

Asymptotic Density of Graphs Excluding Disconnected Minors

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March 12, 2019

Abstract

For a graph H , let

$$c_\infty(H) = \lim_{n \rightarrow \infty} \max \frac{|E(G)|}{n},$$

where the maximum is taken over all graphs G on n vertices not containing H as a minor. Thus $c_\infty(H)$ is the asymptotic maximum density of graphs not containing H as a minor. Employing a structural lemma due to Eppstein, we prove new upper bounds on $c_\infty(H)$ for disconnected graphs H . In particular, we determine $c_\infty(H)$ whenever H is union of cycles. Finally, we investigate the behaviour of $c_\infty(sK_r)$ for fixed r , where sK_r denotes the union of s disjoint copies of the complete graph on r vertices. Improving on a result of Thomason, we show that

$$c_\infty(sK_r) = s(r-1) - 1 \text{ for } s = \Omega\left(\frac{\log r}{\log \log r}\right),$$

and

$$c_\infty(sK_r) > s(r-1) - 1 \text{ for } s = o\left(\frac{\log r}{\log \log r}\right).$$

1 Introduction

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. A well-studied extremal question in graph minor theory is determining the maximum density of graphs G not containing H as a minor. We denote by $v(G)$ and

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$e(G)$ the number of vertices and edges of a graph G , respectively, and by $d(G) = e(G)/v(G)$ the *density* of a non-null graph G . Following Myers and Thomason [MT05] for a graph H with $v(H) \geq 2$ we define the *extremal function* $c(H)$ of H as the supremum of $d(G)$ taken over all non-null graphs G not containing H as a minor. The asymptotic behaviour of $c(K_r)$, where K_r denotes the complete graph on r vertices, was studied in [Kos82, Kos84, Tho84], and was determined precisely by Thomason [Tho01], who has shown that

$$c(K_r) = (\lambda + o_r(1))r\sqrt{\log r}, \quad (1)$$

where

$$\lambda = \max_{\alpha > 0} \frac{1 - e^{-\alpha}}{2\sqrt{\alpha}} = 0.319\dots,$$

is an explicit constant, which we will refer to as *Thomason's constant*.

In [Tho08] Thomason defined an asymptotic variant of the extremal function as

$$c_\infty(H) = \lim_{n \rightarrow \infty} \max_{v(G)=n} d(G)$$

where the maximum is taken over all graphs G on n vertices not containing H as a minor. We refer to $c_\infty(H)$ as the *asymptotic extremal function of H* . Clearly, $c_\infty(H) \leq c(H)$. When H is connected then, as observed in [Tho08], $c(H) = c_\infty(H)$, because in this case one can replace an H -minor free graph G by a disjoint union of many copies of G to obtain arbitrarily large H -minor free graphs with the same density as G . For disconnected graphs H the parameters $c_\infty(H)$ and $c(H)$ frequently differ.

Let lH denote the union of l disjoint copies of a graph H . The following theorem is the main result of [Tho08].

Theorem 1.1 (Thomason [Tho08]).

- a) $c_\infty(lK_r) = (1 + o_r(1))c(K_r)$ for fixed l ,
- b) $c_\infty(lK_r) = l(r - 1) - 1$ for $l \geq 20c(K_r)$

Powerful structural tools of graph minor theory become available when one considers large graphs in minor-closed graph classes, and, in particular, when one investigates $c_\infty(H)$ rather than $c(H)$. The main goal of this paper is to use one such tool, a lemma proved by Eppstein [Epp10], to derive several new bounds on the asymptotic density of graphs excluding disconnected minors. In particular, we improve bounds in Theorem 1.1.

Let us first present a natural lower bound on $c_\infty(H)$. Let $\tau(H)$ denote the *vertex cover number* of the graph H , that is the minimum size of the set $X \subseteq V(H)$ such that $H - X$ is edgeless. Let $\bar{K}_{s,t}$ denote the graph obtained from the disjoint union of a complete graph K_s and an edgeless graph E_t on t vertices by making every vertex of K_s adjacent to every vertex of E_t . Then $\tau(\bar{K}_{s,t}) = s$ for $t \geq 1$, and $\lim_{t \rightarrow \infty} d(\bar{K}_{s,t}) = s$. As the vertex cover of any minor of a graph G does not exceed $\tau(G)$, it follows that H is not a minor of the graph $\bar{K}_{s,t}$ for any $s < \tau(H)$ and any t . Thus

$$c_\infty(H) \geq \tau(H) - 1 \tag{2}$$

for every graph H . We say that a graph H is *well-behaved* if (2) holds with equality. Dirac [Dir64], Mader [Mad68], Jørgensen [Jør94], and Song and Thomas [ST06] proved that $c(K_r) = r - 2$ for $r \leq 5$, $r \leq 7$, $r = 8$ and $r = 9$, respectively. Thus K_r is well-behaved for $r \leq 9$, however (1) implies that K_r is far from being well-behaved for large r . On the other hand, Theorem 1.1 b) implies that lK_r is well-behaved for fixed r and large l . The results of this paper imply that many classes of disconnected graphs are well-behaved, or are close to being well-behaved.

Our first result provides a general upper bound on $c_\infty(H)$ for a disconnected graph H in terms of the asymptotic extremal function and the vertex cover number of its components.

Theorem 1.2. *Let H be the disjoint union of non-null graphs H_1 and H_2 , then*

$$c_\infty(H) \leq \max\{c_\infty(H_2), c_\infty(H_1) + \tau(H_2)\}. \tag{3}$$

In particular,

$$c_\infty(H) \leq c_\infty(H_1) + c_\infty(H_2) + 1. \quad (4)$$

Note that $c_\infty(H) + l - 1 \leq c_\infty(lH)$ for any positive integer l and non-null graph H . Theorem 1.2 together with this observation immediately imply the following corollary, which establishes in a strong form Theorem 1.1 a), and provides upper and lower bounds for $c_\infty(lK_r)$ in terms of $c(K_r)$ which differ at most by a multiplicative factor of two.

Corollary 1.3. *For all positive integers l and r we have*

$$\max\{c(K_r) + l - 1, l(r - 1) - 1\} \leq c_\infty(lK_r) \leq c(K_r) + (l - 1)(r - 1).$$

Theorem 1.2 also implies that if H_1 is a well-behaved graph, and a graph H_2 satisfies $c_\infty(H_2) \leq c_\infty(H_1) + \tau(H_2)$, then the disjoint union of H_1 and H_2 is well-behaved. Thus the disjoint union of cliques of size nine or less is well-behaved.

The inequality (3) does not necessarily hold with c_∞ replaced by c . However, it was conjectured by the third author that (4) still holds. A weaker form of this conjecture has been verified by Cs3ka et al. [CLN⁺17] who have shown the following.

Theorem 1.4 (Cs3ka et al. [CLN⁺17]). *Let H be a disjoint union of 2-connected graphs H_1, H_2, \dots, H_k . Then*

$$c(H) \leq c(H_1) + c(H_2) + \dots + c(H_k) + k - 1.$$

The proof of Theorem 1.4 relies on extremal graph theory techniques, in particular, on a lemma about partitioning graphs into parts with prescribed average degree, the proof of which requires extensive calculations. In contrast, the proof of Theorem 1.2 is very short, modulo the aforementioned lemma by Eppstein.

In [CLN⁺17] Theorem 1.4 is used to prove the following upper bound on the extremal function of the union of cycles, verifying conjectures of Reed and Wood [RW15], and Harvey and Wood [HW15].

Theorem 1.5. *Let H be a disjoint union of cycles. Then*

$$c(H) \leq \frac{v(H) + \text{comp}(H)}{2} - 1. \quad (5)$$

In the case when H is the union of odd cycles, the right side of (5) is equal to $\tau(H) - 1$ and thus the union of odd cycles is well-behaved. In most of the remaining cases for a union of cycles H , the exact value of $c(H)$ remains undetermined, but our next result completely determines $c_\infty(H)$.

Theorem 1.6. *Let H be a 2-regular graph with $\text{odd}(H)$ odd components. Then*

$$c_\infty(H) = \frac{v(H) + \text{odd}(H)}{2} - 1,$$

unless $H = C_{2l}$, in which case $c_\infty(H) = l - \frac{1}{2}$, or $H = kC_4$, in which case $c_\infty(H) = 2k - \frac{1}{2}$.

Next we turn to investigating unions of large cliques. Theorem 1.1 b) and Theorem 1.2 imply that for every r there exists $l_0 = l_0(r) \leq 20c(K_r)$, such that lK_r is not well-behaved for $l < l_0$ and lK_r is well-behaved for $l \geq l_0$. It follows from (1) that $l_0(r) \geq (\lambda + o_r(1))\sqrt{\log r}$. Thomason mentions in [Tho07, Tho08] that it is likely that $l_0(r) = \Theta(\sqrt{\log r})$. This prediction is motivated by the belief that for large enough r and any l , the extremal examples should either be “close” to being K_r -minor free or of the form $\bar{K}_{l(r-1)-1, n}$ for some n .

We show that Thomason’s prediction is almost, but not quite correct, as the next theorem implies that $l_0(r) = \Theta(\log r / \log \log r)$. The main reason for the discrepancy is that for a certain range of l we exhibit extremal examples, which do not have the structure suggested in [Tho08], but are obtained by gluing certain non-uniform random graphs.

Theorem 1.7. *There exist constants $c, C > 0$ such that for every positive*

integer r

$$\begin{aligned} a) \quad c_\infty(lK_r) &> l(r-1) - 1 \quad \text{for} \quad l \leq c \frac{\log r}{\log \log r} \\ b) \quad c_\infty(lK_r) &= l(r-1) - 1 \quad \text{for} \quad l \geq C \frac{\log r}{\log \log r}. \end{aligned}$$

Additionally, the next two theorems provide upper and lower bounds on $c_\infty(lK_r)$, which allow us to approximate the error term $c_\infty(lK_r) - lr$ in the range where this term is substantial, i.e. $l = o(\log r / \log \log r)$.

Theorem 1.8. *Let λ be Thomason's constant. For $l = \omega(\sqrt{\log n})$ and $l = o(\log r / \log \log r)$ we have*

$$c_\infty(lK_r) \geq lr + (1 - o(1)) \frac{\lambda^2 r \log r}{4l}.$$

Theorem 1.9. *There exists a constant $C_u > 0$ such that for all positive integers l, r*

$$c_\infty(lK_r) \leq lr + C_u \frac{r \log r}{l}.$$

As one of the ingredients in the proof of Theorem 1.9 we need an upper bound on the extremal function $c(K_{s,t})$ which is within a constant factor of optimal. This extremal function has been extensively investigated in the past. It follows from the results of Kostochka [Kos84] and Thomason [Tho84] that $c(K_{s,t}) = O(t\sqrt{\log t})$ for all $s \leq t$. Myers [Mye03] considered $c(K_{s,t})$ for $s \ll t$ and conjectured that $c(K_{s,t}) \leq c_s t$ for some constant independent on t . Kühn and Osthus [KO05] and Kostochka and Prince [KP08] have independently proved this conjecture by showing that $c(K_{s,t}) = (1/2 + o(1))t$ for $s \ll t$. Unfortunately, none of the above bounds suffice for our purpose and we prove the following result, which is tighter in the regime $s = \omega(t / \log t)$ and $s = o(t)$.

Theorem 1.10. *Let $t \geq s \geq 2$ be positive integers. Then*

$$c(K_{s,t}) \leq 40(\sqrt{st \log s} + s + t).$$

Theorem 1.10 additionally answers a question of Harvey and Wood [HW16]. They have asked whether there exists a constant $\varepsilon > 0$ such that for every graph H on n vertices we have

$$c(H) \geq \varepsilon n \sqrt{d(H - S)} \quad (6)$$

for some set $S \subseteq V(H)$ such that $|S| \leq \frac{n}{\varepsilon \log n}$. By Theorem 1.10 the answer is negative. Indeed, if $s = \omega(t/\log t)$ then $d(K_{s,t} - S) \geq s/2$ for every set S as above. Therefore, if additionally $s = o(t)$ then the bound given by Theorem 1.10 is smaller than the bound in (6) by a factor of roughly $\sqrt{t/s}$.

The rest of the paper is structured as follows In Section 2 we introduce the lemma of Eppstein [Epp10], which will serve as our main tool and prove several additional preliminary lemmas. We prove Theorem 1.2 in Section 3, and Theorem 1.6 in Section 4. In Section 5 we prove a general lower bound on $c_\infty(lK_r)$ attained by a random construction and derive Theorem 1.7 a) and 1.8 from this bound. In Section 6 we introduce several additional tools we need for proving the upper bounds on $c_\infty(lK_r)$. In particular, we prove Theorem 1.10. In Section 7 we prove Theorem 1.7 a) and 1.8. Section 8 contains the concluding remarks.

2 Blades, fans and Eppstein's lemma

In this section we define blades and fans and present a lemma of Eppstein [Epp10], which will provide the framework for proving our results.

We say that a pair (G, S) is a *blade* if G is a graph and $S \subsetneq V(G)$. Given a blade $\mathcal{B} = (G, S)$ and a positive integer k , let $\text{Fan}(\mathcal{B}, k)$ or $\text{Fan}(G, S, k)$ denote the graph obtained by k copies of G by identifying the vertices in S . For example, $\bar{K}_{s,t}$ can be considered as $\text{Fan}(K_{s+1}, S, t)$, where S is a subset of vertices of K_{s+1} of size s . It is easy to see that

$$\lim_{k \rightarrow \infty} d(\text{Fan}(G, S, k)) = \frac{e(G) - e(G[S])}{v(G) - |S|},$$

and we define the *density* of a blade $\mathcal{B} = (G, S)$ as

$$d(\mathcal{B}) = d(G, S) = \frac{e(G) - e(G[S])}{v(G) - |S|}.$$

We say that a blade (G, S) is *semiregular* if

- $G[S]$ is complete,
- $G \setminus S$ is connected,
- each vertex of S has a neighbor in $V(G) - S$,

We say that a semiregular blade \mathcal{B} is *regular* if $\deg(v) \geq d(G, S)$ for every $v \in V(G \setminus S)$.

Given a graph H and a blade \mathcal{B} , we say that H is a *minor* of \mathcal{B} if H is a minor of $\text{Fan}(\mathcal{B}, k)$ for some k , and we say that \mathcal{B} is *H -minor free*, otherwise.

We are now ready to state the key lemma, which is proven in [Epp10] for general minor-closed classes of graphs. For convenience we state only a weaker version for classes of graphs with a single excluded minor.

Lemma 2.1 (Eppstein [Epp10]). *Let H be a graph. Then for any $\epsilon > 0$ there exists a regular H -minor free blade \mathcal{B} such that $d(\mathcal{B}) \geq c_\infty(H) - \epsilon$.*

(In [Epp10], it is only shown that a semiregular blade as above exists. However, it is easy to see that if a blade (G, S) satisfies the conclusion of the lemma is chosen so that $d(G, S)$ is maximum and subject to that $v(G)$ is minimum, then (G, S) is regular.)

Essentially, Lemma 2.1 allows us to restrict our attention to fans when proving upper bounds on $c_\infty(H)$. The following convenient corollary is immediately implied by Lemma 2.1.

Corollary 2.2. *Let H be a graph, and let $c \in \mathbb{R}$ be such that $d(\mathcal{B}) \leq c$ for every regular H -minor free blade \mathcal{B} . Then $c_\infty(H) \leq c$.*

Conversely, if \mathcal{B} is an H -minor-free blade then $d(\mathcal{B}) \leq c_\infty(H)$.

We finish this section by introducing additional notation and several easy, but useful, lemmas. Let $\mathcal{B} = (G, S)$ be a blade. For $S' \subseteq S$, we denote by $\mathcal{B}[S']$ the blade $(G \setminus (S - S'), S')$ obtained from \mathcal{B} by deleting vertices in $S - S'$.

Lemma 2.3. *Let $\mathcal{B} = (G, S)$ be a blade, and let $S' \subseteq S$. Then $d(\mathcal{B}[S']) \geq d(\mathcal{B}) - |S| + |S'|$*

Proof. We have

$$(e(G) - e(G[S])) - (e(G - (S - S')) - e(G[S'])) \leq (|S| - |S'|)(v(G) - |S|),$$

implying the desired inequality by definition of the blade density. \square

Lemma 2.4. *Let (G, S) be a semiregular blade, and let H be a graph. If $|S| \geq \tau(H)$ then H is a minor of G .*

Proof. Note that $\bar{K}_{|S|,k}$ is a minor of (G, S) for every k . On the other hand H is isomorphic to a subgraph of $\bar{K}_{\tau(H), v(H) - \tau(H)}$. The desired conclusion follows. \square

Showing that a graph G contains a graph H as a minor typically involves constructing a model of H in G , defined as follows. We say that a map μ is a *blueprint of H in G* if μ maps vertices of H to disjoint subsets of vertices of G , called *bags of μ* . We will use $\mu(H)$ to denote $\cup_{v \in V(H)} \mu(v)$.

We say that a blueprint is a *premodel* if for every edge $\{u, v\} \in E(H)$ there exists an edge of G with one end in $\mu(u)$ and another in $\mu(v)$. Finally, we say that a premodel is a *model* if $G[\mu(v)]$ is connected for every $u \in V(H)$. The following useful observation is well known.

Observation 2.5. *A graph H is a minor of a graph G if and only if there exists a model of H in G .*

Observation 2.5 is used, in particular, in the proofs of the next remaining lemmas of this section.

Lemma 2.6. *Let $\mathcal{B} = (G, S)$ be a blade, let H_1, H_2, \dots, H_t be vertex disjoint graphs, and let H be their union. Then the following are equivalent*

1. *H is a minor of \mathcal{B} , and*
2. *there exist disjoint $S_1, \dots, S_t \subseteq S$ such that H_i is a minor of $\mathcal{B}[S_i]$ for every $1 \leq i \leq t$.*

Proof. We start by showing that the first condition implies the second. By Observation 2.5, there exists a model μ of H in $\text{Fan}(G, S, k)$ for some positive integer k . Equivalently there exists models μ_i of H_i in $\text{Fan}(G, S, k)$ for $1 \leq i \leq t$ such that $\mu_i(H_i) \cap \mu_j(H_j) = \emptyset$ for all $i \neq j$. Let $S_i = \mu_i(H_i) \cap S$ then the second condition clearly holds.

The proof of the other implication is similar. \square

Lemma 2.7. *Let $\mathcal{B} = (G, S)$ be a semiregular blade. If K_r is a minor of \mathcal{B} then K_r is a minor of G .*

Proof. Let μ be a model of K_r in $\text{Fan}(G, S, k)$ for some positive integer k . If $|S| \geq r$ then K_r is a subgraph of $G[S]$ and so the lemma holds. Otherwise, there exists a $v \in V(K_r)$ such that $\mu(v) \subseteq V(G') \setminus S$ for some copy G' of G in $\text{Fan}(G, S, k)$. Then $\mu(u) \cap V(G') \neq \emptyset$ for every $u \in V(K_r)$, and it is easy to see that the restriction of μ to $V(G')$ is a model of K_r in G' . \square

3 Proof of Theorem 1.2

Let $c = \max\{c_\infty(H_2), c_\infty(H_1) + \tau(H_2)\}$. By Corollary 2.2 it suffices to show that $d(\mathcal{B}) \leq c$ for every H -minor free regular blade $\mathcal{B} = (G, S)$.

We number the vertices in $S = \{v_1, v_2, \dots, v_s\}$, where $s = |S|$. Let $S_i = \{v_1, \dots, v_i\}$, $\overline{S}_i = S - S_i$. Choose i minimum such that H_1 is a minor of $\mathcal{B}[S_i]$. Thus $\mathcal{B}[\overline{S}_i]$ is H_2 minor-free by Lemma 2.6, and therefore $d(\mathcal{B}[\overline{S}_i]) \leq c_\infty(H_2)$ by Corollary 2.2. In particular, if $i = 0$ then $d(G, S) \leq c_\infty(H_2) \leq c$, as desired. Thus we assume $i > 0$. By Lemma 2.4 we have

$$s - i \leq \tau(H_2) - 1. \tag{7}$$

By minimality of i , $\mathcal{B}[S_{i-1}]$ is H_1 -minor-free. Therefore $c_\infty(H_1) \geq d(\mathcal{B}[S_{i-1}])$. By Lemma 2.3 and (7), we have

$$d(G, S) \leq d(\mathcal{B}[S_{i-1}]) + s - i + 1 \leq c_\infty(H_1) + \tau(H_2),$$

as desired.

4 Proof of Theorem 1.6

A classical result of Erdős and Gallai below implies that

$$c_\infty(C_l) \leq \frac{l-1}{2} \tag{8}$$

for every $l \geq 3$.

Theorem 4.1 (Erdős and Gallai [EG59]). *Let $l \geq 3$ be an integer and let G be a graph with n vertices and more than $(l-1)(n-1)/2$ edges. Then G contains a cycle of length at least l .*

We prove Theorem 1.6 by induction on $v(H)$. By (8) we may assume that H has at least 2 components. Let

$$d_0 = \begin{cases} 2m - \frac{1}{2}, & \text{if } H = mC_4; \\ \frac{v(H) + \text{odd}(H)}{2} - 1, & \text{otherwise.} \end{cases}$$

By Corollary 2.2 it suffices to show that $d(G, S) \leq d_0$ for every H -minor-free regular blade $\mathcal{B} = (G, S)$. Let C be the longest cycle in H , let $l = v(C)$, and let H be the disjoint union of C and a graph H_1 .

If H_1 is a minor of $G \setminus S$ then (G, S) is C -minor-free, and so by (8) we have $d(G, S) \leq \frac{(l-1)}{2} \leq d_0$, as desired. Thus H_1 is not a minor of $G \setminus S$, and by the induction hypothesis

$$d(G \setminus S) \leq \frac{v(H_1) + \text{odd}(H_1)}{2} - \frac{1}{2}.$$

Suppose that $|S| \leq (l-1)/2$, then

$$\begin{aligned} d(G, S) &\leq |S| + d(G \setminus S) \leq |S| + \frac{v(H_1) + \text{odd}(H_1)}{2} - \frac{1}{2} \\ &\leq \frac{v(H_1) + v(C) + \text{odd}(H_1)}{2} - 1 \leq \frac{v(H) + \text{odd}(H)}{2} - 1 \leq d_0. \end{aligned}$$

Thus we assume that $|S| \geq l/2$.

Suppose next that there exists $v \in V(G) - S$ such that v is the only vertex in $V(G) - S$ adjacent to a vertex in S . Then $e(G) - e(G[S]) \leq e(G \setminus S) + |S|$. If $|S| \geq |V(G) - S|$ then

$$\begin{aligned} d(G, S) &\leq d(G \setminus S) + \frac{|S|}{v(G) - |S|} \\ &\leq \frac{v(G) - |S| - 1}{2} + \frac{|S|}{v(G) - |S|} \\ &\leq |S| \leq \tau(H) - 1 \leq d_0, \end{aligned}$$

where second to last inequality uses Lemma 2.4. Otherwise,

$$\begin{aligned} d(G, S) &\leq d(G \setminus S) + \frac{|S|}{v(G) - |S|} \leq d(G \setminus S) + 1 \\ &\leq \frac{v(H_1) + \text{odd}(H_1)}{2} + \frac{1}{2} \leq \frac{v(H) + \text{odd}(H)}{2} - 1 \leq d_0. \end{aligned}$$

Thus we assume that there exists distinct $u_1, u_2 \in S$, $v_1, v_2 \in G \setminus S$ such that $u_1 v_1, u_2 v_2 \in E(G)$.

Let $S' \subseteq S$ be such that $u_1, u_2 \in S'$, and let $k = |S'|$. We show that C_{2k+2} is a minor $\mathcal{B}[S']$. Let $S' = \{u_1, u_2, \dots, u_k\}$. We say that a path P in G is an S' -jump if both ends of P are in S' , and P is otherwise disjoint from S . By taking a path joining v_1 and v_2 in $G \setminus S$ we obtain an S' -jump P_1 with ends u_1 and u_2 and at least 3 edges. If $k = 2$, then taking the union of two copies of P_1 in $\text{Fan}(\mathcal{B}[S'], 2)$ we obtain a cycle of length at least six, as desired. Thus we assume $k \geq 3$. Let v_3 be a neighbor of u_3 in $V(G) - S$, and assume without loss of generality that $v_3 \neq v_2$. Let P_2 be an S' -jump of length at least three with ends u_2 and u_3 . For $i = 3, \dots, k$, let P_i be an S' -jump of length at least two with ends u_i and u_{i+1} , where $u_{k+1} = u_1$ by convention. By taking the union of copies of paths P_1, \dots, P_k , each chosen from a separate copy of G we obtain a cycle of length at least $2k + 2$ in $\text{Fan}(\mathcal{B}[S'], k)$, as desired.

We finish the proof by considering two cases. Suppose first that $H = mC_4$, and let S' with $|S'| = 2$ be as in the previous paragraph. Then

$H_1 = (m-1)C_4$ is not a minor of $\mathcal{B}[S - S']$ and therefore by Corollary 2.2, Lemma 2.3 and the induction hypothesis we have

$$d(\mathcal{B}) \leq c_\infty(H_1) + |S'| \leq 2(m-1) - \frac{1}{2} + 2 = d_0.$$

Thus we assume that at least one cycle in H has length not equal to four. If $l \geq 5$, then by the claim above there exists $S' \subseteq S$ such that $|S'| \leq \lceil l/2 \rceil - 1 = (v(C) + \text{odd}(C))/2 - 1$, and C is a minor of $\mathcal{B}[S']$. Again it follows that

$$\begin{aligned} d(\mathcal{B}) &\leq c_\infty(H_1) + |S'| \leq \frac{v(H_1) + \text{odd}(H_1)}{2} - \frac{1}{2} + |S'| \\ &\leq \frac{v(H) + \text{odd}(H)}{2} - \frac{3}{2} < d_0. \end{aligned}$$

It remains to consider the case $l \leq 4$, but H contains at least one cycle of length not equal to four. It follows that $c_\infty(H_1) \leq \frac{v(H_1) + \text{odd}(H_1)}{2} - 1$ by the induction hypothesis, and choosing $S' \subseteq S$ with $|S'| = 2$, we once again have

$$d(\mathcal{B}) \leq c_\infty(H_1) + |S'| \leq \frac{v(H_1) + \text{odd}(H_1)}{2} + 1 \leq \frac{v(H) + \text{odd}(H)}{2} - 1 = d_0,$$

finishing the proof.

5 A lower bound on $c_\infty(lK_r)$

Our constructions of dense blades with no lK_r minor are random. Let $\mathbf{G}(a, b, p, q)$ be a random graph, with $V(\mathbf{G}(a, b, p, q)) = A \cup B$, where A and B are disjoint sets with $|A| = a$, $|B| = b$, the vertices of B form a clique and the edges are chosen independently at random so that every edge with both ends in A is present with probability p and an edge joining a vertex in A to a vertex in B is present with probability q .

The next lemma is a technical variation of a computation which to the best of our knowledge was first used by Bollobas, Caitlin and Erdős [BCE80] to compute the size of the largest minor in a random graph.

Lemma 5.1. *Let positive integers a, b and r , and reals $\alpha, \beta > 0$ be such that $a + b \leq r^2$, $r \leq 2b$ and*

$$\alpha(r - b)b(\log(r - b) - \log \log r - 3) \geq (\alpha a + \beta b)^2. \quad (9)$$

Then

$$\Pr[K_r \text{ is a minor of } \mathbf{G}(a, b, 1 - e^{-\alpha}, 1 - e^{-\beta})] \leq e^{-2r \log r}.$$

Proof. We denote the random graph $\mathbf{G}(a, b, 1 - e^{-\alpha}, 1 - e^{-\beta})$ by G for brevity. There are at most

$$(a + b)^r \leq r^{2r} = e^{2r \log r}$$

blueprints μ of K_r in G . Thus it suffices to show that the probability that for a fixed blueprint μ is a premodel of K_r is at most $e^{-4r \log r}$.

Let \mathcal{K}_a be the collection of all bags of μ which lie completely in A , and let \mathcal{K}_b be the collection of the remaining bags. Let $\mathcal{K}_a = \{X_1, X_2, \dots, X_s\}$, and let $x_i = |X_i|$ for $1 \leq i \leq s$. Note that $s \geq r - b$. Let $\mathcal{K}_b = \{U_1, U_2, \dots, U_{r-s}\}$, and let $Y_i = U_i \cap A$, $Z_i = U_i \cap B$, $y_i = |Y_i|$, $z_i = |Z_i|$ for $1 \leq i \leq r - s$. Note that the probability that X_i and X_j are adjacent in G is $1 - e^{-\alpha x_i x_j}$, and the probability that X_i is adjacent to U_j is $1 - e^{-\alpha x_i y_j - \beta x_i z_j}$.

Suppose first that $s > b$. We upper bound the probability that μ is premodel of K_r by the probability that the bags in \mathcal{K}_a are pairwise adjacent, which is

$$\prod_{1 \leq i < j \leq s} (1 - e^{-\alpha x_i x_j}) \leq \exp \left(- \sum_{1 \leq i < j \leq s} e^{-\alpha x_i x_j} \right)$$

Thus it suffices to show that

$$\sum_{1 \leq i < j \leq s} e^{-\alpha x_i x_j} \geq 4r \log r.$$

As $b \geq r - b$, the condition (9) implies that

$$b^2(\log b - \log \log r - 3) \geq \alpha a^2. \quad (10)$$

By the AM-GM inequality

$$\begin{aligned} \sum_{1 \leq i < j \leq s} e^{-\alpha x_i x_j} &\geq \binom{s}{2} \exp \left(-\frac{\alpha}{\binom{s}{2}} \sum_{1 \leq i < j \leq s} x_i x_j \right) \geq \frac{b^2}{2} \exp \left(-\alpha \left(\frac{a}{b} \right)^2 \right) \\ &\stackrel{(10)}{\geq} \frac{b^2}{2} \exp(-\log b + \log \log r + 3) \geq \frac{20b \log r}{2} \geq 4r \log r, \end{aligned}$$

as desired.

Thus we assume that $s \leq b$. Now we upper bound the probability that every set in \mathcal{K}_a is adjacent to every set in \mathcal{K}_b . Repeating the beginning of the argument in the previous case we see that it suffices to show that

$$\sum_{1 \leq i \leq s} \sum_{1 \leq j \leq r-s} e^{-x_i(\alpha y_j + \beta z_j)} \geq 4r \log r,$$

Let $x = \sum_{1 \leq i \leq s} x_i$. Applying the AM-GM inequality as before we obtain

$$\begin{aligned} \sum_{1 \leq i \leq s} \sum_{1 \leq j \leq r-s} e^{-x_i(\alpha y_j + \beta z_j)} &\geq s(r-s) \exp \left(-\frac{1}{s(r-s)} \sum_{1 \leq i \leq s} x_i \left(\sum_{1 \leq j \leq r-s} \alpha y_j + \sum_{1 \leq j \leq r-s} \beta z_j \right) \right) \\ &\geq b(r-b) \exp \left(-\frac{x(\alpha(a-x) + \beta b)}{b(r-b)} \right) \geq b(r-b) \exp \left(-\frac{(\alpha a + \beta b)^2}{\alpha b(r-b)} \right) \\ &\stackrel{(9)}{\geq} b(r-b) \exp(-\log(r-b) + \log \log r + 3) = e^3 b \log r \geq 4r \log r, \end{aligned}$$

as desired. \square

Theorem 5.2. *Let λ be the Thomason's constant. There exists $\xi > 0$ so that for every $0 \leq \varepsilon \leq 1/2$, $r \gg 1/\varepsilon$ and $\sqrt{\log r}/\varepsilon \leq l \leq \log r$, we have*

$$c_\infty(lK_r) \geq lr + (1 - \varepsilon) \frac{\lambda^2 r \log r}{4l} - lr \exp \left(-\frac{\xi \varepsilon \log r}{l} \right). \quad (11)$$

Proof. Consider a, b, α and β satisfying the conditions of Lemma 5.1. Let $G = \mathbf{G}(a, l(b+1) - 1, 1 - e^{-\alpha}, 1 - e^{-\beta})$ be a random graph, and let the set of vertices A and B be as in the definition of such random graph. Consider the blade $\mathcal{B} = (G, B)$. If lK_r is a minor of \mathcal{B} then by Lemma 2.6 there

exists $B' \subseteq B$ with $|B'| \leq \lfloor |B|/l \rfloor = b$ such that K_r is a minor of $\mathcal{B}[B']$. From Lemma 2.7 it follows that K_r is a minor of $G[A \cup B']$. However, by Lemma 5.1 the probability that K_r is a minor of $G[A \cup B']$ for some $B' \subseteq B$ with $|B'| = b$ is at most

$$|B|^b \exp(-2r \log r) \leq (lr)^r \exp(-2r \log r) \leq \exp(r(\log \log r - \log r)) \leq e^{-r}.$$

Thus the probability that lK_r is a minor of \mathcal{B} is at most e^{-r} . Let

$$D(a, b, \alpha, \beta) = \frac{a}{2}(1 - e^{-\alpha}) + lb(1 - e^{-\beta}).$$

An easy computation shows that

$$\mathbb{E}[d(\mathcal{B})] = \frac{(a-1)}{2}(1 - e^{-\alpha}) + (l(b+1) - 1)(1 - e^{-\beta}) \geq D(a, b, \alpha, \beta) + 1.$$

As

$$\Pr[d(\mathcal{B}) \geq \mathbb{E}[d(\mathcal{B})] - 1] \geq \frac{1}{a + l(b+1)} \geq \frac{1}{r^2} \geq e^{-r},$$

it follows that there exists an lK_r -minor-free blade \mathcal{B} with density at least $D(a, b, \alpha, \beta)$, i.e. $c_\infty(lK_r) \geq D(a, b, \alpha, \beta)$.

It remains to choose a, b, α and β satisfying the conditions of Lemma 5.1 so that

$$D(a, b, \alpha, \beta) \geq lr + (1 - 2\varepsilon) \frac{\lambda^2 r \log r}{4l} - lr \exp\left(-\frac{2\xi\varepsilon \log r}{l}\right).$$

(Note that we replaced ε by 2ε for later convenience.) Let constant $0 < \alpha < 1$ be chosen to maximize $\frac{1-e^{-\alpha}}{2\sqrt{\alpha}}$, i.e. $\lambda = \frac{1-e^{-\alpha}}{2\sqrt{\alpha}}$, and let

$$\begin{aligned} \gamma &= \frac{\lambda(1-\varepsilon)}{2}, & \sigma &= \frac{\gamma r \log r}{l}, \\ k &= \frac{\gamma^2 r \log r}{l^2} = \frac{\gamma \sigma}{l}, & b &= \lceil r - k \rceil, \\ \beta &= \frac{\varepsilon \sqrt{\alpha} \sigma}{2r}, & a &= \left\lceil \frac{(1-\varepsilon)\sigma}{\sqrt{\alpha}} \right\rceil. \end{aligned}$$

Note that by the choice of l we have

$$\gamma^2 \frac{r}{\log r} \leq k \leq \frac{\gamma^2 \varepsilon r \log r}{\log r} = \frac{\varepsilon r}{2} \quad (12)$$

Let us first verify that a, b, α and β satisfy (9). For $r \gg 1/\varepsilon$, we have

$$\begin{aligned} & \alpha(r-b)b(\log(r-b) - \log \log r - 3) \\ & \geq (1 - \varepsilon/2)^2 \alpha r k \log r \\ & = (\sqrt{\alpha}(1 - \varepsilon/2)\sigma)^2 \end{aligned}$$

Thus it suffices to show that $\alpha a + \beta b \leq \sqrt{\alpha}(1 - \varepsilon/2)\sigma$, which is immediate from the definitions.

We now return to the computation of $D(a, b, \alpha, \beta)$ for a, b, α and β as above. Let $\xi = \sqrt{\alpha}\lambda/16$. We have

$$\begin{aligned} D(a, b, \alpha, \beta) & \geq \frac{(1 - \varepsilon)\sigma}{2\sqrt{\alpha}}(1 - e^{-\alpha}) + l \left(r - \frac{\gamma\sigma}{l} \right) (1 - e^{-\beta}) \\ & \geq 2\gamma\sigma + lr - \gamma\sigma - lr \exp \left(-\frac{\varepsilon\sqrt{\alpha}\lambda(1 - \varepsilon)\log r}{4l} \right) \\ & = lr + \left(\frac{\lambda(1 - \varepsilon)}{2} \right)^2 \frac{r \log r}{l} - lr \exp \left(-\frac{\varepsilon\sqrt{\alpha}\lambda(1 - \varepsilon)\log r}{4l} \right) \\ & \geq lr + (1 - 2\varepsilon) \frac{\lambda^2 r \log r}{4l} - lr \exp \left(-\frac{2\xi\varepsilon \log r}{l} \right), \end{aligned}$$

which finishes the proof of the theorem. \square

Proof of Theorem 1.7 a). Let ξ be as in Theorem 5.2, and let $2\sqrt{\log r} \leq l \leq \frac{\xi \log r}{2 \log \log r}$. Thus $l = c \log r / \log \log r$ for some $c \leq \xi/2$. It suffices to show that $c_\infty(lK_r) \geq lr$. By Theorem 5.2 applied with $\varepsilon = 1/2$ we have

$$\begin{aligned} c_\infty(lK_r) - lr & \geq \frac{\lambda^2 r \log r}{8l} - lr \exp \left(-\frac{\xi \log r}{2l} \right) \\ & = \frac{\lambda^2}{8c} r \log \log r - \frac{cr \log r}{\log \log r} e^{-\frac{\xi \log \log r}{2c}} \\ & \geq \frac{\lambda^2}{8c} r \log \log r - \frac{cr}{\log \log r} \geq 0, \end{aligned}$$

as desired. \square

Proof of Theorem 1.8. The inequality (11) gives the required bound, as long as we show that for every $0 < \varepsilon \leq 1$ there exists $\delta > 0$ so that for $l \leq \delta \log r / \log \log r$ we have

$$lr \exp \left(-\frac{\xi\varepsilon \log r}{l} \right) \leq \varepsilon \frac{r \log r}{l}.$$

Let $\delta = \min\{\xi\varepsilon, \sqrt{\varepsilon}\}$. Then

$$\exp\left(-\frac{\xi\varepsilon \log r}{l}\right) \leq \exp\left(-\frac{\xi\varepsilon \log \log r}{\delta}\right) \leq \frac{1}{\log r} \leq \frac{\varepsilon(\log \log r)^2}{\delta^2 \log r} \leq \varepsilon \frac{\log r}{l^2},$$

as desired. \square

6 Hefty graphs

In this section we introduce the tools which will be subsequently used to upper bound $c_\infty(lK_r)$. These tools are built around the concept of hefty graphs. We say that a graph H is *hefty* if $H = K_2$, or $\deg(v) \geq 0.65|V(H)|$ for every $v \in V(H)$. (Our choice of constant 0.65 is motivated by Lemma 6.1 below.)

Classes of graphs with similar properties are considered in many proofs of upper bounds on the extremal function and the following lemmas demonstrate some of the ways in which they are used.

The first lemma allows one to replace any graph by a hefty graph at a cost of a constant fraction of density. It is a variant of a result first proved by Mader [Mad68], and appears in a slightly stronger form than the one stated below in Reed and Wood [RW15].

Lemma 6.1. *Let G be a graph such that $d(G) \neq 0$. Then there exists a hefty minor H of G such that $|V(H)| \geq d(G)/2$.*

Next lemma shows that if a hefty graph G contains a small model of a graph H and a graph H' is obtained from H by adding a few edges then G contains a model of H' . We say that a set F of pairs of vertices of G is a *completion* of a blueprint μ of a graph H in a graph G if μ is a model of H in a graph obtained from G by adding F to $E(G)$. The *defect* of a blueprint μ is the minimum size of a completion of μ .

Lemma 6.2. *Let G be a hefty graph, and let μ be a blueprint of a graph H in G with defect at most $0.3|V(G)| - |\mu(H)|$. Then μ extends to a model of H in G .*

Proof. We prove the lemma by induction on the defect c of μ . The base case $c = 0$ is immediate. For the induction step, let F be a completion of μ with $|F| = c \geq 1$ and consider arbitrary $f = \{u, v\} \in F$. Note that u and v have at least $0.3|V(G)|$ common neighbors in G , and so there exists $w \in V(G) - \mu(H)$ adjacent to both u and v . Let $x \in V(H)$ be such that $u \in \mu(x)$. Adding w to $\mu(x)$ we obtain a blueprint μ' of H in G such that $F \setminus \{f\}$ is a completion of μ' . By the induction hypothesis μ' extends to a model of H in G as desired. \square

As a first application of the above lemmas we prove Theorem 1.10. The technical part of the proof is contained in the following lemma.

Lemma 6.3. *Let G be a hefty graph on a vertices. Let s, t, k, l be positive integers such that $sk + tl \leq 3a/20$ and $(k - 2)l - 2 \geq \log_2 s$. Then $K_{s,t}$ is a minor of G .*

Proof. Let $d = 0.65$. For every $v \in V(G)$ and a set $X \subseteq V(G) \setminus \{v\}$ of size l chosen uniformly at random the probability that v has no neighbor in X is at most $(1 - d)^l$. Thus for a set X as above the expected number of vertices in $V(G) - X$ with no neighbor in X is at most $a(1 - d)^l$. We say that a set X is *good* if at most $3a(1 - d)^l$ vertices in $V(G) - X$ have no neighbor in X . By Markov's inequality the probability that X is good is at least $2/3$.

Given a good set X if a set Y of size k is selected from $V(G) - X$ uniformly at random then the probability that no vertex of Y is adjacent to a vertex of X is at most $(4(1 - d)^l)^k < (1/2)^{(l-2)k}$.

We now select disjoint subsets $X_1, X_2, \dots, X_{2t}, Y_1, Y_2, \dots, Y_s$ of $V(G)$ such that $|X_i| = l$, $|Y_j| = k$ uniformly at random. We say that a pair (i, j) is *fulfilled* if there exist $\{u, v\} \in E(G)$ with $u \in X_i$, $v \in Y_j$. We say that X_i is *perfect* if (i, j) is fulfilled for every j , and we say that X_i is *flawed* otherwise.

By the calculations above the probability that X_i is good, but flawed is at most $s(1/2)^{(l-2)k} \leq 1/4$. Therefore the probability that X_i is perfect is at least $1/2$. Thus there exists a choice of subsets as above such that at least t

of subsets X_1, X_2, \dots, X_{2t} are perfect. If, say, X_1, \dots, X_t are these subsets then $Y_1, Y_2, \dots, Y_s, X_1, X_2, \dots, X_t$ form a premodel μ of $K_{s,t}$ which can be extended to a model by Lemma 6.2, as $2|\mu(K_{s,t})| \leq 2(sk + tl) \leq 3a/10$. \square

Proof of Theorem 1.10. Let $d = 40(\sqrt{st \log_2 s} + s + t)$, and let G be a graph with $d(G) \geq d$. By Lemma 6.1 there exists a hefty minor H of G with $a = |V(H)| \geq d/2$. Let $p = \sqrt{st \log_2 s}$, $k = \lceil p/s \rceil + 2$, and $l = \lceil p/t \rceil + 2$. Then we have

$$\begin{aligned} (l-2)k-2 &\geq (k-2)(l-2) \geq \frac{p^2}{st} = \log_2 s, \quad \text{and} \\ sk + tl &< s(p/s + 3) + t(p/t + 3) \\ &= 2\sqrt{st \log_2 s} + 3s + 3t \leq 3d/40 \leq 3a/20. \end{aligned}$$

Thus s, t, k and l satisfy the conditions of Lemma 6.3. It follows that $K_{s,t}$ is a minor of H as desired. \square

Next we prove a counterpart of Lemma 5.1. We will show that if a graph has the structure similar to that of the random examples of K_r minor-free graphs considered in that lemma, but is somewhat denser, then it has a K_r minor.

To make the above statement precise we need a definition. We say that a partition (A, B) of the vertices of the graph G is (a, b, δ) -*semicomplete* if $|A| = a$, $|B| = b$, $G[A]$ is hefty, $G[B]$ is complete and every $v \in B$ has at least $(1 - \delta)a$ neighbors in A . We say that G is (a, b, δ) -*semicomplete* if $V(G)$ admits an (a, b, δ) -semicomplete partition. We will investigate the range of parameters which guarantee the presence of a K_r minor in an (a, b, δ) -semicomplete graph. First, we need an easy lemma.

Lemma 6.4. *Let G be a graph, let $d = e(G)/\binom{n}{2}$ and let $X \subseteq V(G)$, $|X| = k$ be chosen uniformly at random. Then*

$$\Pr \left[e(G[X]) \geq \left(d - \frac{1}{2} \right) \binom{k}{2} \right] \geq \frac{1}{2}.$$

Proof. Note that the expected value of $e([G[X]])$ is $d\binom{k}{2}$, and so the lemma follows immediately from Markov's inequality. \square

We are now ready to prove the first of the main results on minors in semicomplete graphs.

Lemma 6.5. *There exists $\varepsilon > 0$ satisfying the following. Let a, k, r be positive integers and $\delta > 0$ be real so that*

$$k \cdot \max \left\{ \sqrt{\log k}, -\frac{\log r}{\log \delta} \right\} < \varepsilon a, \quad (13)$$

then every $(a, r - k, \delta)$ -semicomplete graph has a K_r minor.

Proof. Let (A, B) be an $(a, r - k, \delta)$ -semicomplete partition of vertices of a graph G . Let $0.05 \leq c \leq 0.1$ be such that $s = ca/k$ is an integer. We say that $X \subseteq A$ with $|X| = s$ is *bad* if some vertex of B has no neighbors in X , and *good* otherwise. Then the probability that a set X chosen uniformly at random is bad is at most

$$r\delta^s \leq r\delta^{\frac{a}{20k}} \leq \frac{1}{3},$$

where the last condition follows from (13), when ε is sufficiently small.

We now choose disjoint subsets $X_1, X_2, \dots, X_{3k}, Z$ of A such that $|X_i| = s$, $|Z| = ks$ uniformly at random. By the computation above with probability greater than $1/2$ at least k of the sets X_1, X_2, \dots, X_{3k} are good. By Lemma 6.4 with probability at least $1/2$ we have $d(G[Z]) \geq 0.15 \cdot (a/20 - 1) \geq a/200$.

It follows that for some choice as above, X_1, \dots, X_k are good and $d(G[Z]) \geq a/200$. By (13) and (1) if ε is sufficiently small then there exists a model μ of K_k in $G[Z]$. Assume for convenience that $V(K_k) = \{1, 2, \dots, k\}$, and extend μ to a blueprint μ' of K_k in $G[A]$ by adding X_i to $\mu(i)$. Then $|\mu'(K_k)| \leq 2ca \leq 0.2a$ and the defect of μ' is at most ca . By Lemma 6.2 the blueprint μ' extends to a model μ'' of K_k in $G[A]$, and by the choice of X_1, \dots, X_k every vertex in B has a neighbor in $\mu(i)$ for every i . Therefore

adding each vertex of B as a new bag to μ'' produces a model of K_r in G , as desired. \square

The next lemma differs from Lemma 6.5 by the restriction on parameters and the construction of the model of K_r .

Lemma 6.6. *There exists $\varepsilon > 0$ satisfying the following. Let $a, r \geq k \geq 2$ be positive integers and $\delta > 0$ be real so that*

$$\max\{r, \sqrt{rk \log r}\} < \varepsilon a, \quad (14)$$

then every $(a, r - k, 0.8)$ -semicomplete graph has a K_r minor.

Proof. Let (A, B) be an $(a, r - k, 0.8)$ -semicomplete partition of vertices of a graph G . As in the proof Lemma 6.5 we can find $Z \subseteq A$ such that $d(G[Z]) \geq a/200$ and $|Z| \leq a/10$. (In fact, the constants can be significantly improved, if needed.) By Theorem 1.10 and (14) if ε is sufficiently small then $G[Z]$ contains a model of $K_{k,r}$ and thus a model of $\bar{K}_{k,r}$. Let the vertices of independent set of $\bar{K}_{k,r-k}$ be v_1, v_2, \dots, v_{r-k} and let $B = \{u_1, u_2, \dots, u_{r-k}\}$. As every vertex in B has at least $a/5$ neighbors in A and $|B| \leq a/10$, there exist distinct x_1, \dots, x_{r-k} in $A \setminus \mu(K_{k,r-k})$ such that x_i is adjacent to u_i . By Lemma 6.2 the model μ extends to a model of μ' of $\bar{K}_{k,r-k}$ in $G[A]$ such that $x_i \in \mu'(v_i)$ for $1 \leq i \leq r - k$. Adding u_i to $\mu'(v_i)$ for each i produces the desired model of K_r in G . \square

7 Proof of Theorems 1.7 b) and Theorem 1.9

We start this section by introducing a crucial lemma which will allow us to apply the results of the previous section. Recall that by Lemma 6.1 every graph can be replaced with a hefty minor while losing only constant fraction of density. Given a blade (G, S) , we would like to apply it to the graph $G - S$ while controlling the loss of the density of the blade. We can do this if we first ensure that every vertex of $G - S$ has a large number of neighbors in S . This is accomplished by the next lemma.

First let us recall some standard definitions, which are used in the proof. A *separation* of a graph G is a pair (A, B) such that $A \cup B = V(G)$ and no edge of G has one end in $A - B$ and the other in $B - A$. The *order* of a separation (A, B) is $|A \cap B|$. For $X, Y \subseteq V(G)$ an (X, Y) -*linkage* is a set of vertex disjoint paths, each with one end in X and the other end in Y . By Menger's theorem the maximum order of an (X, Y) -linkage in G is equal to the minimum order of a separation (A, B) of G such that $X \subseteq A, Y \subseteq B$.

Lemma 7.1. *For every graph G there exists a graph H and a model μ of a graph H in G such that for every $v \in V(H)$ there exists $u \in \mu(v)$ such that $\deg_G(u) \leq 96d(H) + 24$.*

Proof. Let $G_1 \subseteq G$ be chosen such that $d(G_1)$ is maximum. Let $d = d(G_1)$. Let R be the set of all vertices of G of degree at most $12d$, and let \mathcal{P} be the $(V(G_1), R)$ -linkage in G of maximum order. Let $x = |\mathcal{P}|$ and $n = |V(G_1)|$. As noted above, by Menger's theorem there exists a separation (A, B) of G such that $V(G_1) \subseteq A, R \subseteq B$ and $|A \cap B| = x$. Let $G_2 = G[A]$ and $n' = |A|$. Then, $|A - B| = n' - x$ and every vertex in $A - B$ has degree at least $12d$. By the choice of G_1 we have

$$d \geq d(G_2) = \frac{e(G_2)}{v(G_2)} \geq \frac{6d(n' - x)}{n'}.$$

Thus $x \geq \frac{5}{6}n' \geq \frac{5}{6}n$.

Let Q be the set of starting vertices of paths \mathcal{P} in $V(G_1)$, then $|Q| = x$. Let $G_3 = G[V(G_1) - Q]$, then

$$e(G_3) \leq dv(G_3) = d(v(G_1) - |Q|) \leq dn/6.$$

Let $G_4 = G_1 \setminus E(G_3)$, then $|E(G_4)| \geq \frac{5}{6}dn$.

Let S be the set of vertices in $V(G_4) - Q$ with degree at least $2d$. We claim that there exists a matching M in G_4 so each vertex of S is joined by an edge of M to a vertex in Q . Suppose not. Then by Hall's theorem there exists a set $S' \subseteq Q$ such that $|S'| \leq |S|$ and all the edges of G_4

incident with vertices of S have their second end in S' . It follows that $|E(G[S \cup S'])| \geq 2d|S| > d|S \cup S'|$ contradicting the choice of d and proving our claim. For every edge of $e \in M$ with an end $q \in Q$ extend the path P in \mathcal{P} which ends in q to include e .

We are now ready to construct the graph H satisfying the lemma. Let $G_5 = G_4[Q \cup S]$, let $V(H) = \mathcal{P}$ and $P', P'' \in H$ are adjacent in H if some edge of G_5 joins a vertex of P' to a vertex of P'' . Then the identity map μ is a model of H in G .

Next we estimate $d(H)$. Note that $|V(P) \cap V(G_5)| \leq 2$ for every $P \in \mathcal{P}$, and every vertex of G_5 is a vertex of some path in \mathcal{P} . It follows that $e(H) \geq \frac{e(G_5) - v(H)}{4}$. Moreover,

$$e(G_5) \geq e(G_4) - 2d(v(G_4) - |Q| - |S|) \geq \frac{5}{6}dn - 2d\frac{n}{6} \geq \frac{dv(H)}{2}.$$

Thus $d(H) \geq \frac{d-2}{8}$. Finally, by the choice of \mathcal{P} , for every $v \in V(H)$ there exists $u \in V(\mu(v))$ such that $\deg_G(u) \leq 12d \leq 96d(H) + 24$. \square

We say that a blade (G, S) is (a, m) -hefty if

- (G, S) is semiregular,
- $G \setminus S$ is hefty,
- $a = |V(G) - S|$,
- there are at least m edges joining vertices of S to vertices of $G \setminus S$.

We say that a blade (G', S') is a *minor* of a blade (G, S) if G' is obtained from G by repeatedly deleting vertices and deleting and contracting edges with both ends in $V(G) \setminus S$. Lemmas 7.1 and 6.1 imply the following.

Lemma 7.2. *There exists a constant D satisfying the following Let $\mathcal{B} = (G, S)$ be a regular blade such that $|V(G) - S| > 1$. Then \mathcal{B} has an $(a, d(\mathcal{B})a - Da^2)$ -hefty minor for some positive integer $a \geq 2$.*

Proof. We show that $D = 204$ satisfies the lemma. By Lemma 7.1 there exists a graph H and a model μ of H in $G \setminus S$ such that for every $v \in V(H)$ there exists $u \in \mu(v)$ such that $\deg_{G \setminus S}(u) \leq 96d(H) + 24$. As \mathcal{B} is regular, it follows that each such vertex u has at least $d(\mathcal{B}) - 96d(H) - 24$ neighbors in S . Contracting the bags of μ to single vertices we obtain a minor (G', S) of \mathcal{B} such that $G' \setminus S$ is isomorphic to H and every vertex in $V(G') \setminus S$ has at least $d(\mathcal{B}) - 96d(H) - 24$ neighbors in S . Applying Lemma 6.1 to $G' \setminus S$ we obtain a minor (G'', S) of \mathcal{B} such that $G'' \setminus S$ is hefty, $a = |V(G'') - S| \geq d(H)/2$ and every vertex $V(G'') \setminus S$ has at least $d(\mathcal{B}) - 96d(H) - 24 \geq d(\mathcal{B}) - 204a$ neighbors in S . Let Z be the set of vertices in S with no neighbors in $V(G'') - S$, then $(G'' - Z, S - Z)$ is $(a, d(\mathcal{B})a - 204a^2)$ hefty, as desired. \square

Lemma 7.2 allows us to restrict our attention to hefty blades during the investigation of $c_\infty(lK_r)$ at the expense of an error term linear to the size of $V(G) \setminus S$ in such a blade (G, S) . Meanwhile, Lemmas 6.5 and 6.6 seem tailored for finding disjoint complete minors in hefty blades. Our next lemma makes the connection explicit.

Lemma 7.3. *Let $a, r \geq k \geq 2$ be positive integers. If every $(a, r - k, \frac{k-1}{r-1})$ -semicomplete graph has a K_r minor. Then for every $l \geq 1$, every $(a, (l(r - k) + k - 1)a)$ -hefty blade has an lK_r minor.*

Proof. Let $\mathcal{B} = (G, S)$ be an $(a, (l(r - k) + k - 1)a)$ -hefty blade. Let $H = G \setminus S$. Let $\delta = (k - 1)/(r - 1)$, and let S' be the set of all vertices in S with at least $a(1 - \delta)$ neighbors in $V(H)$. Then for every subset T of S' with $|T| = r - k$ the underlying graph of the blade $\mathcal{B}[T]$ is $(a, r - k, \delta)$ -semicomplete, and so $\mathcal{B}[T]$ contains a K_r minor by the assumption of the lemma.

Let $x = \lfloor |S'|/(r - k) \rfloor$. Then there exists disjoint $T_1, T_2, \dots, T_x \subseteq S'$ such that $|T_i| = r - k$ for $1 \leq i \leq x$. Let $S'' = S \setminus \cup_{i=1}^x T_i$. Suppose that $|S''| \geq (r - 1)(l - x)$. Then there exist disjoint $T_{x+1}, \dots, T_l \subseteq S''$, such that $|T_i| = r - 1$ for $x + 1 \leq i \leq l$. Contracting H to a single vertex gives a model

of K_r in T_i for $x+1 \leq i \leq l$. Thus by Lemma 2.6 the blade \mathcal{B} has an lK_r minor.

Therefore we may assume for a contradiction that $|S''| \leq (r-1)(l-x)$. We have $|S' \cap S''| \leq r-k-1$ and so the total number of edges of G with one end in S'' and another in $V(H)$ is at most

$$a(r-k) + ((r-1)(l-x) - r-k)a(1-\delta)$$

Adding the edges with one end in $S \setminus S''$, we obtain the following upper bound on the number of edges from S to $V(H)$

$$\begin{aligned} & x(r-k)a + a(r-k) + ((r-1)(l-x) - r-k)a(1-\delta) \\ &= a(x(r-k) - (r-1)(1-\delta)) + l(r-1)(1-\delta) + \delta(r-k) \\ &= a\left(l(r-k) + \frac{(k-1)(r-k)}{r-1}\right) \\ &< a(l(r-k) + k-1), \end{aligned}$$

contradicting the assumption that \mathcal{B} is $(a, a(l(r-k) + k-1))$ -hefty. \square

We now have all the ingredients in place for the proofs of our main theorems.

Proof of Theorem 1.7 b). Let D be as in Lemma 7.2, let ε be as in Lemma 6.5, let λ^* be such that every graph H with $d(H) \geq \lambda^* r \sqrt{\log r}$ contains a K_r minor. Assuming $C \gg \lambda^*, D, 1/\varepsilon$, we will show that $c_\infty(lK_r) \leq l(r-1) - 1$ for all $l \geq C \log r / \log \log r$.

By Corollary 2.2 it suffices to show that if $\mathcal{B}' = (G', S')$ is a regula with $d(\mathcal{B}') > l(r-1) - 1$ then lK_r is a minor of \mathcal{B}' . If $|S'| \geq l(r-1)$, then \mathcal{B}' has an lK_r minor by Lemma 2.4, and so we assume $|S'| < l(r-1)$. Therefore $|V(G') - S| \geq 2$, and by Lemma 7.2, \mathcal{B}' contains an $(a, (l(r-1) - 1 - Da)a)$ -hefty minor $\mathcal{B} = (G, S)$ for some integer $a \geq 2$. We will show that $\mathcal{B} = (G, S)$ contains an lK_r minor.

If $a \geq 2\lambda^* r \sqrt{\log r}$ then $G - S$ has a K_r minor, and so \mathcal{B}' contains an nK_r minor for any integer $n > 0$. Thus we assume

$$\varepsilon a \leq 2\lambda^* r \sqrt{\log r} \quad (15)$$

Suppose next that $l \geq 2Da^3$. Then G contains at least $(l(r-2) + Da^2)a$ edges joining vertices of S to vertices in $V(G) - S$, and so $|S| \geq l(r-2) + Da^2$. Moreover, $|S| \leq l(r-1)$, and therefore at most Da^2 vertices in S have a non-neighbor in $V(G) - S$. Thus there exist a set $S' \subseteq S$ such that $|S'| \geq l(r-2)$ and every $v \in S'$ is adjacent to every vertex of $V(G) - S$. Let S_1, S_2, \dots, S_l be disjoint subsets of S' such that $|S_i| = r-2$ for $1 \leq i \leq l$. Then $\mathcal{B}[S_i]$ contains K_r as a subgraph, and so \mathcal{B} has an lK_r minor by Lemma 2.6. Thus we may assume that $2Da^3 \geq l \geq C$, implying $a \gg 1$, which in turn implies $r \gg 1$ by (15).

Suppose that there exist an integer $2 \leq k \leq r$ such that

$$k \cdot \max \left\{ \sqrt{\log k}, 2 \frac{\log r}{\log r - \log k} \right\} < \varepsilon a \quad (16)$$

$$l(r-1) - 1 - Da \geq l(r-k) + k - 1, \quad (17)$$

Then by Lemma 6.5 every $(a, r-k, (k-1)/(r-1))$ -semicomplete graph has a K_r minor, and thus by Lemma 7.3 every $(a, (l(r-k) + k-1)a)$ -hefty blade has an lK_r minor. Meanwhile, the last condition implies that \mathcal{B} is $(a, (l(r-k) + k-1)a)$ -hefty. Thus it remains to find k satisfying the above.

Let $k = \lceil 2Da/l + 1 \rceil$. Then $(k-1)(l-1) \geq Da + 1$ and so (17) holds. If $k \leq 3$ then (16) also holds $\varepsilon a, r \gg 1$. Otherwise, $k \leq 4Da/l$. By (15), we have

$$\begin{aligned} \log l &\geq \log C + \log \log r - \log \log \log r \\ &\geq \frac{1}{3} \log \log r - \log r + \log a + \log 4D, \end{aligned}$$

and so $\log k \leq \log r - \frac{1}{3} \log \log r$. Thus the left side of (16) is at most

$$a \cdot \frac{4D \log \log r}{C \log r} \cdot \frac{2 \log r}{\frac{1}{3} \log \log r} = \frac{24D}{C} a < \varepsilon a$$

as desired. \square

Proof of Theorem 1.9. The argument is very similar to the proof of Theorem 1.7 b) above, except that we use Lemma 6.6 in place of Lemma 6.5.

Let D be as in Lemma 7.2, let ε be as in Lemma 6.6, and let λ^* be such that every graph H with $d(H) \geq \lambda^* r \sqrt{\log r}$ contains a K_r minor, and let C be as in Theorem 1.7 b). We show that the theorem holds as long as $C_u \gg C, \lambda^*, D, 1/\varepsilon$.

Let $\Delta = C_u r \log r / l$. As in the proof of Theorem 1.7 by Lemma 7.2 it suffices to show that if $\mathcal{B} = (G, S)$ is an $(a, (l(r-1) - 1 + \Delta - Da)a)$ -hefty blade for some integer $a \geq 2$ then \mathcal{B} contains an lK_r minor. By Theorem 1.7 b) we may assume that $l \leq C \log r \log \log r$.

As in the previous proof we may assume that $|S| < l(r-1)$ and that (15) holds. The first of these conditions implies $Da \geq \Delta$, that is

$$a \geq \frac{C_u r \log r}{Dl}. \quad (18)$$

Substituting the upper bound on l , we have $a > r/\varepsilon$. As a consequence of (15) and (18) we have $r \gg 1$ and

$$l > 6D\lambda^* \sqrt{\log r}. \quad (19)$$

(The constants in the above inequalities may seem arbitrary, but are chosen for later use.)

As in the proof of Theorem 1.7 successively applying Lemma 6.6 and Lemma 7.3 we see that it suffices to find a positive integer $k \geq 2$ satisfying

$$\max\{r, \sqrt{rk \log r}\} < \varepsilon a, \quad (20)$$

$$0.2 \leq \frac{k-1}{r-1}, \quad (21)$$

$$l(r-1) - Da \geq l(r-k) + k - 1. \quad (22)$$

Choose $k = cDa/l$ for some $2 < c < 3$. Then $lk \geq 2Da$ and (22) holds. The condition (21) holds by (15) and (19). It remains to show that $\sqrt{rk \log r} < \varepsilon a$, i.e.

$$\frac{cDa}{l} r \log r < \varepsilon^2 a^2,$$

which follows directly from (18). \square

8 Concluding remarks

In this paper we explored applications of the structural lemma of Eppstein [Epp10] to bounds on the asymptotic extremal function $c_\infty(H)$ for disconnected graphs H . In particular, the large portion of the paper is dedicated to proving bounds on $c_\infty(lK_r)$. In this direction the following interesting questions remain open

Question 8.1. *How large is $c_\infty(2K_r) - c_\infty(K_r)$?*

Clearly, $c_\infty(2K_r) - c_\infty(K_r) \geq 1$, and we have $c_\infty(2K_r) - c_\infty(K_r) \leq r - 1$ by Theorem 1.2, but we can not improve on either of the bounds. Giving a precise answer to Question 8.1 might be out of reach of the current techniques, as it seems likely to involve obtaining estimates on $c(K_r)$ with additive error sublinear in r . In contrast, we believe that it is possible that a refinement of the tools presented in this paper is sufficient to answer the following two questions.

Question 8.2. *Give an estimate on $c_\infty(lK_r)$ which is asymptotically tight for all l, r such that $l + r \rightarrow \infty$.*

As noted in the introduction, we have

$$\frac{1}{2} - o(1) \leq \frac{c_\infty(lK_r)}{\lambda r \sqrt{\log r} + l(r-1)} \leq 1 + o(1),$$

but can one improve on the estimate in denominator to remove the gap between the bounds?

Question 8.3. *Give a tight estimate of $c_\infty(lK_r) - l(r-1)$ in the range $l = \omega(\sqrt{\log r})$ and $l = o(\log r / \log \log r)$.*

Theorems 1.8 and 1.9 provide bounds on the above difference which differ by a constant factor. We believe that the lower bound is tight.

There are also many natural questions which could be asked about the behaviour of $c_\infty(lH)$ for non-complete graph H . For example, define the

excess of H by

$$\text{exc}(H) = \lim_{l \rightarrow \infty} (c_\infty(lH) - l\tau(H) + 1).$$

By (2) and Theorem 1.2, $\text{exc}(H)$ is well-defined and is non-negative for every graph H . By Theorem 1.1 we have $\text{exc}(K_r) = 0$ for every r . By Theorem 1.5 we have $\text{exc}(C_l) = 0$ for every $l \neq 4$, while $\text{exc}(C_4) = 1/2$.

Question 8.4. *Describe $\text{exc}(H)$ in terms of other (natural) parameters of the graph H .*

Finally, note once again that $c_\infty((l+1)H) - c_\infty(lH) \leq \tau(H)$ for all H and all $l \geq 1$ by Theorem 1.2. It is possible to show that for fixed H and large enough l the above inequality holds with equality. Hence one might consider the following question.

Question 8.5. *For a fixed graph H is the sequence $c_\infty((l+1)H) - c_\infty(lH)$ unimodular? Is it non-decreasing?*

Note that the answer to Question 8.5 might shed light on Questions 8.1 and 8.2.

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