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On Efficiency of the English Auction

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Abstract

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KEYWORDS: English auction, efficient auction, ex post equilibrium, single-crossing, interdependent values.

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1 Introduction

How to sell a good efficiently—to the buyer who values it the most—is one of the main questions of the theory of auctions. The task becomes harder as the informational environment gets more complex. When the valuations of the buyers are asymmetric and depend on the private information of the others the set of efficient mechanisms is quite limited. Among these is the open ascending price, or English, auction. It is typically modeled as the irrevocable exit clock auction, and this model is known to possess an efficient equilibrium when value functions satisfy certain conditions. What is the minimal (necessary and sufficient) condition for efficiency of the English auction is a long-standing problem. This paper provides a solution.

In a classic paper, Milgrom and Weber (1982) introduce the irrevocable exit model of the English auction. They show that in the setting with symmetric interdependent values the English auction has an efficient equilibrium, and, if signals are affiliated, it generates higher revenues to the seller than other common auction forms. Maskin (1992) indicates that the pairwise single-crossing condition is necessary for efficiency of the asymmetric English auction, and shows that it is also a sufficient condition when the number of bidders is two. Perry and Reny (2001) provide an example with three

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bidders, where the pairwise single-crossing is satisfied and no efficient equilibrium exists. Krishna (2003) presents a pair of sufficient conditions for efficiency of the N-bidder English auction—an average-crossing and a cyclical-crossing conditions.[1]

We introduce the *generalized single-crossing* condition which is a natural extension of the pairwise single-crossing to the case of N bidders. The pairwise single-crossing imposes the following: if starting from a signal profile where the values of two bidders are equal and maximal we slightly increase the signal of one of the bidders, her value becomes the highest. This implies that the private information held by a bidder affects her valuation more than the valuations of her competitors. Our condition requires the following: if starting from a signal profile where the values of a group of bidders are equal and maximal we slightly increase the signals of a subset of the group, no bidder outside of the subset can attain the value higher than the maximal value attained among the bidders from the subset. The generalized single-crossing both implies the pairwise single-crossing and reduces to it in the case of two bidders.

Two main results of this paper are the *necessity*: if the generalized single-crossing condition is violated at some interior signal profile, then no efficient equilibrium in the N-bidder English auction exists; and the *sufficiency*: if value functions satisfy the generalized single-crossing condition both in the interior and on the boundary of the signals' domain, then there exists an efficient ex post equilibrium in the N-bidder English auction. If the generalized single-crossing is violated only on the boundary an efficient equilibrium may or may not exist, see Section 3.2. Given that the gap between the necessity and sufficiency statements is the set of measure zero we simply refer to the generalized single-crossing as to the necessary and sufficient condition.[2]

The English auction is not the only efficient mechanism in the interdependent values setting, and the generalized single-crossing is not the weakest condition for efficiency. What makes the English auction so special, aside from its widespread use, is the strategic simplicity, transparent set of rules, and ease of conducting. In the English auction, even if the values are interdependent, the strategy in the efficient equilibrium is nothing but "...drop out when the price reaches what you believe your value is." The "contingent bid" mechanism of Dasgupta and Maskin (2000) requires each buyer to submit a price she is willing to pay given the realized values of the others—a $(N-1)$-variable function. This auction is efficient if the pairwise single-crossing holds. Utilizing the fact that two-bidder sealed bid and ascending price auctions are efficient Perry and Reny (2002) and Perry and Reny (2001) design two elegant mechanisms that incorporate a concept of "directed bids"—every buyer bids against every other buyer, thus managing $N-1$ bids simultaneously. These

[1] The average-crossing condition requires that, if starting from a signal profile where the values of several bidders are equal and maximal, the signal of one of them is increased, the corresponding increments to the values of the others are lower than the average increment. The cyclical-crossing requires that the increments to the values are ranked in the prespecified cyclical order—the effect on the own value is the largest and decreases for each subsequent bidder in the cycle.

[2] In fact, even in the two-bidder case, the pairwise single-crossing is necessary up to the boundary, see Example 1 in Section 2. In this sense, our results are the exact extenstion of Maskin's.

auctions require the strong form of the pairwise single-crossing for efficiency.[3] The above mechanisms are remarkable constructions, designed to allocate multiple units efficiently. In their single unit version, however, they are significantly more complex than the English auction.[4]

Izmalkov (2003) proposes an alternative model of the English auction. In this model the bidders are allowed to reenter—become active again after they dropped out. Izmalkov shows that the English auction with reentry is efficient under the conditions that are weaker than the generalized single-crossing.[5] At the same time the possibility of reentry substantially enriches the strategy space and provides opportunities to exchange messages, which, potentially, may allow bidders to coordinate on a collusive outcome. In contrast, the irrevocable-exit English auction is robust to collusion within the auction. The fact that the exits are irrevocable implies that the bidders cannot coordinate their actions: the only way a bidder can send a message is by exiting, which makes winning impossible.

The rest of our paper is organized as follows. Section 2 contains the description of the environment and the discussion of the pairwise single-crossing. Section 3 introduces the generalized single-crossing condition and states our main necessity and sufficiency results. Sections 4 and 5 contain the proofs of the results.

2 Preliminaries

There is a single indivisible object to be auctioned among a set $\mathcal{N} = \{1, 2, ..., N\}$ of bidders. Prior to the auction each bidder j receives a real valued signal $s_j \in [0, 1]$. Signal s_j is bidder j's private information. Signals are distributed according to a joint density function $f(\mathbf{s})$, where profile $\mathbf{s} = (s_1, s_2, ..., s_N)$ represents signals of all the bidders. It is assumed that f has full support and is strictly positive on the interior of it.

If the realized signals are $\mathbf{s}$, the value of the object to bidder j is $V_j(\mathbf{s})$—it depends potentially on the information obtained by the other bidders. The sale of an oil track is a typical example of such an environment—a firm's estimate of the worth of the track may depend on the results of the "off-site" drilling conducted by a rival firm that owns an adjacent track, see Porter (1995).

Value functions $\mathbf{V} = (V_1, V_2, ..., V_N)$ are assumed to have the following properties. For any j, and any $i \neq j$: $V_j(\mathbf{0}) = 0$; $V_j(\mathbf{1}) < \infty$; V_j is twice-differentiable in $\mathbf{s}$;

[3] The pairwise single-crossing has to be satisfied for any pair of bidders with equal values, not only when their values are maximal. Thus, the strong pairwise single-crossing and the generalized single-crossing are not comparable.

[4] The generalized Vickrey auction is simple and efficient if the pairwise single-crossing holds, but, being a direct revelation mechanism, it requires the auctioneer to know everything that the bidders know about each other, which is "utterly unworkable in practice" (see Maskin (2003) for further discussion).

[5] The generalized single-crossing would imply that no reentry happens in the efficient equilibrium.

$\frac{\partial V_j}{\partial s_j} > 0$; and $\frac{\partial V_j}{\partial s_i} \geq 0$. Value functions V_j for all j and distribution $f(\mathbf{s})$ are assumed to be commonly known among the bidders.

We denote $\mathbf{s}_{\mathcal{A}} = (s_j)_{j\in\mathcal{A}}$—the signal profile of the bidders from a subset $\mathcal{A} \subset \mathcal{N}$, and $\mathbf{s}_{-\mathcal{A}}$—the signal profile of the bidders from $\mathcal{N} \setminus \mathcal{A}$.[6]

Definition 1 *For a given profile of signals* $\mathbf{s}$ *the* winners circle $\mathcal{I}(\mathbf{s})$ *is the set of bidders with the highest values imputed at* $\mathbf{s}$. *Formally,*

$$(1) \qquad i \in \mathcal{I}(\mathbf{s}) \Longleftrightarrow V_i(\mathbf{s}) = \max_{j\in\mathcal{N}} V_j(\mathbf{s}).$$

Thus, the object is allocated efficiently at s, if the person it goes to—the winner—belongs to the winners circle $\mathcal{I}(\mathbf{s})$.

We require value functions to be *regular*—at every s for any subset of bidders $\mathcal{J} \subset \mathcal{I}(\mathbf{s})$ it is assumed that $\det DV_{\mathcal{J}} \neq 0$, where $DV_{\mathcal{J}} = \left(\frac{\partial V_i(\mathbf{s})}{\partial s_j}\right)_{i,j\in\mathcal{J}}$ is the matrix of partial derivatives (Jacobian).

2.1 Pairwise Single-Crossing

Definition 2 *The* pairwise single-crossing (SC) condition *is* satisfied *if at any* $\mathbf{s}$ *with* $\#\mathcal{I}(\mathbf{s}) \geq 2$, *for any pair of bidders* $i, j \in \mathcal{I}(\mathbf{s})$,

$$(2) \qquad \frac{\partial V_i(s_j, \mathbf{s}_{-j})}{\partial s_j} \leq \frac{\partial V_j(s_j, \mathbf{s}_{-j})}{\partial s_j}.$$

In words, take a group of bidders who have equal and maximal values. If the signal of one of the bidders from the group is increased, the corresponding impact on the value of that bidder is the highest among the group.

We say that *SC* is *violated* at s if there exist bidders $i, j \in \mathcal{I}(\mathbf{s})$ such that

$$\frac{\partial V_i(s_j, \mathbf{s}_{-j})}{\partial s_j} > \frac{\partial V_j(s_j, \mathbf{s}_{-j})}{\partial s_j}.$$

We say that *SC* is *strictly satisfied* if (2) holds with strict inequalities.

2.2 The English Auction

Following Milgrom and Weber (1982) we consider a model that became a standard model for the analysis of English auctions. Specifically, the price of the object rises continuously, and bidders indicate whether they are willing to buy the object at that price or not. A bidder who is willing to buy at the current price is said to be an active bidder. At a price of 0 all the bidders are active, and, as the price rises,

[6] We denote vectors and sets by bold and calligraphic letters correspondingly; $\mathbf{a} \gg \mathbf{b}$ ($\mathbf{a} \geqq \mathbf{b}$) denotes that $a_i > b_i$ ($a_i \geq b_i$) in every component.

bidders can choose to drop out of the auction. The decision to drop out is both public and irrevocable. Thus, if bidder j drops out at a price p_j, both her identity and the exiting price p_j are observed by all the bidders. Furthermore, once bidder j drops out she cannot "reenter" the auction at a higher price. The auction ends at the moment when at most one bidder remains active. The clock stops, the only remaining bidder is the winner. If no bidders remain active the winner is chosen at random among those who exited last. The winner is obliged to pay the price shown on the clock.[7]

At price p all the bidders commonly know who was active at every preceding price. This *public history* $H(p)$ can be effectively summarized as a sequence of prices at which bidders, inactive at p, have exited, $H(p) = \mathbf{p}_{-\mathcal{M}}$, where $\mathcal{M}$ is the set of active bidders just before p. If no bidder exits at $p \in [p', p'')$, then $H(p') = H(p'')$. Denote with $\bar{H}(p)$ the public history $H(p)$ together with all exits that happen at p. Therefore, if $\bar{H}(p) \neq H(p)$, then there exists a bidder who exited at p. All the bidders are assumed to be active just before the clock starts at $p = 0$, so $H(0) = \varnothing$.

In the English auction a strategy of bidder j determines the price at which she would drop out given her signal and public history—given that no other bidder drops out earlier. Formally, following Krishna (2003), a bidding strategy for bidder $j \in \mathcal{M}$ is a collection of functions $\beta_j^{\mathcal{M}} : [0,1] \times \mathbb{R}_+^{N-M} \longrightarrow \mathbb{R}_+$, where $\mathcal{M}$ is the set of active bidders just before p, $M = \#\mathcal{M} > 1$. Function $\beta_j^{\mathcal{M}}$ determines the price $\beta_j^{\mathcal{M}}(s_j; H(p))$ at which bidder j will drop out when the set of active bidders is $\mathcal{M}$, j's own signal is s_j, and the bidders in $\mathcal{N} \setminus \mathcal{M}$ have dropped out at prices $H(p) = \mathbf{p}_{-\mathcal{M}} = \{p_j\}_{j \in \mathcal{N} \setminus \mathcal{M}}$. The rules of the English auction require that $\beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}}) > \max\{p_j : j \in \mathcal{N} \setminus \mathcal{M}\}$. If active bidders are able to extract true signals $\mathrm{s}_{-\mathcal{M}}$ of inactive bidders from their exit prices $\mathbf{p}_{-\mathcal{M}}$, the strategies can be equivalently written as $\beta_j^{\mathcal{M}}(s_j; \mathrm{s}_{-\mathcal{M}})$.

The equilibrium concept we use throughout this paper is a Bayesian-Nash equilibrium. The equilibrium we present in Section 4 is also ex-post and efficient.

Definition 3 *An* ex-post equilibrium *is a Bayesian-Nash equilibrium $\boldsymbol{\beta}$ with the property that $\boldsymbol{\beta}$ remains a Nash equilibrium even if the signals $(s_1, s_2, ..., s_N)$ are commonly known.*

This notion is equivalent to the notion of the robust equilibrium (see Dasgupta and Maskin (2000)), which requires that the strategies remain optimal under any initial distribution of signals.

Definition 4 *An equilibrium in the English auction is* efficient *if the object is allocated to the bidder with the highest value in every realization of signals $(s_1, s_2, ..., s_N)$.*

[7] We should complete the description of the game by specifying the outcome in the case where two or more bidders decide to remain active forever (do not drop out first), and an auction continues indefinitely. In this case we assign to every such bidder a payoff of $-\infty$. Alternatively, it suffices to set that the object is not allocated in this case.

Maskin (1992) establishes that:

Claim 1 (Maskin, sufficiency) *The pairwise single-crossing is sufficient for the existence of an efficient ex-post equilibrium in the English auction with two bidders.*

Claim 2 (Maskin, necessity) *Suppose the pairwise single-crossing is violated at some interior signal profile. Then the English auction with $N \geq 2$ bidders does not possess an efficient equilibrium.*[8]

The following example illustrates that efficient equilibria may exist even when *SC* is violated on the boundary of the signals' domain.

Example 1 *Consider the English auction with two bidders with value functions of the form*

$$\begin{aligned} V_1 &= \tfrac{2}{3}s_1 + \tfrac{1}{3}s_2, \\ V_2 &= s_1 + s_2. \end{aligned}$$

There exists an efficient ex post equilibrium.

At the point $s_1 = s_2 = 0$, $V_1 = V_2$, the pairwise single-crossing is violated, while at any other $\mathbf{s}$ it is vacuously satisfied. Strategies $\beta_1(s_1) = s_1$, $\beta_2(s_2) = \infty$ (bidder 2 never drops out first) form an ex post equilibrium, which is efficient.

3 Generalized Single Crossing

For an arbitrary vector $\mathbf{u}$ consider $\mathbf{u} \cdot \nabla V_k(\mathbf{s})$—the derivative of V_k in the direction $\mathbf{u}$, where $\nabla V_k(\mathbf{s}) = \left(\frac{\partial V_k}{\partial s_1}, \frac{\partial V_k}{\partial s_2}, \ldots, \frac{\partial V_k}{\partial s_N}\right)$ is the gradient of $V_k(\mathbf{s})$.

Definition 5a *(Directional formulation) The* generalized single-crossing (GSC) condition *is* satisfied *if at any* $\mathbf{s}$ *with* $\#\mathcal{I}(\mathbf{s}) \geq 2$, *for any subset of bidders* $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$,

$$\mathbf{u} \cdot \nabla V_k(\mathbf{s}) \leq \max_{j \in \mathcal{A}} \{\mathbf{u} \cdot \nabla V_j(\mathbf{s})\}, \tag{3}$$

for any bidder $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$ *and any direction* $\mathbf{u}$, *such that* $u_i > 0$ *for all* $i \in \mathcal{A}$ *and* $u_j = 0$ *for all* $j \notin \mathcal{A}$.

In words, select any group $\mathcal{A}$ of bidders from $\mathcal{I}(\mathbf{s})$—bidders who have equal and maximal values. Increase the signals of bidders from $\mathcal{A}$ only. Consider the corresponding increments to the values of bidders from $\mathcal{I}(\mathbf{s})$. *GSC* in the directional formulation requires that the increments to the values of bidders from $\mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$ are at most as high as the highest increment among the bidders from $\mathcal{A}$. Or, stated differently, at least one bidder from $\mathcal{A}$ should be in the resulting winners' circle.

Note that in the case of $\mathcal{A} = \{j\}$, *GSC* reduces to the pairwise single-crossing.

[8] This claim is indicated in Maskin (1992). The proof is straightforward.

Definition 5b *(Equal increments formulation) The* generalized single-crossing (GSC) condition *is* satisfied *if at any* s *with* $\#\mathcal{I}(\mathrm{s}) \geq 2$, *for any subset of bidders* $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$,

$$\mathbf{u}^{\mathcal{A}} \cdot \nabla V_k(\mathrm{s}) \leq 1 \tag{4}$$

for any bidder $k \in \mathcal{I}(\mathrm{s}) \setminus \mathcal{A}$, *where vector* $\mathbf{u}^{\mathcal{A}} = (\mathbf{u}^{\mathcal{A}}_{\mathcal{A}}, \mathbf{0}_{-\mathcal{A}})$ *is defined by*

$$\mathbf{u}^{\mathcal{A}} \cdot \nabla V_j(\mathrm{s}) = 1,$$

for all $j \in \mathcal{A}$.

Existence and uniqueness of vector $\mathbf{u}^{\mathcal{A}}$ follows from the fact that $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}}$ solves vector system $DV_{\mathcal{A}}(\mathrm{s}) \cdot \mathbf{u}^{\mathcal{A}}_{\mathcal{A}} = \mathbf{1}_{\mathcal{A}}$ (marginal increments to the values are equal), thus $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} = (DV_{\mathcal{A}}(\mathrm{s}))^{-1} \cdot \mathbf{1}_{\mathcal{A}}$. By regularity assumption, $\det DV_{\mathcal{A}}(\mathrm{s}) \neq 0$. We further refer to $\mathbf{u}^{\mathcal{A}}$ as to *the equal increments vector* corresponding to subset $\mathcal{A}$.

In words, select any group of bidders $\mathcal{A}$ from $\mathcal{I}(\mathrm{s})$. There exists the unique direction of the change of the signals of bidders from $\mathcal{A}$ such that along this direction the values of all the bidders from $\mathcal{A}$ increase uniformly. *GSC* in the equal increments formulation requires that along this direction the value to any bidder from $\mathcal{I}(\mathrm{s}) \setminus \mathcal{A}$ does not increase more rapidly.

Lemma 1 *The formulations of* GSC *given in Definitions 5a and 5b are equivalent.*

Proof. The proof is rather technical and is presented in Appendix A.1. ∎

Thus the equal increments formulation of *GSC* is only seemingly less demanding than the directional formulation. In fact, both of them put the same restriction on the value functions. In the proofs that follow we use these formulations interchangeably, whichever is more convenient for the specific argument.

Now we state our main results.

3.1 Results

Proposition 1 (Sufficiency) *Suppose value functions satisfy* GSC. *Then there exists an* efficient *ex post equilibrium in the* N*-bidder English auction.*

Definition 6 GSC *condition (in the directional formulation) is* violated *at the signal profile* s *for the proper subset* $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$ *and bidder* $k \in \mathcal{I}(\mathrm{s}) \setminus \mathcal{A}$ *if there exists a vector* $\mathbf{u}$, *with* $u_i > 0$ *for all* $i \in \mathcal{A}$, $u_j = 0$ *for all* $j \notin \mathcal{A}$, *such that*

$$\mathbf{u} \cdot \nabla V_k(\mathrm{s}) > \max_{i \in \mathcal{A}} \{\mathbf{u} \cdot \nabla V_i(\mathrm{s})\}. \tag{5}$$

Similarly a violation of *GSC* condition in the equal increments formulation can be defined. Hereafter whenever we say that *GSC* is violated it means that there exist some s, $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$, and $k \in \mathcal{I}(\mathrm{s}) \setminus \mathcal{A}$, such that (5) holds.

Proposition 2 (Necessity) *Suppose* GSC *condition is violated at some interior signal profile. Then no efficient equilibrium in the* N*-bidder English auction exists.*

The proofs of Propositions 1 and 2 are presented in Sections 4 and 5 correspondingly.

3.2 Examples

Here we present three examples to illustrate the link between the generalized single-crossing condition and efficiency. We start with the known example where the English auction fails to allocate efficiently and show that *GSC* is indeed violated there. In the other two examples we show that an efficient equilibrium may or may not exist if *GSC* is violated on the boundary of the signals' domain and satisfied everywhere else.

Example 2 (Perry and Reny (2001)) *Consider the English auction with three bidders with value functions of the form*

$$\begin{aligned} V_1 &= s_1 + s_2 s_3, \\ V_2 &= \tfrac{1}{2} s_1 + s_2, \\ V_3 &= s_3. \end{aligned}$$

There exists no efficient equilibrium.

Perry and Reny (2001) contains the proof of the fact that the three-bidder English auction possesses no efficient equilibrium in this example. It is easy to see that *GSC* is violated here. Notice that at the signal profile $\mathrm{s} = (.3, .6, .75)$ all the values are tied. Now choose a subset $A = \{2, 3\}$ and the direction $\mathbf{u} = (0, 1, 1)$. Then, $\mathbf{u} \cdot \nabla V_2(\mathrm{s}) = \mathbf{u} \cdot \nabla V_3(\mathrm{s}) = 1$, while $\mathbf{u} \cdot \nabla V_1(\mathrm{s}) = \frac{\partial V_1(\mathrm{s})}{\partial s_2} + \frac{\partial V_1(\mathrm{s})}{\partial s_3} = 1.35 > 1$.

The next example generalizes the message of Example 1 and illustrates that the English auction may possess an efficient equilibrium even when value functions violate *GSC* on the boundary of the signals' domain. In Example 3, however, any bidder may have the highest value, hence the fact that an efficient equilibrium exists is not as trivial as it was in Example 1.

Example 3 *Consider the English auction with threc bidders with value functions of the form*

$$\begin{aligned} V_1 &= s_1 + \tfrac{2}{3}(s_2 + s_3), \\ V_2 &= s_2, \\ V_3 &= s_3. \end{aligned}$$

There exists an efficient ex post equilibrium.

It is clear that *GSC* is violated at $\mathrm{s} = (0, 0, 0)$ for $A = \{2, 3\}$, bidder 1 and vector $(0, 1, 1)$. There is no other s at which values of all three bidders are equal. *SC* (or *GSC* for $\#A = 1$) is clearly satisfied everywhere.

The following strategies form an efficient ex post equilibrium. When all the bidders are active, bidders 2 and 3 drop out when the price reaches their private values and bidder 1 never drops out first. After one of bidders 2 and 3, say bidder 2, drops out, bidder 3 stays active until the price reaches her private value. Bidder 1 drops out when the price p reaches $s_1 + \frac{2}{3}(p + s_2)$, where s_2 is the revealed signal of bidder 2, who had dropped first.

Note that if bidders 2 and 3 follow these strategies and drop out simultaneously, the value of the object to bidder 1 is always higher than the price that she has to pay. Thus the "waiting strategy" is "safe" for bidder 1. Bidders 2 and 3 use their dominant strategies.

The next example illustrates that *GSC* being satisfied only in the interior of the signals' domain is not sufficient for the existence of an efficient equilibrium.

Example 4 *Consider the English auction with three bidders with the value functions of the form*

$$\begin{aligned} V_1 &= s_1 + \tfrac{2}{3}\left(s_2 + s_3\right), \\ V_2 &= \varepsilon s_2 + \left(1 - \varepsilon\right) s_3, \\ V_3 &= \left(1 - \varepsilon\right) s_2 + \varepsilon s_3. \end{aligned}$$

When $\varepsilon > 0$ is small enough, there is no efficient equilibrium.

Observe that whenever $V_2 = V_3$ at an interior signal profile, $V_1 > V_2 = V_3$. Therefore, both *SC* and *GSC* are satisfied at every interior signal profile. At $\mathbf{s} = \mathbf{0}$, *GSC* is violated for the subset $\mathcal{A} = \{2, 3\}$ and bidder 1.[9]

Suppose an efficient equilibrium exists. Bidder 2 with $s_2' = 0.9$ never has the highest value. She never wins and her expected payoff is 0. Bidder 2 with $s_2 = 0.1$ has the highest value when s_3 is high enough, therefore she wins with positive probability, and so

$$E_{s_1,s_3}\left(V_2(\mathbf{s}) - p(\mathbf{s}) \mid V_2(\mathbf{s}) \geq \max\{V_1(\mathbf{s}), V_3(\mathbf{s})\}\right) \geq 0.$$

Then $E_{s_1,s_3}\left(V_2(s_2', \mathbf{s}_{-2}) - p(\mathbf{s}) \mid V_2(\mathbf{s}) \geq \max\{V_1(\mathbf{s}), V_3(\mathbf{s})\}\right) > 0$, and so bidder 2 with $s_2' = 0.9$ can profitably deviate by imitating bidder 2 with $s_2 = 0.1$. Thus, there is no efficient equilibrium in the English auction.

The common feature of Examples 3 and 4 is that bidder 1 with $s_1 = 0$ has the highest value whenever $V_2 = V_3 \neq 0$. The difference in predictions of the examples is explained by the properties of the values that are not maximal (those of bidders' 2 and 3). *GSC* does not restrict such values since it pertains only to the bidders with equal and maximal values. If *GSC* is imposed on the values that are not maximal, as in Example 3, an efficient equilibrium exists, and if it is violated, as in Example 4, an efficient equilibrium may not exist.

[9] The fact that *SC* is violated is not crucial here. We can construct an example with 4 or more bidders where *SC* is satisfied. It is important for the example that *GSC* is violated for different subsets of bidders and would have been violated in the interior if one of the bidders were removed from the auction.

By the same argument as in Section 4 it can be shown that *GSC* is a sufficient condition if it is imposed at s on the bidders whose values are equal and maximal among the bidders with $s_i \neq 0$. What is the necessary and sufficient condition for efficiency on the boundary of the signals' domain is still an open question, but we think that addressing it is a mere technicality with a very limited value added. Above examples suggest that *GSC* provides a good approximate answer, and the issue disappears completely if, in addition, valuations satisfy a simple condition.

Remark 1 *Suppose that a bidder with the most pessimistic signal cannot have the highest value whenever some other bidder has a positive signal. Formally, suppose that $V_i(\mathbf{s}) < \max_j V_j(\mathbf{s})$ for every $\mathbf{s} \neq \mathbf{0}$ with $s_i = 0$. Then* GSC *is both a necessary and sufficient condition for efficiency of the N-bidder English auction: an efficient equilibrium exists if and only if* GSC *is satisfied.*[10]

4 Sufficiency

In this section we show that *GSC* is sufficient for the existence of an efficient equilibrium in the N-bidder English auction. The proof is by construction.[11]

In this equilibrium, for a given public history $H(p) = \mathbf{p}_{-\mathcal{M}}$, active bidders from $\mathcal{M}$ calculate $\boldsymbol{\sigma}(p, H(p))$—profile of inverse bidding functions that are used to define the strategies. For bidder j, to decide whether to be active at p or not is sufficient to compare her true signal s_j with $\sigma_j(p)$.[12] If $\sigma_j(p) < s_j$ bidder j is suggested to remain active; at the lowest price level p_j such that $\sigma_j(p_j) \geq s_j$ bidder j is suggested to exit. Once bidder j exits at p_j, her true signal $s_j = \sigma_j(p_j)$ is revealed, $\sigma_j(p) = \sigma_j(p_j)$ for any higher price, $p > p_j$.

Now we define the (equilibrium) strategies.

Suppose there exists a profile of functions $\boldsymbol{\sigma}(p, H(p))$, which we call *inferences*, such that:[13]

1. for an inactive bidder $i \in \mathcal{N} \setminus \mathcal{M}$, $\sigma_i(p) = \sigma_i(p_i, H(p_i))$, that is $\sigma_i(p)$ is fixed after bidder i exits at p_i;

2. for any active bidder $j \in \mathcal{M}$, $\sigma_j(p) \in [0, 1]$ solves $V_j(\sigma_j(p), \boldsymbol{\sigma}_{-j}(p)) = p$, if such a solution exists with $\sigma_j(p) \leq 1$, otherwise $\sigma_j(p) = 1$ with $V_j(\sigma_j(p), \boldsymbol{\sigma}_{-j}(p)) < p$.

[10] Indeed, if *GSC* is violated at $\mathbf{s} = \mathbf{0}$, then for a sufficiently small $\varepsilon > 0$, one can find $\mathbf{s}^\varepsilon \gg \mathbf{0}$ such that $V_j(\mathbf{s}^\varepsilon) = \varepsilon$ for each bidder j (by the above assumption and regularity), and that *GSC* is violated at (interior) $\mathbf{s}^\varepsilon$.

[11] Milgrom and Weber (1982) propose the ideology of constructing efficient equilibria for the English auction with symmetric bidders; Maskin (1992) extends it to the case of two asymmetric bidders, and Krishna (2003) generalizes it to the case of N asymmetric bidders.

[12] To shorten the notation we are omitting $H(p)$ from the set of arguments, whenever the public history can be implied from the context.

[13] Note that $H(p)$ is a collection of exit prices, $H(p) = \mathbf{p}_{-\mathcal{M}}$, where $\mathcal{M}$ is the set of active bidders. Therefore, $\boldsymbol{\sigma}(p, H(p))$ will be defined for all $p \geq \max_{i \notin \mathcal{M}} p_i$.

Thus, for all active bidders, $\boldsymbol{\sigma}_{\mathcal{M}}(p)$ are determined simultaneously, as a solution to

$$\begin{aligned}(6)\qquad \mathbf{V}_{\mathcal{M}}(\boldsymbol{\sigma}_{\mathcal{M}}(p), \boldsymbol{\sigma}_{-\mathcal{M}}(p)) \leqq p\mathbf{1}_{\mathcal{M}}, \quad \boldsymbol{\sigma}_{\mathcal{M}}(p) \leqq \mathbf{1}_{\mathcal{M}}, \\ \forall j : (V_j(\boldsymbol{\sigma}(p)) - p)(\sigma_j(p) - 1) = 0.\end{aligned}$$

For bidder $j \in \mathcal{M}$ *strategy* $\beta_j^{\mathcal{M}} : (s_j, H(p)) \longrightarrow \mathbb{R}_+$ is

$$(7)\qquad \beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}}) = \arg\min_p \{\sigma_j(p) \geq s_j\}.$$

Strategy β_j can be interpreted as follows. If bidder j is active at p, given the public history $H(p) = \mathbf{p}_{-\mathcal{M}}$ of exits of inactive bidders $\mathcal{N} \setminus \mathcal{M}$, bidder j is supposed to exit the auction at $p_j = \beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}})$, provided no other bidder exits before. If the current price p satisfies $p < \beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}})$, bidder j is suggested to maintain an active status; if $p \geq \beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}})$ bidder j is suggested to exit at p.

Once bidder j exits at p_j, other bidders update the public history and, expecting bidder j to follow (7), infer $s_j^* = \sigma_j(p_j)$. If $\sigma_j(\cdot)$ is non-decreasing the inferred s_j^* is unique and coincides with true signal s_j. The strategies can then be reformulated as functions of the own and inferred signals of inactive bidders, $\beta_j^{\mathcal{M}}(s_j; \mathbf{s}_{-\mathcal{M}}) = \beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}})$.

To proceed with the sufficiency result we need the following:

Lemma 2 *Suppose* GSC *is satisfied. Then there exist inferences $\boldsymbol{\sigma}(p, H(p))$, such that each $\sigma_j(\cdot, H(p))$ is continuous and non-decreasing for any $H(p)$, and $\sigma_j(p, \bar{H}(p)) = \sigma_j(p, H(p))$ for all p such that $\bar{H}(p) \neq H(p)$. For any active at $H(p)$ bidder j, $j \in \mathcal{I}(\boldsymbol{\sigma}(p))$ if $\sigma_j(p, H(p)) < 1$.*

Proof. Proof is presented in Appendix A.2. ■

The following Lemma then proves Proposition 1.

Lemma 3 *Suppose value functions satisfy* GSC. *Then $\boldsymbol{\beta}$ defined by (7) constitute an* efficient ex-post *equilibrium in the N-bidder English auction.*

Proof. We first show that $\boldsymbol{\beta}$ are well-defined. For any bidder j, arbitrarily fix exit prices of other bidders, $\mathbf{p}_{-j}$, possibly with $p_i = \infty$ for some bidders. Then one can obtain $\sigma_j(p)$ defined for any $p \geq 0$ as $\sigma_j(p) = \sigma_j(p, H(p))$, where $H(p) = \cup_{p_i < p}\{p_i\}$. Lemma 2 shows that $\sigma_j(p)$ is continuous and non-decreasing for any given $\mathbf{p}_{-j}$. Therefore, $p_j = \arg\min_p \{\sigma_j(p) \geq s_j\}$ is unique, so $\beta_j(s_j; \cdot)$ is well defined.

Next, we show that when all the bidders follow strategies (7), the object is allocated to the bidder with the highest value. Suppose it is bidder j who wins the object at price p^*. Then, $\sigma_i(p^*) = \sigma_i(p_i) = s_i$ for any $i \neq j$, $\sigma_j(p^*) \leq s_j$, and according to Lemma 2

$$(8)\qquad V_j(\sigma_j(p^*), \mathbf{s}_{-j}) = \max_{i \neq j} V_i(\sigma_j(p^*), \mathbf{s}_{-j}) = p^*.$$

The pairwise single-crossing, $\sigma_j(p^*) \leq s_j$, and equation (8) imply that

$$V_j(\mathrm{s}) \geq \max_{i \neq j} V_i(\mathrm{s}) \geq p^*, \tag{9}$$

so bidder j is (one of) the bidder(s) with the highest value. Note that price p^* that bidder j has to pay for the object does not depend on the signal of bidder j.

Finally, we show that $\boldsymbol{\beta}$ form an ex-post equilibrium. Suppose every bidder other than bidder j follows the proposed strategy and bidder j deviates. The payoff of bidder j can change only if the deviation affects whether bidder j obtains the object. If bidder j wins the object as a result of the deviation, she has to pay $p_j^* = \max_{i \neq j} V_i(\sigma_j(p_j^*), \mathrm{s}_{-j})$. If bidder j were not the winner in the equilibrium, $\sigma_j(p_j^*) \geq s_j$ since $\sigma_j(p)$ is non-decreasing, so $V_j(\mathrm{s}) \leq p_j^*$ and the deviation is not profitable. If as a result of the deviation bidder j is not the winner while she is in the equilibrium, she is possibly forfeiting positive profits according to (9). Thus, no profitable deviation exists.

The above is valid even if signals s are commonly known, hence the presented equilibrium is ex-post. ■

5 Necessity

In this section we establish that *GSC* is necessary for the existence of an efficient equilibrium in the N-bidder English auction.

Proposition 2 (Necessity) *Suppose* GSC *is violated at an interior signal profile. Then no efficient equilibrium in the N-bidder English auction exists.*

The proof is quite involved. There is a number of technical complications to be resolved. Before we proceed we would like to illustrate the main ideas behind the proof with a partial analysis of the three-bidder English auction.

5.1 An Illustration

Claim 3 *Suppose there are three bidders in the auction, $\mathcal{N} = \{1, 2, 3\}$. Suppose that* GSC *is violated for $\mathcal{A} = \{2, 3\}$ and bidder 1 at the interior signal profile* s$'$, *such that $V_1(\mathrm{s}') = V_2(\mathrm{s}') = V_3(\mathrm{s}')$. Suppose also that* SC *is strictly satisfied.*

Then, no efficient equilibrium exists with $\beta_2(s_2; \varnothing)$ and $\beta_3(s_3; \varnothing)$ continuous at s_2' and s_3' correspondingly.

Proof. Suppose an efficient equilibrium exists.

Step 1. Consider a stage in the auction when all three bidders are active, $H(p) = \varnothing$. *GSC* is violated for $\mathcal{A} = \{2, 3\}$ and bidder 1 at s$'$; that is, there exists direction $\mathbf{u} = (0, u_2, u_3)$ with $u_2 > 0$, $u_3 > 0$, such that for every small enough $\varepsilon > 0$, if the signals of the bidders 2 and 3 are increased along this direction, bidder 1 has the highest value,

$$V_1(\mathbf{s}' + \varepsilon \mathbf{u}) > \max\{V_2(\mathbf{s}' + \varepsilon \mathbf{u}), V_3(\mathbf{s}' + \varepsilon \mathbf{u})\}. \tag{10}$$

Thus, efficiency prescribes that she must not be the first to drop out,

$$\beta_1(s_1') > \min\{\beta_2(s_2' + \varepsilon u_2), \beta_3(s_3' + \varepsilon u_3)\}. \tag{11}$$

Step 2. We show that $\beta_2(s_2') = \beta_3(s_3')$. Suppose not, without loss of generality consider $\beta_2(s_2') < \beta_3(s_3')$. By continuity of $\beta_2(s_2)$ and of $\beta_3(s_3)$ at s_2' and s_3' correspondingly, for sufficiently small ε, we have $\beta_2(s_2' + \varepsilon u_2) < \beta_3(s_3')$ and $\beta_2(s_2' + \varepsilon u_2) < \beta_3(s_3' + \varepsilon u_3)$. Together with (11), we get $\beta_1(s_1') > \beta_2(s_2)$ for $s_2 > s_2'$ close to s_2', and so $\beta_2(s_2) < \min\{\beta_1(s_1'), \beta_3(s_3')\}$. This contradicts efficiency since the value of bidder 2 is strictly the highest at (s_1', s_2, s_3'), so she must not be the first to drop out. Therefore, $\beta_2(s_2') = \beta_3(s_3') \equiv b$.

Taking limits in (11) we obtain $\beta_1(s_1') \geq b$.

Step 3. Since bidder 2 has strictly the highest value at (s_1', s_2, s_3') for $s_2 > s_2'$ close to s_2', we have $\beta_2(s_2) > \min\{\beta_1(s_1'), \beta_3(s_3')\}$. Similarly for bidder 3. Therefore,

$$\beta_2(s_2) > b, \quad \beta_3(s_3) > b, \tag{12}$$

for $s_2 > s_2'$ and $s_3 > s_3'$ close to s_2' and s_3' correspondingly.

Step 4. Finally, by (10) and by continuity of value functions, for a given $\varepsilon > 0$, there exists $\varepsilon_1 > 0$, such that bidder 1 has the highest value at $(s_1' - \varepsilon_1, s_2' + \varepsilon u_2, s_3' + \varepsilon u_3)$. By efficiency, it must be that

$$\beta_1(s_1' - \varepsilon_1) > \min\{\beta_2(s_2' + \varepsilon u_2), \beta_3(s_3' + \varepsilon u_3)\}.$$

Together with (12) this implies

$$\beta_1(s_1' - \varepsilon_1) > b = \beta_2(s_2') = \beta_3(s_3'). \tag{13}$$

Thus, at $(s_1' - \varepsilon_1, s_2', s_3')$ bidders 2 and 3 drop out simultaneously, and bidder 1 wins. However, since *SC* is strictly satisfied, bidder 1 has the lowest value. ■

In what follows as an intermediate step we basically show that if an efficient equilibrium exists then $\beta_2(s_2; \varnothing)$ and $\beta_3(s_3; \varnothing)$ are almost everywhere continuous in the neighborhoods of s_2' and s_3' correspondingly.

5.2 Proof of Proposition 2

We proceed from the contrary—we assume that an efficient equilibrium exists, while *GSC* is violated at some interior signal profile. Fix an efficient equilibrium $\boldsymbol{\beta}$, $\beta_i(s_i; H(p))$ is the equilibrium strategy of bidder i with signal s_i. No restrictions on the strategies are imposed, they need not be monotonic and can be discontinuous everywhere.

Our proof incorporates the following three main principles.

Principle 1. Suppose that at s exactly one bidder, say bidder j, has the highest value among all the bidders, $\#\mathcal{I}(\mathrm{s}) = 1$. Then, in the equilibrium bidder j must win the object, so at any intermediate history $H(p)$ with $\mathcal{M}$ being the set of active bidders,

$$\beta_j(s_j; H(p)) > \min_{i \in \mathcal{M}} \beta_i(s_i; H(p)); \tag{14}$$

that is, *the bidder who has the highest value must not be the one to drop out first.* And, in particular, bidder j cannot be among the bidders who drop out simultaneously at the end, since then she will not obtain the object with certainty.

When two or more bidders have the highest value at some s, we do not impose any restrictions on who should be the winner among them. In particular, we do not require that each of them has to win the object with positive probability. Therefore, (14) must hold only for the eventual winner, not for any $j \in \mathcal{I}(\mathrm{s})$.

Principle 2. If *GSC* is violated at some interior signal profile, we can find a possibly different interior signal profile s, where *GSC* is violated for bidder k and a *minimal* subset $\mathcal{A}$, that is the subset that contains the fewest possible number of bidders needed to violate *GSC*. Indeed, for all interior profiles s and all pairs k and $\mathcal{A}$ that exhibit a violation, the number of bidders in $\mathcal{A}$ is an integer between 1 and $N-1$, and so min and arg min operators are well-defined. Then we can find s, minimal subset $\mathcal{A}$, and bidder $k = 1$ (after relabeling), such that these are the only bidders in the winners circle, $\mathcal{A}^{+1} \equiv \mathcal{A} \cup \{1\} = \mathcal{I}(\mathrm{s})$. To separate bidders $\mathcal{B} = \mathcal{I}(\mathrm{s}) \setminus \mathcal{A}^{+1}$ when $\mathcal{B} \neq \varnothing$, we can lower the values of all the bidders from $\mathcal{B}$ in some manner, while keeping the values of all the bidders from $\mathcal{A}^{+1}$ fixed and the signals of all the others fixed. If the change is sufficiently small, then by regularity and continuity of value functions and by continuity of their first derivatives the resulting signal profile s will be interior, $\mathcal{I}(\mathrm{s}) = \mathcal{A}^{+1}$, and *GSC* is violated for bidder 1 and minimal $\mathcal{A}$.

Our focus will always be on bidders $\mathcal{A}^{+1}$, *the signals (and so the strategies) of the rest of the bidders are fixed throughout the proof.*

Principle 3. This principle, or to be more exact, convention of how we use notation to make strong statements about bidding functions shortens and simplifies the proof by a lot. It is important then to describe it in detail.

Suppose that at s, *GSC* is violated for bidder 1 and subset $\mathcal{A}$, $\mathcal{A}$ is minimal, $\mathcal{I}(\mathrm{s}) = \mathcal{A}^{+1}$, and $\mathrm{s}_{-\mathcal{I}(\mathrm{s})}$ are fixed. By continuity, there exists an open neighborhood of $\mathrm{s}_{\mathcal{I}(\mathrm{s})}$, $U^{\mathrm{s}}_{\mathcal{I}(\mathrm{s})}$, such that for any $\mathrm{s}' = (\mathrm{s}'_{\mathcal{I}(\mathrm{s})}, \mathrm{s}_{-\mathcal{I}(\mathrm{s})})$, $\mathcal{I}(\mathrm{s}') \subseteq \mathcal{I}(\mathrm{s})$. In other words, if we slightly disturb the signals of the bidders from $\mathcal{I}(\mathrm{s})$ only, all the bidders with the highest value as a result must belong to $\mathcal{A}^{+1}$.

For any such s′, let $p(\mathrm{s}')$ to be the first price at which a bidder from $\mathcal{A}^{+1}$ drops out. Slightly abusing the notation, $H(\mathrm{s}') = H(p(\mathrm{s}'))$ is the history of play just prior to $p(\mathrm{s}')$—the sequence of exits of the bidders not from $\mathcal{A}^{+1}$ up to a moment of the first drop-out of a bidder from $\mathcal{A}^{+1}$. It is not necessary that all the bidders *not* from $\mathcal{A}^{+1}$ exit first, and we allow for a possibility that $H(\mathrm{s}') = \varnothing$.

If it were the case that for any such s' history $H(\mathrm{s}')$ is the same (this would have been the case if all the bidders not from $\mathcal{A}^{+1}$ exited before any of the bidders from $\mathcal{A}^{+1}$ for all s', e.g.), then we would fix this history, H, and consider only parts of the strategies, $\beta_j(s_j; H)$ for all $j \in \mathcal{A}^{+1}$, to derive results and reach a contradiction at the end. Unfortunately, for different s', $H(\mathrm{s}')$ may be different. For example, suppose the first bidder from $\mathcal{A}^{+1}$ to drop out (at $p(\mathrm{s}')$) is bidder j. Then if we change slightly s_j, bidder j may no longer be the first from $\mathcal{A}^{+1}$ to drop out, $H(\mathrm{s}')$ may stay the same or lengthen depending on whether some other bidder from $\mathcal{A}^{+1}$ or a bidder from $\mathcal{N} \setminus \mathcal{A}^{+1}$ exits first instead. Even if bidder j is still the first from $\mathcal{A}^{+1}$ to drop out, the number of bidders who exit before j can decrease or increase.

To avoid dealing with potentially different histories, and, therefore, different parts of strategies, we propose the following. For any bidder $j \in \mathcal{A}^{+1}$ for any signal s_j we calculate $\hat{\beta}_j(s_j)$—the price level at which bidder j with s_j would exit according to her equilibrium strategy if all other bidders from $\mathcal{A}^{+1}$ remained active forever (or do not exit before her) and all the bidders $\mathcal{N} \setminus \mathcal{A}^{+1}$ followed their equilibrium strategies (their signals are fixed). It is possible that $\hat{\beta}_j(s_j) = \infty$ for some bidder j with signal s_j. In the proof we will be deriving results concerning these *bidding functions.*

The fact that the bidder, say j, with strictly the highest value at s' never drops out first implies that

$$\hat{\beta}_j(s'_j) > \min_{i \in \mathcal{A}^{+1}} \hat{\beta}_i(s'_i).$$

Indeed, in equilibrium, there must exist a bidder $i \in \mathcal{A}^{+1}$, who, at s', drops out the first among $\mathcal{A}^{+1}$. But, then the price at which she does so is equal to $\hat{\beta}_i(s_i)$. Bidder j (with the highest value) must at least stay longer.

In what follows, to avoid excessive notation, we are writing simply $\beta_j(s_j)$ in place of $\hat{\beta}_j(s_j)$. We are also omitting the signals of the bidders from $\mathcal{N} \setminus \mathcal{A}^{+1}$ since these are fixed, so s denotes the profile of signals held by the bidders from $\mathcal{A}^{+1}$ only. We write $\mathrm{s}_{\mathcal{N}}$ when referring to the full profile of signals.

Proof of Proposition 2. Suppose that the minimal subset $\mathcal{A}$ contains n bidders, and GSC is violated at some interior signal profile $\mathrm{s}_{\mathcal{N}}$ for $\mathcal{A}$ and bidder 1, with $\mathcal{A}^{+1} = \mathcal{I}(\mathrm{s}_{\mathcal{N}})$. The fact that $n \geq 2$ follows from Claim 2.

Step 1. Consider trajectory $\mathrm{s}(t)$ that for each t solves

$$V_j(\mathrm{s}(t)) = V(\mathrm{s}) + t, \text{ for all } j \in \mathcal{A}^{+1}.$$

Such a trajectory exists and is unique, since it can be found as a solution to the differential equation

$$\frac{d\mathrm{s}}{dt} = (DV(\mathrm{s}))^{-1} \cdot \mathbf{1}. \tag{15}$$

By continuity of value functions and their first derivatives, $\mathcal{A}^{+1} = \mathcal{I}(\mathrm{s}_{\mathcal{N}}(t))$, and *GSC* is violated at $\mathrm{s}_{\mathcal{N}}(t)$ for $\mathcal{A}$ and bidder 1 for all t in some open neighborhood U_t^0 of $t = 0$.

Step 2. Consider $\mathrm{s}' = \mathrm{s}(t)$ for an arbitrary $t \in U_t^0$. Lemma 4 in Appendix A.3 shows that for any $j \in \mathcal{A}$, there exist $b_j(s_j') \equiv \lim_{s_j \downarrow s_j'} \inf \beta_j(s_j)$, and these limits are equal for the bidders from $\mathcal{A}$: for any $j \in \mathcal{A}$, $b_j(s_j') \equiv b(\mathrm{s}') \equiv b(t) < \infty$. In addition, for any $j \in \mathcal{A}$ and $s_j > s_j'$ sufficiently close to s_j', $\beta_j(s_j) \geq b(t)$; and for bidder 1, $\beta_1(s_1') > b(t)$.

Step 3. Corollary 2 in Appendix A.1 shows that either: (i) for any $j \in \mathcal{A}$, $s_j(t') > s_j(t)$ while $s_1(t') < s_1(t)$ for $t' > t$; or (ii) for any $j \in \mathcal{A}$, $s_j(t') < s_j(t)$ while $s_1(t') > s_1(t)$ for $t' > t$. This, together with the results of Step 2, implies that $b(t)$ is (weakly) monotonic in t. In Case (i) it is non-decreasing, in Case (ii) it is non-increasing.

Step 4. Corollary 3 in Appendix A.3 shows that if for some bidder $j \in \mathcal{A}$, $\beta_j(s_j(t)) \neq b(t)$, then t has to be a discontinuity point for $b(t)$. Since $b(t)$ is monotonic it has no more than a countable number of discontinuity points. Hence for almost all $t \in U_t^0$, $\beta_j(s_j(t)) = b(t)$ for every $j \in \mathcal{A}$. That is, when the signals of bidders from $\mathcal{A}$ belong to trajectory $\mathrm{s}(t)$, bidders from $\mathcal{A}$ almost always exit simultaneously.

Step 5. Consider two continuity points for $b(t)$, t and t', such that $b(t') \geq b(t)$. In Case (i), $t' > t$; in Case (ii), $t' < t$. Then, $s_1(t') < s_1(t)$, and $\beta_1(s_1(t')) > b(t') \geq b(t) = \beta_j(s_j(t))$ for all $j \in \mathcal{A}$.

Step 6. By construction, at t, $\mathcal{I}(s_1(t), \mathrm{s}_{\mathcal{A}}(t)) = \mathcal{A}^{+1}$. Since there exists a unique solution to (15), and *SC* is satisfied for bidder 1, we have that at $s_1 \equiv s_1(t)$,

$$\frac{\partial V_1(s_1, \mathrm{s}_{\mathcal{A}}(t))}{\partial s_1} > \max_{j \in \mathcal{A}} \frac{\partial V_j(s_1, \mathrm{s}_{\mathcal{A}}(t))}{\partial s_1}. \tag{16}$$

Therefore, we can find t' sufficiently close to t such that: t' is a continuity point for $b(t)$, $s_1(t') < s_1(t)$, and, as follows from (16), all the bidders with the highest value belong to $\mathcal{A}$. Then by results of Step 5, at the realization $(s_1(t'), \mathrm{s}_{\mathcal{A}}(t))$, bidders from $\mathcal{A}$ drop out simultaneously at $b(t)$, and bidder 1 for sure stays longer. Thus, efficiency is not achieved—a contradiction. ∎

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A Appendix

A.1 Equivalence Lemma

Lemma 1 *The formulations of* GSC *given in Definitions 5a and 5b are equivalent.*

Proof. Fix s. Introduce $\mu_k(\mathbf{u}) \equiv \mathbf{u} \cdot \nabla V_k(\mathrm{s})$—the derivative of V_k along the direction $\mathbf{u}$.

$5a \Longrightarrow 5b$. It is enough to show that every component of the equal increments vector $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}}$ is non-negative for all subsets $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$.

Step 1. Suppose inequalities in the directional formulation (3) are strict for all $\mathcal{A}$. Then we can show that $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \gg \mathbf{0}$ (every component is strictly positive).

This is done by induction on the number of bidders in $\mathcal{A}$. For $\#\mathcal{A} = 1$, $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} = \left(\frac{\partial V_{\mathcal{A}}}{\partial s_{\mathcal{A}}}\right)^{-1} > 0$. Suppose for all $\mathcal{B} \subset \mathcal{I}(\mathrm{s})$ with $\#\mathcal{B} \leq n-1$, $\mathbf{u}_{\mathcal{B}}^{\mathcal{B}} \gg \mathbf{0}$. We want to show the same for an arbitrary subset $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$ with $\#\mathcal{A} = n$.

Suppose, on the contrary, there exists $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$ with $\#\mathcal{A} = n$ such that $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \not\gg \mathbf{0}$. We can partition $\mathcal{A} = \mathcal{B} \sqcup \mathcal{C} \sqcup \mathcal{D}$, where $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are subsets of bidders for which the corresponding components of $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}}$ are negative, equal to zero, and positive correspondingly. By presumption, $\mathcal{B} \cup \mathcal{C} \neq \varnothing$. Obviously, $\mathcal{D}$ is also not empty. Note that $\mathbf{u}_{\mathcal{D}}^{\mathcal{D}} \gg \mathbf{0}$ as $\#\mathcal{D} < n$.

If $\mathcal{B} = \varnothing$, then $\mathcal{C} \neq \varnothing$ and $\mu_j(\mathbf{u}^{\mathcal{D}}) = 1 = \max_{i \in \mathcal{D}}\{\mu_i(\mathbf{u}^{\mathcal{D}})\}$ for any $j \in \mathcal{C}$, which contradicts the supposition that the inequalities in (3) are strict. Thus, $\mathcal{B}$ is not empty. Introduce vector $\mathbf{u} \equiv \mathbf{u}^{\mathcal{A}} - \mathbf{u}'$, where $\mathbf{u}'_{-\mathcal{B}} = \mathbf{0}$ and $\mathbf{u}'_{\mathcal{B}} = \mathbf{u}_{\mathcal{B}}^{\mathcal{A}}$, note that $-\mathbf{u}'_{\mathcal{B}} \gg \mathbf{0}$. By construction, $\mathbf{u}_{\mathcal{D}} \gg \mathbf{0}$ and $\mathbf{u}_{-\mathcal{D}} = \mathbf{0}$. Consider bidder $i \in \mathcal{B}$ with the maximal $\mu_i(-\mathbf{u}')$ and bidder $j \in \mathcal{D}$ with the maximal $\mu_j(\mathbf{u})$. *GSC* in the directional formulation for the set $\mathcal{B}$ dictates that $\mu_i(-\mathbf{u}') > \mu_j(-\mathbf{u}')$. *GSC* in the directional formulation for the set $\mathcal{D}$ dictates that $\mu_i(\mathbf{u}) < \mu_j(\mathbf{u})$. Since $\mathbf{u}^{\mathcal{A}} = \mathbf{u} - (-\mathbf{u}')$, we have $\mu_i(\mathbf{u}^{\mathcal{A}}) < \mu_j(\mathbf{u}^{\mathcal{A}})$. We have a contradiction since $\mu_i(\mathbf{u}^{\mathcal{A}}) = \mu_j(\mathbf{u}^{\mathcal{A}}) = 1$ by definition of $\mathbf{u}^{\mathcal{A}}$. Therefore, $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \gg \mathbf{0}$.

Step 2. Suppose that weak inequalities in (3) are possible. Then we can slightly perturb the Jacobian of value functions at s, $DV_{\mathcal{I}}(\mathrm{s})$, in the following way: add $\varepsilon > 0$ to every diagonal element,

$$DV'_{\mathcal{I}}(\mathrm{s}) = DV_{\mathcal{I}}(\mathrm{s}) + \varepsilon I_{\#\mathcal{I}}.$$

All inequalities in (3) become strict after the perturbation—for any $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$ and $\mathbf{u}$ from Definition 5a we have $\mu'_j(\mathbf{u}) = \varepsilon \frac{\partial V_j}{\partial s_j} + \mu_j(\mathbf{u}) > \mu_j(\mathbf{u})$ for any bidder $j \in \mathcal{A}$, while $\mu'_i(\mathbf{u}) = \mu_i(\mathbf{u})$ for all $i \notin \mathcal{A}$.

If prior to the perturbation, there existed a subset $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$ such that for some $i \in \mathcal{A}$, $u_i^{\mathcal{A}} < 0$, then, by continuity, for sufficiently small ε, $u_i^{\mathcal{A}}$ would still be negative after the perturbation which would contradict the result in Step 1. Therefore, $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \gg \mathbf{0}$ for all subsets $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$.

$5b \Longrightarrow 5a$. Again we use the induction on the number of bidders in $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$. For $\#\mathcal{A} = 1$ the result is obvious. Fix the subset $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$ with $\#\mathcal{A} = n$ and suppose that *GSC* is satisfied at s in the directional formulation for all subsets $\mathcal{B} \subset \mathcal{I}(\mathrm{s})$ with $\#\mathcal{B} < n$, and that *GSC* in the directional formulation is violated at s for $\mathcal{A}$, that is, there exists a vector $\mathbf{u}$ with $\mathbf{u}_{\mathcal{A}} \gg \mathbf{0}$ and $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$, such that for some $k \in \mathcal{I}(s) \setminus \mathcal{A}$, $\mu_k(\mathbf{u}) > \max_{j \in \mathcal{A}} \mu_j(\mathbf{u})$. Clearly, $\mathbf{u} \neq \mathbf{u}^{\mathcal{A}}$.

Consider $\mathcal{B} \subset \mathcal{A}$—the subset of bidders who have the highest increments to their values in the direction $\mathbf{u}$, $i \in \mathcal{B} \Leftrightarrow \mu_i(\mathbf{u}) = \max_{j\in\mathcal{A}} \mu_j(\mathbf{u})$. Since $\mathbf{u} \neq \mathbf{u}^{\mathcal{A}}$, $\mathcal{B} \neq \mathcal{A}$. Consider vector $\mathbf{w}_1(t) = \mathbf{u} - t\mathbf{u}^{\mathcal{B}}$. Note that $\mathbf{u}_{\mathcal{B}} \gg \mathbf{0}$ and, since $\#\mathcal{B} < \#\mathcal{A}$, by the induction hypothesis and the argument above, $\mathbf{u}_{\mathcal{B}}^{\mathcal{B}} \geqq \mathbf{0}$.

At $t = 0$, for any $j \in \mathcal{A}\backslash\mathcal{B}$ and $i \in \mathcal{B}$, we have $\mu_j(\mathbf{w}_1(0)) < \mu_i(\mathbf{w}_1(0)) < \mu_k(\mathbf{w}_1(0))$. Once we start increasing t, that is, decreasing in a special direction the signals of all the bidders from $\mathcal{B}$ only, all $\mu_i(\mathbf{w}_1(t))$, for $i \in \mathcal{B}$, decrease uniformly at rate t, while for any bidder $l \in \mathcal{I}(s) \setminus \mathcal{B}$ (including k) their $\mu_l(\mathbf{w}_1(t))$ decrease at most at the same rate, because *GSC* is satisfied for $\mathcal{B}$. Introduce t_1—the minimal value of $t > 0$ such that: either $\mu_j(\mathbf{w}_1(t)) = \mu_i(\mathbf{w}_1(t))$ for some $j \in \mathcal{A} \setminus \mathcal{B}$ and every $i \in \mathcal{B}$, or $w_{1i}(t) = 0$ for some $i \in \mathcal{B}$. In the latter case, stop. If the former case applies, consider $\mathcal{C}$—a subset that includes $\mathcal{B}$ and all the bidders $j \in \mathcal{A} \setminus \mathcal{B}$ such that $\mu_j(\mathbf{w}_1(t_1)) = \mu_i(\mathbf{w}_1(t_1))$. Define $\mathbf{w}_2(t) = \mathbf{w}_1(t_1) - t\mathbf{u}^{\mathcal{C}}$. Find the smallest $t_2 > 0$ such that: either $\mu_j(\mathbf{w}_2(t_2)) = \mu_i(\mathbf{w}_2(t_2))$ for some bidder $j \in \mathcal{A} \setminus \mathcal{C}$ and every $i \in \mathcal{C}$, or $w_{2i}(t_2) = 0$ for some $i \in \mathcal{C}$, in which case stop. Again, if the former case applies consider $\mathcal{D} \supset \mathcal{C}$. Repeat this procedure until for some bidder $i \in \mathcal{A}$, $w_{mi}(t_m) = 0$. This will take at most $\#\mathcal{A}$ repetitions and may result in all bidders $i \in \mathcal{A}$ having $w_{mi}(t_m) = 0$.

Note that by the induction hypothesis for bidder $k \in \mathcal{I}(s) \setminus \mathcal{A}$, $\mu_k(\mathbf{w}_1(t))$ always decreased at a rate no higher than the corresponding rate for bidders from $\mathcal{B}$, $\mathcal{C}$, ... Thus, at any stage $l \leq m$ of the procedure, $\mu_k(\mathbf{w}_l(t)) > \max_{j\in\mathcal{A}} \mu_j(\mathbf{w}_l(t))$. In particular,

$$\mu_k(\mathbf{w}_m(t_m)) > \max_{j\in\mathcal{A}} \mu_j(\mathbf{w}_m(t_m)). \tag{17}$$

If for all $j \in \mathcal{A}$, $w_{mj}(t_m) = 0$, then, by construction, $\mathbf{w}_m(t_m) = \mathbf{0}$, which makes (17) impossible, otherwise define $\mathcal{A}'$ to be the set of bidders $j \in \mathcal{A}$ with $w_{mj}(t_m) > 0$. Since $\#\mathcal{A}' < n$, *GSC* in the directional formulation is violated for the set $\mathcal{A}'$, vector $\mathbf{w}_m(t_m)$ and bidder $k \in \mathcal{I}(s) \setminus \mathcal{A}$, which contradicts the induction presumption. ■

The following is an obvious corollary to Lemma 1.

Corollary 1 GSC *in the directional formulation is violated at* s *if and only if* GSC *in the equal increments formulation is violated at* s.

It should be noted that for a violation to occur it is not necessary that the same subsets of bidders are involved under both definitions.

The following Corollary is used in the proof of Proposition 2.

Corollary 2 *At a given* s, *consider an arbitrary* $\mathcal{A} \subset \mathcal{I}(\mathrm{s})$ *with* $\#\mathcal{A} = n \geq 2$. *Suppose* GSC *is satisfied at* s *for any subset* $\mathcal{B} \subset \mathcal{I}(\mathrm{s})$ *with* $\#\mathcal{B} < n$. GSC *is violated at* s *for* $\mathcal{A}$ *and bidder* $k \in \mathcal{I}(\mathrm{s}) \setminus \mathcal{A}$ *if and only if (i):* $u_k^{\mathcal{C}} < 0$ *and* $u_j^{\mathcal{C}} > 0$ *for all* $j \in \mathcal{A}$ *or (ii):* $u_k^{\mathcal{C}} > 0$ *and* $u_j^{\mathcal{C}} < 0$ *for all* $j \in \mathcal{A}$, *where* $\mathcal{C} \equiv \mathcal{A} \cup \{k\}$ *and* $\mathbf{u}^{\mathcal{C}}$ *is an equal increment vector for the subset* $\mathcal{C}$.

Proof. By the above conditions one needs at least $n+1$ bidders to violate *GSC*. Therefore, as follows from the proof of Lemma 1, $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \gg \mathbf{0}$.

($\Longrightarrow$). Suppose first that $u_k^{\mathcal{C}} < 0$, while $\mathbf{u}_{\mathcal{A}}^{\mathcal{C}} \gg \mathbf{0}$. Consider vector $\mathbf{u}'$ such that $u'_k = 0$, $\mathbf{u}'_{-k} = \mathbf{u}_{-k}^{\mathcal{C}}$. Since *GSC* is satisfied for $\mathcal{B} = \{k\}$, $\mu_k(\mathbf{u}') > \max_{i \in \mathcal{A}} \mu_i(\mathbf{u}')$. Therefore, *GSC* is violated for k, $\mathcal{A}$, and $\mathbf{u}'$. Now suppose $u_k^{\mathcal{C}} > 0$, while $\mathbf{u}_{\mathcal{A}}^{\mathcal{C}} \ll \mathbf{0}$. Consider vector $\mathbf{u}'$ such that $u'_k = 0$, $\mathbf{u}'_{-k} = -\mathbf{u}_{-k}^{\mathcal{C}}$. Then *GSC* is violated for k, $\mathcal{A}$, and $\mathbf{u}'$.

($\Longleftarrow$). If $u_k^{\mathcal{C}} = 0$, then $\mathbf{u}^{\mathcal{C}} = \mathbf{u}^{\mathcal{A}}$. In this case *GSC* is satisfied with equality for bidder k and $\mathcal{A}$. If $u_k^{\mathcal{C}} > 0$ or $u_k^{\mathcal{C}} < 0$, we can suppose that all the inequalities in (4) are strict since by disturbing the Jacobian, as we did in Step 2 of the proof of Lemma 1, we eliminate all equalities. If ε is small enough, whether *GSC* is violated for bidder k or not and the sign of $u_k^{\mathcal{C}}$ remain unchanged. As a result, for any $i \in \mathcal{A}$, $u_i \neq 0$.

Step 1. We show that $\mathbf{u}_{\mathcal{C}}^{\mathcal{C}}$ can have either 1 or n positive components. Partition $\mathcal{C} = \mathcal{B} \sqcup \mathcal{D}$ (inequalities are strict), $i \in \mathcal{B}$ ($i \in \mathcal{D}$) if $u_i^{\mathcal{C}} < 0$ ($u_i^{\mathcal{C}} > 0$). Given that $\mathcal{D} \neq \varnothing$, suppose that $\#\mathcal{D} \notin \{1, n\}$. Consider vector $\mathbf{w}_1(t) = \mathbf{u}^{\mathcal{C}} - t\mathbf{u}^{\mathcal{D}}$. We have $\mathbf{u}_{\mathcal{D}}^{\mathcal{D}} \gg \mathbf{0}$, $\mu_j(\mathbf{u}^{\mathcal{D}}) > \mu_i(\mathbf{u}^{\mathcal{D}})$ for all $j \in \mathcal{D}$ and $i \in \mathcal{B}$ since $\#\mathcal{D} \leq \#\mathcal{C} - 2 < n$. Then $\mu_j(\mathbf{w}_1(t)) < \mu_i(\mathbf{w}_1(t))$ for all $t > 0$. There exists the minimal $t_1 > 0$ such that for some $j \in \mathcal{D}$, $\mathbf{w}_{1j}(t) = 0$. Consider the subset $\mathcal{E}$ of bidders $l \in \mathcal{D}$ with $w_{1l}(t) > 0$, and vector $\mathbf{w}_2(t) = \mathbf{w}_1(t_1) - t\mathbf{u}^{\mathcal{E}}$. Increase t until for some bidder $j \in \mathcal{E}$, $w_{2j}(t_2) = 0$. Again, $\mu_j(\mathbf{w}_2(t_2)) < \mu_i(\mathbf{w}_2(t_2))$ for all $j \in \mathcal{E}$ and $i \in \mathcal{B}$. Repeating this procedure we eventually obtain vector $\mathbf{w}_m(t_m)$ such that for all $j \in \mathcal{D}$, $w_{mj}(t_m) = 0$. Introduce vector $\mathbf{w}'_m(t_m) \equiv -\mathbf{w}_m(t_m)$. Note that for all $i \in \mathcal{B}$, $w'_{mi}(t_m) = -u_i^{\mathcal{C}} > 0$. Fix bidder $j \in \mathcal{D}$ with $w_{mj}(0) > 0$. Clearly $\mu_j(\mathbf{w}'_m(t_m)) > \mu_i(\mathbf{w}'_m(t_m))$ for all $i \in \mathcal{B}$. Therefore *GSC* is violated for bidder j, subset $\mathcal{B}$, and vector $\mathbf{w}'_m(t_m)$, which is a contradiction since $\#\mathcal{B} < n$.

Step 2. Suppose $u_k^{\mathcal{C}} > 0$. We show that $\mathbf{u}_{\mathcal{A}}^{\mathcal{C}} \ll \mathbf{0}$. Suppose the otherwise and consider vector $\mathbf{u}'$ such that $u'_k = 0$ and $\mathbf{u}'_{-k} = \mathbf{u}_{-k}^{\mathcal{C}}$. Clearly, $\mathbf{u}' \neq \mathbf{u}^{\mathcal{A}}$. Since $\mu_k(\mathbf{u}^{\mathcal{C}}) = \mu_i(\mathbf{u}^{\mathcal{C}}) = 1$ for all $i \in \mathcal{A}$ and *GSC* is satisfied for $\mathcal{B} = \{k\}$, we have $\mu_k(\mathbf{u}') \leq \min_{i \in \mathcal{A}} \mu_i(\mathbf{u}')$. Since *GSC* is violated for $\mathcal{A}$ and bidder k, there exists vector $\mathbf{u}$, with $\mathbf{u}_{\mathcal{A}} \gg \mathbf{0}$ and $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$, such that $\mu_k(\mathbf{u}) > \max_{i \in \mathcal{A}} \mu_i(\mathbf{u})$. Consider vector $\mathbf{w}(t) = \mathbf{u} - t\mathbf{u}'$. It follows that $\mu_k(\mathbf{w}(t)) > \max_{i \in \mathcal{A}} \mu_i(\mathbf{w}(t))$ for $t \geq 0$. Note that $\mathbf{w}_{-\mathcal{C}}(t) = \mathbf{0}$ for any t and $\mathbf{w}_{\mathcal{A}}(0) \gg \mathbf{0}$. Since at least one of the components of $\mathbf{u}'_{\mathcal{A}} = \mathbf{u}_{\mathcal{A}}^{\mathcal{C}}$ is positive, there exist the smallest $t' > 0$ such that for some $i \in \mathcal{A}$, $w_i(t') = 0$. Then, *GSC* is violated for bidder k, subset $\mathcal{B} = \mathcal{A} \setminus \{i\}$ and vector $\mathbf{w}(t')$, which is a contradiction since $\#\mathcal{B} < n$.

Suppose $u_k^{\mathcal{C}} < 0$. Then, similarly to the above, $\mathbf{u}_{\mathcal{A}}^{\mathcal{C}} \gg 0$.

Step 3. It remains to be shown that, once equalities in (4) are allowed, if *GSC* is violated for $\mathcal{A}$ and bidder k, then for any $i \in \mathcal{C}$, $u_i^{\mathcal{C}} \neq 0$. Suppose $u_i^{\mathcal{C}} = 0$ for bidders $i \in \mathcal{B} \subset \mathcal{A}$. Obviously $\mathcal{A} \setminus \mathcal{B} \neq \varnothing$. By the arguments similar to the above, *GSC* is violated for the subset $\mathcal{A} \setminus \mathcal{B}$ and bidder k, which contradicts the supposition of the Lemma since $\#\{\mathcal{A} \setminus \mathcal{B}\} < n$. ■

A.2 Sufficiency

Lemma 3 *Suppose* GSC *is satisfied. Then there exist inferences* $\boldsymbol{\sigma}(p, H(p))$*, such that each* $\sigma_j(\cdot, H(p))$ *is continuous and non-decreasing for any* $H(p)$*, and* $\sigma_j(p, \bar{H}(p)) = \sigma_j(p, H(p))$ *for all* p *such that* $\bar{H}(p) \neq H(p)$*. For any active at* $H(p)$ *bidder* j*, if* $\sigma_j(p, H(p)) < 1$ *then* $j \in \mathcal{I}(\boldsymbol{\sigma}(p, H(p)))$*.*

Proof. First, we construct inferences $\boldsymbol{\sigma}(p, H(p))$, such that each $\sigma_j(\cdot, H(p))$ is continuous at p for any $H(p)$, and any bidder j active at p with if $\sigma_j(p, H(p)) < 1$ has the highest value at $\boldsymbol{\sigma}(p)$. Define $\mathcal{A}(\boldsymbol{\sigma}(p, H(p)))$ to be the set of active bidders at price p with $\sigma_j(p, H(p)) < 1$.

Suppose at some p^0 with $H(p^0) = \bar{H}(p^0)$ there exists a profile $\boldsymbol{\sigma}^0(p^0)$ that satisfies (6), and $\mathcal{A} = \mathcal{A}(\boldsymbol{\sigma}^0(p^0)) \subset \mathcal{I}(\boldsymbol{\sigma}^0(p^0))$. Fix $\boldsymbol{\sigma}_{-\mathcal{A}}(p) = \boldsymbol{\sigma}^0_{-\mathcal{A}}(p^0)$ for $p \geq p^0$. Consider a profile of functions $\boldsymbol{\sigma}(p) = (\boldsymbol{\sigma}_{\mathcal{A}}(p), \boldsymbol{\sigma}_{-\mathcal{A}}(p))$ such that $\boldsymbol{\sigma}_{\mathcal{A}}(p)$ satisfies (6) for every $p \in [p^0, p^*]$ for some $p^* > p^0$. Finding a solution $\boldsymbol{\sigma}_{\mathcal{A}}(p)$ to the system

$$\mathbf{V}_{\mathcal{A}}(\boldsymbol{\sigma}_{\mathcal{A}}(p), \boldsymbol{\sigma}^0_{-\mathcal{A}}(p^0)) = p\mathbf{1}_{\mathcal{A}} \tag{18}$$

is equivalent to solving the system of differential equations

$$\frac{d\boldsymbol{\sigma}_{\mathcal{A}}}{dp} = (DV_{\mathcal{A}})^{-1}\mathbf{1}_{\mathcal{A}}. \tag{19}$$

By the Caushy-Peano theorem, there exists a unique continuous solution $\boldsymbol{\sigma}_{\mathcal{A}}(p)$ to the system (19) with initial condition $\boldsymbol{\sigma}_{\mathcal{A}}(p^0) = \boldsymbol{\sigma}^0_{\mathcal{A}}(p^0)$, and this solution extends to all $p \leq p^*_{\mathcal{A}}$, where $p^*_{\mathcal{A}}$ is the lowest price at which $\sigma_j(p^*_{\mathcal{A}}) = 1$ for some bidder $j \in \mathcal{A}$.

Suppose *GSC* is satisfied. As long as $\mathcal{A} \subset \mathcal{I}(\boldsymbol{\sigma}(p))$, $\frac{d\boldsymbol{\sigma}_{\mathcal{A}}}{dp} = \mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \geqq \mathbf{0}$ (this follows from the proof of Lemma 1), and $\frac{\partial V_i}{\partial \boldsymbol{\sigma}_{\mathcal{A}}} \frac{d\boldsymbol{\sigma}_{\mathcal{A}}}{dp} \leq 1$ for any $i \in \mathcal{I}(\boldsymbol{\sigma}(p)) \setminus \mathcal{A}$ by (4). Since $\mathcal{A} \subset \mathcal{I}(\boldsymbol{\sigma}(p^0))$, it follows that $\mathcal{A} \subset \mathcal{I}(\boldsymbol{\sigma}(p))$ and $\frac{d\boldsymbol{\sigma}_{\mathcal{A}}}{dp} \geqq \mathbf{0}$ for all $p \in [p^0, p^*_{\mathcal{A}}]$. We have constructed $\boldsymbol{\sigma}(p, H(p))$ for $p \in [p^0, p^*_{\mathcal{A}}]$. To extend $\boldsymbol{\sigma}(p, H(p))$ beyond $p^*_{\mathcal{A}}$, we have to solve a new system (18) for $\mathcal{A}' = \mathcal{A}(\boldsymbol{\sigma}(p^*_{\mathcal{A}}, H(p^*_{\mathcal{A}}))) \subsetneq \mathcal{A}$ with initial condition $\boldsymbol{\sigma}_{\mathcal{A}'}(p^*_{\mathcal{A}}) = \boldsymbol{\sigma}_{\mathcal{A}}(p^*_{\mathcal{A}})$. This is repeated until no bidder remains with $\sigma_j(p) < 1$, thereafter $\boldsymbol{\sigma}(p)$ is fixed.

To provide $\boldsymbol{\sigma}(p, H(p))$ for all prices and histories we need to specify for each $H(p)$ initial $\boldsymbol{\sigma}^0(p^0)$, where $p^0 = \max_{p_j \in H(p)} p_j$. At $p = 0$ set $\boldsymbol{\sigma}^0(0, \varnothing) = \mathbf{0}$, then $\boldsymbol{\sigma}(p)$ are calculated as above with $\mathcal{A} = \mathcal{N}$, for $p \in [0, p^*_{\mathcal{N}}]$.

At p^0 such that $H(p^0) \neq \bar{H}(p^0)$, define $\boldsymbol{\sigma}^0(p^0, \bar{H}(p^0)) = \boldsymbol{\sigma}(p^0, H(p^0))$. Obviously, if $\mathcal{A}(\boldsymbol{\sigma}(p, H(p^0))) \subset \mathcal{I}(\boldsymbol{\sigma}^0(p^0))$, then $\mathcal{A}(\boldsymbol{\sigma}^0(p^0, \bar{H}(p^0))) \subset \mathcal{A}(\boldsymbol{\sigma}(p, H(p^0)))$ and $\mathcal{A}(\boldsymbol{\sigma}^0(p^0)) \subset \mathcal{I}(\boldsymbol{\sigma}^0(p^0))$. Then, we can define $\boldsymbol{\sigma}(p, \bar{H}(p^0))$. Note that proceeding this way allows us to maintain continuity of $\boldsymbol{\sigma}$, or more formally, to link $\boldsymbol{\sigma}(\cdot, H(p^0))$ and $\boldsymbol{\sigma}(\cdot, \bar{H}(p^0))$ at the price p^0 where bidders exit the auction. ■

A.3 Necessity

Throughout this section we follow our notational convention and assume that *GSC* is violated for $\mathcal{A}$ and bidder 1, $\mathcal{I}(s) = \mathcal{A}^{+1}$, $\mathcal{A}$ is minimal, $\#\mathcal{A} = n \geq 2$, the set of signals considered is limited to $U^{\mathrm{s}}_{\mathcal{I}(\mathrm{s})}$. All additional definitions and supporting results are located in Appendix A.3.1.

Lemma 4 *Consider* $\mathrm{s}' = \mathrm{s}(t)$ *for an arbitrary* $t \in U^0_t$, $\mathrm{s}(t)$ *is the trajectory defined in Step* 1 *of the proof of Proposition 2. For any* $j \in \mathcal{A}$, *there exist* $b_j(s'_j) \equiv \lim_{s_j \downarrow s'_j} \inf \beta_j(s_j)$, *and these limits are equal, for any* $j \in \mathcal{A}$, $b_j(s'_j) \equiv b < \infty$. *In addition,* $\beta_1(s'_1) > b$ *and for any* $j \in \mathcal{A}$ *and* $s_j > s'_j$ *sufficiently close to* s'_j, $\beta_j(s_j) \geq b$.

Proof. Consider trajectory $\mathrm{s}^{\mathcal{A}}(\tau) \equiv \mathrm{s}^{\mathbf{u}^{\mathcal{A}}\mathcal{A}}(\tau)$ with $\mathrm{s}^{\mathbf{u}^{\mathcal{A}}\mathcal{A}}(0) = \mathrm{s}'$ as in Definition 7 in Appendix A.3.1. Along this trajectory the values of the bidders from $\mathcal{A}$ are increasing uniformly while s'_1 is fixed. Since *GSC* is violated for $\mathcal{A}$ and 1, for any sufficiently small $\tau > 0$, $V_1(\mathrm{s}^{\mathcal{A}}(\tau)) > \max_{j\in\mathcal{A}} V_j(\mathrm{s}^{\mathcal{A}}(\tau))$, and therefore,

$$\beta_1(s'_1) > \min_{j\in\mathcal{A}} \beta_j(s^{\mathcal{A}}_j(\tau)). \tag{20}$$

By continuity, for any s_1 sufficiently close to s'_1, $s_1 < s'_1$,

$$\beta_1(s_1) > \min_{j\in\mathcal{A}} \beta_j(s^{\mathcal{A}}_j(\tau)). \tag{21}$$

Since $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \gg \mathbf{0}$, $s^{\mathcal{A}}_j(\tau)$ is strictly increasing for any τ.

Define $\mathcal{B}$ as $j \in \mathcal{B}$ if and only if $j \in \mathcal{A}$ and $b_j(s'_j) = \min_{i\in\mathcal{A}} b_i(s'_i)$, define $b_{\mathcal{B}} \equiv \min_{i\in\mathcal{A}} b_i(s'_i)$ and $b_{-\mathcal{B}} \equiv \min_{i\in\mathcal{A}\setminus\mathcal{B}} b_i(s'_i)$. Since we are not imposing any *a priori* restrictions on the bidding functions, we allow for $b_{\mathcal{B}} = \infty$ and $b_{-\mathcal{B}} = \infty$.[14] Clearly, when $\#\mathcal{B} < n$, by definition, $b_{\mathcal{B}} < \infty$.

We show by induction on $\#\mathcal{B}$ that unless $\#\mathcal{B} = n$ efficiency is violated.

Step 1. Suppose first that $\#\mathcal{B} = 1$, $\mathcal{B} = \{j\}$. In words, we show that if we increase s_j, and on a much smaller scale decrease s'_1 and increase the signals of the other bidders from $\mathcal{A}$, then bidder j has the highest value, she must not be the first to exit, and so there must exist some other bidder with a lower bid. It is not bidder 1, so it must be some other bidder from $\mathcal{A}$. Then for a sequence of bids $\beta_j(s_{jm}) \to_{s_{jm}\downarrow s'_j} b_j$, there must exist bidder i and a sequence $s_{il} \downarrow s'_i$, such that $\beta_i(s_{il}) < \beta_j(s_{jm(l)})$, and so $b_i(s'_i) \leq b_j$, contradicting $\#\mathcal{B} = 1$.

Formally, for any $\varepsilon > 0$, we can find $\delta_\tau > 0$, such that for any $\tau \in (0, \delta_\tau)$, $\beta_i(s^{\mathcal{A}}_i(\tau)) > b_{-j} - \varepsilon$ for any $i \in \mathcal{A} \setminus \{j\}$ and (20) holds. Consider $\mathrm{s}^*(r) = \mathrm{s}' + r\mathbf{v}^j$, where vector $\mathbf{v}^j \equiv \mathbf{v}^j(\boldsymbol{\lambda})$ is defined in Lemma 5. Then we can find $\delta_\lambda \in (0,1)$, such that for all $\boldsymbol{\lambda}$ with $\lambda_1 \in (\delta_\lambda, 1)$, for all $r \in (0, \delta_j)$, where $\delta_j = s^{\mathcal{A}}_j(\delta_\tau) - s'_j$, for any $i \in \mathcal{A} \setminus \{j\}$ such that $v^j_i > 0$, we have $\beta_i(s^*_i(r)) > b_{-j} - \varepsilon$ as well. The existence of δ_λ follows from the fact that $\mathbf{u}^{\mathcal{A}}_{\mathcal{A}} \gg \mathbf{0}$.

[14] In what follows a statement $a > b$ together with $b = \infty$ implies $a = \infty$.

Fix $\varepsilon = (b_{-j} - b_j)/2$, and consider an element s_{jm} of the sequence $s_{jm} \downarrow s'_j$ with $\beta_j(s_{jm}) \to b_j(s'_j)$, such that $\beta_j(s_{jm}) < b_j + \varepsilon$, and r_m, the solution to $s^*_j(r_m) = s_{jm}$, belongs to $(0, \delta_j)$. Find τ_m, such that $s^{\mathcal{A}}_j(\tau_m) = s^*_j(r_m)$. Then, by (20) and by the above,

$$\beta_1(s'_1) > \beta_j(s^*_j(r_m)) = \min_{j \in \mathcal{A}} \beta_j(s^{\mathcal{A}}_j(\tau_m)), \tag{22}$$

and, similarly to (21), there exist $\delta_1 > 0$ such that for any $s_1 \in (s'_1 - \delta_1, s'_1)$,

$$\beta_1(s_1) > \beta_j(s_j(r_m)). \tag{23}$$

By continuity and Lemma 5, we can find sufficiently large m, and so sufficiently small $r_m > 0$, such that $\mathcal{I}(\mathbf{s}^*(r_m)) \subset \{j\} \cup \{1\}$. Since $\mathcal{A}$ is minimal, $\{1\} \neq \mathcal{I}(\mathbf{s}^*(r))$. By continuity and Lemma 6, whenever $r_m > 0$ is sufficiently small, there exists $\delta'_1 > 0$ such that for any $s_1 \in (s'_1 - \delta'_1, s'_1)$,

$$\{j\} = \mathcal{I}(s_1, \mathbf{s}^*_{\mathcal{A}}(r_m)). \tag{24}$$

Pick $r_m \in (0, \delta_j)$ and s_1 such that both (23) and (24) hold. Then, even if we slightly increase the signals of all the bidders $i \in \mathcal{A}$ with $s^*_i(r_m) = s'_i$, by continuity, $\{j\} = \mathcal{I}(\mathbf{s})$, where $\mathbf{s}$ is the disturbed profile. Since, $\beta_i(s_i) > b_{-j} - \varepsilon$ for any $s_i \in (s'_i, s^{\mathcal{A}}_i(\delta_\tau))$, $\beta_j(s_{jm}) = \beta_j(s_j) = \min_{i \in \mathcal{A}^{+1}} \beta_i(s_i)$. We reached a contradiction; bidder j has the highest value but drops out the first.

Step 2. Here we show that no matter what $\mathcal{B}$ is, for any $j \in \mathcal{A}$, for s_j sufficiently close to s'_j, $\beta_j(s_j) \geq b_{\mathcal{B}}$. Suppose otherwise. For any $j \in \mathcal{A}$ pick trajectory $\mathbf{s}^j(r) = \mathbf{s}' + r\mathbf{v}^j(\boldsymbol{\lambda}^j)$, where $\mathbf{v}^j(\boldsymbol{\lambda}^j)$ satisfies conditions of Lemma 5 and $\lambda^j_1 \in (1 - \delta_\lambda, 1)$ for all $j \in \mathcal{A}$, for $\delta_\lambda \in (0, 1)$ and arbitrarily close to 1.

For any given $j \in \mathcal{A}$, there exists δ_{rj}, such that for any $r \in (0, \delta_{rj})$, $\{j\} \cup \{1\} \supset \mathcal{I}(\mathbf{s}^j(r))$ *and* $\{j\} = \mathcal{I}(s_1, \mathbf{s}^j_{\mathcal{A}}(r))$ for any s_1 sufficiently close to s'_1 (this may depend on a particular r), $s_1 \leq s'_1$. Pick $\delta_r > 0$ with $\delta_r \leq \min_j \delta_{rj}$ and such that for any $i, j \in \mathcal{A}$, for any $r \in (0, \delta_r)$, $s^j_i(r)$ is such that whenever $s^j_i(r) > 0$, (20) holds for τ that solves $s^j_i(r) = s^{\mathcal{A}}_i(\tau)$.

By our presumption, there exists bidder $j \in \mathcal{A}$, such that $\beta_j(s_j) < b_{\mathcal{B}}$ for s_j close to s'_j, $s_j > s'_j$. We can always find bidder j with s_j such that r_j, the solution to $s_j = s^j_j(r)$, satisfies $r_j \in (0, \delta_r)$. Define $\varepsilon = b_{\mathcal{B}} - \beta_j(s_j)$. Then, $\{j\} \cup \{1\} \supset \mathcal{I}(\mathbf{s}^j(r))$ and by the procedure similar to the one in Step 1, by slightly reducing the signal of bidder 1 and slightly increasing signals of all $i \in \mathcal{A}$ with $s^j_i(r) = s'_i$, we obtain profile $\mathbf{s}$ at which bidder j has strictly the highest value. There must be some other bidder $i \in \mathcal{A}^{+1}$ with $\beta_i(s_i) < \beta_j(s_j)$. If it is always bidder 1, no matter how slight is the decrease in her signal, then $\lim_{s_1 \uparrow s'_1} \sup \beta_1(s_1) \leq b_{\mathcal{B}} - \varepsilon$, which contradicts (21). Otherwise, there exists bidder $i \in \mathcal{A}$ with $s_i > s'_i$, such that $\beta_i(s_i) < b_{\mathcal{B}} - \varepsilon$, and r_i, the solution to $s_i = s^i_i(r)$, satisfies $r_i < Cr_j$. By choosing δ_λ as close to 1 as necessary, we can make C as close to zero as necessary. It suffices to have $C < 1$.

Repeating this procedure we either find a contradiction involving bidder 1, or find a bidder $i \in \mathcal{A}$, and a converging sequence of $r_{im} \downarrow 0$, such that $s_{im} = s_i^i(r_{im}) \downarrow s_i'$ and $\beta_i(s_{im}) < b_{\mathcal{B}} - \varepsilon$ for any $m \geq 2$. But then, $b_i(s_i') \leq b_{\mathcal{B}} - \varepsilon$.

Step 3. Suppose now that $\#\mathcal{B} = k \geq 2$. In words, we show that we can find a trajectory $\mathrm{s}(\rho)$ along which bidders from $\mathcal{B}$ have the highest value and are dropping simultaneously for almost all ρ. Then, as above, after slightly increasing signals of bidders from $\mathcal{A} \setminus \mathcal{B}$, there will exist some other bidder who drops earlier, and so there will be a bidder $i \in \mathcal{A} \setminus \mathcal{B}$, for whom $b_i(s_i') \leq b_{\mathcal{B}}$.

Formally, consider trajectory $\mathrm{s}^{\mathcal{B}}(\rho) = \mathrm{s}^{\mathbf{v}^{\mathcal{B}}\mathcal{D}}(\rho)$, $\rho \geq 0$, defined in Definition 7 for $\mathbf{v}^{\mathcal{B}}$ and subset $\mathcal{D}$ from Lemma 5. Note that it is possible that for some $j \in \mathcal{B}$, $v_j^{\mathcal{B}} = 0$, and so $s_j^{\mathcal{B}}(\rho)$ is not necessarily increasing. Along this trajectory, for sufficiently small $\rho > 0$, by construction and since GSC is satisfied for bidder 1 and $\mathcal{D}$, $\mathcal{I}(\mathrm{s}^{\mathcal{B}}(\rho)) \subset \mathcal{B} \cup \{1\}$. By continuity and by Lemma 6, for sufficiently small $\rho > 0$, if we slightly decrease the signal of bidder 1, all the bidders with the highest value as a result belong to $\mathcal{B}$.

Consider $b_j(\rho) \equiv b_j(s_j^{\mathcal{B}}(\rho))$, clearly, $\lim_{\rho \to 0} b_j(\rho) = b_j(s_j')$ for all $j \in \mathcal{B}$. Then, for sufficiently small ρ, $\max_{j \in \mathcal{B}} b_j(\rho) < b_{-\mathcal{B}}$.

Proceeding from the contrary, by the arguments similar to the one made in Step 1 applied to $\mathrm{s} = \mathrm{s}^{\mathcal{B}}(\rho)$ for a sufficiently small $\rho > 0$, and by induction (any subset of $\mathcal{B}$ has less than k elements), we have that $b_j(\rho) = b_{\mathcal{B}}(\rho)$ for all $j \in \mathcal{B}$.

In general, as long as we stay sufficiently close to s', we can, by slightly moving in appropriate directions away from s', possibly in several steps, separate bidders from $\mathcal{A}$ in any given order. This implies that whenever two or more bidders from $\mathcal{A}$ have equal and maximal values, the limits of their bids from the right have to be equal.

Similarly, by the argument as in Step 2, $\beta_j(s_j) \geq b_{\mathcal{B}}(\rho)$ for any $j \in \mathcal{A}$ and s_j sufficiently close to $s_j^{\mathcal{B}}(\rho)$, $s_j > s_j^{\mathcal{B}}(\rho)$. Therefore, since $s_j^{\mathcal{B}}(\rho)$ is strictly increasing for some $j \in \mathcal{B}$, $b_{\mathcal{B}}(\rho)$ is weakly increasing.

Suppose that for any $\delta_\rho > 0$ we can find $\rho \in (0, \delta_\rho)$ and a bidder $j \in \mathcal{B}$, such that $\beta_j(s_j^{\mathcal{B}}(\rho)) > b_{\mathcal{B}}(\rho)$. Then, fixing signal $s_j^{\mathcal{B}}(\rho)$ and bid $\beta_j(s_j^{\mathcal{B}}(\rho))$ for bidder j, by induction and arguments similar to the above applied to $\mathcal{B}' = \mathcal{B} \backslash \{j\}$ and $\mathcal{A}' = \mathcal{A} \backslash \{j\}$, we obtain a contradiction. If it is to the presumption that an efficient equilibrium exists or to $\#\mathcal{B} = k < n$, then we are done immediately. Otherwise, we have shown that for any sufficiently small $\rho > 0$ and any $j \in \mathcal{B}$, $\beta_j(s_j^{\mathcal{B}}(\rho)) \leq b_{\mathcal{B}}(\rho)$. Combined with monotonicity of $b_{\mathcal{B}}(\rho)$ and the fact that for all j, $\beta_j(s_j) \geq b_{\mathcal{B}}(\rho)$ for $s_j > s_j^{\mathcal{B}}(\rho)$ (locally), we have that whenever $\beta_j(s_j^{\mathcal{B}}(\rho)) < b_{\mathcal{B}}(\rho)$ for some $j \in \mathcal{B}$, ρ is a discontinuity point for $b_{\mathcal{B}}(\rho)$. Since a monotonic function can have only a countable number of discontinuity points, we have that for almost all ρ, for all j, $\beta_j(s_j^{\mathcal{B}}(\rho)) = b_{\mathcal{B}}(\rho)$.

Now, we add bidder 1 into the picture. First, consider the case, when for some $j \in \mathcal{B}$, there exists $\delta_\rho > 0$, such that $s_j^{\mathcal{B}}(\rho) = s_j'$ for $\rho \in (0, \delta_\rho)$. Then, $b_{\mathcal{B}}(\rho) = b_{\mathcal{B}}$, and so for any $i \in \mathcal{B}$, with $s_i^{\mathcal{B}}(\rho)$ strictly increasing, $\beta_i(s_i^{\mathcal{B}}(\rho)) = b_{\mathcal{B}}$ for all $\rho \in (0, \delta_\rho)$. Consider $\mathrm{s}^{\mathcal{A}}(\tau)$, for a sufficiently small $\tau > 0$ we can separate bids of the bidders from $\mathcal{A} \backslash \mathcal{B}$ away from $b_{\mathcal{B}}$ and for each $j \in \mathcal{B}$, $\beta_j(s_j^{\mathcal{A}}(\tau)) \geq b_{\mathcal{B}}$, while at least for some $i \in \mathcal{B}$,

$\beta_i(s_i^{\mathcal{A}}(\tau)) = b_{\mathcal{B}}$. Therefore, from (20), $\beta_1(s_1') > b_{\mathcal{B}} = \min_{j\in\mathcal{A}} \beta_j(s_j^{\mathcal{A}}(\tau))$. Let $\mathcal{B}'$ be a subset of bidders $i \in \mathcal{B}$, for whom the bids $\beta_i(s_i) = b_{\mathcal{B}}$ in the right neighborhood of s_i'. Consider a trajectory $\mathrm{s}^*(r) = \mathrm{s}' + r\mathbf{v}^{\mathcal{B}'}$. Along this trajectory, the set of bidders with the highest value is a subset of $\mathcal{B}' \cup \{1\}$. By continuity and Lemma 6, for a sufficiently small $r > 0$, (21) holds as well and, after slightly reducing the signal of bidder 1, all the bidders with the highest value as a result belong to $\mathcal{B}'$. After slightly increasing the signal of each $j \in \mathcal{A}$ with $s_j^*(r) = s_j'$, we obtain signal profile s, at which all the bidders from $\mathcal{B}'$ drop out simultaneously at $b_{\mathcal{B}} = \min_{j\in\mathcal{A}^{+1}} \beta_j(s_j)$—a contradiction.

In the remaining case, $s_j^{\mathcal{B}}(\rho)$ are strictly increasing for all $j \in \mathcal{B}$, therefore $\beta_j(\cdot)$ for all $j \in \mathcal{B}$ are monotonic in the right neighborhood of s_j'. For any small $\tau > 0$, since $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \gg \mathbf{0}$ and $s_j^{\mathcal{B}}(\rho)$ are strictly increasing, we can define ρ_j for each $j \in \mathcal{B}$, such that $s_j^{\mathcal{B}}(\rho_j) = s_j^{\mathcal{A}}(\tau)$, let $\rho' \equiv \min_{j\in\mathcal{B}} \rho_j$. For any $\varepsilon > 0$ there exist $\delta_\tau > 0$, such that for any $\tau \in (0, \delta_\tau)$, we have: (i) for any $i \in \mathcal{A} \setminus \mathcal{B}$, $\beta_i(s_i^{\mathcal{A}}(\tau)) > b_{-\mathcal{B}}(\mathrm{s}') - \varepsilon/2$; (ii) for all $j \in \mathcal{B}$, ρ_j is such that $|b_{\mathcal{B}}(\rho) - b_{\mathcal{B}}| < \varepsilon/2$ and the above results hold; that is, in particular, that the bidders from $\mathcal{B}$ have the highest value at $\mathrm{s}^{\mathcal{B}}(\rho)$, $b_{\mathcal{B}}(\rho)$ is monotonic, and for all j, $\beta_j(s_j^{\mathcal{B}}(\rho)) \le b_{\mathcal{B}}(\rho)$; (iii) starting from $\mathrm{s}^{\mathcal{B}}(\rho')$ after reducing s_1' slightly all the bidders with the highest value as a result belong to $\mathcal{B}$.

Pick any $\tau \in (0, \delta_\tau)$ such that for all $j \in \mathcal{B}$, $b_{\mathcal{B}}(\rho)$ is continuous at ρ'. Then, consider $i \in \mathcal{B}$ with $\rho_i = \rho'$. From (21) we obtain for any s_1 sufficiently close to s_1', $s_1 < s_1'$,

$$\beta_1(s_1) > \min_{j\in\mathcal{A}} \beta_j(s_j^{\mathcal{A}}(\tau)) = \min_{j\in\mathcal{B}} \beta_j(s_j^{\mathcal{A}}(\tau)) = \min_{j\in\mathcal{B}} \beta_j(s_j^{\mathcal{B}}(\rho_j)) = \beta_i(s_i^{\mathcal{B}}(\rho')) = b_{\mathcal{B}}(\rho').$$

Then, starting from $\mathrm{s}^{\mathcal{B}}(\rho')$, reducing slightly s_1' and increasing slightly s_j for each $j \in \mathcal{A}$ with $s_j^{\mathcal{B}}(\rho') = s_j'$ we obtain signal profile s, at which $\mathcal{I}(\mathrm{s}) \subset \mathcal{B}$, but all the bidders from $\mathcal{B}$ exit first simultaneously at $b_{\mathcal{B}}(\rho')$—a contradiction.

Step 4. We have shown that $\#\mathcal{B} = n$, and so $\mathcal{B} = \mathcal{A}$. Let $b \equiv b_{\mathcal{A}}$. Since for all $j \in \mathcal{A}$, $\beta_j(s_j) \ge b(\mathrm{s}')$ for all s_j close to s_j', $s_j > s_j'$, from (20) we have $\beta_1(s_1') > b$.

It remains to be shown that $b < \infty$. If $b = \infty$, then for each bidder $j \in \mathcal{A}^{+1}$ there exists a range of signals with $\beta_j(s_j) = \infty$. As a result, the equilibrium payoff to each of the bidders is equal to $-\infty$, which cannot happen in an equilibrium since each bidder can exit instead at $p = 0$ and assure herself the payoff of 0. ∎

Corollary 3 *If for some bidder $j \in \mathcal{A}$, $\beta_j(s_j(t)) \neq b(t)$, then t is a discontinuity point for $b(t)$.*

Proof. If for some $j \in \mathcal{A}$, $\beta_j(s_j(t)) > b(t)$, then by the argument similar to the one made in Step 3 of the proof of Lemma 4, considering $\mathcal{A}' = \mathcal{A} \setminus \{j\}$, we can find a profile s, at which all the bidders from $\mathcal{A}'$ exit simultaneously prior to bidder 1 and j, while the bidder or bidders with the highest value belong to $\mathcal{A}'$.

Monotonicity of $b(t)$ is established in Step 3 of the proof of Proposition 2. From Lemma 4 it follows that for all $j \in \mathcal{A}$, whenever $s_j(t') > s_j(t)$, $\beta_j(s_j(t')) \ge b(t)$, for t and t' from the considered neighborhood U_t^0. Therefore, if for some $j \in \mathcal{A}$, $\beta_j(s_j(t)) < b(t)$, then $b(t'') \leq \beta_j(s_j(t))$ whenever $s_j(t'') < s_j(t)$, so t is a discontinuity point for $b(t)$. ■

A.3.1 Supporting results for value functions

Definition 7 *For a given* $\mathbf{s}'$, $\mathcal{B} \subset \mathcal{I}(\mathbf{s}')$, *and vector* $\mathbf{x}$ *with* $\mathbf{x}_{-\mathcal{B}} = \mathbf{0}$, *define* $\mathbf{y}_{\mathcal{B}} \equiv \mathbf{x} \cdot \nabla \mathbf{V}_{\mathcal{B}}(\mathbf{s}')$*—the derivative of* $\mathbf{V}_{\mathcal{B}}$ *along* $\mathbf{x}$*. Define trajectory* $\mathbf{s}^{\mathbf{x}\mathcal{B}}(\tau)$ *with* $\mathbf{s}^{\mathbf{x}\mathcal{B}}(0) = \mathbf{s}'$ *and* $\mathbf{s}^{\mathbf{x}\mathcal{B}}_{-\mathcal{B}}(\tau) = \mathbf{s}'_{-\mathcal{B}}$ *as a solution to the system*

$$\mathbf{V}_{\mathcal{B}}(\mathbf{s}^{\mathbf{x}\mathcal{B}}(\tau)) = (V + \tau)\mathbf{y}_{\mathcal{B}},$$

where $V = \max_{j \in \mathcal{N}} V_j(\mathbf{s}')$*. Clearly,* $\left.\frac{d\mathbf{s}^{\mathbf{x}\mathcal{B}}_{\mathcal{B}}(\tau)}{d\tau}\right|_{\tau=0} = \mathbf{x}_{\mathcal{B}}$.

Lemma 5 *For any proper subset* $\mathcal{B} \subsetneq \mathcal{A}$ *there exists subset* $\mathcal{D} = \mathcal{D}(\mathcal{B})$, $\mathcal{B} \subseteq \mathcal{D} \subsetneq \mathcal{A}$, *such that for any* $k \in \mathcal{A} \setminus \mathcal{D}$, $\mathbf{u}^{\mathcal{D}} \cdot \nabla V_k(\mathbf{s}) < 1$*. Moreover for any* $\varepsilon > 0$*, there exist vector* $\mathbf{v}^{\mathcal{B}} \geqq \mathbf{0}$ *such that* $\left\|\mathbf{v}^{\mathcal{B}} - \mathbf{u}^{\mathcal{B}}\right\| < \varepsilon$, $\mathbf{v}^{\mathcal{B}}_k = 0$ *for any* $k \in \mathcal{A} \setminus \mathcal{D}$, $\mathbf{v}^{\mathcal{B}} \cdot \nabla V_i(\mathbf{s}) < 1$ *for any* $i \in \mathcal{A} \setminus \mathcal{B}$ *and* $\mathbf{v}^{\mathcal{B}} \cdot \nabla V_j(\mathbf{s}) = 1$ *for all* $j \in \mathcal{B}$.

Proof. The proof is by induction on the number of bidders in $\mathcal{B}$.

Define $\mathcal{C}$ as the set of bidders $k \in \mathcal{A} \setminus \mathcal{B}$ such that $\mathbf{u}^{\mathcal{B}} \cdot \nabla V_k(\mathbf{s}) = 1$. Since *GSC* is satisfied for $\mathcal{B}$, $\mathbf{u}^{\mathcal{B}} \neq \mathbf{u}^{\mathcal{A}}$ and $\mathcal{C} \neq \mathcal{A} \setminus \mathcal{B}$. If $\mathcal{C} = \varnothing$, then set $\mathcal{D} \equiv \mathcal{B}$, and $\mathbf{v}^{\mathcal{B}} \equiv \mathbf{u}^{\mathcal{B}}$. Whenever $\#\mathcal{B} = \#\mathcal{A} - 1 = n - 1$ we have the result. Suppose that for all $\mathcal{B}'$, with $\#\mathcal{B}' > \#\mathcal{B}$ the result holds.

If $\mathcal{C} \neq \varnothing$ define $\mathcal{B}' = \mathcal{A} \setminus \mathcal{C}$, then $\mathbf{u}^{\mathcal{B}} \cdot \nabla V_k(\mathbf{s}) < 1$ for any $k \in \mathcal{B}' \setminus \mathcal{B}$. Pick $\mathcal{D} \equiv \mathcal{D}(\mathcal{B}')$. Consider $\mathbf{v}^{\mathcal{B}} = \lambda_1 \mathbf{u}^{\mathcal{B}} + (1 - \lambda_1)\mathbf{v}^{\mathcal{B}'}$ with $\lambda_1 \in (0, 1)$. When $\lambda_1 \to 1$, $\mathbf{v}^{\mathcal{B}} \to \mathbf{u}^{\mathcal{B}}$. By induction, for all $j \in \mathcal{A} \setminus \mathcal{B}'$, $\mathbf{v}^{\mathcal{B}'} \cdot \nabla V_i(\mathbf{s}) < 1$. Therefore, for all $j \in \mathcal{A} \setminus \mathcal{B}$, $\mathbf{v}^{\mathcal{B}} \cdot \nabla V_i(\mathbf{s}) < 1$ as long as $\lambda_1 \in (0, 1)$. ■

Remark 2 *As follows from the proof of Lemma 5 we can find a finite sequence* $\mathcal{B} \subsetneq \mathcal{B}' \subsetneq \mathcal{B}'' \subsetneq \ldots \subsetneq \mathcal{A}$*, such that* $\mathbf{v}^{\mathcal{B}}$ *can be represented as* $\mathbf{v}^{\mathcal{B}} = \lambda_1 \mathbf{u}^{\mathcal{B}} + \lambda_2 \mathbf{u}^{\mathcal{B}'} + \lambda_3 \mathbf{u}^{\mathcal{B}''} + \ldots$*, where* $\sum_i \lambda_i = 1$*, for any* i*,* $\lambda_i \in (0, 1)$*. Lemma states that* $\mathbf{v}^{\mathcal{B}} \equiv \mathbf{v}^{\mathcal{B}}(\boldsymbol{\lambda})$ *with required properties can be found with* λ_1 *being arbitrarily close to* 1.

Lemma 6 *For any* $\mathcal{B} \subsetneq \mathcal{A}$*, it is either* $\mathbf{v}^{\mathcal{B}} \cdot \nabla V_1(\mathbf{s}) < 1$*, or* $\frac{\partial V_1}{\partial s_1}(\mathbf{s}') > \min_{j \in \mathcal{B}} \frac{\partial V_j}{\partial s_1}(\mathbf{s}')$.

Proof. If the inequality in (4) is strict for $\mathcal{B}$ and bidder 1, $\mathbf{u}^{\mathcal{B}} \cdot \nabla V_1(\mathbf{s}) < \max_{j \in \mathcal{B}} \mathbf{u}^{\mathcal{B}} \cdot \nabla V_j(\mathbf{s})$, and so $\mathbf{v}^{\mathcal{B}} \cdot \nabla V_1(\mathbf{s}) < 1$. Otherwise, since $\#\mathcal{B} < n$, the only remaining case is $\mathbf{u}^{\mathcal{B}} \cdot \nabla V_1(\mathbf{s}') = 1$ or $DV_{\mathcal{C}} \mathbf{u}^{\mathcal{B}}_{\mathcal{C}} = \mathbf{1}_{\mathcal{C}}$, where $\mathcal{C} = \mathcal{B} \cup \{1\}$. Since $\det DV_{\mathcal{C}} \neq 0$ we have $DV_{\mathcal{C}} \mathbf{e}^1_{\mathcal{C}} \neq \mathbf{1}_{\mathcal{C}}$, therefore, by *SC*, $\frac{\partial V_1}{\partial s_1}(\mathbf{s}') > \min_{j \in \mathcal{B}} \frac{\partial V_j}{\partial s_1}(\mathbf{s}')$. ■

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