

Set-Rationalizable Choice and Self-Stability

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A common assumption in modern microeconomic theory is that choice should be rationalizable via a binary preference relation, which Sen showed to be equivalent to two consistency conditions, namely α (contraction) and γ (expansion). Within the context of *social* choice, however, rationalizability and similar notions of consistency have proved to be highly problematic, as witnessed by a range of impossibility results, among which Arrow's is the most prominent. Since choice functions select *sets* of alternatives rather than single alternatives, we propose to rationalize choice functions by preference relations over sets (set-rationalizability). We also introduce two consistency conditions, $\hat{\alpha}$ and $\hat{\gamma}$, which are defined in analogy to α and γ , and find that a choice function is set-rationalizable if and only if it satisfies $\hat{\alpha}$. Moreover, a choice function satisfies $\hat{\alpha}$ and $\hat{\gamma}$ if and only if it is *self-stable*, a new concept based on earlier work by Dutta. The class of self-stable social choice functions contains a number of appealing Condorcet extensions such as the minimal covering set and the essential set.

1 Introduction

Arguably the most basic model of individual and collective choice is a *choice function*, which associates with each set A of feasible alternatives a non-empty subset $S(A) \subseteq A$. Apparently, not every choice function complies with our intuitive understanding of rationality. Consider, for example, the choice function S with $S(\{a, b\}) = \{a\}$ and $S(\{a, b, c\}) = \{b\}$. Doubts as to an agent's rationality could be raised, if, when offered the choice between apple pie and brownies, he were to choose the former, but the latter, when told that chocolate mousse is also an option.¹ In microeconomic theory, the

¹Sen (1993, 1997) gives examples where rational choosers actually make choices as described and generally argues against imposing internal consistency conditions on rational choice. His examples usually involve a kind of context-dependence. For instance, a modest person may be unwilling to take the largest piece of cake and thus his choice depends on the other pieces that are available. Our view is

existence of a binary relation R on all alternatives such that S returns precisely the maximal elements according to R from any feasible set is commonly taken as a minimal rationality condition on choice functions. Choice functions for which this is the case are called *rationalizable* (see, e.g., Richter, 1966; Herzberger, 1973; Blair et al., 1976; Moulin, 1985a).² Rationalizable choice functions have been characterized using two *consistency* conditions that relate choices within feasible sets of variable size, namely conditions α and γ (Sen, 1971). Clearly, acyclicity of the strict part P of R is necessary and sufficient for S to be rationalizable if every finite set of alternatives is feasible. Stronger rationality conditions can be obtained by requiring the rationalizing relation R to satisfy certain structural restrictions, such as completeness, transitivity, or quasi-transitivity (i.e., transitivity of P).

The above considerations have had a profound impact on the theory of social choice, in particular on the interpretation of Arrow's general impossibility theorem (Arrow, 1951), which states the impossibility of social choice functions that satisfy four intuitive criteria, including rationalizability via a transitive preference relation. An obvious way around Arrow's disturbing result is to try to relax this condition, e.g., by requiring social choice functions to be merely rationalizable. Although this approach does allow for some social choice functions that also meet the remaining three criteria, these functions turned out to be highly objectionable, usually on grounds of involving a weak kind of dictatorship or violating other conditions deemed to be indispensable for rational social choice (for an overview of the extensive literature, see Blair et al., 1976; Kelly, 1978; Schwartz, 1986; Sen, 1977, 1986; Campbell and Kelly, 2002). Sen (1995, page 5) concludes that

[...] the arbitrariness of power of which Arrow's case of dictatorship is an extreme example, lingers in one form or another even when transitivity is dropped, so long as *some* regularity is demanded (such as the absence of cycles).

One possibility to escape the haunting impossibility of rationalizable social choice is to require only α or γ but not both at the same time. It turns out that α (and even substantially weakened versions of α) give rise to impossibility results that retain Arrow's spirit (Sen, 1977). By contrast, there are a number of social choice functions that satisfy γ . An attractive one among these based on majority rule is the *uncovered set* (Fishburn, 1977; Miller, 1980; Moulin, 1986).

In this paper, we approach the matter from a slightly different angle. Choice functions are defined so as to select *subsets* of alternatives from each feasible set, rather than a single alternative. Still, the consistency and rationality conditions on choice functions

that violation of internal consistency conditions need not necessarily point at irrational behavior as such but can just as well indicate the presence of situational features that affect choice but are not (appropriately) represented in the mathematical model. In some cases the set of alternatives can be redefined so as to capture all aspects that affect their choice. For instance, the choice of the modest person above is arguably not between mere pieces of cake, but rather between tuples that consist of a piece of cake and the pieces left for others to choose from.

²Rationalizable choice functions have also been referred to as *binary* (Schwartz, 1976), *normal* (Sen, 1977), and *reasonable* (Allingham, 1999).

have been defined in terms of alternatives. Taking cue from this observation, we propose an alternative notion of rationality called *set-rationalizability*. A choice function S is *set-rationalizable* if a binary relation R on all non-empty subsets of alternatives can be found such that for each feasible subset A , $S(A)$ is maximal with respect to R among all non-empty subsets of A .

We find that set-rationalizable choice functions can be characterized by $\hat{\alpha}$, a natural variant of α defined in terms of sets rather than alternatives. Despite its intuitive appeal, $\hat{\alpha}$ has played a remarkably small role in (social) choice theory (Chernoff, 1954; Aizerman and Aleskerov, 1995). Yet, it differentiates quite a number of well-known choice functions. In particular, we will show that various prominent social choice functions—such as all scoring rules, all scoring runoff rules, and all weak Condorcet extensions—do not satisfy $\hat{\alpha}$, whereas several Condorcet extensions—such as weak closure maximality, the minimal covering set, and the essential set—do.

For our second result, we introduce a new property $\hat{\gamma}$, which varies on γ in an analogous way as $\hat{\alpha}$ varies on α . It turns out that $\hat{\alpha}$ and $\hat{\gamma}$ characterize the class of *self-stable* choice functions, whose definition is inspired by earlier work of Dutta (1988) and Brandt (2009). Despite the logical independence of $\hat{\alpha}$ and $\hat{\gamma}$, the class of self-stable social choice functions also contains the Condorcet extensions mentioned above. These Condorcet extensions also satisfy all conditions typically appearing in Arrovian impossibility results except rationalizability, i.e., α and γ . Accordingly, by replacing α and γ with $\hat{\alpha}$ and $\hat{\gamma}$, the impossibility of rationalizable social choice can be avoided and turned into a possibility result.

2 Preliminaries

We assume there to be a universe U of at least three *alternatives*. Any subset of U from which alternatives are to be chosen is a *feasible set* (sometimes also called an *issue* or *agenda*). Throughout this paper we assume the set of feasible subsets of U to be given by $\mathcal{F}(U)$, the set of finite and non-empty subsets of U , and generally refer to finite non-empty subsets of U as feasible sets. Our central object of study are *choice functions*, i.e., functions $S : \mathcal{F}(U) \rightarrow \mathcal{F}(U)$ such that $S(A) \subseteq A$ for all feasible sets A .

A choice function S is called *rationalizable* if there exists a binary relation R on U such that for each feasible set A ,

$$S(A) = \{a \in A : x P a \text{ for no } x \in A\}$$

where P is the strict part of R . Observe that acyclicity of P is required to guarantee that S invariably returns a non-empty set.

Two typical candidates for the rationalizing relation are the *base relation* \overline{R}_S (Herzberger, 1973) and the *revealed preference relation* R_S (Samuelson, 1938), which, for all alternatives x and y , are given by

$$\begin{aligned} & a \overline{R}_S b \text{ if and only if } a \in S(\{a, b\}), \text{ and} \\ & a R_S b \text{ if and only if } a \in S(X) \text{ for some } X \text{ with } b \in X. \end{aligned}$$

Thus, the revealed preference relation relates a to b if a is chosen in the presence of b and possibly other alternatives, whereas the base relation only relates a to b if a is chosen in the exclusive presence of b .

Rationalizable choice functions are characterized by a consistency axiom, which Schwartz (1976) defined such that for all feasible sets A and B and all alternatives $x \in A \cap B$,

$$x \in S(A \cup B) \text{ if and only if } x \in S(A) \text{ and } x \in S(B).$$

The above equivalence can be factorized into two implications, viz. the conditions α and γ (Sen, 1971) for feasible sets A and B and alternatives $x \in A \cap B$,³

$$\text{if } x \in S(A \cup B) \text{ then } x \in S(A) \text{ and } x \in S(B), \quad (\alpha)$$

$$\text{if } x \in S(A) \text{ and } x \in S(B) \text{ then } x \in S(A \cup B). \quad (\gamma)$$

Axiom α is a *contraction* consistency property, which states that alternatives that are chosen in a feasible set are still chosen in feasible subsets. By contrast, γ is an *expansion* consistency property, which states that alternatives chosen in two feasible sets are also chosen in their union. Sen (1971) proved that a choice function S is rationalizable if and only if it satisfies both α and γ , with the witnessing relations \overline{R}_S and R_S , which are identical in the presence of α .

Theorem 1 (Sen, 1971). *A choice function is rationalizable if and only if it satisfies both α and γ .*

Similar results can also be obtained if stronger requirements are imposed on the rationalizing relation (see, e.g., Sen, 1977; Moulin, 1985a; Schwartz, 1976). For instance, Arrow (1959) showed that a choice function can be rationalized by a complete and transitive relation if and only if it satisfies the *weak axiom of revealed preference (WARP)*—a consistency condition, first proposed by Samuelson (1938), which is stronger than the conjunction of α and γ and central to large parts of microeconomic theory. Formally, WARP is defined such that for all feasible sets A and B with $B \subseteq A$,

$$\text{if } S(A) \cap B \neq \emptyset \text{ then } S(A) \cap B = S(B). \quad (\text{WARP})$$

3 Set-Rationalizable Choice

In analogy to the definitions of Section 2, we now define the concept of set-rationalizability, the base and revealed preference relations over sets of alternatives, and properties $\widehat{\alpha}$ and $\widehat{\gamma}$. The main result of this section is that set-rationalizable choice is completely characterized by $\widehat{\alpha}$.

We say a choice function is *set-rationalizable* if it can be rationalized via a preference relation on sets of alternatives.

³The definitions of α and γ given here are equivalent, but not syntactically identical, to Sen's original ones. They are chosen so as they reveal their similarity to $\widehat{\alpha}$ and $\widehat{\gamma}$ below.

Definition 1. A choice function S is *set-rationalizable* if there exists a binary relation $R \subseteq \mathcal{F}(U) \times \mathcal{F}(U)$ such that for each feasible set A there is no $X \in \mathcal{F}(A)$ with $X P S(A)$ where P is the strict part of R .

We define the *base relation* \overline{R}_S and the *revealed preference relation* \widehat{R}_S of a choice function S on sets as follows:⁴

$$\begin{aligned} A \overline{R}_S B &\text{ if and only if } A = S(A \cup B), \\ A \widehat{R}_S B &\text{ if and only if } A = S(X) \text{ for some } X \text{ with } B \subseteq X. \end{aligned}$$

3.1 Set-contraction consistency

Condition $\widehat{\alpha}$ is defined as a natural variant of α that makes reference to the entire set of chosen alternatives rather than its individual elements.

Definition 2. A choice function S satisfies $\widehat{\alpha}$, if for all feasible sets A , B , and X with $X \subseteq A \cap B$,

$$\text{if } X = S(A \cup B) \text{ then } X = S(A) \text{ and } X = S(B). \quad (\widehat{\alpha})$$

$\widehat{\alpha}$ is not implied by the standard contraction consistency condition α (see Example 2). Moreover, $\widehat{\alpha}$ is not a contraction consistency property according to Sen's original terminology (see, e.g., Sen, 1977). It does not only require that chosen alternatives remain in the choice set when the feasible set is reduced, but also that unchosen alternatives remain outside the choice set. Thus, it has the flavor of both contraction and expansion consistency. $\widehat{\alpha}$ can be split into two conditions that fall in Sen's categories: an expansion condition known as ϵ^+ (Bordes, 1983) and *Aizerman* (Moulin, 1986), which requires that $S(B) \subseteq S(A)$ for all $S(A) \subseteq B \subseteq A$, and a corresponding expansion condition. Similarly, $\widehat{\gamma}$ can be factorized into two conditions.

In this paper, however, we are concerned with the choice set as a whole and $\widehat{\alpha}$ merely says that the set $S(A)$ chosen from a feasible set A is also chosen from any subset B of A , provided the former contains $S(A)$. This reading is reflected by the useful characterization of $\widehat{\alpha}$ given in the following lemma, which reveals that $\widehat{\alpha}$ is equivalent to such established notions as Chernoff's *postulate 5** (Chernoff, 1954), the *strong superset property* (Bordes, 1979), and *outcast* (Aizerman and Aleskerov, 1995).

Lemma 1. A choice function S satisfies $\widehat{\alpha}$ if and only if for all feasible sets A and B ,

$$\text{if } S(A) \subseteq B \subseteq A \text{ then } S(A) = S(B).$$

Proof. For the direction from left to right, let $S(A) \subseteq B \subseteq A$. Then, both $A \cup B = A$ and $B = A \cap B$. Hence, $S(A \cup B) = S(A) \subseteq B = A \cap B$. Since S satisfies $\widehat{\alpha}$, $S(A) = S(B)$.

For the opposite direction, assume for an arbitrary non-empty set X , both $X \subseteq A \cap B$ and $X = S(A \cup B)$. Then, obviously, both $S(A \cup B) \subseteq A \subseteq A \cup B$ and $S(A \cup B) \subseteq B \subseteq A \cup B$. It follows that $S(A \cup B) = S(A)$ and $S(A \cup B) = S(B)$. \square

⁴Given a choice function S , the base relation on sets is a natural extension of the base relation on alternatives and, hence, both are denoted by \overline{R}_S .

As a corollary of Lemma 1, we have that choice functions S satisfying $\hat{\alpha}$, like those satisfying α , are *idempotent*, i.e., $S(S(A)) = S(A)$ for all feasible sets A .

An influential and natural consistency condition that also has the flavor of both contraction and expansion is *path independence* (Plott, 1973). Choice function S satisfies path independence if $S(A \cup B) = S(S(A) \cup S(B))$ for all feasible sets A and B . Aizerman and Malishevski (1981) have shown that path independence is equivalent to the conjunction of α and ϵ^+ . Since α is the strongest contraction consistency property and implies the contraction part of $\hat{\alpha}$, we obtain the following alternative characterization of path independence.

Proposition 1. *A choice function satisfies path independence if and only if it satisfies α and $\hat{\alpha}$.*

It can easily be verified that the revealed preference relation on sets \hat{R}_S of any choice function S that satisfies $\hat{\alpha}$ is closed under intersection, i.e., for all feasible sets X , Y , and Z such that $Y \cap Z \neq \emptyset$,

$$X \hat{R}_S Y \text{ and } X \hat{R}_S Z \text{ imply } X \hat{R}_S Y \cap Z.$$

3.2 Set-expansion Consistency

We define $\hat{\gamma}$ in analogy to γ as follows.

Definition 3. A choice function S satisfies $\hat{\gamma}$ if for all feasible sets A , B , and X ,

$$\text{if } X = S(A) \text{ and } X = S(B) \text{ then } X = S(A \cup B). \quad (\hat{\gamma})$$

Thus, a choice function satisfies $\hat{\gamma}$, if whenever it chooses X from two different sets, it also chooses X from their union.

Example 2 shows that $\hat{\alpha}$ is not a weakening of α (and not even of the conjunction of α and γ). To see that $\hat{\gamma}$ is not implied by γ , consider the following choice function over the universe $\{a, b, c\}$, which satisfies γ but not $\hat{\gamma}$:

X	$S(X)$
$\{a, b\}$	$\{a\}$
$\{a, c\}$	$\{a\}$
$\{b, c\}$	$\{b\}$
$\{a, b, c\}$	$\{a, b, c\}$

However, $\hat{\gamma}$ is implied by the conjunction of α and γ .

Proposition 2. *Every rationalizable choice function satisfies $\hat{\gamma}$.*

Proof. Assume both α and γ to hold for an arbitrary choice function S and consider an arbitrary feasible sets X , A , and B with $X = S(A)$ and $X = S(B)$. The inclusion of X in $S(A \cup B)$ follows immediately from γ . To appreciate that also $S(A \cup B) \subseteq X$, consider an arbitrary $x \notin X$ and assume for contradiction that $x \in S(A \cup B)$. Then, either $x \in A$ or $x \in B$. Without loss of generality we may assume the former. Clearly, $x \in (A \cup B) \cap A$ and α now implies that $x \in S(A)$, a contradiction. \square

Schwartz (1976) has shown that quasi-transitive rationalizability is equivalent to the conjunction of α , γ , and ϵ^+ . Since α implies the contraction part of $\hat{\alpha}$ and α and γ imply $\hat{\gamma}$, we obtain the following alternative characterization of quasi-transitive rationalizability. As a consequence, WARP implies both $\hat{\alpha}$ and $\hat{\gamma}$.

Proposition 3. *A choice function is quasi-transitively rationalizable if and only if it satisfies α , $\hat{\alpha}$, and $\hat{\gamma}$.*

$\hat{\gamma}$ is reminiscent of the generalized Condorcet condition (see, e.g., Blair et al., 1976), which requires that for all feasible sets A and all $a \in A$,

$$\text{if } S(\{a, b\}) = \{a\} \text{ for all } b \in A \text{ then } S(A) = \{a\}.$$

Choice functions that satisfy this condition we will refer to as *generalized Condorcet extensions*. It is easily appreciated that $\hat{\gamma}$ implies the generalized Condorcet condition. In the setting of social choice, *Condorcet extensions* are commonly understood to be social choice functions for which additionally choice over pairs is determined by majority rule (see Section 5.3.1).

In analogy to the relationship between closure under intersection of \hat{R}_S and $\hat{\alpha}$, \hat{R}_S of a choice function S that satisfies $\hat{\gamma}$ is closed under union,⁵ i.e., for all feasible sets X , Y , and Z ,

$$X \hat{R}_S Y \text{ and } X \hat{R}_S Z \text{ imply } X \hat{R}_S Y \cup Z.$$

3.3 Set-Rationalizability

As in the case of α and γ , a single intuitive consistency condition summarizes the conjunction of $\hat{\alpha}$ and $\hat{\gamma}$: for all feasible sets A , B , and X with $X \subseteq A \cap B$,

$$X = S(A) \text{ and } X = S(B) \text{ if and only if } X = S(A \cup B).$$

For illustrative purposes consider the following examples.

Example 1. Let the choice function S over the universe $\{a, b, c\}$ be given by the table in Figure 1. For S the revealed preference relation on sets \hat{R}_S and the base relation on sets \bar{R}_S coincide and are depicted in the graph on the right. A routine check reveals that this choice function satisfies both $\hat{\alpha}$ and $\hat{\gamma}$ (while it fails to satisfy α). Also observe that each feasible set X contains a subset that is maximal (with respect to \hat{R}_S) among the non-empty subsets of X , e.g., $\{a, b, c\}$ in $\{a, b, c\}$ and $\{a\}$ in $\{a, b\}$. Theorem 2, below, shows that this is no coincidence.

Example 1 also shows that the revealed preference relation over sets need not be complete. Some reflection reveals that the relation is always incomplete.

Example 2. The table in Figure 2 summarizes a choice function that is rationalizable by any acyclic relation P with $a P c P b$. Nevertheless, the revealed preference relation

⁵This condition is also known as *robustness* (Arlegi, 2003, see also Barberà et al. (2004)).

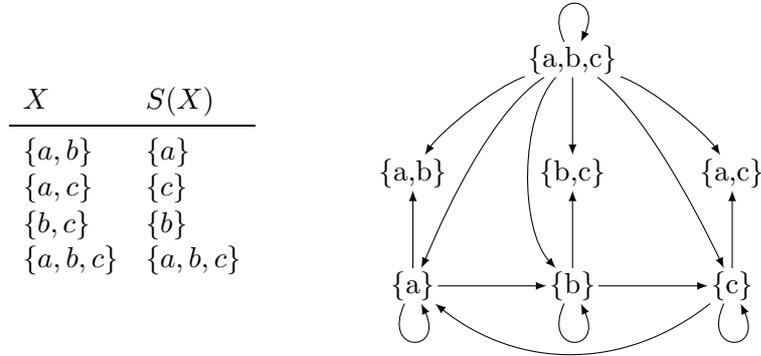


Figure 1: The revealed preference relation \widehat{R}_S of the choice function S as in Example 1.

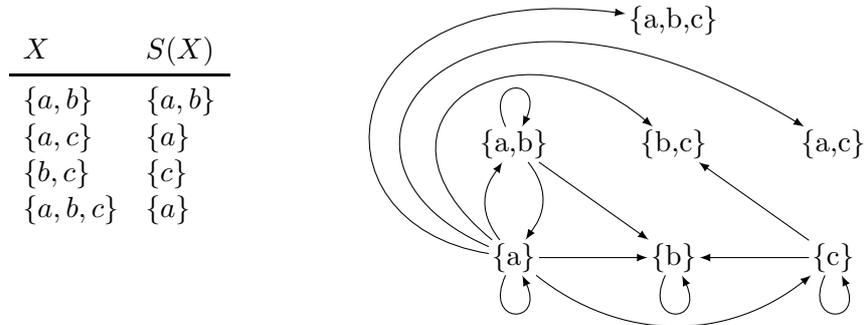


Figure 2: The revealed preference relation \widehat{R}_S of the choice function S as in Example 2.

over sets does not set-rationalize this choice function. Observe that both $\{a\}$ and $\{a, b\}$ are maximal in $\{a, b\}$ with respect to the strict part of \widehat{R}_S . As $S(\{a, b, c\}) = \{a\}$ and $S(\{a, b\}) = \{a, b\}$, S clearly does not satisfy $\widehat{\alpha}$. Again Theorem 2, below, shows that this is no coincidence.

By definition, the base relation \overline{R}_S of any choice function S is anti-symmetric, i.e., $X \overline{R}_S Y$ and $Y \overline{R}_S X$ imply $X = Y$. In the presence of $\widehat{\alpha}$, \widehat{R}_S and \overline{R}_S coincide and are thus both anti-symmetric.

Set-rationalizable choice functions are characterized by $\widehat{\alpha}$.⁶

Theorem 2. *A choice function is set-rationalizable if and only if it satisfies $\widehat{\alpha}$.*

Proof. For the direction from left to right, assume S is set-rationalizable and let R be the witnessing binary relation on sets. Now consider arbitrary feasible sets A, B and X with $X \subseteq A \cap B$ and assume that $X = S(A \cup B)$. Then, $S(A \cup B) \subseteq A \cap B$. Hence, $Y P S(A \cup B)$, for no non-empty subset $Y \subseteq A \cup B$. It follows that there is no non-empty subset Y of A such that $Y P S(A \cup B)$ either. Hence, $S(A \cup B)$ is maximal with respect to R within A . As S is set-rationalizable, $S(A \cup B)$ has also to be the unique such subset in A . The argument that $S(A \cup B)$ is also the unique maximal element of R in B runs along analogous lines. It follows that both $S(A) = S(A \cup B)$ and $S(B) = S(A \cup B)$.

For the opposite direction assume S to satisfy $\widehat{\alpha}$ and consider arbitrary feasible sets A and B such that $B \subseteq A$ and $S(A) \neq B$. Then,

$$S(A) \subseteq B \cup S(A) \subseteq A.$$

Hence, by virtue of Lemma 1,

$$S(A) = S(B \cup S(A)),$$

which implies that $S(A) \overline{R}_S B$. Since \overline{R}_S is anti-symmetric, it is thus impossible that $B \overline{R}_S S(A)$. A similar argument holds for \widehat{R}_S , which coincides with \overline{R}_S in the presence of $\widehat{\alpha}$. \square

In the proof of Theorem 2, it is precisely the revealed preference relation on sets that is witness to the fact that choice functions satisfying $\widehat{\alpha}$ are set-rationalizable. In contrast to Sen's Theorem 1, however, the revealed preference relation on sets is not the unique relation that can achieve this. It is also worth observing that the proof shows that for each feasible set X and choice function S satisfying $\widehat{\alpha}$, the selected set $S(X)$ is not merely a maximal set but also the unique *maximum* set within X given \widehat{R}_S , i.e., $S(X) \widehat{R}_S Y$ for all non-empty subsets Y of X .

4 Self-Stability

It turns out that the notions of set consistency introduced in the previous section bear a strong relationship to the stability of choice sets as introduced by Dutta (1988) and

⁶Moulin shows a similar statement for single-valued choice functions (Moulin, 1985a).

generalized by Brandt (2009). Stability of choice sets is based on the notions of internal and external stability by von Neumann and Morgenstern (1944), which can be merged in the following fixed-point characterization.

Definition 4. Let A, X be feasible sets and S a choice function. X is S -stable in A if

$$X = \{a \in A : a \in S(X \cup \{a\})\}.$$

Alternatively, X is S -stable in A if it satisfies both *internal* and *external S -stability* in A :

$$S(X) = X, \quad (\text{internal } S\text{-stability})$$

$$a \notin S(X \cup \{a\}) \text{ for all } a \in A \setminus X. \quad (\text{external } S\text{-stability})$$

The intuition underlying stable sets is that there should be no reason to restrict the selection by excluding some alternative from it and, secondly, there should be an argument against each proposal to include an outside alternative into the selection.

For some choice functions S , a unique inclusion-minimal S -stable set generally exists. If that is the case, we use \widehat{S} to denote the choice function that returns the unique minimal S -stable set in each feasible set and say that \widehat{S} is *well-defined*. Within the setting of social choice, a prominent example is Dutta's *minimal covering set* MC (Dutta, 1988; Dutta and Laslier, 1999), which is defined as the unique minimal stable set with respect to the uncovered set UC , i.e., $MC = \widehat{UC}$. Proving that a choice function \widehat{S} is well-defined frequently turns out to be highly non-trivial (Brandt, 2009).

We find that there is a close connection between $\widehat{\gamma}$ and minimal S -stable sets.

Lemma 2. *Let S be a choice function such that \widehat{S} is well-defined. Then \widehat{S} satisfies $\widehat{\gamma}$.*

Proof. Consider arbitrary feasible sets A, B, X and assume that $\widehat{S}(A) = \widehat{S}(B) = X$. Trivially, as X is internally S -stable in A , so is X in $A \cup B$. To appreciate that X is also externally S -stable in $A \cup B$, consider an arbitrary $x \in (A \cup B) \setminus X$. Then, $x \in A \setminus X$ or $x \in B \setminus X$. In either case, $S(X \cup \{x\}) = X$, by external S -stability of X in A if the former, and by external S -stability of X in B if the latter. Also observe that any subset of X that is S -stable in $A \cup B$, would also have been S -stable in both A and B . Hence, X is minimal S -stable in $A \cup B$. Having assumed that \widehat{S} is well-defined, we may conclude that $\widehat{S}(A \cup B) = X$. \square

We now introduce the notion of self-stability. A choice function S is said to be self-stable if for each feasible set A , $S(A)$ is the unique (minimal) S -stable set in A .

Definition 5. A choice function S is *self-stable* if \widehat{S} is well-defined and $S = \widehat{S}$.

In the next section we argue that self-stability defines an interesting class of choice functions, containing a number of well-known and important social choice functions. First, and on a more abstract level, however, we establish that the class of self-stable choice functions is characterized by the conjunction of $\widehat{\alpha}$ and $\widehat{\gamma}$.

Theorem 3. *A choice function is self-stable if and only if it satisfies both $\hat{\alpha}$ and $\hat{\gamma}$.*

Proof. For the direction from left to right, assume S to be self-stable. Lemma 2 implies that S satisfies $\hat{\gamma}$. For $\hat{\alpha}$, consider arbitrary feasible sets A, B such that $S(A) \subseteq B \subseteq A$. By virtue of Lemma 1, it suffices to show that $S(B) = S(A)$. As, moreover, $\hat{S}(B) = S(B)$ and \hat{S} is well-defined, $S(B)$ is the unique S -stable set in B . Hence, it suffices to show that $S(A)$ is both internally and externally S -stable in B . Internal S -stability of $S(A)$ in B is trivial since $S(S(A)) = S(A)$ by $S(A)$'s being internally S -stable in A . To appreciate that $S(A)$ is also externally S -stable in B , consider an arbitrary $x \in B \setminus S(A)$. Then also $x \in A \setminus S(A)$ and by $S(A)$'s being externally S -stable in A , we obtain $S(A) = S(S(A) \cup \{x\})$. It follows that $S(A)$ is also externally S -stable in B .

For the other direction, assume S satisfies both $\hat{\alpha}$ and $\hat{\gamma}$ and consider an arbitrary feasible set A . To show that S satisfies internal S -stability, observe that trivially $S(A) \subseteq S(A) \subseteq A$. Hence, by Lemma 1, $S(A) = S(S(A))$. To appreciate that S also satisfies external S -stability, let $x \in A \setminus S(A)$. Then, $S(A) \subseteq S(A) \cup \{x\} \subseteq A$ and, again by Lemma 1, $S(A) = S(S(A) \cup \{x\})$. To see that \hat{S} is well-defined, consider an arbitrary S -stable set Y in A and let $A \setminus Y = \{x_1, \dots, x_k\}$, i.e., $A = Y \cup \{x_1, \dots, x_k\}$. By external S -stability of Y , then $S(Y \cup \{x_i\}) = S(Y)$ for each i with $1 \leq i \leq k$. Thus by $k - 1$ applications of $\hat{\gamma}$, we obtain $S(Y) = S(Y \cup \{x_1, \dots, x_k\}) = S(A)$. \square

As an immediate consequence of Theorem 3 and the observation that $\hat{\gamma}$ implies the generalized Condorcet condition, we have the following corollary.

Corollary 1. *Every self-stable choice function is a generalized Condorcet extension.*

Examples of self-stable Condorcet extensions will be given in Section 5.3.1.

5 Social Choice

In this section, we assess the consequences of the reflections in the previous two sections on the theory of social choice. Before we do so, however, we introduce some additional terminology and notation.

5.1 Social Choice Functions

We consider a finite set $N = \{1, \dots, n\}$ of at least two agents. Each agent i entertains preferences over the alternatives in U , which are represented by a transitive and complete preference relation R_i . In some cases, we will assume preferences to be linear, i.e., also satisfying anti-symmetry, but otherwise we impose no further restrictions on preference relations. We have $a R_i b$ denote that agent i values alternative a at least as much as alternative b . We write P_i for the strict part of R_i , i.e., $a P_i b$ if $a R_i b$ but not $b R_i a$. Similarly, I_i denotes i 's indifference relation, i.e., $a I_i b$ if both $a R_i b$ and $b R_i a$. The set of all preference relations over the universal set of alternatives U will be denoted by $\mathcal{R}(U)$. The set of *preference profiles*, with typical element $R = (R_1, \dots, R_n)$, is then given by $\mathcal{R}(U)^N$.

The central object of study in this section are *social choice functions*, i.e., functions that map the individual preferences of the agents and a feasible set to a set of socially preferred alternatives.

Definition 6. A *social choice function (SCF)* is a function $f : \mathcal{R}(U)^N \times \mathcal{F}(U) \rightarrow \mathcal{F}(U)$ such that $f(R, A) \subseteq A$ for all preference profiles R and feasible sets A .

Clearly, every SCF f together with a preference profile R in $\mathcal{R}(U)$ defines a choice function $S_{f,R}$ on U in a natural way by letting for each feasible set A , $S_{f,R}(A) = f(R, A)$. We say that f satisfies WARP, rationalizability, or any other condition defined for choice functions, if $S_{f,R}$ does for every preference profile R . *Pareto-optimality*, *independence of irrelevant alternatives*, and *non-dictatorship* are conditions that are more specifically defined for SCFs.

Pareto-optimality requires that an alternative should not be chosen if there exists another alternative that *all* agents unanimously prefer to the former.

Definition 7. An SCF f satisfies (pairwise) *Pareto-optimality* if for all preference profiles R and all alternatives a, b , if $b P_i a$ for all $i \in N$ then $a \notin f(R, \{a, b\})$.

Independence of irrelevant alternatives reflects the idea that choices from a set of feasible alternatives should not depend on preferences over alternatives that are not contained in this set.

Definition 8. An SCF f satisfies *independence of irrelevant alternatives (IIA)* if $f(R, A) = f(R', A)$ for all feasible sets A and preference profiles R, R' such that $R|_A = R'|_A$.

In the context of SCFs, IIA constitutes no more than a framework requirement for social choice.

Another minimal requirement for any SCF is that it should be sensitive to the preferences of more than one agent. In particular, there should not be a single agent who can enforce the inclusion of alternatives in the choice set no matter which preferences the other agents have. Such an agent is usually called a (weak) *dictator*.⁷

Definition 9. An SCF f is (pairwise) *non-dictatorial* if there is no agent i such that for all preference profiles R and alternatives a, b , if $a P_i b$ then $a \in f(R, \{a, b\})$.

Definition 7 through Definition 9 are also referred to as the *Arrovian conditions*. Other useful and frequently imposed requirements on SCFs are *neutrality*, *anonymity*, and *positive responsiveness*.

Neutrality can be seen as a strengthening of IIA and requires SCFs to be invariant under renaming alternatives, i.e., all alternatives are to be treated equally.

Definition 10. An SCF f is *neutral* if $\pi(f(R, A)) = f(R', A)$ for all feasible sets A , preference profiles R, R' , and permutations $\pi : A \rightarrow A$ such that $a R'_i b$ if and only if $\pi(a) R_i \pi(b)$ for all alternatives a, b and agents i .

⁷For presentational purposes we employ the notion of a *weak dictator* or *vetoer* in all impossibility theorems, although Theorem 4 holds for an even weaker notion of non-dictatorship.

By contrast, anonymity says that SCFs be invariant under renaming agents and as such is a strong variant of non-dictatorship.

Definition 11. An SCF f is *anonymous* if $f(R, A) = f(R', A)$ for all feasible sets A , preference profiles R and R' , and permutations $\pi : N \rightarrow N$ such that $R'_i = R_{\pi(i)}$ for all agents i .

It also appears reasonable to demand that SCFs are monotonic in the sense that increased support may not hurt an alternative.

Definition 12. An SCF f is (pairwise) *positive responsive* if for all alternatives a, b and all preference profiles R, R' , there is some agent i such that $R_j = R'_j$ for all agents $j \neq i$ and either both $a I_i b$ and $a P'_i b$ or both $b P_i a$ and $a R'_i b$,

$$\text{if } a \in f(R, \{a, b\}) \text{ then } f(R', \{a, b\}) = \{a\}.$$

5.2 Impossibility Results

Famously, Arrow's general impossibility theorem, as formulated for SCFs, states that no SCF that satisfies all of the Arrovian conditions exists.

Theorem 4 (Arrow, 1951; 1959). *No SCF satisfies Pareto-optimality, IIA, WARP, and non-dictatorship.*

As the Arrovian conditions cannot be satisfied by any SCF, at least one of them needs to be excluded or relaxed to obtain positive results. Clearly, dropping non-dictatorship is unacceptable and, as already mentioned, IIA merely states that the SCF represents a reasonable model of preference aggregation (see, e.g., Schwartz, 1986; Bordes and Tideman, 1991). Wilson (1972) has shown that without Pareto-optimality only SCFs that are constant (i.e., completely unresponsive) or fully determined by the preferences of a single agent are possible. Moreover, it could be argued that not requiring Pareto-optimality runs counter to the very idea of *social* choice. Accordingly, the only remaining possibility is to exclude WARP.

Imposing weaker rationality conditions than WARP, however, offers little relief as it turns out that the vicious essence of Arrow's impossibility remains. There is a range of results stating the impossibility of SCFs satisfying weaker versions of WARP in a satisfactory way (see, e.g., Kelly, 1978; Schwartz, 1986; Campbell and Kelly, 2002; Banks, 1995). Among these, the results by Mas-Colell and Sonnenschein (1972) and Blau and Deb (1977) deserve special mention as they concern rationalizability instead of WARP. We will employ a variant of Blau and Deb's theorem due to Austen-Smith and Banks (2000).

Theorem 5 (Mas-Colell and Sonnenschein, 1972). *No SCF satisfies Pareto-optimality, positive responsiveness, IIA, rationalizability, and non-dictatorship, provided that $n > 3$.*

By strengthening IIA to neutrality and assuming that the number of alternatives exceeds the number of agents, positive responsiveness is no longer required.

1	1	1	$\{x, y\}$	$f(R, \{x, y\})$
a	c	b	$\{a, b\}$	$\{a\}$
b	a	c	$\{b, c\}$	$\{b\}$
c	b	a	$\{a, c\}$	$\{c\}$

Table 1: On the left a preference profile (figures indicate numbers of agents) leading to the Condorcet paradox. On the right the corresponding choice function on pairs if determined by majority rule.

Theorem 6 (Austen-Smith and Banks, 2000). *No SCF satisfies Pareto-optimality, neutrality, rationalizability, and non-dictatorship, provided that $|U| > n$.*

For further characterizations of rationalizable social choice the reader be referred to Moulin (1985b), Banks (1995), and Austen-Smith and Banks (2000).

5.3 Condorcet Extensions and Scoring Rules

In light of the severe problems that α and γ entail in social choice, we now investigate which of the well-known SCFs satisfy $\hat{\alpha}$ and $\hat{\gamma}$. We focus on two types of SCFs, namely *Condorcet extensions* and *scoring rules*.

5.3.1 Condorcet Extensions

Despite the Arrovian impossibility results, social choice over two alternatives is unproblematic. May (1952) has shown that the *simple majority rule*—choosing the alternative that a majority prefers to the other alternative, and in case of a tie, choose both—can be characterized by neutrality, anonymity, and positive responsiveness. Thus, it seems reasonable to require of SCFs f that they reflect majority rule on pairs. Extending any such SCF f to feasible sets with more than two alternatives, one immediately runs into arguably one of the earliest Arrovian impossibility results, viz. the *Condorcet paradox* (de Condorcet, 1785). Consider the preference profile depicted in Table 1. Then, if f on pairs is determined by majority rule, the base relation $\overline{R}_{S_f, R}$ fails to be acyclic, and therefore $S_{f, R}$ does not satisfy α . Observe, moreover, that $f(R, \{a, b, c\}) = \{a, b, c\}$, if f satisfies $\hat{\alpha}$. For suppose otherwise, then, without loss of generality, we may assume that either $f(R, \{a, b, c\}) = \{a\}$ or $f(R, \{a, b, c\}) = \{a, b\}$. Assuming that f satisfies $\hat{\alpha}$, however, the former is at variance with $f(R, \{a, c\}) = \{c\}$, and the latter with $f(R, \{a, b\}) = \{a\}$. Thus, $S_{f, R}$ coincides with the choice function S of Example 1 and Figure 1 depicts its weak revealed preference relation $\widehat{R}_{S_{f, R}}$.

By Theorem 3, the class of SCFs that satisfy both $\hat{\alpha}$ and $\hat{\gamma}$ consists precisely of all self-stable SCFs. By virtue of Theorem 2 and Corollary 1 the SCFs in this class are all set-rationalizable generalized Condorcet extensions. May’s characterization furthermore implies that all self-stable SCFs that satisfy anonymity, neutrality, and positive responsiveness are Condorcet extensions. Among them are well-known rules like *weak closure*

3	2	1
a	b	c
c	a	b
b	c	a

Table 2: A preference profile (figures indicate numbers of agents) showing that no weak Condorcet extension and scoring rule satisfies $\hat{\alpha}$. For every weak Condorcet extension and every scoring rule the choice function for this profile is as in Example 2 (also compare Figure 2).

maximality (also known as the *top cycle*, *GETCHA*, or the *Smith set*), the *minimal covering set*, the *essential set*, and their generalizations (Bordes, 1976; Dutta and Laslier, 1999; Laslier, 2000).⁸

Interestingly, well-known Condorcet extensions that satisfy only one of $\hat{\alpha}$ and $\hat{\gamma}$ appear to be less common. Still, Schwartz’s *strong* closure maximality (Schwartz, 1972), which he refers to as GOCHA, is an example of an SCF that satisfies $\hat{\gamma}$ but not $\hat{\alpha}$. By contrast, the *iterated uncovered set* (e.g., Dutta, 1988) satisfies $\hat{\alpha}$ but not $\hat{\gamma}$.

When pairwise choice is determined via majority rule, the set of *weak Condorcet winners* for a given preference profile R and feasible set A is defined as $\{a \in A : a \in f(R, \{a, b\}) \text{ for all } b \in A\}$. An SCF is called a *weak Condorcet extension* if it returns the set of weak Condorcet winners whenever this set is non-empty. Clearly, every weak Condorcet extension is a Condorcet extension. The converse is not generally the case, but many Condorcet extensions (such as Kemeny’s rule, Dodgson’s rule, Nanson’s rule, and the minimax rule) are also weak Condorcet extensions (see Fishburn, 1977). It turns out that no weak Condorcet extension is set-rationalizable.

Theorem 7. *No weak Condorcet extension satisfies $\hat{\alpha}$.*

Proof. Let f be a weak Condorcet extension and consider the linear preference profile R with preferences over $A = \{a, b, c\}$ as given in Table 2. Since alternative a is the unique weak Condorcet winner—three out of six agents prefer it to b and all agents but one prefer it over c — $f(R, \{a, b, c\}) = \{a\}$. Now observe that the preferences of the same agents over the subset $\{a, b\}$ are such that three agents prefer a to b and three b to a . Accordingly, $f(R, \{a, b\}) = \{a, b\}$. As $c \notin f(R, \{a, b, c\})$ but $f(R, \{a, b, c\}) \neq f(R, \{a, b\})$, we may conclude that f does not satisfy $\hat{\alpha}$. \square

5.3.2 Scoring Rules

Scoring rules are based on the idea that the voters each rank the alternatives in a feasible set according to their preferences, which, for technical convenience we will here assume

⁸Brandt (2009) defines an infinite hierarchy of self-stable SCFs. If we assume an odd number of agents with linear preferences, the class of self-stable SCFs is also conjectured to contain the *tournament equilibrium set* (Schwartz, 1990) and the *minimal extending set*. Whether this is indeed the case depends on a certain graph-theoretic conjecture (Laffond et al., 1993; Brandt, 2009).

to be linear. Each time an alternative is ranked m th by some voter it gets a particular score s_m . The scores of each alternative are then added and the alternatives with the highest cumulative score are selected. The class of scoring rules includes several well-known SCFs, like the *Borda rule*—alternative a gets k points from agent i if i prefers a to k other alternatives—and the *plurality rule*—the cumulative score of an alternative equals the number of agents by which it is ranked first.

Formally, we define a *score vector of length k* as a vector $s = (s_1, \dots, s_k)$ in \mathbb{R}^k such that $s_1 \geq \dots \geq s_k$ and $s_1 > s_k$. For example, $(1, 0, 0)$, $(2, 1, 0)$, and $(1, 1, 0)$ are the score vectors of length 3 for the *plurality rule*, the *Borda rule*, and the *anti-plurality rule*, respectively. Given a feasible set X of k alternatives, an $x \in X$, and a linear preference profile R , we have $s(x, i)$ denote the score alternative x obtains from voter i , i.e., $s(x, i) = s_m$ if and only if x is ranked m th by i within X . Then, the (cumulative) score $s(x)$ of an alternative x within X given R is then defined such that

$$s(x) = \sum_{i \in N} s(x, i).$$

A *scoring rule* is an SCF that selects from each feasible set X for each preferences profile the set of alternatives x in X with the highest score $s(x)$ according to some score vector s of length $|X|$. Observe that no restrictions are imposed on how the score vectors for different lengths are to be related.

As every scoring rule fails to select the Condorcet winner for some preference profile (Fishburn, 1973) and coincides with majority rule on two alternatives, they generally do not satisfy $\hat{\gamma}$. We find that no scoring rule can satisfy $\hat{\alpha}$ either. It follows that no scoring rule is set-rationalizable.

Theorem 8. *No scoring rule satisfies $\hat{\alpha}$.*

Proof. Let f be a scoring rule. Let further $s = (s_1, s_2, s_3)$ and $s' = (s'_1, s'_2)$ be its associated score vectors of lengths 3 and 2, respectively. Without loss of generality we may assume that $s_1 = 1$ and $s_3 = 0$. Consider the preference profile R with preferences over $A = \{a, b, c\}$ as depicted in Table 2. Then, $s(a) = 3 + 2s_2$, $s(b) = 2 + s_2$ and $s(c) = 1 + 3s_2$. Since $1 \leq s_2 \leq 0$, it can easily be appreciated that $s(a) > s(b)$ as well as $s(a) > s(c)$. Hence, $f(R, \{a, b, c\}) = \{a\}$. As in the proof of Theorem 7, $f(R, \{a, b\}) = \{a, b\}$ and we may conclude that f does not satisfy $\hat{\alpha}$. \square

Using the same example as in the proof of Theorem 8, the reader can easily verify that all scoring run-off rules—such as single transferable vote (STV)—also fail to satisfy $\hat{\alpha}$ and as such are not set-rationalizable.

An interesting question in this context is whether the impossibility shown in Theorem 8 can be generalized to *rank-based* SCFs, i.e., SCFs that merely take into account the positions of alternatives in the individual rankings (Laslier, 1996). It turns out this is not the case because, for instance, the (rather unattractive) SCF that chooses all alternatives that are ranked first by at least one voter is rank-based and satisfies $\hat{\alpha}$.⁹ It

⁹This SCF has also been mentioned by Gärdenfors (1976) and Kelly (1977). Taylor (2005) calls it the *omnination rule*.

might also be worth observing that this SCF happens to satisfy $\hat{\gamma}$ and Pareto-optimality as well.

6 Summary and Conclusion

Problems relating to the possibility of reasonable social choice functions have proved to be rather tenacious. In particular, attempts to circumvent Arrow's impossibility result by replacing the weak axiom of revealed preference (WARP), which requires a transitive and complete preference relation on alternatives underlying choice, by weaker conditions on the underlying preference relation have generally failed to deliver.

By weakening WARP to set-rationalizability, we have shown that social choice functions that also satisfy the other Arrovian postulates do exist. These social choice functions are generally characterized by their satisfying $\hat{\alpha}$. This condition, which also goes by the names of strong superset property, 5^* , and outcast, is no stranger in choice theory, but nevertheless has played a surprisingly small role therein. In an early publication, Chernoff writes that "*postulate 5^* is not imposed in our definition of a rational solution*" (Chernoff, 1954, page 430) and Aizerman and Aleskerov chime in by stating that "*this property $[\hat{\alpha}]$ did not find wide use in the choice theory literature*" (Aizerman and Aleskerov, 1995, page 21). The characterization of set-rationalizable choice functions via $\hat{\alpha}$ can be interpreted as a strong argument for this postulate.

By also imposing $\hat{\gamma}$, the set-expansion property that forms the counterpart of $\hat{\alpha}$, we obtain the class of self-stable generalized Condorcet extensions. In the context of social choice, this class comprises appealing social choice functions like the minimal covering set and the essential set, yet excludes other well-known rules like all scoring rules, all scoring runoff rules, and all weak Condorcet extensions. As such, self-stability defines a fascinating class of social choice functions, which offers an interesting way around the impossibility results that have haunted social choice theory for such a long time.

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