture, which is commonly referred as *Goldberg's conjecture*.

Conjecture 1. *For any graph G , if $\chi' \geq \Delta + 2$ then G is elementary.*

An immediate consequence of Conjecture 1 is that χ' can be computed in polynomial time for graphs with $\chi' \geq \Delta + 2$. So the NP-complete problem of computing the chromatic indices lies in determining whether $\chi' = \Delta$, $\Delta + 1$, or $\geq \Delta + 2$, which strengthens Vizing's classic result $\chi' \leq \Delta + \mu$ tremendously when μ is big.

Following $\chi' \leq \frac{3\Delta}{2}$ of the classic result of Shannon [16], we can assume that, for every Δ , there exists the least positive number ζ such that if $\chi' > \Delta + \zeta$ then G is elementary. Conjecture 1 indicates that $\zeta \leq 1$. Asymptotically, Kahn [10] showed $\zeta = o(\Delta)$. Scheide [14], and Chen, Yu, and Zang [2] independently proved that $\zeta \leq \sqrt{\Delta/2}$. In this paper, we show that $\zeta \leq \sqrt[3]{\Delta/2} - 1$ as stated below.

Theorem 1.1. *For any graph G , if $\chi' \geq \Delta + \sqrt[3]{\Delta/2}$, then G is elementary.*

Jakobsen [9] conjectured that $\zeta \leq 1 + \frac{\Delta-2}{m-1}$ for every positive integer $m(\geq 3)$, which gives a reformulation of Conjecture 1 as stated below.

Conjecture 2. *Let m be an integer with $m \geq 3$ and G be a graph. If $\chi' > \frac{m}{m-1}\Delta + \frac{m-3}{m-1}$, then G is elementary.*

Since $\frac{m}{m-1}\Delta + \frac{m-3}{m-1}$ decreases as m increases, it is sufficient to prove Jakobsen's conjecture for all odd integers m (in fact, for any infinite sequence of positive integers), which has been confirmed slowly for $m \leq 15$ by a series of papers over the last 40 years:

- $m = 5$: Three independent proofs given by Andersen [1] (1977), Goldberg [5] (1973), and Sørensen (unpublished, page 158 in [17]), respectively.
- $m = 7$: Two independent proofs given by Andersen [1] (1977) and Sørensen (unpublished, page 158 in [17]), respectively.
- $m = 9$: By Goldberg [6] (1984).
- $m = 11$: Two independent proofs given by Nishizeki and Kashiwagi [13] (1990) and by Tashkinov [18] (2000), respectively.
- $m = 13$: By Favrholt, Stiebitz and Toft [4] (2006).
- $m = 15$: By Scheide [14] (2010).

In this paper, we show that Jakobsen's conjecture is true up to $m = 23$.

Theorem 1.2. *If G is a graph with $\chi' > \frac{23}{22}\Delta + \frac{20}{22}$, then G is elementary.*

Corollary 1.1. *If G is a graph with $\Delta \leq 23$ or $|V(G)| \leq 23$, then $\chi' \leq \max\{\Delta + 1, \lceil \chi_f \rceil\}$.*

Note that in Corollary 1.1, $|V(G)| \leq 23$ does not imply $\Delta \leq 23$, as G may have multiple edges. The remainder of this paper is organized as follows. In Section 2, we introduce some definitions and notation for edge-colorings, Tashkinov trees, and several known results which are useful for the proofs of Theorems 1.1 and 1.2; in Section 3, we give an extension of Tashkinov trees and prove several properties of the extended Tashkinov trees; and in Section 4, we prove Theorem 1.1, Theorem 1.2 and Corollary 1.1 based on the results in Section 3.

2 Preliminaries

2.1 Basic definitions and notation

Let G be a graph with vertex set V and edge set E . Denote by $|G|$ and $||G||$ the number of vertices and the number of edges of G , respectively. For any two sets $X, Y \subseteq V$, denote by $E(X, Y)$ the set of

edges with one end in X and the other one in Y and denote by $\partial(X) := E(X, V - X)$ the boundary edge set of X , that is, the set of edges with exactly one end in X . Moreover, let $E(x, y) := E(\{x\}, \{y\})$ and $E(x) := \partial(\{x\})$. Denote by $G[X]$ the subgraph induced by X and $G - X$ the subgraph induced by $V(G) - X$. Moreover, let $G - x = G - \{x\}$. For any subgraph H of G , we let $G[H] = G[V(H)]$ and $\partial(H) = \partial(V(H))$. Let $V(e)$ be the set of the two ends of an edge e .

A path P is usually denoted by an alternating sequence $P = (v_0, e_1, v_1, \dots, e_p, v_p)$ with $V(P) = \{v_0, \dots, v_p\}$ and $E(P) = \{e_1, \dots, e_p\}$ such that $e_i \in E_G(v_{i-1}, v_i)$ for $1 \leq i \leq p$. The path P defined above is called a (v_0, v_p) -path. For any two vertices $u, v \in V(P)$, denote by uPv or vPu the unique subpath connecting u and v . If u is an end of P , then we obtain a *linear order* $\preceq_{(u,P)}$ of the vertices of P in a natural way such that $x \preceq_{(u,P)} y$ if $x \in V(uPy)$.

The set of all k -edge-colorings of a graph G is denoted by $\mathcal{C}^k(G)$. Let $\varphi \in \mathcal{C}^k(G)$. For any color α , let $E_\alpha = \{e \in E : \varphi(e) = \alpha\}$. More generally, for each subgraph $H \subseteq G$, let

$$E_\alpha(H) = \{e \in E(H) : \varphi(e) = \alpha\}.$$

For any two distinct colors α and β , denote by $G_\varphi(\alpha, \beta)$ the subgraph of G induced by $E_\alpha \cup E_\beta$. The components of $G_\varphi(\alpha, \beta)$ are called (α, β) -chains. Clearly, each (α, β) -chain is either a path or a cycle of edges alternately colored with α and β . For each (α, β) -chain P , let φ/P denote the k -edge-coloring obtained from φ by exchanging colors α and β on P , that is, for each $e \in E$,

$$\varphi/P(e) = \begin{cases} \varphi(e), & e \notin E(P); \\ \beta, & e \in E(P) \text{ and } \varphi(e) = \alpha; \\ \alpha, & e \in E(P) \text{ and } \varphi(e) = \beta. \end{cases}$$

For any $v \in V$, let $P_v(\alpha, \beta, \varphi)$ denote the unique (α, β) -chain containing v . Notice that, for any two vertices $u, v \in V$, either $P_u(\alpha, \beta, \varphi) = P_v(\alpha, \beta, \varphi)$ or $P_u(\alpha, \beta, \varphi) \cap P_v(\alpha, \beta, \varphi) = \emptyset$. For any $v \in V$, let $\varphi(v) := \{\varphi(e) : e \in E(v)\}$ denote the set of colors presented at v and $\overline{\varphi}(v)$ the set of colors not assigned to any edge incident to v , which are called *missing* colors at v . For any vertex set $X \subseteq V$, let $\varphi(X) = \cup_{x \in X} \varphi(x)$ and $\overline{\varphi}(X) = \cup_{x \in X} \overline{\varphi}(x)$ be the set of colors presenting and missing at some vertices of X , respectively. For any edge set $F \subseteq E$, let $\varphi(F) = \cup_{e \in F} \varphi(e)$.

2.2 Elementary sets and closed sets

Let G be a graph. An edge $e \in E(G)$ is called *critical* if $\chi'(G - e) < \chi'(G)$, and the graph G is called *critical* if $\chi'(H) < \chi'(G)$ for any proper subgraph $H \subseteq G$. A graph G is called *k -critical* if it is critical and $\chi'(G) = k + 1$. In the proofs, we will consider a graph G with $\chi'(G) = k + 1 \geq \Delta + 2$, a critical edge $e \in E(G)$, and a coloring $\varphi \in \mathcal{C}^k(G - e)$. We call them together a *k -triple* (G, e, φ) .

Definition 1. Let G be a graph and $e \in E(G)$ such that $\mathcal{C}^k(G - e) \neq \emptyset$ and let $\varphi \in \mathcal{C}^k(G - e)$. Let $X \subseteq V(G)$ contain two ends of e .

- We call X *elementary* (with respect to φ) if all missing color sets $\overline{\varphi}(x)$ ($x \in X$) are mutually disjoint.
- We call X *closed* (with respect to φ) if $\varphi(\partial(X)) \cap \overline{\varphi}(X) = \emptyset$, i.e., no missing color of X appears on the edges in $\partial(X)$. If additionally, each color in $\varphi(X)$ appears at most once in $\partial(X)$, we call X *strongly closed* (with respect to φ).

Moreover, we call a subgraph $H \subseteq G$ *elementary*, *closed*, and *strongly closed* if $V(H)$ is elementary, closed, and strongly closed, respectively. If a vertex set $X \subseteq V(G)$ containing two ends of e is both elementary and strongly closed, then $|X|$ is odd and $k = \frac{2(|E(G[X])|-1)}{|X|-1}$, so $k+1 = \left\lceil \frac{2|E(G[X])|}{|X|-1} \right\rceil = \lceil \chi'_f \rceil$. Therefore, if $V(G)$ is elementary then G is elementary, i.e., $\chi'(G) = k+1 = \lceil \chi'_f \rceil$.

2.3 Tashkinov trees

Definition 2. A *Tashkinov tree* of a k -triple (G, e, φ) is a tree T , denoted by $T = (e_1, e_2, \dots, e_p)$, induced by a sequence of edges $e_1 = e, e_2, \dots, e_p$ such that for each $i \geq 2$, e_i is a boundary edge of the tree induced by $\{e_1, e_2, \dots, e_{i-1}\}$ and $\varphi(e_i) \in \overline{\varphi}\left(V\left(\bigcup_{j=1}^{i-1} e_j\right)\right)$.

For each $e_j \in \{e_1, \dots, e_p\}$, we denote by Te_j the subtree $T[\{e_1, \dots, e_j\}]$ and denote by e_jT the subgraph induced by $\{e_j, \dots, e_p\}$. For each edge e_i with $i \geq 2$, the end of e_i in Te_{i-1} is called the *in-end* of e_i and the other one is called the *out-end* of e_i .

Algorithmically, a Tashkinov tree is obtained incrementally from e by adding a boundary edge whose color is missing in the previous tree. Vizing-fans (stars) (used in the proof of Vizing's classic theorem [19]) and Kierstead-paths (used in [11]) are special Tashkinov trees.

Theorem 2.1. [Tashkinov [18]] For any given k -triple (G, e, φ) with $k \geq \Delta + 1$, all Tashkinov trees are elementary.

For a graph G , a Tashkinov tree is associated with an edge $e \in E(G)$ and a k -edge-coloring of $G - e$ with $k \geq \Delta + 1$. We distinguish the following three different types of maximality.

Definition 3. Let (G, e, φ) be a k -triple with $k \geq \Delta + 1$, and T be a Tashkinov tree of (G, e, φ) .

- We call T (e, φ) -maximal if there is no Tashkinov tree T^* of (G, e, φ) containing T as a proper subtree, and denote by $\mathcal{T}_{e, \varphi}$ the set of all (e, φ) -maximal Tashkinov trees.
- We call T e -maximal if there is no Tashkinov tree T^* of a k -triple (G, e, φ^*) containing T as a proper subtree, and denote by \mathcal{T}_e the set of all e -maximal Tashkinov trees.
- We call T maximum if $|T|$ is maximum over all Tashkinov trees of G , and denote by \mathcal{T} the set of all maximum Tashkinov trees.

Let T be a Tashkinov tree of a k -triple (G, e, φ) . Then, T is (e, φ) -maximal if and only if $V(T)$ is closed. Moreover, the vertex sets are the same for all $T \in \mathcal{T}_{e, \varphi}$. We call colors in $\varphi(E(T))$ *used* and colors not in $\varphi(E(T))$ *unused* on T , call an unused missing color in $\overline{\varphi}(V(T))$ a *free color* of T and denote the set of all free colors of T by $\Gamma^f(T)$. For each color α , let $E_\alpha(\partial(T))$ denote the set of edges with color α in boundary $\partial(T)$. A color α is called a *defective color* of T if $|E_\alpha(\partial(T))| \geq 2$. The set of all defective colors of T is denoted by $\Gamma^d(T)$. Note that if $T \in \mathcal{T}_{e, \varphi}$, then $V(T)$ is strongly closed if and only if T does not have any defective colors.

The following corollary follows immediately from the fact that a maximal Tashkinov tree is elementary and closed.

Corollary 2.1. *For each $T \in \mathcal{T}_{e, \varphi}$, the following properties hold.*

- (1) $|T| \geq 3$ is odd.
- (2) For any two missing colors $\alpha, \beta \in \overline{\varphi}(V(T))$, we have $P_u(\alpha, \beta, \varphi) = P_v(\alpha, \beta, \varphi)$, where u and v are the two unique vertices in $V(T)$ such that $\alpha \in \overline{\varphi}(u)$ and $\beta \in \overline{\varphi}(v)$, respectively. Furthermore, $V(P_u(\alpha, \beta, \varphi)) \subseteq V(T)$.
- (3) For every defective color $\delta \in \Gamma^d(T)$, $|E_\delta(\partial(T))| \geq 3$ and is odd.
- (4) There are at least four free colors. More specifically,

$$|\Gamma^f(T)| \geq |T|(k - \Delta) + 2 - |\varphi(E(T))| \geq |T| + 2 - (|T| - 2) \geq 4.$$

The following lemma was given in [17].

Lemma 2.1. *Let $T \in \mathcal{T}_e$ be a Tashkinov tree of a k -triple (G, e, φ) with $k \geq \Delta + 1$. For any free color $\gamma \in \Gamma^f(T)$ and any $\delta \notin \overline{\varphi}(V(T))$, the (γ, δ) -chain $P_u(\gamma, \delta, \varphi)$ contains all edges in $E_\delta(\partial(T))$, where u is the unique vertex of T missing color γ .*

Proof. Otherwise, consider the coloring $\varphi_1 = \varphi/P_u(\gamma, \delta, \varphi)$. Since δ and γ are both unused on T with respect to φ , T is still a Tashkinov tree and δ is a missing color with respect to φ_1 . But $E_\delta(\partial(T)) \neq \emptyset$, which gives a contradiction to T being an e -maximal tree. \square

Following the notation in Lemma 2.1, we consider the case of δ being a defective color. Then $P := P_u(\gamma, \delta, \varphi)$ is a path with u as one end. Since u is the unique vertex in T missing γ by Theorem 2.1, the other end of P is not in T . In the linear order $\preceq_{(u,P)}$, the last vertex v with $v \in V(T) \cap V(P)$ is called an *exit vertex* of T . Applying Lemma 2.1, Scheide [14] obtained the following result.

Lemma 2.2. *Let $T \in \mathcal{T}_e$ be a Tashkinov tree of a k -triple (G, e, φ) with $k \geq \Delta + 1$. If v is an exit vertex of T , then every missing color in $\overline{\varphi}(v)$ must be used on T .*

Let $T \in \mathcal{T}_{e,\varphi}$ be a Tashkinov tree of (G, e, φ) and $V(e) = \{x, y\}$. By keeping odd number of vertices in each step of growing a Tashkinov tree from e , Scheide [14] showed that there is another $T^* \in \mathcal{T}_{e,\varphi}$, named a *balanced Tashkinov tree*, such that $V(T^*) = V(T)$ constructed incrementally from e by the following steps:

- **Adding a path:** Pick two missing colors α and β with $\alpha \in \overline{\varphi}(x)$ and $\beta \in \overline{\varphi}(y)$, and let $T^* := \{e\} \cup (P_x(\alpha, \beta, \varphi) - y)$ where $P_x(\alpha, \beta, \varphi)$ is the (α, β) -chain containing both x and y .
- **Adding edges by pairs:** Repeatedly pick two boundary edges f_1 and f_2 of T^* with $\varphi(f_1) = \varphi(f_2) \in \overline{\varphi}(V(T^*))$ and redefine $T^* := T^* \cup \{f_1, f_2\}$ until T^* is closed.

The path $P_x(\alpha, \beta, \varphi)$ in the above definition is called the *trunk* of T^* and $h(T^*) := |V(P_x(\alpha, \beta, \varphi))|$ is called the *height* of T^* .

Lemma 2.3. [Scheide [14]] *Let G be a k -critical graph with $k \geq \Delta + 1$ and $T \in \mathcal{T}$ be a balanced Tashkinov tree of a k -triple (G, e, φ) with $h(T)$ being maximum. Then, $h(T) \geq 3$ is odd. Moreover, if $h(T) = 3$ then G is elementary.*

Corollary 2.2. *Let G be a non-elementary k -critical graph with $k \geq \Delta + 1$ and $T \in \mathcal{T}$ be a balanced Tashkinov tree of a k -triple (G, e, φ) with $h(T)$ being maximum. Then $|T| \geq 2(k - \Delta) + 1$.*

Proof. Since G is not elementary, T is not strongly closed with respect to φ . There is an exit vertex v by Lemma 2.1, so $\overline{\varphi}(v) \subseteq \varphi(E(T))$ by Lemma 2.2. Since T is balanced and $h(T) \geq 5$ by Lemma 2.3, each used color is assigned to at least two edges of $E(T)$. Thus,

$$|T| = ||T|| + 1 \geq 2|\overline{\varphi}(v)| + 1 \geq 2(k - \Delta) + 1. \quad \square$$

Working on balanced Tashkinov trees, Scheide proved the following result.

Lemma 2.4. [Scheide [14]] *Let G be a k -critical graph with $k \geq \Delta + 1$. If $|T| < 11$ for all Tashkinov trees T , then G is elementary.*

3 An extension of Tashkinov trees

3.1 Definitions and basic properties

In this section, we always assume that G is a *non-elementary* k -critical graph with $k \geq \Delta + 1$ and $T_0 \in \mathcal{T}$ is a maximum Tashkinov tree of G . Moreover, we assume that T_0 is a Tashkinov tree of the k -triple (G, e, φ) .

Definition 4. *Let $\varphi_1, \varphi_2 \in \mathcal{C}^k(G - e)$ and $H \subseteq G$ such that $e \in E(H)$. We say that H is (φ_1, φ_2) -stable if $\varphi_1(f) = \varphi_2(f)$ for every $f \in E(G[V(H)]) \cup \partial(H)$, that is, $\varphi_1(f) \neq \varphi_2(f)$ implies that $f \in E(G - V(H))$.*

Following the definition, if a Tashkinov tree T_0 of (G, e, φ_1) is (φ_1, φ_2) -stable, then it is also a Tashkinov tree of (G, e, φ_2) . Moreover, the sets of missing colors of T_0 , used colors of T_0 , and free colors of T_0 are the same in both colorings φ_1 and φ_2 .

The following definition of *connecting edges* will play a critical role in our extension based on a maximum Tashkinov tree.

Definition 5. *Let $H \subseteq G$ be a subgraph such that $T_0 \subseteq H$. A color δ is called a defective color of H if H is closed, $\delta \notin \overline{\varphi}(V(H))$ and $|E_\delta(\partial(H))| \geq 2$. Moreover, an edge $f \in \partial(H)$ is called a connecting edge if $\delta := \varphi(f)$ is a defective color of H and there is a missing color $\gamma \in \overline{\varphi}(V(T_0)) - \varphi(E(H))$ of T_0 such that the following two properties hold.*

- *The (γ, δ) -chain $P_u(\delta, \gamma, \varphi)$ contains all edges in $E_\delta(\partial(H))$, where u is the unique vertex in $V(T_0)$ such that $\gamma \in \overline{\varphi}(u)$;*
- *Along the linear order $\preceq_{(u, P_u(\gamma, \delta, \varphi))}$, f is the first boundary edge on $P_u(\gamma, \delta, \varphi)$ with color δ .*

In the above definition, we call the successor f^s of f along $\preceq_{(u, P_u(\gamma, \delta, \varphi))}$ the *companion* of f , (f, f^s) a *connecting edge pair* and (δ, γ) a *connecting color pair*. Since $P_u(\gamma, \delta, \varphi)$ contains all edges in $E_\delta(\partial(H))$, we have that f^s is not incident to any vertex in H and $\varphi(f^s) = \gamma$.

Definition 6. *We call a tree T an **Extension of a Tashkinov Tree (ETT)** of (G, e, φ) based on T_0 if T is incrementally obtained from $T := T_0$ by repeatedly adding edges to T according to the following two operations subject to $\Gamma^f(T_0) - \varphi(E(T)) \neq \emptyset$:*

- **ET0:** If T is closed, add a connecting edge pair (f, f^s) , where $\varphi(f)$ is a defective color and $\varphi(f^s) \in \Gamma^f(T_0) - \varphi(E(T))$, and rename $T := T \cup \{f, f^s\}$.
- **ET1:** Otherwise, add an edge $f \in \partial(T)$ with $\varphi(f) \in \overline{\varphi}(V(T))$ being a missing color of T , and rename $T := T \cup \{f\}$.

Note that the above extension algorithm ends with $\Gamma^f(T_0) \subseteq \varphi(E(T))$. Let T be an ETT of (G, e, φ) . Since T is defined incrementally from T_0 , the edges added to T follow a linear order \prec_ℓ . Along the linear order \prec_ℓ , for any initial subsequence S of $E(T)$, $T_0 \cup S$ induces a tree; we call it a *premier segment* of T provided that when a connecting edge is in S , its companion must be in S . Let f_1, f_2, \dots, f_{m+1} be all connecting edges with $f_1 \prec_\ell f_2 \prec_\ell \dots \prec_\ell f_{m+1}$. For each $1 \leq i \leq m+1$, let T_{i-1} be the premier subtree induced by T_0 and edges before f_i in the ordering \prec_ℓ . Clearly, we have $T_0 \subset T_1 \subset T_2 \subset \dots \subset T_m \subset T$. We call T_i a *closed segment* of T for each $0 \leq i \leq m$, $T_0 \subset T_1 \subset T_2 \subset \dots \subset T_m \subset T$ the *ladder* of T , and T an *ETT with m -rungs*. We use $m(T)$ to denote the number of rungs of T . For each edge $f \in E(T)$ with $f \neq e$, following the linear order \prec_ℓ , the end of f is called the *in-end* if it is in T before f and the other one is called the *out-end* of f . For any edge $f \in E(T)$, the subtree induced by T_0 , f and all its predecessors is called an *f -segment* and denoted by Tf .

Let \mathbb{T} denote the set of all ETTs based on T_0 . We now define a binary relation \prec_t of \mathbb{T} such that for two $T, T^* \in \mathbb{T}$, we call $T \prec_t T^*$ if either $T = T^*$ or there exists s with $1 \leq s \leq \min\{m+1, m^*+1\}$ such that $T_h = T_h^*$ for every $0 \leq h < s$ and $T_s \subsetneq T_s^*$, where $T_0 \subset T_1 \subset \dots \subset T_s \subset \dots \subset T_m \subset T_{m+1}(= T)$ and $T_0^*(= T_0) \subset T_1^* \subset \dots \subset T_s^* \subset \dots \subset T_{m^*+1}^*(= T^*)$ are the ladders of T and T^* , respectively. Notice that in this definition, we only consider the relations of T_h and T_h^* for $h \leq s$. Clearly, for any three ETTs T, T' and T^* , $T \prec_t T'$ and $T' \prec_t T^*$ give $T \prec_t T^*$. So, \mathbb{T} together with \prec_t forms a poset, which is denoted by (\mathbb{T}, \prec_t) .

Lemma 3.1. *In the poset (\mathbb{T}, \prec_t) , if T is a maximal tree over all ETTs with at most $|T|$ vertices, then any premier segment T' of T is also a maximal tree over all ETTs with at most $|T'|$ vertices.*

Proof. Suppose on the contrary: there is a premier segment T' of T and an ETT T^* with $|T^*| \leq |T'|$ and $T' \prec_t T^*$. We assume that $T' \neq T^*$. Let $T_0 \subset T_1 \subset \dots \subset T_{m'} \subset T'$ and $T_0 \subset T_1^* \subset \dots \subset T_{m^*}^* \subset T^*$ be the ladders of T' and T^* , respectively. Since $T' \prec_t T^*$, there exists s with $1 \leq s \leq \min\{m'+1, m^*+1\}$ such that $T_j = T_j^*$ for each $0 \leq j \leq s-1$ and $T_s \subsetneq T_s^*$, where $T_{m'+1}' = T'$ and $T_{m^*+1}^* = T^*$. Since $|T^*| \leq |T'|$, we have $s < m'+1$. Since T' is a premier segment of T , $T_0 \subset T_1 \subset \dots \subset T_{m'}$ is a part of the ladder of T . So, we have $T \prec_t T^*$, giving a contradiction to the maximality of T . \square

Lemma 3.2. *Let T be a maximal ETT in (\mathbb{T}, \prec_t) over all ETTs with at most $|T|$ vertices, and let $T_0 \subset T_1 \subset \dots \subset T_m \subset T$ be the ladder of T . Suppose T is an ETT of (G, e, φ_1) . Then for every*

$\varphi_2 \in \mathcal{C}^k(G - e)$ such that T_m is (φ_1, φ_2) -stable, T_m is an ETT of (G, e, φ_2) . Furthermore, if T_m is elementary, then for every $\gamma \in \Gamma^f(T_0) - \varphi_1(E(T_m))$ and $\delta \notin \overline{\varphi}_1(V(T_m))$, $P_u(\gamma, \delta, \varphi_2) \supseteq \partial_\delta(T_m)$ where $u \in V(T_0)$ such that $\gamma \in \overline{\varphi}_1(u)$.

Proof. Suppose on the contrary: let T be a counterexample to Lemma 3.2 with minimum number of vertices. Let $T_0 \subset \dots \subset T_m \subset T$ be the ladder of T and let $\varphi_1, \varphi_2 \in \mathcal{C}^k(G - e)$ be two edge colorings such that T is an ETT of (G, e, φ_1) , T_m is (φ_1, φ_2) -stable and either

- (1) T_m is not an ETT of (G, e, φ_2) or
- (2) T_m is elementary and there exist $\gamma \in \Gamma^f(T_0) - \varphi_1(E(T_m))$ and $\delta \notin \overline{\varphi}_1(V(T_m))$ such that $P_u(\gamma, \delta, \varphi_2) \not\supseteq \partial_\delta(T_m)$ where $u \in V(T_0)$ such that $\gamma \in \overline{\varphi}_1(u)$.

By the minimality of T , we observe that $|T| = |T_m| + 2$. Furthermore, since $T_0 \in \mathcal{T}$ is a maximum Tashkinov tree of G , it follows that $m \geq 1$ by Lemma 2.1.

First, we show that (1) does not hold, in other words, T_m is an ETT of (G, e, φ_2) . Since colors for edges incident to vertices in T_m are the same in both φ_1 and φ_2 , we only need to show that each connecting edge pair in coloring φ_1 is still a connecting edge pair in coloring φ_2 . For $0 \leq j \leq m - 1$ let (f_j, f_j^s) be the connecting edge pair of T_j and let (δ_j, γ_j) be the corresponding connecting color pair with respect to φ_1 . Since T_{j+1} is (φ_1, φ_2) -stable and an ETT of (G, e, φ_1) and $T_{j+1} \subsetneq T$, by the minimality of T , it follows that $P_{u_j}(\gamma_j, \delta_j, \varphi_2)$ contains $\partial_{\delta_j}(T_j)$ where u_j is the unique vertex in $V(T_0)$ with $\gamma_j \in \overline{\varphi}_1(u_j)$. Moreover, since T_{j+1} is (φ_1, φ_2) -stable, it follows that f_j is the first boundary edge on $P_{u_j}(\gamma_j, \delta_j, \varphi_2)$ with color δ_j and f_j^s being its companion. So (f_j, f_j^s) is still a connecting edge pair in φ_2 . We point out that $P_{u_j}(\gamma_j, \delta_j, \varphi_1)$ and $P_{u_j}(\gamma_j, \delta_j, \varphi_2)$ may be different in (G, e, φ_1) and (G, e, φ_2) .

Thus (2) holds and there exist $\gamma \in \Gamma^f(T_0) - \varphi_1(E(T_m))$ and $\delta \notin \overline{\varphi}_1(V(T_m))$ such that $P_u(\gamma, \delta, \varphi_2) \not\supseteq \partial_\delta(T_m)$. Let $P = P_u(\gamma, \delta, \varphi_2)$. Since T_m is both elementary and closed and u is one of the two ends of P , the other end of P must be in $V \setminus V(T_m)$. So, $E(P) \cap E_\delta(\partial(T_m)) \neq \emptyset$. Let Q be another (γ, δ) -chain such that $E(Q) \cap E_\delta(\partial(T_m)) \neq \emptyset$. Let $\varphi_3 := \varphi_2/Q$ be a coloring of $G - e$ obtained from φ_2 by interchanging colors assigned on $E(Q)$.

Let (f, f^s) be the connecting edge pair of T_{m-1} , and $T' = T_{m-1} \cup \{f, f^s\}$. We claim that $E(T') \cap E(Q) = \emptyset$. By the minimality of T , P contains every edge of $E_\delta(\partial(T_{m-1}))$, and so $E(T_{m-1}) \cap E(Q) = \emptyset$. If $\varphi_2(f) \neq \delta$ then $f \notin E(Q)$ and if $\varphi_2(f) = \delta$ then $f \in E(P)$ so $f \notin E(Q)$. Thus $f \notin E(Q)$. Lastly, $\varphi_2(f^s) \neq \delta$ since $\delta \in \overline{\varphi}_2(V(T_m))$ and $\varphi_2(f^s) \neq \gamma$ since $\gamma \notin \varphi_2(E(T_m))$, so $f^s \notin E(Q)$.

Observe that T' is an ETT of (G, e, φ_1) with ladder $T_0 \subset \dots \subset T_{m-1}$ and is (φ_1, φ_3) -stable. Moreover $|T'| \leq |T_m| < |T|$. Therefore, by the minimality of T , T_{m-1} is an ETT of (G, e, φ_3) , and

because we do not use any edge in Q when we extend T_{m-1} to T_m , T_m is also an ETT of (G, e, φ_3) which is not closed. However, it is a contradiction that T is a maximal ETT. \square

In Lemma 3.2, by taking $\varphi_1 = \varphi_2$, we easily obtain the following lemma.

Lemma 3.3. *Let T be a maximal ETT in (\mathbb{T}, \prec_t) over all ETTs with at most $|T|$ vertices, and let $T_0 \subset T_1 \subset \dots \subset T_m \subset T$ be the ladder of T . Suppose T is an ETT of (G, e, φ) . If T_m is elementary and $\Gamma^f(T_0) - \varphi(E(T)) \neq \emptyset$, then for any $\gamma \in \Gamma^f(T_0) - \varphi(E(T))$ and $\delta \notin \overline{\varphi}(V(T_m))$, $P_u(\gamma, \delta, \varphi) \supset E_\delta(\partial(T_i))$ for every i with $0 \leq i \leq m$, where $u \in V(T_0)$ such that $\gamma \in \overline{\varphi}(u)$.*

Lemma 3.4. *For every ETTT of (G, e, φ) based on T_0 , if T is elementary such that $|\Gamma^f(T_0)| > m(T)$ and $|E(T) - E(T_0)| - m(T) < |\overline{\varphi}(V(T_0))|$, then there exists an ETT T^* containing T as a premier segment.*

Proof. Let T be an ETT of (G, e, φ) and $m = m(T)$. Since $\varphi(f_i) \notin \overline{\varphi}(V(T_0))$ for each connecting edge f_i , where $i \in \{1, 2, \dots, m\}$, we have $|\varphi(E(T) - E(T_0)) \cap \overline{\varphi}(V(T_0))| \leq |E(T) - E(T_0)| - m < |\overline{\varphi}(V(T_0))|$. So, $\overline{\varphi}(V(T_0)) - \varphi(E(T) - E(T_0)) \neq \emptyset$. Let $\gamma \in \overline{\varphi}(V(T_0)) - \varphi(E(T) - E(T_0))$.

We may assume $\gamma \notin \varphi(E(T_0))$, i.e., $\gamma \in \Gamma^f(T_0)$. Since $m < |\Gamma^f(T_0)|$, there exists a color $\beta \in \Gamma^f(T_0) - \{\gamma_1, \gamma_2, \dots, \gamma_m\}$. Since T_0 is closed, a (β, γ) -chain is either in $G[V(T_0)]$ or vertex disjoint from T_0 . Let φ_1 be obtained from φ by interchanging β and γ for edges in $E_\beta(G - V(T_0)) \cup E_\gamma(G - V(T_0))$. Clearly, T_0 is (φ, φ_1) -stable. So, T is also an ETT of (G, e, φ_1) . Since $\gamma \notin \varphi(E(T) - E(T_0))$, we have $\beta \notin \varphi_1(E(T))$, so the claim holds.

We can apply **ET0** and **ET1** to extend T to a larger tree T^* unless T is closed and does not have a connecting edge. In this case, T is both elementary and closed. Since G itself is not elementary, T is not strongly closed. Thus, T has a defective color δ . Since T does not have a connecting edge, $P_v(\gamma, \delta, \varphi)$ does not contain all edges of $E_\delta(\partial(T))$, where $v \in V(T_0)$ is the unique vertex with $\gamma \in \overline{\varphi}(v)$. Let Q be another (γ, δ) -chain containing some edges in $E_\delta(\partial(T))$ and let $\varphi_2 = \varphi/Q$. By Lemma 3.3, Q is disjoint from T_m , where T_m is the largest closed segment of T . So, T_m is (φ, φ_2) -stable. By Lemma 3.2, T_m is an ETT of (G, e, φ_2) , which in turn gives that T is also an ETT of (G, e, φ_2) . Applying **ET1**, we extend T to a larger ETT T^* , which contains T as a premier segment. \square

3.2 The major result

The following result is fundamental for both Theorems 1.1 and 1.2.

Theorem 3.1. *Let G be a k -critical graph with $k \geq \Delta + 1$ and T be a maximal ETT over all ETTs with at most $|T|$ vertices in the poset (\mathbb{T}, \prec_t) . Suppose T is an ETT of (G, e, φ) . If $|E(T) - E(T_0)| - m(T) < |\overline{\varphi}(V(T_0))| - 1$ and $m(T) < |\Gamma^f(T_0)| - 1$, then T is elementary.*

Proof. Suppose on the contrary: let T be a counterexample to Theorem 3.1 with minimum number of vertices. And we assume that (G, e, φ) is the triple in which T is an ETT.

By Theorem 2.1, we have $T \supsetneq T_0$. For any premier segment T' of T , by Lemma 3.1, T' is maximal over all ETTs with at most $|T'|$ vertices. Additionally, following the definition, we can verify that $|E(T') - E(T_0)| - m(T') \leq |E(T) - E(T_0)| - m(T)$ and $m(T') \leq m(T)$. So, every premier segment of T satisfies the conditions of Theorem 3.1. Hence, Theorem 3.1 holds for all premier segments of T which are proper subtrees of T . Let $T_0 \subset T_1 \subset \dots \subset T_m \subset T$ be the ladder of T .

Let v_1, v_2 be two distinct vertices in T such that there is a color $\alpha \in \overline{\varphi}(v_1) \cap \overline{\varphi}(v_2)$. For each connecting edge f_i with $1 \leq i \leq m$, let $(\delta_i, \gamma_{\delta_i})$ denote the corresponding color pair, where $\varphi(f_i) = \delta_i$. According to the definition of ETT, $\gamma_{\delta_1}, \gamma_{\delta_2}, \dots, \gamma_{\delta_m}$ are pairwise distinct while $\delta_1, \delta_2, \dots, \delta_m$ may not be. Let $L = \{\gamma_{\delta_1}, \gamma_{\delta_2}, \dots, \gamma_{\delta_m}\}$. In the paper [2] by Chen et al., the condition $\overline{\varphi}(v) \not\subseteq L$ is needed for any $v \in V(T) - V(T_0)$. In the following proof, we overcome this constraint. We make the following assumption.

Assumption 1: We assume that over all colorings in $\mathcal{C}^k(G - e)$ such that T is a minimum counterexample, the coloring $\varphi \in \mathcal{C}^k(G - e)$ is one such that $|\overline{\varphi}(V(T_0)) - (\varphi(E(T) - E(T_0)) \cup \{\alpha\})|$ is minimum.

The following claim states that we can use other missing colors of T_0 before using free colors of T_0 except those in L .

Claim 3.1. *We may assume that if $\varphi(E(T) - E(T_0)) \cap (\Gamma^f(T_0) - (L \cup \{\alpha\})) \neq \emptyset$, then $\varphi(E(T) - E(T_0)) \supset \overline{\varphi}(V(T_0)) - \Gamma^f(T_0)$.*

Proof. Assume that there is a color $\gamma \in \varphi(E(T) - E(T_0)) \cap (\Gamma^f(T_0) - (L \cup \{\alpha\}))$ and there is a color $\beta \in (\overline{\varphi}(V(T_0)) - \Gamma^f(T_0)) - \varphi(E(T) - E(T_0))$. Since T_0 is closed, a (β, γ) -chain is either in $G[V(T_0)]$ or disjoint from $V(T_0)$. Let φ_1 be obtained from φ by interchanging colors β and γ on all (β, γ) -chains disjoint from $V(T_0)$. It is readily seen that T_0 is (φ, φ_1) -stable. Since both γ and β are in $\overline{\varphi}(V(T_0)) - L$, T is also an ETT of (G, e, φ_1) . In coloring φ_1 , we still have $\gamma \in \Gamma^f(T_0) - (L \cup \{\alpha\})$ and $\beta \in \overline{\varphi}_1(V(T_0)) - \Gamma^f(T_0)$. However, γ is not used on $T - T_0$ while β is used. Additionally, Assumption 1 holds since $|\overline{\varphi}(V(T_0)) - (\varphi(E(T) - E(T_0)) \cup \{\alpha\})| = |\overline{\varphi}_1(V(T_0)) - (\varphi_1(E(T) - E(T_0)) \cup \{\alpha\})|$. By repeatedly applying this argument, we show that Claim 3.1 holds. \square

Since $m(T) < |\Gamma^f(T_0)| - 1$, we have $\Gamma^f(T_0) - (L \cup \{\alpha\}) \neq \emptyset$. Since $|E(T) - E(T_0)| - m(T) < |\overline{\varphi}(V(T_0))| - 1$, we have $\overline{\varphi}(V(T_0)) - (\varphi(E(T) - E(T_0)) \cup \{\alpha\}) \neq \emptyset$. By Claim 3.1, we have the following claim.

Claim 3.2. *We may assume that $\Gamma^f(T_0) - (\varphi(E(T)) \cup \{\alpha\}) \neq \emptyset$.*

We consider two cases to complete the proof according to the type of the last operation in adding edge(s) to extend T_0 to T .

Case 1: The last operation is **ET0**, i.e., the two edges in the connecting edge pair (f, f^s) are the last two edges in T following the linear order \prec_ℓ .

Let x be the in-end of f , y be the out-end of f (in-end of f^s), and z be the out-end of f^s . In this case, we have $V(T) = V(T_m) \cup \{y, z\}$, i.e., $T' = T_m$. Let $\delta = \varphi(f)$ be the defective color and $\gamma_\delta \in \Gamma^f(T_0) - \varphi(E(T_m))$ such that f is the first edge in $\partial(E(T_m))$ along $P := P_u(\gamma_\delta, \delta, \varphi)$ with color δ , where $u \in V(T_0)$ such that $\gamma_\delta \in \overline{\varphi}(u)$. Recall that v_1 and v_2 are the two vertices in T such that $\alpha \in \overline{\varphi}(v_1) \cap \overline{\varphi}(v_2)$. We have $\{v_1, v_2\} \cap \{y, z\} \neq \emptyset$. We consider the following three subcases to lead a contradiction.

Subcase 1.1: $\{v_1, v_2\} = \{y, z\}$.

Assume, without loss of generality, $y = v_1$ and $z = v_2$. Since f^s is the successor of f along the linear order $\preceq_{(u, P)}$, $\varphi(f^s) = \gamma_\delta$. So, f^s is an (α, γ_δ) -chain. Let $\varphi_1 = \varphi/f^s$, a coloring obtained from φ by changing color on f^s from γ_δ to α . Then T_m is (φ, φ_1) -stable. By Lemma 3.2, T_m is an ETT of (G, e, φ_1) and γ_δ is missing at y in φ_1 , which in turn gives that $P_u(\gamma_\delta, \delta, \varphi_1) := uPy$ only contains one edge $f \in E_\delta(\partial(T_m))$, giving a contradiction to Lemma 3.3.

Subcase 1.2: $\alpha \in (\overline{\varphi}(y) - \overline{\varphi}(z)) \cap \overline{\varphi}(V(T_m))$.

Since $\delta, \gamma_\delta \in \varphi(y)$ and $\alpha \in \overline{\varphi}(y)$, $\alpha \notin \{\delta, \gamma_\delta\}$. We may assume that $\alpha \in \Gamma^f(T_0) - \varphi(E(T))$. Otherwise, let $\beta \in \Gamma^f(T_0) - \varphi(E(T))$ and consider the (α, β) -chain $P_1 := P_y(\alpha, \beta, \varphi)$. Since $\alpha, \beta \in \overline{\varphi}(V(T_m))$ and $V(T_m)$ is closed with respect to φ by the assumption, we have $V(P_1) \cap V(T_m) = \emptyset$. Let $\varphi_1 = \varphi/P_1$. Since $\{\alpha, \beta\} \cap \{\delta, \gamma_\delta\} = \emptyset$, we have $f^s \notin E(P_1)$. Hence T_m is (φ, φ_1) -stable, which gives that T_m is an ETT of (G, e, φ_1) , so is T . The claim follows from $\beta \in \overline{\varphi}_1(y) \cap (\Gamma^f(T_0) - \varphi_1(E(T)))$.

Consider the (α, γ_δ) -chain $P_2 := P_y(\alpha, \gamma_\delta, \varphi)$. Since $\alpha, \gamma_\delta \in \overline{\varphi}(V(T_0))$ and T_m is closed, $V(P_2) \cap V(T_m) = \emptyset$. Let $\varphi_2 = \varphi/P_2$. Clearly, T_m is (φ, φ_2) -stable. By Lemma 3.2, T_m is an ETT of (G, e, φ_2) , so is T . Then $P_u(\gamma_\delta, \delta, \varphi_2)$ is the subpath of $P_u(\gamma_\delta, \delta, \varphi)$ from u to y . So, it does not contain all edges in $E_\delta(\partial(T_m))$, which gives a contradiction to Lemma 3.3.

Subcase 1.3: $\alpha \in (\overline{\varphi}(z) - \overline{\varphi}(y)) \cap \overline{\varphi}(V(T_m))$.

Since $P_u(\gamma_\delta, \delta, \varphi)$ contains all the edges in $E_\delta(\partial(T_m))$ and $\alpha \in \overline{\varphi}(z)$, we have $\alpha \notin \{\delta, \gamma_\delta\}$. Following a similar argument given in Subcase 1.2, we may assume that $\alpha \in \Gamma^f(T_0) - \varphi(E(T))$. Let v be the unique vertex in $V(T_0)$ with $\alpha \in \overline{\varphi}(v)$. Let $\beta \in \overline{\varphi}(y)$, $P_v := P_v(\alpha, \beta, \varphi)$, $P_y := P_y(\alpha, \beta, \varphi)$ and $P_z := P_z(\alpha, \beta, \varphi)$. We claim that $P_v = P_y$. Suppose, on the contrary, that $P_v \neq P_y$. By Lemma 3.3,

$E(P_v) \supset E_\beta(\partial(T_m))$. Therefore, $V(P_y) \cap V(T_m) = \emptyset$. Let $\varphi_1 = \varphi/P_y$. In (G, e, φ_1) , T is an ETT and $\alpha \in \overline{\varphi}_1(y) \cap \overline{\varphi}_1(V(T_0))$. This leads back to either Subcase 1.1 or Subcase 1.2. Hence, $P_v = P_y$ and it is vertex disjoint with P_z . Let $\varphi_2 = \varphi/P_z$. By Lemma 3.3, $E(P_v) \supset E_\beta(\partial(T_m))$. So, $V(P_z) \cap V(T_m) = \emptyset$, which in turn gives that T is an ETT of (G, e, φ_2) and $\beta \in \overline{\varphi}_2(y) \cap \overline{\varphi}_2(z)$. This leads back to Subcase 1.1.

Case 2: The last edge f is added to T by **ET1**.

Let y and z be the in-end and out-end of f , respectively, and let $T' = T - z$. Clearly, T' is a premier segment of T and $T_m \subsetneq T'$. In this case, we assume that $z = v_2$, i.e., $\alpha \in \overline{\varphi}(z) \cap \overline{\varphi}(v_1)$ and $v_1 \in V(T')$. Recall that v_1 and v_2 are the two vertices in T such that $\alpha \in \overline{\varphi}(v_1) \cap \overline{\varphi}(v_2)$.

Claim 3.3. *For any color $\gamma \in \Gamma^f(T_0)$ and any color $\beta \in \overline{\varphi}(V(T'))$, let $u \in V(T_0)$ such that $\gamma \in \overline{\varphi}(u)$ and $v \in V(T')$ such that $\beta \in \overline{\varphi}(v)$. Denote by $e_v \in E(T)$ the edge containing v as the out-end and $e_v \prec_\ell e^*$ for every $e^* \in E(T)$ with $\varphi(e^*) = \gamma$, then u and v are on the same (β, γ) -chain.*

Proof. Since T_m is both elementary and closed, u and v are on the same (β, γ) -chain if $v \in V(T_m)$. Suppose $v \in V(T) - V(T_m)$ and, on the contrary, $P_u := P_u(\gamma, \beta, \varphi)$ and $P_v := P_v(\gamma, \beta, \varphi)$ are vertex disjoint. By Lemma 3.3, $E(P_u) \supset E_\beta(\partial(T_m))$, so $V(P_v) \cap V(T_m) = \emptyset$. Let $\varphi_1 = \varphi/P_v$ be the coloring obtained by interchanging the colors β and γ on $P_v(\gamma, \beta, \varphi)$. Clearly, T_m is (φ, φ_1) -stable. By Lemma 3.2, T_m is an ETT of (G, e, φ_1) . As $e_v \prec_\ell e^*$ for every $e^* \in E(T)$ with $\varphi(e^*) = \gamma$, we can extend T_m to Te_v such that Te_v is still an ETT of (G, e, φ_1) . But, in the coloring φ_1 , $\gamma \in \overline{\varphi}_1(u) \cap \overline{\varphi}_1(v)$, which gives a contradiction to the minimality of $|T|$. \square

Claim 3.4. *We may assume $\alpha \in \Gamma^f(T_0) - \varphi(E(T_m))$.*

Proof. Otherwise, by Claim 3.2, let $\gamma \in \Gamma^f(T_0) - (\varphi(E(T)) \cup \{\alpha\})$. Let φ_1 be obtained from φ by interchanging colors α and γ for edges in $E_\alpha(G - V(T_m)) \cup E_\gamma(G - V(T_m))$. Since T_m is closed, φ_1 exists. Clearly, T_m is (φ, φ_1) -stable. By Lemma 3.2, T_m is an ETT of (G, e, φ_1) , so is T . In the coloring φ_1 , $\gamma \in \overline{\varphi}_1(z)$ but is not used on T_m . \square

Applying Claim 3.2 again if it is necessary, we assume both Claim 3.2 and Claim 3.4 hold. Recall that z is the out-end of f and y is the in-end of f , and $\alpha \in \overline{\varphi}(v_1) \cap \overline{\varphi}(z)$.

Subcase 2.1: $y \in V(T') - V(T_m)$, i.e., $f \notin \partial(T_m)$.

Claim 3.5. *Color α is used in $E(T - T_m)$, i.e., $\alpha \in \varphi(E(T - T_m))$.*

Proof. Suppose on the contrary that $\alpha \notin \varphi(E(T - T_m))$. By Claim 3.4, we may assume that $\alpha \notin \varphi(E(T_m))$, so $\alpha \notin \varphi(E(T))$. Let $\varphi(f) = \theta$ and $\beta \in \overline{\varphi}(y)$ be a missing color of y . We consider the

following two cases according to whether y is the last vertex of $T' = T - z$.

We first assume that y is the last vertex of T' . Let $P_{v_1} := P_{v_1}(\alpha, \beta, \varphi)$, $P_y := P_y(\alpha, \beta, \varphi)$ and $P_z := P_z(\alpha, \beta, \varphi)$ be (α, β) -chains containing vertices v_1 , y and z , respectively. By Claim 3.3, we have $P_{v_1} = P_y$, so it is disjoint from P_z . By Lemma 3.3, $E(P_{v_1}) \supset E_\beta(\partial(T_m))$, so $V(P_z) \cap V(T_m) = \emptyset$. Let $\varphi_1 = \varphi/P_z$ be the coloring obtained from φ by interchanging colors α and β on P_z . Since $\alpha \notin \varphi(E(T - T_m))$ and $\beta \in \overline{\varphi}(y) - \overline{\varphi}(V(T'))$, $\beta \notin \varphi_1(E(T - T_m))$. Clearly, T_m is (φ, φ_1) -stable. By Lemma 3.2, T_m is an ETT of (G, e, φ_1) , so is T . In the coloring φ_1 , $\theta = \varphi_1(f)$ and f itself is a (β, θ) -chain. Let $\varphi_2 = \varphi_1/f$ be the coloring obtained from φ_1 by changing color θ to β on f . Since f is disjoint from T_m , we can verify that T is an ETT of (G, e, φ_2) by applying Lemma 3.2. Since f is not a connecting edge, $\theta \in \overline{\varphi}(V(T'))$, which in turn shows that T' is not elementary with respect to φ_2 , giving a contradiction to the minimality of $|T|$.

We now assume that y is not the last vertex of T' ; and let x be the last one. Recall $\theta = \varphi(f)$. If $\theta \in \varphi(x)$ then $T - x$ is not an elementary ETT of (G, e, φ) , which contradicts the minimality of $|T|$. Hence we assume $\theta \in \overline{\varphi}(x)$. Clearly $\alpha \in \varphi(x)$. Let $P_{v_1} := P_{v_1}(\alpha, \theta, \varphi)$, $P_x := P_x(\alpha, \theta, \varphi)$ and $P_z := P_z(\alpha, \theta, \varphi)$ be (α, θ) -chains containing vertices v_1 , x and z , respectively. By Claim 3.3 we have $P_{v_1} = P_x$ which is disjoint with P_z . Furthermore Lemma 3.3 implies that $E(P_{v_1}) \supset E_\theta(\partial(T_m))$, together with the assumption that $\alpha \in \Gamma^f(T_0)$, we get $V(P_z) \cap V(T_m) = \emptyset$. Let $\varphi_1 = \varphi/P_z$ be the coloring obtained from φ by interchanging colors α and θ along P_z . Observe that θ is only used on f for $E(T - (T_m \cup \partial(T_m)))$ since $\theta \in \overline{\varphi}(x)$, f is colored by α in φ_1 . Clearly T_m is (φ, φ_1) stable. By Lemma 3.2, T_m is an ETT of (G, e, φ_1) , so is T . By Claim 3.2, let $\gamma \in \Gamma^f(T_0) - (\varphi_1(E(T)) \cup \{\theta\})$. Say $\gamma \in \overline{\varphi}(v_2)$ for $v_2 \in V(T_0)$. By Claim 3.3 the (γ, θ) -chain $P'_{v_2} := P_{v_2}(\gamma, \theta, \varphi_1)$ is the same with $P'_x := P_x(\gamma, \theta, \varphi_1)$, hence it is disjoint with $P'_z := P_z(\gamma, \theta, \varphi_1)$. Now we consider T_{zx} obtained from T by switching the order of adding vertices x and z . Clearly T_{zx} is an ETT of (G, e, φ_1) since f is colored by α in φ_1 . Similarly by Claim 3.3 the (γ, θ) -chain $P'_{v_2} := P_{v_2}(\gamma, \theta, \varphi_1)$ is the same with $P'_z := P_z(\gamma, \theta, \varphi_1)$. Now we reach a contradiction. \square

We now prove the following claim which gives a contradiction to **Assumption 1** and completes the proof of this subcase.

Claim 3.6. *There is a coloring $\varphi_1 \in \mathcal{C}^k(G - e)$ such that T is a non-elementary ETT of (G, e, φ_1) , T_m is (φ, φ_1) -stable, and $|\overline{\varphi}_1(V(T_0)) \cap \varphi_1(E(T) - E(T_0))| > |\overline{\varphi}(V(T_0)) \cap \varphi(E(T) - E(T_0))|$.*

Proof. Following the linear order \prec_ℓ , let e_1 be the first edge in $E(T - T_m)$ with $\varphi(e_1) = \alpha$, and let y_1 be the in-end of e_1 . Pick a missing color $\beta_1 \in \overline{\varphi}(y_1)$. Note that, since $\varphi(e_1) = \alpha$ and $\alpha \in \Gamma^f(T_0) - \varphi(E(T_m))$, $e_1 \notin \partial(T_m)$. Hence $y_1 \in V(T) - V(T_m)$. Let $P_{v_1} := P_{v_1}(\alpha, \beta_1, \varphi)$, $P_{y_1} := P_{y_1}(\alpha, \beta_1, \varphi)$, and $P_z := P_z(\alpha, \beta_1, \varphi)$ be (α, β_1) -chains containing v_1 , y_1 and z , respectively. By Claim 3.3, $P_{v_1} = P_{y_1}$, which in turn shows that it is disjoint from P_z . By Lemma 3.3, $E(P_{v_1}) \supset$

$E_{\beta_1}(\partial(T_m))$, so $V(P_z) \cap V(T_m) = \emptyset$.

Consider the coloring $\varphi_1 = \varphi/P_z$. Since $V(P_z) \cap V(T_m) = \emptyset$, T_m is (φ, φ_1) -stable. By Lemma 3.2, T_m is an ETT of (G, e, φ_1) . Since e_1 is the first edge colored with α along \prec_ℓ , we have that $e_1 \prec_\ell e^*$ for all edges e^* colored with β_1 . So, T is an ETT of (G, e, φ_1) . Note that $e_1 \in E(P_{y_1}) = E(P_{v_1})$, which in turn gives $\varphi_1(e_1) = \alpha$. We also note that $\beta_1 \in \overline{\varphi}_1(z) \cap \overline{\varphi}_1(y_1)$.

By Claim 3.2, there is a color $\gamma \in \Gamma^f(T_0) - \varphi(E(T))$. Let $u \in V(T_0)$ such that $\gamma \in \overline{\varphi}(u)$. Let $Q_u := P_u(\gamma, \beta_1, \varphi_1)$, $Q_{y_1} := P_{y_1}(\gamma, \beta_1, \varphi_1)$ and $Q_z := P_z(\gamma, \beta_1, \varphi_1)$ be (γ, β_1) -chains containing u , y_1 and z , respectively. By Claim 3.3, $Q_u = Q_{y_1}$, so Q_u and Q_z are disjoint. By Lemma 3.3, $E(Q_u) \supset E_{\beta_1}(\partial(T_m))$, so $V(Q_z) \cap V(T_m) = \emptyset$. Let $\varphi_2 = \varphi_1/Q_z$ be a coloring obtained from φ_1 by interchanging colors on Q_z . Since $V(Q_u) \cap V(T_m) = \emptyset$, T_m is an ETT of (G, e, φ_2) . Since $\gamma \in \overline{\varphi}(V(T_0)) - \varphi(E(T))$, T_m can be extended to T as an ETT in φ_2 . Since $\gamma \in \overline{\varphi}_2(z) \cap \overline{\varphi}_2(u)$, by Claim 3.5, we have $\gamma \in \varphi_2(E(T - T_m))$. Since $e_1 \in Q_{y_1} = Q_u$, the color α assigned to e_1 is unchanged. Thus,

$$\overline{\varphi}_2(V(T_0)) \cap \varphi_2(E(T) - E(T_0)) \supseteq (\overline{\varphi}(V(T_0)) \cap \varphi(E(T) - E(T_0))) \cup \{\gamma\},$$

and $\alpha \in \overline{\varphi}(V(T_0)) \cap \varphi(E(T))$. So, Claim 3.6 holds. \square

Subcase 2.2: $y \in V(T_m)$, i.e. $f \in \partial(T_m)$.

The following two claims are similar to Claims 3.5 and 3.6 in Subcase 2.1, which lead to a contradiction to **Assumption 1**. Their proofs respectively are similar to those of the previous two claims. However, for the completeness, we still give the details.

Claim 3.7. *Color α is used in $E(T - T_m)$, i.e., $\alpha \in \varphi(E(T - T_m))$.*

Proof. Suppose on the contrary $\alpha \notin \varphi(E(T - T_m))$. By Claim 3.4, we assume that $\alpha \notin \varphi(E(T_m))$, so $\alpha \notin \varphi(E(T))$. Let $\varphi(f) = \theta$. As $f \in \partial(T_m)$ is not a connecting edge and T_m is closed, we know that there exists $w \in V(T - T_m)$ such that $\theta \in \overline{\varphi}(w)$. Consider the (α, θ) -chain $P_{v_1} := P_{v_1}(\alpha, \theta, \varphi)$. By Lemma 3.3, $E(P_{v_1}) \supset E_\theta(\partial(T_m))$. So, $f \in E(P_{v_1})$ and z is the other end of P_{v_1} . Then, $P_w := P_w(\alpha, \theta, \varphi)$ is disjoint from P_{v_1} , which in turn shows $V(P_w) \cap V(T_m) = \emptyset$. Let $\varphi_1 = \varphi/P_w$. Since $V(P_w) \cap V(T_m) = \emptyset$, T_m is (φ, φ_1) -stable. By Lemma 3.2, T_m is an ETT of (G, e, φ_1) . Since α is not used in $T - T_m$, T_m can be extended to T' as an ETT of (G, e, φ_1) . Note that $\alpha \in \overline{\varphi}_1(v_1) \cap \overline{\varphi}_1(w)$. So, T' is not elementary, which gives a contradiction to the minimality of $|T|$. \square

Claim 3.8. *There is a coloring $\varphi_1 \in \mathcal{C}^k(G - e)$ such that T is a non-elementary ETT of (G, e, φ_1) , T_m is (φ, φ_1) -stable, and $|\overline{\varphi}_1(V(T_0)) \cap \varphi_1(E(T) - E(T_0))| > |\overline{\varphi}(V(T_0)) \cap \varphi(E(T) - E(T_0))|$.*

Proof. Following the linear order \prec_ℓ , let e_1 be the first edge in $E(T - T_m)$ with $\varphi(e_1) = \alpha$, and let y_1 be the in-end of e_1 . Pick a missing color $\beta_1 \in \overline{\varphi}(y_1)$. Since $\varphi(e_1) = \alpha \in \overline{\varphi}(V(T_0))$ and T_m is closed,

$e_1 \notin \partial(T_m)$. Hence, $y_1 \in V(T) - V(T_m)$. Let $P_{v_1} := P_{v_1}(\alpha, \beta_1, \varphi)$, $P_{y_1} := P_{y_1}(\alpha, \beta_1, \varphi)$, and $P_z := P_z(\alpha, \beta_1, \varphi)$ be (α, β_1) -chains containing v_1 , y_1 and z , respectively. By Claim 3.3, $P_{v_1} = P_{y_1}$, which in turn shows that it is disjoint from P_z . By Lemma 3.3, $E(P_{v_1}) \supset E_{\beta_1}(\partial(T_m))$, so $V(P_z) \cap V(T_m) = \emptyset$.

Consider the coloring $\varphi_1 = \varphi/P_z$. Since $V(P_z) \cap V(T_m) = \emptyset$, T_m is (φ, φ_1) -stable. By Lemma 3.2, T_m is an ETT of (G, e, φ_1) . Since e_1 is the first edge colored with α along \prec_ℓ , we have that $e_1 \prec_\ell e^*$ for all edges e^* with $\varphi_1(e^*) = \beta_1$. So, T is an ETT of (G, e, φ_1) . Note that $e_1 \in E(P_{y_1}) = E(P_{v_1})$, which in turn gives $\varphi_1(e_1) = \alpha$. We also note that $\beta_1 \in \overline{\varphi}_1(z) \cap \overline{\varphi}_1(y_1)$.

By Claim 3.2, there is a color $\gamma \in \Gamma^f(T_0) - \varphi(E(T))$. Let $u \in V(T_0)$ such that $\gamma \in \overline{\varphi}(u)$. Let $Q_u := P_u(\gamma, \beta_1, \varphi_1)$, $Q_{y_1} := P_{y_1}(\gamma, \beta_1, \varphi_1)$ and $Q_z := P_z(\gamma, \beta_1, \varphi_1)$ be (γ, β_1) -chains containing u , y_1 and z , respectively. By Claim 3.3, $Q_u = Q_{y_1}$, so Q_u and Q_z are disjoint. By Lemma 3.3, $E(Q_u) \supset E_{\beta_1}(\partial(T_m))$, so $V(Q_z) \cap V(T_m) = \emptyset$. Let $\varphi_2 = \varphi_1/Q_z$ be the coloring obtained from φ_1 by interchanging colors on Q_z . Since $V(Q_u) \cap V(T_m) = \emptyset$, T_m is an ETT of (G, e, φ_2) . Since $\gamma \in \overline{\varphi}(V(T_0)) - \varphi(E(T))$, T_m can be extended to T as an ETT in φ_2 . Since $\gamma \in \overline{\varphi}_2(z) \cap \overline{\varphi}_2(u)$, by Claim 3.5, we have $\gamma \in \varphi_2(E(T - T_m))$. Since $e_1 \in Q_{y_1} = Q_u$, $\varphi_1(e_1) = \varphi(e_1) = \alpha$. Thus,

$$\overline{\varphi}_2(V(T_0)) \cap \varphi_2(E(T) - E(T_0)) \supseteq (\overline{\varphi}(V(T_0)) \cap \varphi(E(T) - E(T_0))) \cup \{\gamma\},$$

and $\alpha \in \overline{\varphi}(V(T_0)) \cap \varphi(E(T))$. So, Claim 3.8 holds. \square

We now complete the proof of Theorem 3.1. \square

Combining Theorem 3.1 and Lemma 3.4, we obtain the following result.

Corollary 3.1. *Let G be a k -critical graph with $k \geq \Delta + 1$. If G is not elementary, then there is an ETT T based on $T_0 \in \mathcal{T}$ with m -rungs such that T is elementary and*

$$|T| \geq |T_0| - 2 + \min\{m + |\overline{\varphi}(V(T_0))|, 2(|\Gamma^f(T_0)| - 1)\}.$$

4 Proofs of Theorems 1.1 and 1.2

4.1 Proof of Theorem 1.1

Clearly, we only need to prove Theorem 1.1 for critical graphs.

Theorem 4.1. *If G is a k -critical graph with $k \geq \Delta + \sqrt[3]{\Delta/2}$, then G is elementary.*

Proof. Suppose on the contrary that G is not elementary. By Corollary 3.1, let T be an ETT of a k -triple (G, e, φ) based on $T_0 \in \mathcal{T}$ with m -rungs such that $V(T)$ is elementary and

$$|T| \geq |T_0| - 2 + \min\{m + |\overline{\varphi}(V(T_0))|, 2(|\Gamma^f(T_0)| - 1)\}.$$

Since $m \geq 1$ and $|\overline{\varphi}(V(T_0))| \geq (k - \Delta)|T_0| + 2$, we have $|T_0| - 2 + m + |\overline{\varphi}(V(T_0))| \geq (k - \Delta + 1)|T_0| + 1$. Following Scheide [14], we may assume that T_0 is a balanced Tashkinov tree with height $h(T_0) \geq 5$. So, $|\varphi(E(T_0))| \leq \frac{|T_0| - 1}{2}$, which in turn gives

$$|\Gamma^f(T_0)| = |\overline{\varphi}(V(T_0))| - |\varphi(E(T_0))| \geq (k - \Delta - \frac{1}{2})|T_0| + \frac{5}{2}.$$

Hence

$$|T_0| - 2 + 2(|\Gamma^f(T_0)| - 1) \geq 2(k - \Delta)|T_0| + 1 \geq (k - \Delta + 1)|T_0| + 1.$$

Therefore, in any case, we have the following inequality

$$|T| \geq (k - \Delta + 1)|T_0| + 1. \quad (1)$$

By Corollary 2.2, $|T_0| \geq 2(k - \Delta) + 1$. Following (1), we get the inequality below.

$$|T| \geq (k - \Delta + 1)(2(k - \Delta) + 1) + 1 = 2(k - \Delta)^2 + 3(k - \Delta) + 2. \quad (2)$$

Since T is elementary, we have $k \geq |\overline{\varphi}(V(T))| \geq (k - \Delta)|T| + 2$. Plugging into (2), we get the following inequality.

$$k \geq 2(k - \Delta)^3 + 3(k - \Delta)^2 + 2(k - \Delta) + 2.$$

Solving the above inequality, we obtain that $k < \Delta + \sqrt[3]{\Delta/2}$, giving a contradiction to $k \geq \Delta + \lceil \sqrt[3]{\Delta/2} \rceil$. \square

4.2 Proofs of Theorem 1.2 and Corollary 1.1

We will need the following observation from [17]. For completeness, we give its proof here.

Lemma 4.1. *Let $s \geq 2$ be a positive integer and G be a k -critical graph with $k > \frac{s}{s-1}\Delta + \frac{s-3}{s-1}$. For any edge $e \in E(G)$, if $X \subseteq V(G)$ is an elementary set with respect to a coloring $\varphi \in \mathcal{C}^k(G - e)$ such that $V(e) \subseteq X$, then $|X| \leq s - 1$.*

Proof. Otherwise, assume $|X| \geq s$. Since X is elementary, $k \geq |\overline{\varphi}(X)| \geq (k - \Delta)|X| + 2 \geq s(k - \Delta) + 2$, which in turn gives

$$\Delta \geq (s - 1)(k - \Delta) + 2 > (\Delta + (s - 3)) + 2 = \Delta + s - 1 > \Delta,$$

a contradiction. \square

Clearly, to prove Theorem 1.2, it is sufficient to restrict our consideration to critical graphs.

Theorem 4.2. *If G is a k -critical graph with $k > \frac{23}{22}\Delta + \frac{20}{22}$, then G is elementary.*

Proof. Suppose, on the contrary, G is not elementary. By Corollary 3.1, let T be an ETT of a k -triple (G, e, φ) based on $T_0 \in \mathcal{T}$ with m -rungs such that $V(T)$ is elementary and

$$|T| \geq |T_0| - 2 + \min\{m + |\overline{\varphi}(V(T_0))|, 2(|\Gamma^f(T_0)| - 1)\}.$$

By Lemma 4.1, $|T| \leq 22$. We will show that $|T| \geq 23$ to lead a contradiction. By Lemma 2.4, we have $|T_0| \geq 11$. Since G is not elementary, $V(T_0)$ is not strongly closed, so $T \supsetneq T_0$. In particular, we have $m \geq 1$. Since $e \in E(T_0)$, we have $|\overline{\varphi}(V(T_0))| \geq |T_0| + 2$. Thus,

$$|T_0| - 2 + m + |\overline{\varphi}(V(T_0))| \geq 2|T_0| + 1 \geq 2 \times 11 + 1 = 23. \quad (3)$$

Following Scheide [14], we may assume that T_0 is a balanced Tashkinov tree with height $h(T_0) \geq 5$, which in turn gives $|\varphi(E(T_0))| \leq (|T_0| - 1)/2$. So, $|\Gamma^f(T_0)| \geq |T_0| + 2 - (|T_0| - 1)/2 \geq (|T_0| + 5)/2$. Thus,

$$|T_0| - 2 + 2(|\Gamma^f(T_0)| - 1) \geq 2|T_0| + 1 \geq 23. \quad (4)$$

Combining (3) and (4), we get $|T| \geq 23$, giving a contradiction. \square

We now give a proof of Corollary 1.1 and recall that Corollary 1.1 is stated as follows.

Corollary 4.1. *If G is a graph with $\Delta \leq 23$ or $|G| \leq 23$, then $\chi' \leq \max\{\Delta + 1, \lceil \chi'_f \rceil\}$.*

Proof. We assume that G is critical. Otherwise, we prove the corollary for a critical subgraph of G instead. If $\Delta \leq 23$, then $\lfloor \frac{23}{22}\Delta + \frac{20}{22} \rfloor = \lfloor \Delta + \frac{\Delta+20}{22} \rfloor \leq \Delta + 1$. If $\chi' \leq \Delta + 1$, we are done. Otherwise, we assume that $\chi' \geq \Delta + 2 \geq \frac{23}{22}\Delta + \frac{20}{22}$. By Theorem 1.2, we have $\chi' = \lceil \chi'_f \rceil$.

Assume that $|G| \leq 23$. If $\chi' \leq \Delta + 1$, then we are done. Otherwise, $\chi' = k + 1$ for some integer $k \geq \Delta + 1$. By Corollary 3.1, let T be an ETT of a k -triple (G, e, φ) based on $T_0 \in \mathcal{T}$ with m -rungs such that $V(T)$ is elementary and

$$|T| \geq |T_0| - 2 + \min\{m + |\overline{\varphi}(V(T_0))|, 2(|\Gamma^f(T_0)| - 1)\}.$$

By Lemma 2.4, we have $|T_0| \geq 11$. Suppose that G is not elementary, then $V(T_0)$ is not strongly closed, so $T \supsetneq T_0$. In particular, we have $m \geq 1$. Since $e \in E(T_0)$, we have $|\overline{\varphi}(V(T_0))| \geq |T_0| + 2$. Thus,

$$|T_0| - 2 + m + |\overline{\varphi}(V(T_0))| \geq 2|T_0| + 1 \geq 2 \times 11 + 1 = 23. \quad (5)$$

Following Scheide [14], we may assume that T_0 is a balanced Tashkinov tree with height $h(T_0) \geq 5$, which in turn gives $|\varphi(E(T_0))| \leq (|T_0| - 1)/2$. So, $|\Gamma^f(T_0)| \geq |T_0| + 2 - (|T_0| - 1)/2 \geq (|T_0| + 5)/2$. Thus,

$$|T_0| - 2 + 2(|\Gamma^f(T_0)| - 1) \geq 2|T_0| + 1 \geq 23. \quad (6)$$

Combining (5) and (6), we get $|T| \geq 23$. Then $|G| \geq |T| \geq 23$. Therefore, $|G| = 23$ and G is elementary, giving a contradiction. \square

5 Acknowledgement

We thank Guangming Jing for comments that greatly improved the manuscript.

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