# SUBDIVISIONS OF DIGRAPHS IN TOURNAMENTS 

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#### Abstract

We show that for every positive integer $k$, any tournament with minimum outdegree at least $(2+o(1)) k^{2}$ contains a subdivision of the complete directed graph on $k$ vertices, which is best possible up to a factor of 8 . This may be viewed as a directed analogue of a theorem proved by Bollobás and Thomason, and independently by Komlós and Szemerédi, concerning subdivisions of cliques in graphs with sufficiently high average degree. We also consider the following problem: given $k$, what is the smallest positive integer $f(k)$ such that any $f(k)$-vertex tournament contains a 1-subdivision of the transitive tournament on $k$ vertices? We show that $f(k)=O\left(k^{2} \log ^{3} k\right)$ which is best possible up to the logarithmic factors.


## 1. Introduction

The complete directed graph on $k$ vertices, denoted by $\vec{K}_{k}$, is a directed graph in which every pair of vertices is connected by an edge in each direction. As usual, we say that a tournament $T$ contains a subdivision of $\vec{K}_{k}$ if it contains a set $B$ of $k$ vertices and a collection of $2\binom{k}{2}$ pairwise internally vertex disjoint directed paths joining every ordered pair of vertices in $B$. We denote such a subdivision by $T \vec{K}_{k}$, and the vertices in $B$ are called the branch vertices of the subdivision. Our main aim in this note is to investigate the density conditions under which a tournament must contain a $T \vec{K}_{k}$, where by 'density' we mean specifically minimum out-degree.

The undirected analogue of this line of research has been studied extensively. The story begins with Mader [9], who showed that any graph with sufficiently large average degree contains a subdivision of the complete graph on $k$ vertices. Later, in [10] he showed that average degree at least $c 2^{k}$ suffices. It is not hard to show that the random graph $G(n, 1 / 2)$ with high probability contains a subdivision of a clique with $\sqrt{n} / 10$ vertices (and with high probability does not contain a subdivision of a clique on at least $10 \sqrt{n}$ vertices). This motivated Mader [9], and independently Erdős and Hajnal [3], to conjecture that any graph with average degree at least $c k^{2}$, for some constant $c$, should necessarily contain a subdivision of $K_{k}$. This was later established by Bollobás and Thomason [2], and independently by Komlós and Szemerédi [7].

Given these results, it is natural to consider the problem in the directed setting, with a suitable density condition. It is not hard to see that average degree is not the right density condition: a transitive tournament has large average degree, yet clearly cannot even contain a subdivision of $\vec{K}_{2}$. But what about large minimum out/in-degree? This, again, does not hold, but the reason is more subtle. Indeed, Thomassen [14] constructed digraphs on $n$ vertices with minimum outdegree at least $c \log n$ which contain no directed cycles of even length; since any subdivision of $\vec{K}_{3}$ must contain an even cycle, these digraphs do not contain $T \vec{K}_{k}$ for any $k \geq 3$. On the other

[^0]hand, Kühn, Osthus and Young [8] showed that any digraph on $n$ vertices with minimum outdegree $d$ contains a subdivision of a complete digraph of order $\left\lfloor d^{2} /\left(8 n^{3 / 2}\right)\right\rfloor$, implying that any digraph on $n$ vertices with minimum out-degree $\sqrt{8 k} n^{3 / 4}$ contains a subdivision of a complete digraph on $k$ vertices.

The above discussion has left out the case of tournaments: Is it true that tournaments with large enough minimum out-degree contain a subdivision of the complete directed graph? The first and last author [5] answered this question in the affirmative: for every positive integer $k$ there is an $m(k)$ such that any tournament with minimum out-degree at least $m(k)$ contains a subdivision of the complete directed graph on $k$ vertices. This result was an important step in the proof of a partial resolution of a conjecture of Pokrovskiy [12]. They proved this with $m(k)$ doubly-exponential in $k^{2}$. Our main result in this paper is to show that we may actually take $m(k)$ to be merely quadratic in $k$. To state our main theorem, let us introduce the following function defined for every integer $k \geq 2$ :

$$
d(k)=\min \left\{m: \text { any tournament } T \text { with } \delta^{+}(T) \geq m \text { contains a } T \vec{K}_{k}\right\}
$$

For example, observe that $d(2)=1$. We are able to determine $d(k)$ for all $k \geq 3$ up to a factor of 8 .

## Theorem 1.1. We have that

$$
k^{2} / 4 \leq d(k) \leq(2+o(1)) k^{2}
$$

where the o(1) term goes to zero as $k \rightarrow \infty$.
The lower bound is simple: any $k^{2} / 4$-regular tournament on $k^{2} / 2$ vertices cannot contain a $T \vec{K}_{k}$ simply because such a subdivision has at least $\binom{k}{2}+k>k^{2} / 2$ vertices. This is true, for example, of a random tournament on $k^{2} / 2$ vertices, as such a tournament with high probability has minimum out-degree $(1-o(1)) k^{2} / 4$. We do not know if there are better constructions, and we leave the exact determination of $d(k)$ as an open problem.

Finally, we consider a similar problem for embedding subdivisions of transitive tournaments. Recall that a tournament is transitive if there is an ordering of the vertices such that every edge goes in the same direction. We denote by $T_{k}$ the transitive tournament on $k$ vertices, and we denote by $T T_{k}$ any subdivision of $T_{k}$. In the context of embedding subdivisions of transitive tournaments in general directed graphs, Scott [13], answering a question of Jagger [6], showed that for $r \geq 2$ and $n \geq n(r)$ every directed graph on $n$ vertices with more edges than the $r$-partite Turán graph $T(r, n)$ contains a $T T_{r+1}$. As for minimum degree conditions, Mader [11] conjectured that for all $k$ there is $f(k)$ such that any digraph with minimum outdegree $f(k)$ contains a subdivision of $T_{k}$. This conjecture remains open to this day, even for $k=5$.

Let $T(k)$ denote the smallest integer such that any tournament on $T(k)$ vertices contains a transitive tournament of order $k$. A well-known theorem of Erdős and Moser [4] states that $2^{(k-1) / 2} \leq T(k) \leq 2^{k-1}$. In particular, any tournament on at least $2^{k-1}$ vertices contains a transitive subtournament on $k$ vertices. If instead of finding a copy of a transitive tournament we allow each edge to be replaced by a directed path of length at most 3 , then the following result holds.

Theorem 1.2. There is a constant $C>0$ such that the following holds. For all $k \geq 2$, any tournament on at least $C k^{2}$ vertices contains a $T T_{k}$, where each directed path in the subdivision has length at most 3. Moreover, this is tight up to the multiplicative constant.

It is natural to ask if a similar lower bound on the number of vertices allows us to embed 1 -subdivisions: subdivisions where each edge is replaced by a directed path of length 2 . An old conjecture of Erdős, confirmed by Alon, Krivelevich and Sudakov [1], states that any graph on $n$ vertices and at least $\varepsilon n^{2}$ edges contains a 1-subdivision of a complete graph on $c(\varepsilon) \sqrt{n}$ vertices (in fact, they show that this holds with $c(\varepsilon)=O(\varepsilon)$ ). We obtain a partial directed analogue of this result, up to log factors.

Theorem 1.3. Any tournament on at least $C k^{2} \log ^{3} k$ vertices contains a 1 -subdivision of $T_{k}$.
We are able to prove this with $C=10^{7}$, but no attempt is made to optimize this constant, as we believe that the same result should hold after removing the log factors (see Section 4 for a conjecture along these lines).
1.1. Notation and Organization. Our notation is standard. Thus, for a vertex $v$ in a directed graph $G$, we let $N_{G}^{+}(v), N_{G}^{-}(v)$ denote the out-neighbourhood and in-neighbourhood of $v$, respectively. Moreover, we let $d_{G}^{+}(v)=\left|N_{G}^{+}(v)\right|$ denote the out-degree of $v$, and analogously $d_{G}^{-}(v)$ the in-degree of $v$. We often omit the subscript ' $G$ ' when the underlying digraph is clear. We denote by $\delta^{+}(G)$ the minimum out-degree of $G$; further, if $X \subset V(G)$, we write $\delta^{+}(X)$ to mean the minimum out-degree of $G[X]$. For a subset $X \subset V(G)$ we let $N^{+}(X)$ denote the set $\bigcup_{x \in X} N^{+}(x)$. Lastly, if $X, Y \subset V(G)$, we write $X \rightarrow Y$ if every edge of $G$ between $X$ and $Y$ is directed from $X$ to $Y$.

The remainder of this paper is organized as follows. In Section 2, we prove our main theorem, Theorem 1.1. In fact, we shall establish a quantitative version that implies Theorem 1.1 (see Theorem 2.4). The proof requires two preparatory lemmas, which we state and prove first. In Section 3, we establish our results Theorem 1.2 and Theorem 1.3 concerning embedding subdivisions of transitive tournaments in large enough tournaments. Finally, we conclude in Section 4 with a further consequence of the general method of this paper, and collect a few open problems.

## 2. Subdivisions of complete directed graphs

Our aim in this section is to prove the upper bound $d(k) \leq(2+o(1)) k^{2}$ in Theorem 1.1. The proof relies on two lemmas, which we prove first. The first lemma allows us to find $k$ vertices whose in-degrees do not differ by much; such vertices will serve as the branch vertex set of our potential subdivision. Our second lemma yields a dichotomy: either we can find a partial subdivision which contains many paths of length 2 or 3 , or we can disconnect the tournament in a particularly nice way. We first isolate the following simple fact, as it will be used elsewhere.

Fact 2.1. Let $T$ be a tournament. Then for every positive integer $\ell$ there are at most $2 \ell+1$ vertices in $T$ of in-degree (out-degree) at most $\ell$.

Proof. If $L$ is the set of vertices in $T$ of in-degree at most $\ell$, then

$$
\ell|L| \geq \sum_{v \in L} d^{-}(v) \geq\binom{|L|}{2}
$$

implying the bound $|L| \leq 2 \ell+1$, as claimed. The proof for 'in-degree' replaced by 'out-degree' is identical.

Lemma 2.2. Suppose $k \geq 3$ is an integer and let $\alpha>0$. If $T$ is a tournament with at least $2 \alpha k^{2}+(20 \alpha+4) k^{7 / 4}$ vertices, then there exists a set $B$ of $k$ vertices and a number $m$ such that for every $v \in B$ :

- $d^{-}(v) \geq \alpha k^{2}+2 k^{7 / 4}$.
- $d^{-}(v) \in\left[m-k^{7 / 4}, m+k^{7 / 4}\right]$.

Additionally, if $|T|=2 \alpha k^{2}+(20 \alpha+4) k^{7 / 4}$, then $d^{+}(v) \leq m+(20 \alpha+1) k^{7 / 4}$ for every $v \in B$.
Proof. By Fact 2.1 there must exist at least $|T|-2 \alpha k^{2}-4 k^{7 / 4}$ vertices in $T$ whose in-degree is at least $\alpha k^{2}+2 k^{7 / 4}$. If we partition the interval $\left[\alpha k^{2}+2 k^{7 / 4},|T|\right]$ into consecutive intervals of size $k^{7 / 4}$, then there must exist at least

$$
k^{7 / 4} \cdot \frac{|T|-2 \alpha k^{2}-4 k^{7 / 4}}{|T|-\alpha k^{2}-2 k^{7 / 4}} \geq k
$$

vertices in the same interval. Note that the above inequality holds since it is equivalent to

$$
|T|\left(k^{3 / 4}-1\right) \geq 2 \alpha k^{11 / 4}+4 k^{10 / 4}-\alpha k^{2}-2 k^{7 / 4}
$$

and it is not hard to verify that this is true for $k \geq 3$, using the assumption that $|T| \geq 2 \alpha k^{2}+$ $(20 \alpha+4) k^{7 / 4}$. Finally, if $v$ is one of the $k$ vertices found above and $|T|=2 \alpha k^{2}+(20 \alpha+4) k^{7 / 4}$, then $d^{+}(v) \leq|T|-\alpha k^{2}-2 k^{7 / 4}=\alpha k^{2}+k^{7 / 4}+(20 \alpha+1) k^{7 / 4}$. Therefore,

$$
m \geq \alpha k^{2}+k^{7 / 4} \geq d^{+}(v)-(20 \alpha+1) k^{7 / 4}
$$

establishing the last claim of the lemma.
We say that a subset $B$ of vertices is $(\alpha, m, k)$-balanced if it satisfies the two properties guaranteed by Lemma 2.2. Additionally, $T \vec{K}_{k}\left(\ell_{1}, \ell_{2}\right)$ denotes a partial subdivision of $\vec{K}_{k}$ with precisely $\ell_{1}$ paths of length $2, \ell_{2}$ paths of length 3 , and no paths of length greater than 3 . If $U \subset V(T)$ disconnects $T$, then $T \backslash U$ decomposes as $S \cup T^{\prime}$ where $S \cap T^{\prime}=\varnothing, S, T^{\prime} \neq \varnothing$, and $S \rightarrow T^{\prime}$. In this situation, we call $S$ the source component, and $T^{\prime}$ the sink. The following key lemma says that either we can find a suitable $T \vec{K}_{k}\left(\ell_{1}, \ell_{2}\right)$, or there exists a subset $U$ of vertices which disconnects $T$, and such that the source component of the remaining tournament is quite large.

Lemma 2.3. Suppose $k \geq 3$ is an integer, $T$ is a tournament with $\delta^{+}(T) \geq k^{2}+2 k^{7 / 4}$, and suppose $B \subset V(T)$ is an $(\alpha, m, k)$-balanced subset of $k$ vertices for some $\alpha, m>0$. Then one of the following must occur:
(1) There is a copy of $T \vec{K}_{k}\left(\ell_{1}, \ell_{2}\right)$ in $T$ with branch vertex set $B$ such that

$$
4\left(\ell_{1}+\ell_{2}\right)+6 k^{7 / 4}>m
$$

(2) There is a subset $U \subset V(T)$ that disconnects $T$ such that the source component $S$ of $T \backslash U$ satisfies $|S| \geq|U|+k$. Moreover, the $\operatorname{sink} T \backslash(U \cup S)$ has size at least $k$.

Proof. Let $B$ be an $(\alpha, m, k)$-balanced $k$-set of vertices in $T$, and suppose (1) fails in the statement of the lemma. List the edges $e_{1}, \ldots, e_{N}$ of $T^{*}[B]$ where $N:=\binom{k}{2}$, and $T^{*}$ is the tournament obtained from $T$ by reversing $T$ 's edges. Then for any permutation $\sigma:[N] \rightarrow[N]$ there is an index $f=f(\sigma)$ such that the edges $e_{\sigma(1)}, \ldots, e_{\sigma(f-1)}$ can be successfully embedded as paths of length 2 or 3 , but $e_{\sigma(f)}$ cannot, and the resulting copy of $T \vec{K}_{k}\left(\ell_{1}, \ell_{2}\right)$ satisfies

$$
m \geq 4\left(\ell_{1}+\ell_{2}\right)+6 k^{7 / 4}
$$

Pick an ordering $\sigma$ such that the number of paths $\ell_{1}$ of length 2 in the partial subdivision $\mathcal{S}$ with embedded edges $e_{\sigma(1)}, \ldots, e_{\sigma(f-1)}$ is maximized. Without loss of generality, we may assume $\sigma$ is the identity permutation, and let $e_{f}=x y$. Since $e_{f}$ fails to embed we must have that every edge is directed from $N^{-}(y) \backslash V(\mathcal{S})$ to $N^{+}(x) \backslash V(\mathcal{S})$. Similarly, we have that $W=N^{+}(x) \cap N^{-}(y) \subset V(\mathcal{S})$. Let $A$ denote the set of $\ell_{1}$ non-branch vertices that are on paths of length 2 in $\mathcal{S}$. We claim that, in fact, $W \subset A \cup B$. Indeed, if there is $e_{i}=u v, i<f$, and a $u-v$ path $u z w v$ with, say, $z \in W$, then consider the embedding order where we swap $e_{i}$ and $e_{f}$ and embed $e_{f}$ using the 2-path $x z y$. This is a legal embedding of $e_{f}$, as $z$ does not belong to any of the subdivided edges $e_{j}$ with $j \neq i, f$. But now we have an embedding order with more directed paths of length 2 in the partial subdivision, contradicting our choice of $\sigma$.
Now since $B$ is $(\alpha, m, k)$-balanced, all in-degrees differ by at most $k^{7 / 4}$, and so

$$
\begin{aligned}
\left|N^{-}(x) \backslash N^{-}(y)\right| & \leq\left|N^{-}(y) \backslash N^{-}(x)\right|+k^{7 / 4} \\
& =|W|+k^{7 / 4} . \\
& \leq \ell_{1}+k+k^{7 / 4} .
\end{aligned}
$$

Note that the last inequality holds since $W \subset A \cup B$. Now, let $U=V(\mathcal{S}) \cup\left(N^{-}(x) \backslash N^{-}(y)\right)$ and observe that $|U|$ satisfies the upper bound

$$
\begin{aligned}
|U| & \leq 2 \ell_{2}+\ell_{1}+k+\left(\ell_{1}+k+k^{7 / 4}\right) \\
& =2\left(\ell_{1}+\ell_{2}\right)+2 k+k^{7 / 4}
\end{aligned}
$$

Then $T \backslash U$ is disconnected with source component $S=N^{-}(y) \backslash U$ and $\operatorname{sink} N^{+}(x) \backslash V(\mathcal{S})$. By the minimum out-degree condition on $T$, the sink has at least $k^{2}+2 k^{7 / 4}-\left(2\binom{k}{2}+k\right)>k$ vertices. Finally, as $|S| \geq\left|N^{-}(y)\right|-|U|$, and recalling that $\left|N^{-}(y)\right| \geq m-k^{7 / 4}$ and $m \geq 4\left(\ell_{1}+\ell_{2}\right)+6 k^{7 / 4}$, we have

$$
\begin{aligned}
|S| & \geq\left(m-k^{7 / 4}\right)-\left(2\left(\ell_{1}+\ell_{2}\right)+2 k+k^{7 / 4}\right) \\
& \geq 2\left(\ell_{1}+\ell_{2}\right)+4 k^{7 / 4}-2 k \\
& \geq|U|+\left(3 k^{7 / 4}-4 k\right) \\
& \geq|U|+k,
\end{aligned}
$$

where the last inequality follows since $k \geq 3$. This completes the proof of the lemma.
With the above lemma complete, we are ready to prove the main result of this section.
Theorem 2.4. Let $k \geq 2$ be an integer. Any tournament $T$ with $\delta^{+}(T) \geq 2 k^{2}+147 k^{7 / 4}$ contains $a T \vec{K}_{k}$. Moreover, this subdivision has the property that each edge is subdivided at most twice.

The rough idea of the proof is as follows. We shall iteratively apply Lemma 2.3: if at some point we find a partial subdivision with many paths of length 2 or 3 , then we stop. Otherwise, we obtain a cut set $U$ with source component $S$ satisfying $|S| \geq|U|$, and look to apply the lemma again to $T \backslash(U \cup S)$. Eventually we either obtain a partial subdivision $\mathcal{S}$, or reach a subtournament $T^{\prime}$ that is quite small. In the first case, we show how to extend this partial subdivision to a full subdivision. In the latter case, since $T^{\prime}$ is small and the minimum outdegree is large, every vertex in $T^{\prime}$ has many out-neighbours outside of $T^{\prime}$. We use this fact,
together with some structural features of the cut sets and source components, to embed the requisite paths. Let us now make these ideas more precise.

Proof of Theorem 2.4. We may assume that $k \geq 3$ as any tournament with $\delta^{+}(T) \geq 1$ contains a directed cycle (i.e., a subdivision of $\vec{K}_{2}$ ). We shall apply Lemma 2.3 repeatedly to obtain subtournaments $T_{1}:=T, T_{2}, \ldots$, and subsets $U_{0}:=\varnothing, U_{1}, U_{2}, \ldots$, such that for every $i \geq 1$

- $U_{i} \subset T_{i}$.
- $\left|T_{i}\right| \geq k$.
- $T_{i} \subset T_{j}$ if $i>j$.
- $T_{i} \backslash U_{i}$ is disconnected with source component $S_{i}$ satisfying $\left|S_{i}\right| \geq\left|U_{i}\right|$ and such that $\left|T_{i} \backslash\left(U_{i} \cup S_{i}\right)\right| \geq k$.
Indeed, set $T_{1}=T, U_{0}=\varnothing$ and suppose $T_{i}, U_{i-1}$ have already been defined for some $i \geq 1$. We shall show how to obtain $T_{i+1}$ and $U_{i}$ as follows. We claim that either $T_{i}$ contains a subtournament $T_{i}^{\prime}$ which has large minimum out-degree, or we can find $k$ vertices of small minimum out-degree in $T_{i}$. To make this precise, initialize $R=\varnothing$. If there is a vertex $v$ in $T_{i}^{\prime}=T_{i}$ with $d_{T_{i}^{\prime}}^{+}(v)<k^{2}+12 k^{7 / 4}$, then add it to the set $R$. Looking at $T_{i}^{\prime}=T_{i} \backslash\{x\}$, we repeat the same process to $T_{i} \backslash\{x\}$ and so on. Either we obtain $|R|=k$ or $T_{i}^{\prime}=\left(T_{i} \backslash R\right) \neq \emptyset$ with $|R|<k$ (as $\left|T_{i}\right| \geq k$ ) and $\delta^{+}\left(T_{i}^{\prime}\right) \geq k^{2}+12 k^{7 / 4}$. As $\delta^{+}\left(T_{i}^{\prime}\right) \geq k^{2}+12 k^{7 / 4}$ we easily have $\left|T_{i}^{\prime}\right| \geq 2 k^{2}+24 k^{7 / 4}$. Choose a real number $\alpha \geq 1$ such that

$$
\left|T_{i}\right|=2 \alpha k^{2}+(20 \alpha+4) k^{7 / 4},
$$

and apply Lemma 2.2 to $T_{i}$. We obtain an $(\alpha, m, k)$-balanced subset $B_{i} \subset V\left(T_{i}^{\prime}\right)$ of $k$ vertices, for some $m$. Now apply Lemma 2.3 to $T_{i}^{\prime}$ and $B_{i}$. If condition (1) holds from the lemma, then we terminate the procedure at step $i$ and obtain a partial subdivision $\mathcal{S}=T \vec{K}_{k}\left(\ell_{1}, \ell_{2}\right)$ in $T_{i}$ on branch vertex set $B_{i}$ satisfying

$$
4\left(\ell_{1}+\ell_{2}\right)+6 k^{7 / 4}>m .
$$

Otherwise, (2) holds. Let $U_{i}^{\prime}$ be the cut set and $S_{i}^{\prime}$ the source component, and moreover, let $C_{i}=U_{i}^{\prime} \cup R$. Note that by (2) of Lemma 2.3 we have $\left|S_{i}^{\prime}\right| \geq\left|U_{i}^{\prime}\right|+k \geq\left|C_{i}\right|$, and the sink, namely $T_{i} \backslash\left(C_{i} \cup S_{i}^{\prime}\right)$, has size at least $k$.

It follows that we may choose a set $U_{i} \subset T_{i}$ of minimum possible size such that $T_{i} \backslash U_{i}$ is disconnected with source component $S_{i}$ satisfying $\left|S_{i}\right| \geq\left|U_{i}\right|$, and such that the sink $T_{i+1}$ has size at least $k$. We can continue applying the same argument to $T_{i+1}$. Note that eventually this process must terminate. Indeed, for each $i$ we have that $\left|T_{i+1}\right|<\left|T_{i}\right|$ (as $\left|U_{i} \cup S_{i}\right| \geq 1$ ). So we must reach a stage $t$ where for which $T_{t+1}$ either contains a partial subdivision $\mathcal{S}$ as per (1) of Lemma 2.3, or we find $k$ vertices all of which have less than $k^{2}+12 k^{7 / 4}$ out-neighbours in $T_{t+1}$. Thus we have established the following:

Claim 1. The above procedure terminates at some stage $t \geq 1$ with either
(1) a partial subdivision $\mathcal{S}=T \vec{K}_{k}\left(\ell_{1}, \ell_{2}\right) \subset T_{t+1}$ satisfying $4\left(\ell_{1}+\ell_{2}\right)+6 k^{7 / 4}>m$, or
(2) a subset $B \subset T_{t+1}$ of $k$ vertices with $d_{T_{t+1}}^{+}(v)<k^{2}+12 k^{7 / 4}$ for every $v \in B$.

We shall denote by $B$ either the branch vertex set of $\mathcal{S}$ in the first case of Claim 1, or the $k$-set obtained in the second case. This set $B$ will play the role of the branch vertex set of the full subdivision we wish to embed. Let $U=\bigcup_{i=1}^{t} U_{i}$ and $S=\bigcup_{i=1}^{t} S_{i}$. The following claim asserts that for each $i \in[t]$ all subsets of $U_{i}$ send many out-edges to $S_{i}$.

Claim 2. For each $i \in[t]$ the following holds: for every non-empty subset $X \subseteq U_{i}$

$$
\left|N^{+}(X) \cap S_{i}\right| \geq|X| / 2
$$

Proof. Let $X$ be a non-empty subset of $U_{i}$. If $S_{i} \subset N^{+}(X)$, then

$$
\left|N^{+}(X) \cap S_{i}\right|=\left|S_{i}\right| \geq\left|U_{i}\right| \geq|X|
$$

where the first inequality holds by definition of the sets $U_{i}$ and $S_{i}$. So we may assume that $S_{i}$ is not contained in $N^{+}(X)$. If the claim is false, then replace $U_{i}$ by $U_{i}^{\prime}=\left(U_{i} \backslash X\right) \cup\left(N^{+}(X) \cap S_{i}\right)$ and set $S_{i}^{\prime}=S_{i} \backslash N^{+}(X)$. The set $U_{i}^{\prime}$ has size strictly smaller than the size of $U_{i}$ and still disconnects $T_{i}$. Moreover,

$$
\begin{aligned}
\left|S_{i}^{\prime}\right| & =\left|S_{i}\right|-\left|N^{+}(X) \cap S_{i}\right| \\
& \geq\left|U_{i}\right|-|X| / 2 \\
& \geq\left|U_{i}\right|-|X|+\left|N^{+}(X) \cap S_{i}\right|=\left|U_{i}^{\prime}\right|,
\end{aligned}
$$

which contradicts the minimal choice of $U_{i}$. This completes the proof of the claim.
The next lemma asserts that, as long as vertices in $B$ send enough out-neighbours outside of $T_{t+1}$, then we may embed the required internally vertex disjoint directed paths joining prescribed pairs in $B$.

Lemma 2.5. Let $\ell \geq 1$ be an integer and let $\left(x_{1}, y_{1}\right), \ldots,\left(x_{\ell}, y_{\ell}\right)$ be distinct pairs of vertices in $B$ with $x_{i} \neq y_{i}$ for each $i \in[\ell]$. If every vertex in $B$ has at least $2 \ell$ out-neighbours in $T \backslash T_{t+1}$, then there exist pairwise internally disjoint directed paths $P_{i}$ of length 3 joining $x_{i}$ to $y_{i}$ for every $i \in[\ell]$.

Before proving the lemma, we record the following simple consequence of Hall's theorem that we need.

Proposition 2.6. Suppose $G$ is a bipartite graph with vertex sets $U, V$ such that $|N(X)| \geq|X| / 2$ for every $X \subset U$. Then there is a set $M \subset E(G)$ with the property that every vertex in $U$ is incident to exactly one edge in $M$, and every vertex in $V$ is incident to at most two edges of $M$.

Proof. For every $v \in V$ add a new vertex $v^{\prime}$ and join $v^{\prime}$ to all of $v$ 's neighbours; call the resulting graph $G^{\prime}$. Then for every $X \subset U$ we have $\left|N_{G^{\prime}}(X)\right| \geq|X|$, so by Hall's theorem there is a matching of $U$ in $G^{\prime}$. The result follows by identifying vertices in $V$ with their duplicates.
Proof of Lemma 2.5. Combining Claim 2 with Proposition 2.6, it follows that for each $i \in[t]$ there is a partition $U_{i}=U_{i}^{\prime} \cup U_{i}^{\prime \prime}$ such that $U_{i}^{\prime}$ and $U_{i}^{\prime \prime}$ are both matched into $S_{i}$. Let $U^{\prime}=\bigcup_{i=1}^{t} U_{i}^{\prime}$ and $U^{\prime \prime}=\bigcup_{i=1}^{t} U_{i}^{\prime \prime}$ so that $U=U^{\prime} \cup U^{\prime \prime}$, and fix a directed matching $M^{\prime}$ from $U^{\prime}$ to $S$, and a directed matching $M^{\prime \prime}$ from $U^{\prime \prime}$ to $S$. Additionally, for each $i \in[\ell]$ let $N_{i}$ denote the outneighbourhood of $x_{i}$ in $T \backslash T_{t+1}$. Observe that some of these $N_{i}$ 's may repeat (as some of the $x_{i}$ 's may repeat among the $\ell$ pairs). Also, since $S \rightarrow T_{t+1}$,

$$
\left|N_{i}\right|=\left|N^{+}\left(x_{i}\right) \cap U\right| \geq 2 \ell,
$$

for each $i=1, \ldots, \ell$. Let $X^{\prime} \subset[\ell]$ be those indices $i$ for which $\left|N_{i} \cap U^{\prime}\right| \geq \ell$ and $X^{\prime \prime}=[\ell] \backslash X^{\prime}$. Note that by our assumption that each $x_{i}$ has at least $2 \ell$ out-neighbours outside of $T_{t+1}$, it follows that $\left|N_{i} \cap U^{\prime \prime}\right|>\ell$ for each $i \in X^{\prime \prime}$. Now, we may pick $\left|X^{\prime}\right| \leq \ell$ distinct vertices in $U^{\prime}$ such that each vertex is an out-neighbour of one of the $x_{i}$ 's with $i \in X^{\prime}$. Thus we obtain
a collection $\mathcal{P}$ of directed paths of length 3 by using the appropriate matching edge from $M^{\prime}$, and the fact that $S \rightarrow\left\{y_{1}, \ldots, y_{\ell}\right\}$. It remains to find the analogous directed paths joining $\left(x_{i}, y_{i}\right)$ for $i \in X^{\prime \prime}$. Let $A$ denote the set of $\left|X^{\prime}\right|$ vertices in $S$ used in paths in $\mathcal{P}$. Remove from $U^{\prime \prime}$ every vertex which is matched by $M^{\prime \prime}$ to a vertex of $A$; obviously we remove at most $\left|X^{\prime}\right|$ vertices, so each $x_{i}$ with $i \in X^{\prime \prime}$ has more than $\ell-\left|X^{\prime}\right|=\left|X^{\prime \prime}\right|$ suitable out-neighbours left in $U^{\prime \prime}$. Therefore we can pick $\left|X^{\prime \prime}\right|$ distinct vertices in $U^{\prime \prime}$ with the property that each such vertex is an out-neighbour of one of these $x_{i}$ 's, and use the appropriate matching edges from $M^{\prime \prime}$ (avoiding $A)$ as before. Hence, we have found paths joining all pairs, completing the proof.

Now we are ready to complete the proof of Theorem 2.4. Suppose first that we are in Case (2) of Claim 1; that is, $d_{T_{t+1}}^{+}(v)<k^{2}+12 k^{7 / 4}$ for every $v \in B$. Then by the minimum out-degree condition on $T$, we have that each vertex in $B$ has more than $k^{2}$ out-neighbours outside of $T_{t+1}$, and as $k^{2}>2\binom{k}{2}$, Lemma 2.5 implies that we can embed all the required $\binom{k}{2}$ paths.

So we may assume that we are in Case (1). Then we have a partial subdivision $\mathcal{S}=$ $T \vec{K}_{k}\left(\ell_{1}, \ell_{2}\right)$ on branch vertex set $B$, where $B$ is $(\alpha, m, k)$-balanced for some $\alpha \geq 1$. As $4\left(\ell_{1}+\ell_{2}\right)+6 k^{7 / 4}>m$, one has

$$
\alpha k^{2}+k^{7 / 4} \leq m<4\binom{k}{2}+6 k^{7 / 4}
$$

so crudely we have $\alpha \leq 7$. As we need to embed $\binom{k}{2}-\ell_{1}-\ell_{2}$ more paths, in view of Lemma 2.5 and the minimum out-degree condition $\delta^{+}(T) \geq 2 k^{2}+147 k^{7 / 4}$, we are done provided

$$
2 k^{2}+147 k^{7 / 4}-d_{T_{t+1}}^{+}(v) \geq 2\left(\binom{k}{2}-\ell_{1}-\ell_{2}\right)
$$

holds for every $v \in B$. But this is true since by Lemma 2.2 we have $d_{T_{t+1}}^{+}(v) \leq m+(20 \alpha+$ 1) $k^{7 / 4} \leq m+141 k^{7 / 4}$ for every $v \in B$, and so

$$
\begin{aligned}
2\binom{k}{2}-2\left(\ell_{1}+\ell_{2}\right)+\left(m+141 k^{7 / 4}\right) & <k^{2}+2\left(\ell_{1}+\ell_{2}\right)+147 k^{7 / 4} \\
& \leq 2 k^{2}+147 k^{7 / 4}
\end{aligned}
$$

where the first inequality follows using the bound $m<4\left(\ell_{1}+\ell_{2}\right)+6 k^{7 / 4}$, and the last inequality holds since always $\ell_{1}+\ell_{2} \leq\binom{ k}{2}$. Thus we may embed all remaining paths yielding a $T \vec{K}_{k}$ in $T$.

Observe that our proof shows that we can embed a $T \vec{K}_{k}$ where each path in the subdivision has length at most 3 . We remark that this is best possible in the sense that there exist tournaments with large minimum out-degree which cannot contain copies of $T \vec{K}_{k}$ where each path has length at most 2. For example, it is routine to check that a blow-up of a cyclic triangle where each class is a copy of the transitive tournament on $10 k^{2}$ vertices has this property.

## 3. Subdivisions of transitive tournaments

Our aim in this section is to prove Theorem 1.2 and Theorem 1.3. We begin with a lemma similar in spirit to Lemma 2.2, but which is tailored to our specific needs in this section. To state it, we say that a subset $B \subset V(T)$ is $C$-nearly-regular if either $d^{-}(v) \leq d^{+}(v) \leq C d^{-}(v)$ for every $v \in B$, or $d^{+}(v) \leq d^{-}(v) \leq C d^{+}(v)$ for every $v \in B$. Further, $B$ is $(C, m, k)$-nearlyregular if it is $C$-nearly-regular and additionally $d^{-}(v) \in[m-10 k, m+10 k]$ for every $v \in B$. The following lemma allows us to find ( $4, m, k$ )-nearly-regular $k$-element subsets in tournaments.

Lemma 3.1. Any tournament $T$ contains a 4-nearly-regular subset of size $|T| / 10$, and a $(4, m, k)$-nearly-regular subset of size $k$, for some $m$.

Proof. We first claim that $T$ contains a 4-nearly-regular subset of size at least $|T| / 10$. Indeed, let $|T|=n$ and let $R \subset V(T)$ be the vertices for which either the ratio between the outneighborhood and in-neighbourhood (or vice-versa) is between 1 and 4 . If $|R| \geq n / 5$, then we are done, as we may pass to a subset $A \subset R$ of at least half the size for which the property is satisfied for one or the other. If not, then let $T^{\prime}=T \backslash R$, so that $\left|T^{\prime}\right| \geq 4 n / 5$. Let $T_{1}^{\prime}$ be the set of vertices $v \in V\left(T^{\prime}\right)$ for which $d_{T}^{+}(v)>4 d_{T}^{-}(v)$ and $T_{2}^{\prime}$ be those vertices $v \in V\left(T^{\prime}\right)$ for which $d_{T}^{-}(v)>4 d_{T}^{+}(v)$. Suppose without loss of generality that $\left|T_{1}^{\prime}\right| \geq\left|T_{2}^{\prime}\right|$, so that $\left|T_{1}^{\prime}\right| \geq 2 n / 5$. This implies that there is a vertex $u$ in $T_{1}^{\prime}$ which has in-degree inside $T_{1}^{\prime}$ at least $n / 5$. But then

$$
n / 5 \leq d_{T}^{-}(u)<\frac{1}{4} d_{T}^{+}(u) \leq n / 5
$$

a contradiction.
Thus, we can always find a 4-nearly-regular subset $A$ of size at least $|T| / 10$. As in the proof of Lemma 2.2, partition the interval $[1, \ldots,|T|]$ into consecutive intervals of size $10 k$, and distribute the vertices of $A$ according to their in-degrees in $T$. By the pigeonhole principle, there must exist at least

$$
10 k \cdot \frac{|A|}{|T|} \geq 10 k \cdot \frac{1}{10}=k
$$

vertices in the same interval. These $k$ vertices form a (4, $m, k$ )-nearly-regular subset for some $m$.

We are now in a position to prove Theorem 1.2 , which we restate here for convenience. The proof is not very different from that of Lemma 2.3: either we can find what we are looking for, or we can pass to a 'nice' subtournament which allows us to embed the required subdivision by induction.

Theorem 1.2. There is a constant $C>0$ such that the following holds. For all $k \geq 2$, any tournament on at least $C k^{2}$ vertices contains a $T T_{k}$, where each directed path in the subdivision has length at most 3. Moreover, this is tight up to the multiplicative constant.

Proof. First, observe that with high probability a uniformly random tournament $T$ on $k^{2} / 10$ vertices does not induce a set of size $k$ whose distance to a transitive tournament is smaller than $k^{2} / 6$. This implies $T$ can not contain a $T T_{k}$ since any such subdivision must span at least $k^{2} / 6$ vertices. Let $C=150$. We shall apply induction on $k$. For $k=2$, the statement holds trivially since any tournament with at least 2 vertices contains a subdivision of a transitive tournament on 2 vertices. Suppose we want to prove the statement for $k$. Let $T$ be a tournament on $C k^{2}$ vertices. Applying Lemma 3.1 to $T$, we obtain a $(4, m, k)$-nearly-regular set $S \subset T$ consisting of $k$ vertices. Without loss of generality, assume that $d^{-}(v) \leq d^{+}(v) \leq 4 d^{-}(v)$ holds for every $v \in S$. We shall iteratively try to embed a subdivision on these $k$ branch vertices. Observe that we may always choose an ordering $\sigma$ of $S$ for which we just need to embed $\binom{k}{2} / 2$ extra paths to find a transitive subdivision. Suppose we are at step $i<\binom{k}{2}$ and we have already found $i$ paths of the subdivision; we may assume $i$ is maximal. Let $P \subset T$ be the set consisting of the inner vertices of the paths already found. Note that $|P| \leq 2 \cdot k^{2} / 4$, since each path we have embedded has at most 2 inner vertices. Suppose now we want to find a directed path from $x$ to $y$ (where $x$ lies before $y$ in the ordering $\sigma$ of $S$ ). By the maximality of $i$, we must
have $N^{+}(x) \cap N^{-}(y) \subset P \cup S$, so $\left|N^{+}(x) \cap N^{-}(y)\right| \leq k^{2} / 2+k \leq k^{2}$. Furthermore, since $S$ is (4, $m, k)$-nearly-regular,

$$
\begin{aligned}
\left|N^{-}(x) \cap N^{+}(y)\right| & \leq\left|N^{+}(x) \cap N^{-}(y)\right|+10 k \\
& \leq k^{2}+10 k \\
& \leq 4 k^{2} .
\end{aligned}
$$

Delete the set $\left(N^{+}(x) \cap N^{-}(y)\right) \cup\left(N^{-}(x) \cap N^{+}(y)\right)$ from $T$ and denote by $T^{\prime}$ the remaining tournament. Then $T^{\prime}$ splits into two disjoint sets $A, B$ where $A$ is the common out-neighborhood of $x, y$, and $B$ is the common in-neighborhood of $x, y$. We claim that the partition $V\left(T^{\prime}\right)=A \cup B$ satisfies the following two properties:
(1) $|A|+|B| \geq(1-1 / 30)|T|$.
(2) $\min \{|A|,|B|\} \geq|T| / 6$.

To see the the first property, simply observe that we have removed at most $5 k^{2}$ vertices to obtain $T^{\prime}$, and therefore $|A|+|B| \geq(1-1 / 30) 150 k^{2}=(1-1 / 30)|T|$. To see the second property, since $S$ is $(4, m, k)$-nearly-regular, for every $v \in S$

$$
d^{-}(v) \geq d^{+}(v) / 4=\left(|T|-d^{-}(v)\right) / 4,
$$

implying $d^{-}(v) \geq|T| / 5$. Then also $d^{+}(v) \geq d^{-}(v) \geq|T| / 5$. It follows that $\min \{|A|,|B|\} \geq$ $|T| / 5-5 k^{2} \geq|T| / 6$ (using our choice of $C$ ), as claimed

Without loss of generality suppose that $|A| \leq|B|$. If there is a directed edge from $A$ to $B$, we may find a directed path from $x$ to $y$ of length 3 , which contradicts the maximality of i. Accordingly, $B \rightarrow A$. Since $|A| \geq|T| / 6=C k^{2} / 6 \geq C(2 k / 5)^{2}$, the induction hypothesis guarantees a subdivision of a transitive tournament on $2 k / 5$ vertices in $T[A]$, where each path has length at most 3 . Similarly, $T[B]$ contains a subdivision of a transitive tournament on $3 k / 5$ vertices, because $|B| \geq(1 / 2-1 / 60) C k^{2} \geq C(3 k / 5)^{2}$. As $B \rightarrow A$, these two subdivisions may be put together to form a subdivision of a transitive tournament on $k$ vertices where each path has length at most 3 .

We close this section by proving Theorem 1.3 (recall that a 1-subdivision of $T_{k}$ is a subdivision of the transitive tournament of order $k$ where each directed path has length 2 ).

Theorem 1.3. Any tournament on at least $C k^{2} \log ^{3} k$ vertices contains a 1 -subdivision of $T_{k}$.
Before proving this theorem we need a lemma. Given a graph $G$ and a vertex $x \in V(G)$ we denote by $B_{r}(x)=\left\{v \in V(G): d_{G}(x, v) \leq r\right\}$ the ball of radius $r$ in $G$ around $x$. The following states that if $G$ has the property that every ball of radius $C \log ^{2} n$ is small, then $G$ can be disconnected by $o(n)$ vertices into many small components. Recall that $\log n$ denotes the logarithm of $n$ to the base $e$.

Lemma 3.2. Let $G$ be an n-vertex graph with the property that for any $r \leq 10 \log ^{2} n$ and any vertex $x$ we have $\left|B_{r}(x)\right| \leq \frac{n}{5 \log n}$. Then $G$ contains a set $S \subset V(G)$ of size at most $\frac{n}{5 \log n}$ such that $G-S$ is the union of connected components each of which has size at most $\frac{n}{5 \log n}$.

Proof. Pick a vertex $x \in V(G)$ and perform a breadth-first-search from $x$, obtaining levels $L_{0}, L_{1}, \ldots, L_{n}$ such that $L_{i}=\{v \in V(G): d(x, v)=i\}$. Denote by $k=10 \log ^{2} n$. We claim that there is a $k^{\prime}<10 \log ^{2} n$ for which $\left|L_{k^{\prime}}\right|<\left|B_{k^{\prime}-1}(x)\right| /(5 \log n)$. If not, for any $k^{\prime}<10 \log ^{2} n$, we
have

$$
\left|B_{k^{\prime}}(x)\right| \geq\left(1+\frac{1}{5 \log n}\right)\left|B_{k^{\prime}-1}(x)\right|
$$

and hence by induction $\left|B_{k}(x)\right| \geq\left(1+\frac{1}{5 \log n}\right)^{k}$. Using the inequality $1+x \geq e^{\frac{x}{x+1}}$ (valid for any $x>-1$ ) with $x=1+1 / 5 \log n$ we obtain

$$
1+\frac{1}{5 \log n} \geq e^{\frac{1}{1+5 \log n}} \geq e^{\frac{1}{10 \log n}}
$$

Hence, since $k=10 \log ^{2} n$, this yields $\left|B_{k}(x)\right|>\left(e^{\frac{1}{10 \log n}}\right)^{10 \log ^{2} n}>n$, which is clearly a contradiction.

So we may choose $k^{\prime}<k$ such that $\left|L_{k^{\prime}}\right|<\frac{\left|B_{k^{\prime}-1}(x)\right|}{5 \log n}$; remove $L_{k}$ from $G$. Pick a vertex from each connected component of size larger than $\frac{n}{5 \log n}$, and perform the same procedure as above. Eventually this process must terminate, and the components we are left with are all balls of radius less than $10 \log ^{2} n$, so by assumption have at most $\frac{n}{5 \log n}$ vertices. Moreover, the union of the sets removed $S$ has size $|S| \leq \frac{n}{5 \log n}$ by construction, completing the proof of the lemma.

We are now in a position to prove Theorem 1.3. The proof goes roughly as follows. We define an auxiliary graph on $V(T)$ where $x \sim y$ if and only if the symmetric difference of their outneighbourhoods has size less than $2 k^{2}$ (it is helpful to think of this as being 'bad' for embedding 1-subdivisions since, roughly speaking, this implies that $\left|N^{+}(x) \backslash N^{+}(y)\right|=\left|N^{+}(x) \cap N^{-}(y)\right|$ is 'small'). It turns out that $G$ satisfies the properties required to apply Lemma 3.2. So $G$ splits into many small components, and therefore every pair of vertices $x, y$ in different components have $\left|N^{+}(x) \Delta N^{+}(y)\right| \geq 2 k^{2}$. Moreover, if $d^{+}(x) \geq d^{+}(y)$, then it is not hard to show that actually $\left|N^{+}(x) \cap N^{-}(y)\right| \geq\binom{ k}{2}+k$. So order the vertices of the components according to nonincreasing out-degree. Finally, we show that enough vertices from the components intersect the first half and second half of the order, enough that we may apply induction to embed a 1-subdivision of $T_{k / 2}$ in each half. Then we can greedily embed the remaining directed paths of length 2 between these partial 1-subdivisions.

Proof of Theorem 1.3. The proof will be by induction on $k$ with $C=10^{7}$. For $k \leq 3$ the theorem follows: $T$ contains a transitive tournament on at least $\log _{2}|T|>6$ vertices, which contains a 1-subdivision of $T_{3}$. So let $k>3$ and let $T$ be a tournament with $|T|:=n=$ $C k^{2} \log ^{3} k$ vertices. Construct an auxiliary graph $G$ on $V(T)$ in the following way: join $x$ to $y$ if $\left|N^{+}(x) \Delta N^{+}(y)\right|<2 k^{2}$. Now, apply Lemma 3.2 to $G$. To see that $G$ satisfies the property needed for the lemma, suppose there is a vertex $x$ which sees at least $\frac{n}{5 \log n}$ vertices in the ball $B_{r}(x)$ of radius $r=10 \log ^{2} n$. It is not hard to check that $\log n \leq 20 \log k$. Now, there is a path in $G$ of length at most $20 \log ^{2} n \leq 8000 \log ^{2} k$ between every pair of vertices in $B$. It follows that every such pair has the property that the symmetric difference between their out-neighborhoods is at most $16000 k^{2} \log ^{2} k$, by the definition of $G$. But this is impossible because $B_{r}(x)$ has order at least $n / 5 \log n \geq 10^{5} k^{2} \log ^{2} k$. By looking at the tournament $T^{\prime}=T\left[B_{r}(x)\right]$, we observe that it contains a vertex $y$ whose out-neighborhood $N_{T^{\prime}}^{+}(y)$ has size at least $\left(10^{5} k^{2} \log ^{2} k\right) / 2=$ $50000 k^{2} \log ^{2} k$, and by the same reasoning, inside the tournament induced on $N_{T^{\prime}}^{+}(y)$ there must exist a vertex $z$ whose in-neighborhood has size at least $25000 k^{2} \log ^{2} k$. Accordingly, in $T$ we have

$$
\left|N^{+}(y) \Delta N^{+}(z)\right| \geq 25000 k^{2} \log ^{2} k
$$

But this contradicts the fact that we must have $\left|N^{+}(y) \Delta N^{+}(z)\right| \leq 16000 k^{2} \log ^{2} k$, as we established earlier.

So we may apply Lemma 3.2 to $G$. This yields a set $S$ of vertices such that $|S| \leq \frac{n}{5 \log n}$ and $G^{\prime}=G-S$ consists of connected components $C_{1}, C_{2}, \ldots, C_{t}$. We claim that for any two vertices belonging to different components, there are many directed paths of length two between them.

Claim 3. Let $u, v$ be any two vertices belonging to different components such that $d^{+}(u) \geq d^{+}(v)$. Then $\left|N^{+}(u) \cap N^{-}(v)\right| \geq k^{2}-1 \geq\binom{ k}{2}+k$.

Proof. Observe first that as $u, v$ belong to different components, they satisfy

$$
\left|N^{+}(u) \cap N^{-}(v)\right|+\left|N^{+}(v) \cap N^{-}(u)\right|=\left|N^{+}(u) \Delta N^{+}(v)\right|-1 \geq 2 k^{2}-1 .
$$

Moreover, since $d^{+}(u) \geq d^{+}(v)$ we have that $\left|N^{+}(u) \cap N^{-}(v)\right| \geq\left|N^{+}(v) \cap N^{-}(u)\right|-1$. Therefore $\left|N^{+}(u) \cap N^{-}(v)\right| \geq\left(2 k^{2}-2\right) / 2=k^{2}-1$, as claimed.

Let $\sigma$ be an ordering of the vertices in $V\left(G^{\prime}\right)$ so that their out-degrees are non-increasing. Furthermore, let $m=\left|V\left(G^{\prime}\right)\right|=|V(G-S)| \geq\left(1-\frac{1}{5 \log n}\right) n$. We assume that $m=\left(1-\frac{1}{5 \log n}\right) n$ by possibly removing some vertices from $G^{\prime}$. Let $A_{1}$ denote the initial segment (according to $\sigma$ ) of $V\left(G^{\prime}\right)$ with $\lfloor m / 2\rfloor$ vertices, and $A_{2}$ the remaining $\lceil m / 2\rceil$ vertices. The following lemma allows us to partition the components $C_{1}, \ldots, C_{t}$ into two families $\mathcal{X}$ and $\mathcal{Y}$ such that the components in $\mathcal{X}$ intersect $A_{1}$ in a set $X$ of 'many' vertices, and the components in $\mathcal{Y}$ intersect $A_{2}$ in a set $Y$ of 'many' vertices. By Claim 3 and the definition of the ordering $\sigma$, we guarantee that there are many directed paths of length two between each pair $x, y$ with $x \in X$ and $y \in Y$. Thus if $X$ and $Y$ are large enough, we may apply induction to $T[X]$ and $T[Y]$, and then embed the remaining paths in-between greedily. To spell out the details more carefully:

Lemma 3.3. There exists a partition $\mathcal{X} \cup \mathcal{Y}=\left\{C_{1}, \ldots, C_{t}\right\}$ of the components such that the following holds. If $X=\bigcup \mathcal{X}$ and $Y=\bigcup \mathcal{Y}$, then

$$
\left|X \cap A_{1}\right| \geq\left(1-\frac{1}{2 \log n}\right) m / 4 \text { and }\left|Y \cap A_{2}\right| \geq\left(1-\frac{1}{2 \log n}\right) m / 4 .
$$

Proof. Let $C_{i}^{1}=C_{i} \cap A_{1}$ and similarly $C_{i}^{2}=C_{i} \cap A_{2}$. Denote by $C_{1}, C_{2}, \ldots, C_{t^{\prime}}$ the set of connected components for which $\left|C_{i}^{1}\right| \geq\left|C_{i}\right| / 2$; this implies that $\left|C_{t^{\prime}+\ell}^{2}\right| \geq\left|C_{t^{\prime}+\ell}\right| / 2$, for every $\ell \in\left[t-t^{\prime}\right]$. If $\left|\cup_{i=1}^{t^{\prime}} C_{i}^{1}\right| \geq m / 4$ and also $\left|\cup_{i=t^{\prime}}^{t} C_{i}^{2}\right| \geq m / 4$, then we may take the partition $\mathcal{X}=\left\{C_{i}: i \in\left[t^{\prime}\right]\right\}$ and $\mathcal{Y}=\left\{C_{i}: i \in[t] \backslash\left[t^{\prime}\right]\right\}$. So assume that $\left|\cup_{i=1}^{t^{\prime}} C_{i}^{1}\right|<m / 4$. Choose a set $B \subset\left\{t^{\prime}+1, \ldots, t\right\}$ (perhaps empty) as large as possible for which

$$
L=\left|\bigcup_{j \in\left[t^{\prime}\right] \cup B} C_{j}^{1}\right| \leq m / 4 .
$$

Recall that $\left|C_{i}\right| \leq n / 5 \log n$ for each $i \in[t]$. Thus if $j \in[t] \backslash\left(B \cup\left[t^{\prime}\right]\right)$, then the maximality of $B$ implies that

$$
m / 4<L+\left|C_{j}^{1}\right| \leq L+\left|C_{j}\right| / 2 \leq L+\frac{n}{10 \log n},
$$

and using $m=\left(1-\frac{1}{5 \log n}\right) n$, we have

$$
\begin{aligned}
L \geq m / 4-\frac{n}{10 \log n} & =\left(1-\frac{3}{5 \log n}\right) n / 4 \\
& \geq\left(1-\frac{1}{2 \log n}\right)\left(1-\frac{1}{5 \log n}\right) n / 4
\end{aligned}
$$

$$
=\left(1-\frac{1}{2 \log n}\right) m / 4
$$

Moreover, by assumption we must have that $\left|\cup_{j \in\left\{t^{\prime}+1, \ldots, t\right\} \backslash B} C_{j}^{2}\right| \geq m / 4>\left(1-\frac{1}{2 \log n}\right) m / 4$. Hence the partition $\mathcal{X}=\left\{C_{i}: i \in\left[t^{\prime}\right] \cup B\right\}$ and $\mathcal{Y}=\left\{C_{j}: j \in\left\{t^{\prime}+1, \ldots, t\right\} \backslash B\right\}$ satisfies the conclusion of the lemma.

Thus Lemma 3.3 furnishes $\mathcal{X}$ and $\mathcal{Y}$ with the stated properties. Consider $T^{\prime}=T\left[X \cap A_{1}\right]$ and $T^{\prime \prime}=T\left[Y \cap A_{2}\right]$. The following claim asserts that these subtournaments are large enough to apply induction. The proof is routine calculation. In the following, recall that $C=10^{7}$.

Claim 4. Both $T^{\prime}$ and $T^{\prime \prime}$ have size at least $C\left(\frac{k}{2}\right)^{2} \log ^{3}\left(\frac{k}{2}\right)$.
Proof. We prove this for $T^{\prime}$; the proof for $T^{\prime \prime}$ is identical. By Lemma 3.3 we have that

$$
\left|T^{\prime}\right| \geq\left(1-\frac{1}{2 \log n}\right) m / 4 \geq\left(1-\frac{1}{2 \log n}\right)\left(1-\frac{1}{5 \log n}\right)\left(C k^{2} / 4\right) \log ^{3} k
$$

so in order to show that $\left|T^{\prime}\right| \geq C(k / 2)^{2} \log ^{3}(k / 2)$ we must prove that

$$
\begin{equation*}
\left(1-\frac{1}{2 \log n}\right)\left(1-\frac{1}{5 \log n}\right) \log ^{3} k \geq \log ^{3}\left(\frac{k}{2}\right) . \tag{3.1}
\end{equation*}
$$

Expanding the left-hand side, one sees that the left-hand side of (3.1) is at least

$$
\left(1-\frac{7}{10 \log n}\right) \log ^{3} k \geq\left(1-\frac{7}{20 \log k}\right) \log ^{3} k
$$

where we have used the inequality $\log n \geq \log \left(k^{2}\right)=2 \log k$. Now it is not difficult to check that $(1-7 /(20 \log k)) \log ^{3} k \geq \log ^{3}(k / 2)=(\log k-\log 2)^{3}$. Indeed, after taking cube roots and cancelling the $\log k$ factor, this is equivalent to showing that

$$
\left(1-\frac{7}{20 \log k}\right)^{1 / 3} \geq 1-\frac{\log 2}{\log k}
$$

which is easily seen to be true since $\log 2>7 / 20$. Accordingly, (3.1) holds.
So by the previous claim, $T^{\prime}$ and $T^{\prime \prime}$ are large enough to find a 1-subdivision of $T_{k / 2}$ in each of them, by induction; denote the branch vertex set of each of these by $B^{\prime}, B^{\prime \prime}$, respectively. As $B^{\prime}$ lies entirely before $B^{\prime \prime}$ in the ordering $\sigma$, for every $x \in B^{\prime}, y \in B^{\prime \prime}$ we have that $d^{+}(x) \geq d^{-}(y)$. Hence it follows from Claim 3 that $\left|N^{+}(x) \cap N^{-}(y)\right| \geq\binom{ k}{2}+k$ and therefore we may greedily embed directed paths of length 2 between every such pair. Thus we have found a 1 -subdivision of $T_{k}$ in $T$.

## 4. Concluding remarks and open problems

Observe that our methods for embedding subdivisions of complete digraphs straightforwardly generalize to obtain the following result concerning embedding subdivisions of general digraphs in tournaments with large minimum out-degree.

Theorem 4.1. There exists an absolute constant $C>0$ such that the following holds. Let $D$ be a digraph with $m$ edges and no isolated vertices, and suppose $T$ is a tournament with $\delta^{+}(T) \geq C m$. Then $T$ contains a subdivision of $D$. Moreover, each edge is subdivided at most two times.

Recall that $d(k)$ is the minimum $m$ such that any tournament $T$ with $\delta^{+}(T) \geq m$ contains a subdivision of $\vec{K}_{k}$. We have determined $d(k)$ up to a factor of 8 , and it is natural to ask whether or not the trivial lower bound is the correct answer.

Question 4.2. Is it true that $d(k)=\left(\frac{1}{4}+o(1)\right) k^{2}$ ?
Earlier, we mentioned that Alon, Krivelevich and Sudakov [1] proved that any graph on $n$ vertices and with at least $\varepsilon n^{2}$ edges contains a 1 -subdivision of a complete graph on $c(\varepsilon) \sqrt{n}$ vertices. We conjecture that the following analogue for tournaments is true (recall that $T_{k}$ denotes the transitive tournament on $k$ vertices).

Conjecture 4.3. There is a constant $C>0$ such that any tournament with at least $C k^{2}$ vertices contains a 1-subdivision of $T_{k}$.

Our Theorems 1.2 and 1.3 provide some evidence for this conjecture. Yet, it seems new ideas are needed to resolve the conjecture in full.

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    The first author wishes to acknowledge support by the EPSRC, grant. no. EP/N019504/1.

