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Uniqueness of stationary equilibrium payoffs in coalitional bargaining

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Uniqueness of Stationary Equilibrium Payoffs in Coalitional Bargaining[1]

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Abstract

We study a model of sequential bargaining in which, in each period before an agreement is reached, the proposer's identity (and whether there is a proposer) are randomly determined; the proposer suggests a division of a pie of size one; each other agent either approves or rejects the proposal; and the proposal is implemented if the set of approving agents is a winning coalition for the proposer. The theory of the fixed point index is used to show that stationary equilibrium expected payoffs of this coalitional bargaining game are unique. This generalizes Eraslan (2002) insofar as: (a) there are no restrictions on the structure of sets of winning coalitions; (b) different proposers may have different sets of winning coalitions; (c) there may be a positive probability that no proposer is selected.

Running Title: **Coalitional Bargaining**

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1 Introduction

Baron and Ferejohn (1989) study a model in which a group of n risk neutral agents divide a fixed pie. In each period a *proposer* (or "formateur") is selected randomly, the proposer suggests a division of the pie, and this division is implemented if it is approved by a *winning coalition* of the agents. Otherwise the process is repeated until agreement is achieved, with payoffs discounted geometrically. They show that when the model is symmetric in the sense that all agents have the same *recognition probability* (probability of being selected as the proposer) and discount factor, and a winning coalition is any set of k agents, in all stationary subgame perfect equilibria agreement is reached in the first period with probability one and each agent's ex ante expected utility is $1/n$.

For the application motivating Baron and Ferejohn (bargaining among parties in a legislature or parliament) it is natural to suppose that recognition probabilities differ across agents, with larger parties typically having higher recognition probabilities. In a committee one would normally expect that the chair's recognition probability is higher than the recognition probabilities of other members. For several years it was unknown whether there could be multiple stationary subgame perfect equilibria yielding different expected utilities when recognition probabilities or discount factors differ across agents. Eraslan (2002) resolved this problem, showing that, even with unequal recognition probabilities and discount factors, there is a single vector of expected utilities common to all stationary subgame perfect equilibria.

Her analysis is restricted to k-majority rule for $1 \leq k \leq n$, but in legislative settings it is also natural to allow different agents to have different weights in the voting over approval of a proposal. This direction of generalization is also of interest from the point of view of other applications. In corporate bankruptcies governed by Chapter 11, the voting over approval of a proposed reorganization is asymmetric with respect to different seniority classes of debt, and creditors who are owed more money have greater power. Other examples are described in the next section.

Here we show that, under more general conditions than those considered by Eraslan (2002), there is a unique vector of expected payoffs that is generated by all of the game's stationary subgame perfect equilibria. Specifically, in addition to allowing different agents to have different recognition probabilities and discount factors, we allow the set of winning coalitions to be arbitrary, and to depend on the proposer, and we allow the sum of the recognition probabilities to be less than one.

Our argument has an interesting mathematical structure. Suppose that C is a nonempty compact convex set and $F : C \to C$ is an upper semicontinuous convex valued correspondence. Roughly, the *fixed point index* assigns an integer to each compact set of fixed points of F that has a neighborhood containing no other fixed points. For any partition of the set of fixed points into such sets, the sum of the indices of the sets must be one. We show that each connected component of the set of fixed points of the relevant correspondence has a neighborhood that has no other fixed points, and that its index is one. Consequently the set of fixed points must consist of a single connected component. We also show that the vector of continuation payoffs is constant in each connected component, so our main result follows. As has been noted (cf. p. 615 of Mas-Colell et al. (1995)) this method of proving uniqueness is widely applicable, but in all earlier cases we know of elementary methods were also available.

The organization of the remainder is as follows. Section 2 describes some of the extensive literature descended from Baron and Ferejohn (1989) as it relates to our work. The model, and a "reduced equilibrium" concept motivated by stationary subgame perfect equilibrium, are explained in Section 3. Section 4 shows that the set of reduced equilibria is the set of fixed points of a correspondence satisfying the hypotheses of Kakutani's theorem. Section 5 presents the axioms that characterize the fixed point index, along with their relevant consequences, and a general version of our uniqueness result, emphasizing its mathematical structure. Section 6 verifies that the hypotheses of the general uniqueness theorem are satisfied by the model, thereby proving the main result. Some possible topics for further research on this topic are sketched in Section 7. An Appendix gives a precise description of the bargaining protocol and shows that every stationary subgame perfect equilibrium has an associated reduced equilibrium, and that every reduced equilibrium is derived from some stationary subgame perfect equilibrium.

2 Related Literature

Since Baron and Ferejohn's paper, an extensive body of work has grown out of their model[1]. There are many variants of the basic model, and many

[1] With apologies for the inevitable omissions, a fairly comprehensive list is: Baron (1989), Baron (1991), McKelvey and Riezman (1992), Okada (1993), Merlo and Wilson (1995), Baron (1996), Calvert and Dietz (1996), Okada (1996), Winter (1996), Chari et al. (1997), Persson (1998), Baron (1998), Diermeier and Feddersen (1998), Banks and Duggan (2000), McCarty (2000b), McCarty (2000a), Bennedsen and Feldmann (2002), Eraslan (2002), Eraslan and Merlo (2002), Jackson and Moselle (2002), Norman (2002),

ways in which the bargaining model might be embedded into some periods of larger dynamic models. The number and diversity of applications in the literature suggest that the methodology pioneered by Baron and Ferejohn (1989) has established itself as an important tool for addressing a central issue of political science: the relationship between the rules governing political institutions and the outcomes they produce.

Perhaps the main alternatives would be the power indices of cooperative game theory. In Section 7 we describe how our result opens the way to similar power indices with explicit noncooperative foundations.

As in other types of social scientific modelling, models with unique (or perhaps finitely many) predictions are preferred for many reasons. As a practical matter, tractable empirical methodologies typically limit attention to models with unique predictions because available statistical methods have little to say about selection of an equilibrium when more than one are available. Indeed, there are already several studies (e.g., Diermeier et al. (2003), Adachi and Watanabe (2008), Diermeier and Merlo (2004), Ansolabehere et al. (2005), Coscia (2005)) taking the Baron-Ferejohn model to data, and our result has direct application to several previous papers (e.g., Winter (1996), McCarty (2000b), Ansolabehere et al. (2003), Snyder et al. (2005)) either strengthening the work by providing uniqueness results that were not available to the authors, or allowing uniqueness to be proved under weaker hypotheses. In addition, the literature contains models (e.g., McCarty (2000a)) that would become instances of our framework after small modifications. It seems natural to expect that further development of the literature will lead to additional applications, and that our uniqueness result will influence model selection in some instances; indeed, Montero (2007) has already applied our result in an analysis of the Council of Ministers of the European Union.

The introduction has already described some of the history of uniqueness results for the Baron-Ferejohn model. In addition to Eraslan (2002), papers concerned with uniqueness of equilibrium expected payoffs include Norman (2002), Yan (2002), Cho and Duggan (2003), Cardona and Ponsati (2007), and Montero (2006). Norman (2002) shows that equilibrium payoffs may fail to be unique when there are finitely many bargaining periods. Cho and Duggan (2003) and Cardona and Ponsati (2007) consider models in

Yan (2002), Ansolabehere et al. (2003), Cho and Duggan (2003), Diermeier et al. (2003), Diermeier and Merlo (2004), Coscia (2005), Knight (2005), Snyder et al. (2005), Banks and Duggan (2006), Kalandrakis (2006), Battaglini and Coate (2007), Cardona and Ponsati (2007), Montero (2007), Predtetchinski (2007), Yıldırım (2007), Adachi and Watanabe (2008), Battaglini and Coate (2008), Yıldırım (2008).

which the space of outcomes is one dimensional, modelling policy concerns rather than private rewards. Cho and Duggan (2003) establish uniqueness when utility functions are quadratic and provide an example with multiple equilibria when the utility functions are not quadratic. Cardona and Ponsati (2007) establish uniqueness in the case of unanimity and asymptotic uniqueness as the discount factor goes to one. Yıldırım (2007, 2008) analyzes models in which recognition probabilities are determined by agents' efforts, either in each period (for unanimity rule and k-majority rule) or persistently (for unanimity rule) in the sense that effort in the initial period determines a single vector of recognition probabilities governing the process in all subsequent periods.

Recall that a *cooperative game with transferable utility* (TU game) consists of a set of agents $N = \{1, \dots, n\}$ together with a specification of a payoff $v(S) \in \mathbb{R}$ for each coalition $S \subset N$. Generalizing results in Okada (1993) that did not appear in Okada (1996), Yan (2002) studies the bargaining protocol analyzed here, with general recognition probabilities, applied to a TU game with a nonempty core, showing that an allocation in the core is realized as the vector of continuation values if and only if it coincides with the vector of recognition probabilities, in which case there are no other stationary subgame perfect equilibrium payoffs.

A TU game is said to be *simple* if each coalition's payoff is either zero or one. That is, a simple game is essentially a specification of a system of winning coalitions. Since we study simple games, which rarely have a nonempty core, the overlap of her results with ours is small. A TU game is *proper* if it is simple and $v(S) = 1$ implies that $v(T) = 0$ for all $T \subset N \setminus S$. Montero (2006) generalizes the restriction of Yan's theorem to proper games, as follows: if the vector of recognition probabilities is in the nucleus[2] and the sum of recognition probabilities over the members of each winning coalition is at least $1/2$, then the vector of recognition probabilities is the unique vector of stationary subgame perfect equilibrium payoffs. The uniqueness assertion is a special case of our result. Okada (2007) extends some of this analysis to a model in which bargaining among the remaining players continues after an agreement by a coalition other than the grand coalition.

Merlo and Wilson (1995) and Eraslan and Merlo (2002) study a generalization of the Baron-Ferejohn model in which the size of the pie varies stochastically; Merlo and Wilson (1995) demonstrate uniqueness under una-

[2]The nucleus is the set of imputations minimizing the maximum, over all coalitions, of the difference between the coalition's worth and its aggregate allocation in the imputation. It coincides with the core when the core is nonempty, and the nucleolus (cf. Schmeidler (1969)) is always an element.

nimity rule, and Eraslan and Merlo (2002) demonstrate nonuniqueness under majority rule. Ali (2006) examines a variant of the Baron-Ferejohn model in which agents have divergent, and optimistic, beliefs about the recognition probabilities, finding that there are unique continuation payoffs even though agreement need not be reached in the first period.

3 The Model and Result

The agents in the set $N := \{1, \ldots, n\}$ bargain over the division of a pie of size 1 according to the following protocol. At the beginning of each period until agreement is reached there is a random determination of whether there will be a *proposer* and, if so, who that will be. The probability that agent i is selected to be the proposer, denoted by p_i, is called i's *recognition probability.* Let $p := (p_1, \ldots, p_n)$ be the vector of recognition probabilities. Of course we require that each p_i is nonnegative, and also that $p_1 + \cdots + p_n \leq 1$. Then $p_0 := 1 - (p_1 + \cdots + p_n)$ is the probability that there is no proposer.

If there is a proposer she selects a proposal from the set

$$\Pi := \{\, x \in [0,1]^n : \sum_{i=1}^{n} x_i \leq 1 \,\}$$

of feasible allocations. There is a random determination of an ordering of the agents, after which the agents each vote for or against the proposal, with each agent seeing the votes of other agents in the ordering before selecting her own vote. For each i there is a collection $\mathcal{S}_i = \{S_{i1}, \ldots, S_{iK_i}\}$ of subsets of N, called *winning coalitions for i*. If i is the proposer and the set of agents voting in favor is an element of $\mathcal{S}_i$, then the proposal is implemented, ending the game. Otherwise the process is repeated in the next period. As a matter of convention we assume that i is a member of every coalition in $\mathcal{S}_i$. The utility for agent i if the proposal x is implemented in period t is $\delta_i^t x_i$ where $\delta = (\delta_1, \ldots, \delta_n) \in (0,1)^n$ is a vector of discount factors. If agreement is never reached, then each agent's utility is zero.

We are interested in the expected payoffs resulting from stationary subgame perfect equilibria of this game. Below we will define a notion called "reduced equilibrium" which imposes conditions on the agents' equilibrium expected payoffs and their probabilities of including each other in minimal winning coalitions. In the Appendix we define a notion of stationary subgame perfect equilibrium for the bargaining game. We show that if any superset of a winning coalition for an agent is also a winning coalition for that agent, then each stationary subgame perfect equilibrium induces a reduced equilibrium, and each reduced equilibrium is induced by at least one stationary subgame perfect equilibrium. This is technically demanding, but unsurprising (the voting protocol was carefully designed with these results in mind) and its primary purpose is to provide one noncooperative foundation for our interest in reduced equilibrium.

Although the intuitive justification for the reduced equilibrium concept depends on supersets of winning coalitions being winning coalitions, the definition itself, and the subsequent analysis, does not. It is, at least hypothetically, possible that there is a derivation of this concept from noncooperative analysis that does not depend on that assumption.

We now give an intuitive description of the reduced equilibrium concept. Prior to the beginning of the initial period, agent i has an expected payoff for the game, which is denoted by v_i. This consists of three parts:

(a) the probability p_0 that no proposer is selected times the discounted value $\delta_i v_i$ of the game beginning in the next period;

(b) $\delta_i v_i$, which is the minimum payment required to induce a vote in favor of the proposal, times the sum over all agents (including herself) of that agent's recognition probability times the probability that agent i is included in that proposer's coalition;

(c) the recognition probability p_i times the surplus w_i net of payments to coalitions members (including herself) in that event.

If she is the proposer she will make a proposal that consists of offering $\delta_j v_j$ (or, intuitively, slightly more to insure acceptance) to each member of one of her winning coalitions, the rest of the pie to herself, and zero to everyone else. In particular, equilibrium is efficient in the sense that agreement is reached in the first period in which there is a proposer. Each agent has a probability distribution over winning coalitions when she is the proposer, and of course this distribution assigns positive probability only to those coalitions that minimize expenditures on coalition partners.

For each agent the crucial characteristics of an equilibrium are the minimal cost of forming a winning coalition when she is the proposer and the aggregate probability of being included in another agent's winning coalition. The latter probabilities can be achieved by various assignments of probability to the proposers' winning coalitions, so the probabilities that the proposers assign to winning coalitions are indeterminate, and it is natural to try to reduce the analysis to conditions on the agents' aggregate probabilities of being included. Unfortunately this does not work, because it does not include enough information to analyze the minimal cost of a winning coalition. Instead our analysis operates at an intermediate level of aggregation. Specifically, we study the probabilities that each proposer assigns to each possible coalition partner, which have the following description. For each i and $S_{ik} \in \mathcal{S}_i$ let $\tilde{S}_{ik} \in \{0,1\}^n$ be the vector whose j^{th} component is 1 if $j \in S_{ik}$ and 0 otherwise. Let $\mathcal{Y}_i \subset [0,1]^n$ be the convex hull of $\{\, \tilde{S}_{ik} : S_{ik} \in \mathcal{S}_i \,\}$, and let

$$\mathcal{Y} := \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n,$$

regarded as a space of $n \times n$ matrices Y whose i^{th} column y_i is the element of $\mathcal{Y}_i$. Then each $Y \in \mathcal{Y}$ has entries in $[0,1]$ and ones on the diagonal. Let

$$\mathcal{V} := \{\, v \in [0,1]^n : \sum_i v_i \leq 1 \,\};$$

elements of $\mathcal{V}$ are thought of as possible expected payoff vectors of the game.

We can now define a *reduced equilibrium* to be a pair $(v, Y) \in \mathcal{V} \times \mathcal{Y}$ such that for each i:

(a) $v_i = (p_0 + \sum_{j=1}^n p_j y_{ij})\delta_i v_i + p_i(1 - \sum_{j=1}^n y_{ji}\delta_j v_j)$;

(b) $y_i \in \operatorname{argmin}_{y_i' \in \mathcal{Y}_i} \sum_{j=1}^n y_{ji}' \delta_j v_j$.

Condition (a) expresses the decomposition of the equilibrium expected payoff described informally below. Condition (b), which requires that each proposer minimizes the cost of coalition partners, has some important implications for our analysis. First of all, because $\mathcal{Y}$ is the cartesian product of the $\mathcal{Y}_i$, condition (b) holds for all i if and only if Y minimizes the total expenditure on coalition partners, or the total expected sum, or in fact any positive weighted sum of the agents' expenditures.

Let $\mathbf{m} : \mathcal{Y} \to [0,1]^n$ be the function $\mathbf{m}(Y) := Yp$ whose i^{th} component

$$\mathbf{m}_i(Y) := \sum_{j=1}^n p_j y_{ij}$$

is the total probability that i is included in a proposer's coalition, given the matrix of inclusion probabilities Y. Suppose that (v, Y) is a reduced equilibrium. If $Y' \in \mathcal{Y}$ with $\mathbf{m}(Y') = \mathbf{m}(Y)$, then each Y and Y' have each agent included in winning coalitions with the same probability, in which case the total expenditure on coalition partners is the same, so each proposer is minimizing expenditure. Therefore, for each agent, the sum of the discounted continuation value, times the probability of receiving it, plus expected proposer surplus, is the same under (v, Y') and (v, Y), so (v, Y') is also a reduced equilibrium. The converse also holds:

Lemma 1. *If (v, Y) is a reduced equilibrium and $Y' \in \mathcal{Y}$, then (v, Y') is a reduced equilibrium if and only if $\mathbf{m}(Y') = \mathbf{m}(Y)$.*

Proof. For each i let

$$w_i = 1 - \sum_{j=1}^{n} y_{ji}\delta_j v_j \quad \text{and} \quad w'_i = 1 - \sum_{j=1}^{n} y'_{ji}\delta_j v_j.$$

Then condition (a) becomes $v_i = (p_0 + \mathbf{m}_i(Y))\delta_i v_i + p_i w_i$. By the observation above, if $\mathbf{m}(Y') = \mathbf{m}(Y)$, then (v, Y') satisfies (b), and consequently $w_i = w'_i$, which implies that $v_i = (p_0 + \mathbf{m}_i(Y'))\delta_i v_i + p_i w'_i$ for all i, which is to say that (a) holds.

Conversely, suppose that (v, Y') is also a reduced equilibrium. Then condition (b) implies that $w_i = w'_i$ for all i, and condition (a) implies that

$$(p_0 + \mathbf{m}_i(Y))\delta_i v_i + p_i w_i = v_i = (p_0 + \mathbf{m}_i(Y'))\delta_i v_i + p_i w_i$$

for all i, so that $\mathbf{m}(Y') = \mathbf{m}(Y)$. □

Our main result characterizes the set of reduced equilibria.

Theorem 1. *The set of reduced equilibria is nonempty, and there is a single vector $v \in \mathcal{V}$ that is the first component of every reduced equilibrium. If (v, Y) is a reduced equilibrium, then the set of all reduced equilibria is $\{v\} \times \mathbf{m}^{-1}(\mathbf{m}(Y))$. In particular, the set of reduced equilibria is a convex polytope.*

The lemma above implies that the set of reduced equilibria is $\bigcup \mathbf{m}^{-1}(v)$ where the union is over all v such that there is some Y such that (v, Y) is a reduced equilibrium. The rest of our analysis is concerned with showing that there is a unique such v.

4 Reduced Equilibria as Fixed Points

This section describes a correspondence whose fixed points correspond to reduced equilibria. For the time being we fix $v \in \mathbb{R}^n$ and $Y \in \mathcal{Y}$. Our first goal is to show that if condition (a) of the definition of reduced equilibrium holds, then v is determined by Y.

We begin by introducing some notational conventions that will simplify the algebra to come. If $\nu \in \mathbb{R}^n$, $\nabla(\nu)$ will denote the $n \times n$ diagonal matrix whose diagonal entries are the components of ν. Often we will denote such a diagonal matrix with the capital letter corresponding to the lower case letter denoting the vector. Thus $P := \nabla(p)$ and $\Delta := \nabla(\delta)$. For $Y \in \mathcal{Y}$ and $v \in [0,1]^n$ let $M(Y) := \nabla(\mathbf{m}(Y))$.

Fix $Y \in \mathcal{Y}$. We will write M in place of $M(Y)$. Let I be the $n \times n$ identity matrix. Condition (a) in the definition of reduced equilibrium can be rewritten as

$$v = \big(M + p_0 I - PY^T\big)\Delta v + p$$

and then as $Av = p$ where, for $Y \in \mathcal{Y}$,

$$A = A(Y) := I - \big(M + p_0 I - PY^T\big)\Delta.$$

Let $A = (a_{ij})$; then

$$a_{ij} = \begin{cases} 1 - (m_i + p_0 - p_i)\delta_i, & j = i, \\ p_j y_{ij} \delta_i, & j \neq i. \end{cases}$$

Proposition 1. *A is invertible.*

In preparation for the proof, and for a key result later, we quickly review the theory of nonsingular M-matrices. A square matrix is a *nonsingular M-matrix* if it has positive entries on the diagonal and nonpositive off-diagonal entries, and is dominant diagonal, meaning that for each column, the sum of the entries is positive. The main properties of these matrices are as follows:

Lemma 2. *If B is a nonsingular M-matrix, then B is invertible, and the entries of its inverse are nonnegative. The determinants of the principal minors are nonnegative, so if B is symmetric, then it is positive definite.*

Proof. E.g., Theorem 2.3 of Chapter 6 of Berman (1979). □

Lemma 3. *$B := A - p\delta^T$ is a nonsingular M-matrix.*

Proof. Let $B = (b_{ij})$. Then for $j \neq i$ we have $b_{ji} = p_j(y_{ij} - 1)\delta_i$, and of course this is nonpositive because $y_{ij} \leq 1$. We also have $b_{ii} = 1-(m_i+p_0)\delta_i$. Therefore

$$\begin{aligned} -\sum_{j\neq i} b_{ji} = (\sum_{j\neq i} p_j - \sum_{j\neq i} p_j y_{ij})\delta_i &= (\sum_{j=1}^n p_j - \sum_{j=1}^n p_j y_{ij})\delta_i \\ &= (1 - p_0 - m_i)\delta_i < b_{ii}. \end{aligned}$$

□

Proof of Proposition 1. In view of the last result, the entries of B^{-1} are nonnegative, so $1 + \delta^T B^{-1} p > 0$, and consequently the Sherman-Morrison formula[3] implies that $A = B + p\delta^T$ is invertible. $\square$

Now that Proposition 1 is established, for $Y \in \mathcal{Y}$ we may define

$$\mathbf{v}(Y) := A(Y)^{-1} p.$$

For each i and $v \in \mathcal{V}$ let

$$\mathcal{Y}_i(v) := \mathrm{argmin}_{y_i \in \mathcal{Y}_i} \sum_{j=1}^{n} y_{ji} \delta_j v_j = \mathrm{argmin}_{y_i \in \mathcal{Y}_i} y_i^T \Delta v,$$

and let

$$\mathcal{Y}(v) := \mathcal{Y}_1(v) \times \cdots \times \mathcal{Y}_n(v).$$

Then (v, Y) satisfies condition (a) of the definition of reduced equilibrium if and only if $v = \mathbf{v}(Y)$, and it satisfies condition (b) if and only if $Y \in \mathcal{Y}(v)$.

Let $F : \mathcal{Y} \to \mathcal{Y}$ be the correspondence $F(Y) := \mathcal{Y}(\mathbf{v}(Y))$.

Proposition 2. *The correspondence F is upper semicontinuous and convex valued. A point $Y \in \mathcal{Y}$ is a fixed point of F if and only if $(\mathbf{v}(Y), Y)$ is a reduced equilibrium.*

Proof. Since each $\mathcal{Y}_i(v)$ is the subset of $\mathcal{Y}_i$ that minimizes a linear function, $\mathcal{Y}(v)$ is compact and convex, so F is convex valued. The minimization problem varies continuously with v, and $\mathbf{v}(Y)$ is a continuous function of Y, so Berge's theorem of the maximum implies that F is upper semicontinuous. Condition (a) in the definition of a reduced equilibrium is equivalent to the requirement that $v = \mathbf{v}(Y)$, and condition (b) is equivalent to $Y \in \mathcal{Y}(v)$. $\square$

5 The General Uniqueness Result

This section explains the general mathematical principle underlying the uniqueness asserted in Theorem 1. We first review the relevant results concerning the fixed point index, then describe the general class of correspondences for which we are able to show that the set of fixed points has a single connected component.

[3] If B is a nonsingular $n \times n$ matrix, $u, v \in \mathbb{R}^n$ are column vectors, and $\lambda := v^T B^{-1} u \neq -1$, then the formula $(B + uv^T)^{-1} = B^{-1} - B^{-1} uv^T B^{-1}/(1 + \lambda)$ can be verified by multiplying the right hand side by $B + uv^T$. (Cf., p. 124 of Meyer (2001).)

Let $D \subset \mathbb{R}^m$ be a nonempty compact convex set, and let $F : D \to D$ be an upper semicontinuous convex valued correspondence. Between Brouwer's (1910) proof of his fixed point theorem and the middle of the last century, there emerged a theory of a fixed point index that assigns an integer to each closed set of fixed points of F that is isolated, in the sense of having a neighborhood containing no other fixed points. More generally, an *index admissible correspondence* is an upper semicontinuous convex valued correspondence $F : \overline{U} \to D$ where $U \subset D$ is open in the relative topology of D, $\overline{U}$ is its (relative) closure, and F has no fixed points in $\partial U := \overline{U} \setminus U$. Let $\mathcal{C}$ be the set of index admissible correspondences.

Proposition 3. *There is a unique function $\Lambda : \mathcal{C} \to \mathbb{Z}$ satisfying the following conditions:*

(A) (Normalization) If $c : D \to D$ is a constant function, then $\Lambda(c) = 1$.

(B) (Additivity) If $F : \overline{U} \to D$ is index admissible, $U_1, \ldots, U_r \subset U$ are open and pairwise disjoint, and $\overline{U} \setminus (U_1 \cup \cdots \cup U_r)$ contains no fixed points of F, then

$$\Lambda(F) = \sum_{i=1}^{r} \Lambda(F|_{\overline{U}_i}).$$

(C) (Homotopy) If $h : \overline{U} \times [0,1] \to D$ is a homotopy (i.e., a continuous function) such that for each $0 \leq t \leq 1$, $h_t := h(\cdot, t) : \overline{U} \to D$ is an index admissible function, then $\Lambda(h_0) = \Lambda(h_1)$.

(D) (Continuity) If $F : \overline{U} \to D$ is index admissible, then there exists a neighborhood $\mathcal{V} \subset \overline{U} \times D$ of the graph of F such that $\Lambda(F') = \Lambda(F)$ whenever $F' : \overline{U} \to D$ is an upper semicontinuous convex valued correspondence whose graph is contained in $\mathcal{V}$.

It can be shown (e.g., McLennan (1989)) that if $F \in \mathcal{C}$ and the domain of F is all of D, then $\Lambda(F) = 1$. (Since any two continuous functions from D to itself are homotopic, because D is convex, its validity for functions follows from Normalization and Homotopy; the main difficulty is to show that any upper semicontinuous convex valued correspondence can be approximated by a continuous function, so that its validity for correspondences follows from Continuity.) In particular, for any partition of the set of fixed points into isolated sets, the sum of the indices of the sets must be one.

We now introduce the framework of the general uniqueness result, which is slightly more general than the framework of the last two sections. Let

$$\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$$

be a given bilinear pairing, let $\mathcal{Y} \subset \mathbb{R}^m$ be a compact, convex polytope, and for $v \in \mathbb{R}^k$ let

$$\mathcal{Y}(v) := \mathrm{argmin}_{Y \in \mathcal{Y}} \langle Y, v \rangle.$$

Let $\mathbf{v} : \mathcal{Y} \to \mathbb{R}^k$ be a C^1 function, and let $F : \mathcal{Y} \to \mathcal{Y}$ be the correspondence

$$F(Y) := \mathcal{Y}(\mathbf{v}(Y)).$$

The argument used to prove Proposition 2 shows that this F also satisfies the hypotheses of Kakutani's fixed point theorem.

We will say that $\mathcal{E} \subset \mathcal{Y}$ is an *attracting set* for $\mathcal{Y}$ and $\mathbf{v}$ if it is nonempty and (i)-(iv) below are satisfied.

(i) $\mathbf{v}$ is constant on $\mathcal{E}$.

Let v be the constant value of $\mathbf{v}$ on $\mathcal{E}$.

(ii) $\mathcal{E} \subset \mathcal{Y}(v)$.

Note that, by (i) and (ii), $\mathcal{E}$ is contained in the set of fixed points of F.

(iii) $\mathcal{E}$ is the intersection of $\mathcal{Y}$ with an affine subspace of $\mathbb{R}^m$.

Let K be the linear subspace of $\mathbb{R}^m$ that is parallel to the affine hull of $\mathcal{Y}(v)$. That is, K is the smallest linear subspace of $\mathbb{R}^m$ such that $\mathcal{Y}(v) = (Y + K) \cap \mathcal{Y}$ for all $Y \in \mathcal{Y}(v)$. Similarly, let L be the linear subspace of $\mathbb{R}^m$ that is parallel to the affine hull of $\mathcal{E}$, so L is the smallest linear subspace of $\mathbb{R}^m$ such that $\mathcal{E} = (Y + L) \cap \mathcal{Y}$ for all $Y \in \mathcal{E}$. Concretely, K is the span of all vectors of the form $Y' - Y''$ where $Y', Y'' \in \mathcal{Y}(v)$, and L is the span of all vectors of the form $Y' - Y''$ where $Y', Y'' \in \mathcal{E}$. Note that $L \subset K$. We say that $Z \in K$ is *inward pointing* if there is $Y_0 \in \mathcal{E}$ and $Y \in \mathcal{Y}(v)$ such that Z is a positive scalar multiple of $Y - Y_0$. We say that Z is *strictly inward pointing* if there are such Y_0 and Y with $Y \notin \mathcal{E}$.

(iv) $\langle Z, D\mathbf{v}(Y_0)Z \rangle > 0$ for all $Y_0 \in \mathcal{E}$ and all strictly inward pointing Z.

Our main goal in this section is:

Theorem 2. *If each fixed point of F is contained in an attracting set, then the set of fixed points of F is a single attracting set.*

We explain the argument by repeatedly reducing the claim to a simpler and more technical assertion. To begin with:

Proposition 4. *If $\mathcal{E}$ is an attracting set, then there is a neighborhood $U \subset \mathcal{Y}$ such that the set of fixed points of F in U is $\mathcal{E}$, and $\Lambda(F|_{\overline{U}}) = 1$.*

Proof of Theorem 2. Proposition 4 implies that each attracting set of fixed points has a neighborhood U as above. Since (by upper semicontinuity) the set of fixed points is compact, it consists of finitely many attracting sets. Since $\Lambda(F) = 1$, and, by Additivity, $\Lambda(F)$ is equal to the number of attracting sets, there is exactly one attracting set. $\square$

We now need to prove Proposition 4. Fix an attracting set $\mathcal{E}$ and let v be the constant value of $\mathbf{v}$ on $\mathcal{E}$. Let K be the linear subspace of $\mathbb{R}^m$ that is parallel to the affine hull of $\mathcal{Y}(v)$, and let L be the linear subspace of $\mathbb{R}^m$ that is parallel to the affine hull of $\mathcal{E}$.

The general idea is to "deform" the given correspondence F to one whose set of fixed points is known. We fix a particular point $Y_0 \in \mathcal{E}$ and define a function $\mathbf{u} : \mathcal{Y} \times [0,1] \to \mathbb{R}^k$ by setting

$$\mathbf{u}(Y,t) := (1-t)\mathbf{v}(Y) + tD\mathbf{v}(Y_0)(Y - Y_0),$$

and define the correspondence $H : \mathcal{Y} \times [0,1] \to \mathcal{Y}$ by

$$H(Y,t) := \mathrm{argmin}_{Y' \in \mathcal{Y}(v)} \langle Y', \mathbf{u}(Y,t) \rangle.$$

Following notational conventions that are standard for homotopies, let H_t denote the correspondence $H(\cdot,t) : \mathcal{Y} \to \mathcal{Y}$ "at time t."

There are now three results concerning, respectively, the fixed points of H_0, H_1, and H_t for all t. The proof of the last of these is harder, and is deferred until after the proof of Proposition 4.

Lemma 4. *There is a neighborhood $U \subset \mathcal{Y}$ of $\mathcal{E}$ with $H_0(Y) = F(Y)$ for all $Y \in U$.*

Proof. The difference between F and H_0 is that F is defined by minimizing over $\mathcal{Y}$ while H_0 is defined by minimizing over $\mathcal{Y}(v)$, but, by continuity (and because $\mathcal{Y}$ is a polytope) $\mathrm{argmin}_{\hat{Y} \in \mathcal{Y}} \langle \hat{Y}, \mathbf{v}(Y) \rangle \subset \mathcal{Y}(v)$ for all Y in some neighborhood of $\mathcal{E}$. $\square$

Lemma 5. *The set of fixed points of H_1 is $\mathcal{E}$.*

Proof. Fix a point $Y_0 \in \mathcal{E}$. First suppose $Y \in \mathcal{E}$. Then $Y \in \mathcal{Y}(v)$ by (ii). In addition, since $\mathbf{v}$ is constant on $\mathcal{E}$, the kernel of $D\mathbf{v}(Y_0)$ contains L, and consequently

$$Y \in H_1(Y) := \mathrm{argmin}_{Y' \in \mathcal{Y}(v)} \langle Y', D\mathbf{v}(Y_0)(Y - Y_0) \rangle = \mathcal{Y}(v).$$

Thus $\mathcal{E}$ is contained in the set of fixed points of H_1.

Now suppose that $Y \in \mathcal{Y} \setminus \mathcal{E}$. We wish to show that Y is not a fixed point of H_1, so we may assume that $Y \in \mathcal{Y}(v)$ because $H_1(Y)$ is contained in this set. Then $Y - Y_0$ is strictly inward pointing, so condition (iv) implies that

$$\big\langle Y, D\mathbf{v}(Y_0)(Y - Y_0)\big\rangle > \big\langle Y_0, D\mathbf{v}(Y_0)(Y - Y_0)\big\rangle.$$

Since $\mathbf{u}(Y, 1) = D\mathbf{v}(Y_0)(Y - Y_0)$, it follows that $Y \notin H_1(Y)$. □

Lemma 6. *There is a neighborhood $U \subset \mathcal{Y}$ of $\mathcal{E}$ such that for all $0 \le t \le 1$ the set of fixed points of H_t in U is $\mathcal{E}$.*

Proof of Proposition 4. Let U be a neighborhood of $\mathcal{E}$ with the properties given by Lemmas 4 and 6. Replacing U with a smaller neighborhood of $\mathcal{E}$ if need be, we may assume that for all $0 \le t \le 1$ the set of fixed points of H_t in $\overline{U}$ is $\mathcal{E}$. Then Continuity implies that $\Lambda(H_t|_{\overline{U}})$ is constant as a function of t, and Lemma 5 implies that $\Lambda(H_1|_{\overline{U}}) = 1$, so $\Lambda(H_0|_{\overline{U}}) = 1$. Since F agrees with H_0 on U, it follows that $\Lambda(F|_{\overline{U}}) = 1$. □

The remaining task is the proof of Lemma 6. The intuition underlying local uniqueness in coalitional bargaining is that as we change Y, say near Y_0, the agents who are included more frequently in minimal winning coalitions should have higher continuation values, which increases the expense of coalitions including many of them, in comparison with others that might be used. The net effect is to encourage change in the opposite direction, and for this reason there must be disequilibrium at points near $\mathcal{E}$ that are not actually in $\mathcal{E}$. The next result "integrates" condition (iv) in the definition of an attracting set, arriving at an algebraic expression of this idea that is valid in some neighborhood of $\mathcal{E}$.

Lemma 7. *There is a neighborhood $W \subset \mathcal{Y}(v)$ of $\mathcal{E}$ in the relative topology of $\mathcal{Y}(v)$ such if $Y \in W \setminus \mathcal{E}$ and Y_1 is the point in $\mathcal{E}$ closest to Y, then*

$$\langle Y - Y_1, \mathbf{v}(Y) - \mathbf{v}(Y_1)\rangle > 0.$$

Proof. Let $M := K \cap L^\perp$; since L is a subspace of K, every element of K has an orthogonal decomposition as a sum of an element of L and an element of M. Since $\mathcal{Y}(v)$ is a convex polytope, there is a number $\delta > 0$ such that $\|X\| \le \delta \|T\|$ whenever $Y \in \mathcal{Y}(v) \setminus \mathcal{E}$, Y_1 is the point in $\mathcal{E}$ closest to Y, and $Y - Y_1 = T + X$ where $X \in L$ and $T \in M$. Let

$$C = \{\, (X, T) \in L \times M : \|X\| \le \delta \text{ and } \|T\| = 1 \,\}.$$

Then C is compact, so condition (iv) and continuity imply that there is a $\kappa > 0$ such that $\big|\langle Z, D\mathbf{v}(Y_1)Z\rangle\big| \ge \kappa \|Z\|^2$ whenever $(X, T) \in C$ and

$Z = T + X$. More generally, this inequality holds whenever $Z = T + X$ with $T \in M$, $X \in L$, and $\|X\| \le \delta\|T\|$. Choose $\varepsilon > 0$ with $\varepsilon < \kappa$, and let W be a convex neighborhood of $\mathcal{E}$ in $\mathcal{Y}(v)$ that is small enough that

$$\|D\mathbf{v}(Y) - D\mathbf{v}(Y_1)\| := \max_{\|U\|=1} \|(D\mathbf{v}(Y) - D\mathbf{v}(Y_1))U\| < \varepsilon$$

whenever $Y \in W$ and Y_1 is the point in $\mathcal{E}$ that is closest to Y.

Now fix $Y \in W \setminus \mathcal{E}$ and let Y_1 be the point in $\mathcal{E}$ closest to Y. Since

$$\mathbf{v}(Y) - \mathbf{v}(Y_1) = \int_0^1 D\mathbf{v}((1-t)Y_1 + tY)(Y - Y_1)\, dt$$

we have

$$\big\langle Y - Y_1, \mathbf{v}(Y) - \mathbf{v}(Y_1)\big\rangle = \int_0^1 \big\langle Y - Y_1, D\mathbf{v}((1-t)Y_1 + tY)(Y - Y_1)\big\rangle\, dt$$

$$= \big\langle Y - Y_1, D\mathbf{v}(Y_1)(Y - Y_1)\big\rangle + \int_0^1 \big\langle Y - Y_1, (D\mathbf{v}((1-t)Y_1 + tY) - D\mathbf{v}(Y_1))(Y - Y_1)\big\rangle\, dt$$

$$\ge \kappa\|Y - Y_1\|^2 - \int_0^1 \|Y - Y_1\| \cdot \big\|\big(D\mathbf{v}((1-t)Y_1 + tY) - D\mathbf{v}(Y_1)\big)(Y - Y_1)\big\|\, dt$$

$$\ge (\kappa - \varepsilon)\|Y - Y_1\|^2 > 0.$$

$\square$

We now combine the last result with the definition of H_t.

Proof of Lemma 6. Let $W \subset \mathcal{Y}(v)$ be as in Lemma 7. Let $U \subset \mathcal{Y}$ be a neighborhood of $\mathcal{E}$ such that $U \cap \mathcal{Y}(v) \subset W$. Fix a particular $t \in [0,1]$. We already know, from Lemma 5, that the set of fixed points of H_1 is $\mathcal{E}$, so we may assume that $t < 1$.

First suppose that $Y \in \mathcal{E}$. Then $D\mathbf{v}(Y_0)(Y - Y_0) = 0$ because $\mathbf{v}$ is constant on $\mathcal{E}$. Thus $\mathbf{u}(Y,t) = (1-t)\mathbf{v}(Y) = (1-t)v$, and Y is an element of $H_t(Y) = \mathcal{Y}(v)$. Thus $\mathcal{E}$ is contained in the set of fixed points of H_t.

We need to show that $U \setminus \mathcal{E}$ does not contain any fixed points of H_t. Fix $Y \in U \setminus \mathcal{E}$. The image of H_t is contained in $\mathcal{Y}(v)$, so we may assume that $Y \in \mathcal{Y}(v)$. Let Y' be the point in $\mathcal{E}$ closest to Y. Then $\mathbf{v}(Y') = v$, and $Y - Y' \in K$ so $\langle Y - Y', \mathbf{v}(Y')\rangle = 0$. Therefore Lemma 7 implies that

$$\langle Y - Y', \mathbf{v}(Y)\rangle = \langle Y - Y', \mathbf{v}(Y) - \mathbf{v}(Y')\rangle > 0.$$

We have $D\mathbf{v}(Y_0)(Y_0 - Y') = 0$, as explained above, so the definition of an attracting set gives

$$\langle Y - Y', D\mathbf{v}(Y_0)(Y - Y_0)\rangle = \langle Y - Y', D\mathbf{v}(Y_0)(Y - Y')\rangle > 0.$$

Multiplying the last two inequalities by $1 - t$ and t, then summing, gives

$$\langle Y - Y', \mathbf{u}(Y, t)\rangle > 0.$$

That is, $\langle Y', \mathbf{u}(Y, t)\rangle < \langle Y, \mathbf{u}(Y, t)\rangle$, and consequently $Y \notin H_t(Y)$. $\square$

6 The Proof of Theorem 1

We now return to the framework of Section 4, so $\mathbf{v}(Y) = A(Y)^{-1}p$. For $Y \in \mathcal{Y}$ and $v \in \mathbb{R}^n$ let

$$\langle Y, v\rangle = p^T Y^T \Delta v,$$

and for $v \in \mathbb{R}^n$ let $\mathcal{Y}(v) = \text{argmin}_{Y \in \mathcal{Y}} \langle Y, v\rangle$. As before $F : \mathcal{Y} \to \mathcal{Y}$ is the correspondence $F(Y) = \mathcal{Y}(\mathbf{v}(Y))$. Fix a particular fixed point Y_0 of F, and let:

$$v := \mathbf{v}(Y_0), \quad m := \mathbf{m}(Y_0), \quad \mathcal{E} := \mathbf{m}^{-1}(m).$$

Lemma 1 implies that for all $Y \in \mathcal{E}$, (v, Y) is a reduced equilibrium, so $\mathbf{v}$ is constant on $\mathcal{E}$, and $\mathcal{E} \subset \mathcal{Y}(v)$. Since $\mathbf{m}$ is a linear function, $\mathcal{E}$ is the intersection of $\mathcal{Y}$ with the affine hull of $\mathcal{E}$. Thus conditions (i)-(iii) of the definition of an attracting set are satisfied. We will show that (iv) also holds, so that $\mathcal{E}$ is an attracting set for $\mathcal{Y}$ and $\mathbf{v}$, and since Y_0 is an arbitrary fixed point of F, Theorem 2 will then imply that it is the entire set of fixed points of F. As we pointed out earlier, the other assertions of Theorem 1 follow from this, so that result will be established.

The remainder of this section is devoted to the proof that (iv) also holds. Let K be the linear subspace of the Euclidean space containing $\mathcal{Y}$ that is parallel to the affine hull of $\mathcal{Y}(v)$, and let L be the linear subspace that is parallel to the affine hull of $\mathcal{E}$. Condition (iv) follows from Lemma 8 below.

Lemma 8. *If $Z \in K$ is strictly inward pointing, then*

$$\langle Z, D\mathbf{v}(Y_0)Z\rangle = p^T Z^T \Delta D\mathbf{v}(Y_0)Z > 0.$$

The rest of the section is devoted to the proof of this. We begin with two technical facts.

Lemma 9. *If $Z \in K$ is inward pointing and $Zp = 0$, then Z is not strictly inward pointing.*

Proof. We know that Z is a positive scalar multiple of $Y - \tilde{Y}_0$ for some $Y \in \mathcal{Y}(v)$ and $\tilde{Y}_0 \in \mathcal{E}$. We have $\mathbf{m}(\tilde{Y}_0 + tZ) = (\tilde{Y}_0 + tZ)p = \tilde{Y}_0 p = \mathbf{m}(\tilde{Y}_0)$ for all t, so $\tilde{Y}_0 + tZ \in \mathcal{E}$ for all $t \geq 0$ such that $\tilde{Y}_0 + tZ \in \mathcal{Y}(v)$. In particular, $Y \in \mathcal{E}$. $\square$

Let $V := \nabla(v)$.

Lemma 10. *For all $Z \in K$, $D\mathbf{v}(Y_0)Z = A^{-1} V \Delta Z p$.*

Proof. Differentiating the equation $A(Y)v(Y) = p$ gives

$$AD\mathbf{v}(Y_0)Z + (DA(Y_0)Z)v = 0,$$

so the assertion is equivalent to $-(DA(Y_0)Z)v = V\Delta Zp$. Evaluating the derivative of A shows that this is equivalent to

$$(\nabla(Zp) - PZ^T)\Delta v = V\Delta Zp.$$

Since $Z^T \Delta v = 0$, this is true if and only if $\nabla(Zp)\Delta v = V\Delta Zp$. But commutativity of multiplication of diagonal matrices, and the fact that $\nabla(x)y = \nabla(y)x$ for all $x, y \in \mathbb{R}^n$, gives

$$\nabla(Zp)\Delta v = \Delta\nabla(Zp)v = \Delta V Zp = V\Delta Zp.$$

$\square$

Our general approach is inspired by Theorem 3 of Debreu (1952), which asserts that if C is an $n \times n$ matrix, B is an $m \times n$ matrix, and $x^T C x > 0$ for all nonzero x such that $Bx = 0$, then there is some $\lambda > 0$ such that $C + \lambda B^T B$ is positive definite.

Fix a strictly inward pointing $Z \in K$. Then $Z^T \Delta v = 0$. (Observe that Z is a scalar multiple of a difference $Y' - Y''$ of two vectors in $\mathcal{Y}(v)$, and that these are both minimizers of $p^T Y^T \Delta v$. Since the columns of Y can be chosen independently, it follows that Y'' and Y' both minimize each component of $Y^T \Delta v$.) The last result gives

$$p^T Z^T \Delta D\mathbf{v}(Y_0)Z = p^T Z^T \Delta A^{-1} V \Delta Z p.$$

We first put this is a symmetric form:

$$\begin{aligned} p^T Z^T \Delta D\mathbf{v}(Y_0)Z &= \tfrac{1}{2} p^T Z^T \Delta A^{-1} V \Delta Z p + \tfrac{1}{2}\big(p^T Z^T \Delta A^{-1} V \Delta Z p\big)^T \\ &= \tfrac{1}{2} p^T Z^T \Delta \big(A^{-1} V + V^T (A^{-1})^T\big) \Delta Z p \\ &= \tfrac{1}{2} p^T Z^T \Delta A^{-1} \big(V A^T + A V^T\big)(A^{-1})^T \Delta Z p. \end{aligned}$$

Since $A^{-1}p = v$ and $Z^T \Delta v = 0$, we have

$$p^T Z^T \Delta A^{-1} p = p^T Z^T \Delta v = 0.$$

Therefore, for any $\gamma \in \mathbb{R}$,

$$p^T Z^T \Delta D\mathbf{v}(Y_0) Z = \tfrac{1}{2} p^T Z^T \Delta A^{-1} G(\gamma)(A^{-1})^T \Delta Z p$$

where

$$G(\gamma) := VA^T + AV^T - \gamma p v^T - \gamma v p^T.$$

To prove Lemma 8 it suffices (since $Zp \neq 0$, by Lemma 9, and A^{-1} and Δ are nonsingular) to find γ such that $G(\gamma)$ is positive definite.

The final step in the proof is:

Lemma 11. *If* $\max_i \delta_i \leq \gamma < 1$*, then* $G(\gamma)$ *is a symmetric nonsingular* M*-matrix, so it is positive definite.*

Proof. Fix an $i = 1, \ldots, n$. Recall that

$$m_i = \sum_{j=1}^{n} p_j y_{ij} \leq \sum_{j=1}^{n} p_j = 1 - p_0,$$

so that $a_{ii} = 1 - \delta_i(m_i - p_i + p_0) \geq 1 - \delta_i(1 - p_i)$. Therefore

$$g_{ii}(\gamma) = 2a_{ii}v_i - 2\gamma p_i v_i = 2(1 - \delta_i(m_i - p_i + p_0))v_i - 2\gamma p_i v_i. \qquad (*)$$

In particular,

$$g_{ii}(\gamma) \geq 2v_i(1 - \delta_i(1 - p_i) - \gamma p_i),$$

so $g_{ii}(\gamma) > 0$ because $\delta_i, \gamma < 1$,

When $i \neq j$ we have $a_{ij} = p_i \delta_j y_{ji}$, so that

$$g_{ij}(\gamma) = a_{ij}v_j + a_{ji}v_i - \gamma(p_i v_j + p_j v_i) = (\delta_j y_{ji} - \gamma)p_i v_j + (\delta_i y_{ij} - \gamma)p_j v_i. \quad (**)$$

Therefore $g_{ij}(\gamma) \leq 0$ when $i \neq j$ because $\gamma \geq \delta_i$ for all i.

We have shown that $g_{ij}(\gamma)$ is positive when $i = j$ and nonpositive when $i \neq j$. The remaining condition that we need to verify is that $G(\gamma)$ is dominant diagonal. Summing equations $(*)$ and $(**)$ above, then recognizing that $y_{ii} = 1$, yields

$$\sum_{j=1}^{n} g_{ij}(\gamma) = C - \gamma\Big(p_i \sum_{j=1}^{n} v_j + v_i \sum_{j=1}^{n} p_j\Big)$$

where

$$C = 2(1 - \delta_i(m_i + p_0))v_i + p_i \sum_{j=1}^{n} v_j \delta_j y_{ji} + \delta_i v_i \sum_{j=1}^{n} p_j y_{ij}.$$

Substituting $m_i = \sum_{j=1}^{n} p_j y_{ij}$ yields

$$C = 2(1 - \delta_i p_0)v_i + p_i \sum_{j=1}^{n} v_j \delta_j y_{ji} - \delta_i v_i \sum_{j=1}^{n} p_j y_{ij}.$$

Part (a) of the definition of reduced equilibrium gives

$$p_i \sum_{j=1}^{n} v_j \delta_j y_{ji} = \delta_i v_i (\sum_{j=1}^{n} p_j y_{ij}) + (p_0 \delta_i - 1)v_i + p_i,$$

so $A = (1 - \delta_i p_0)v_i + p_i$, and consequently

$$\sum_{j=1}^{n} g_{ij}(\gamma) = (1 - \delta_i p_0)v_i + p_i - \gamma\Big(p_i \sum_{j=1}^{n} v_j + v_i \sum_{j=1}^{n} p_j\Big).$$

Since $\sum_{j=1}^{n} p_j = 1 - p_0$ we have

$$\sum_{j=1}^{n} g_{ij}(\gamma) = (2 - (1 + \delta_i)p_0)v_i + p_i\Big(1 - \gamma \sum_{j=1}^{n} v_j\Big).$$

In order to show that $\sum_{j=1}^{n} g_{ij}(\gamma)$ is positive it now suffices to show that $\sum_{j=1}^{n} v_j \leq 1$. Summing condition (a) of the definition of reduced equilibrium over i, simplifying, and substituting $\sum_{i=1}^{n} p_i = 1 - p_0$, leads to

$$\sum_{i=1}^{n} v_i = p_0 \sum_{i=1}^{n} \delta_i v_i + \sum_{i=1}^{n} p_i = 1 - p_0(1 - \sum_{i=1}^{n} \delta_i v_i) \leq 1 - p_0(1 - \sum_{i=1}^{n} v_i).$$

Therefore $(1-p_0)\sum_{i=1}^{n} v_i \leq 1-p_0$, which implies the desired conclusion. $\square$

7 Future Research

Many questions and issues remain unresolved. An algorithm for computing the vector v of stationary subgame perfect equilibrium expected payoffs could be an important tool supporting theoretical and empirical work based on this model. The definition of reduced equilibrium has a combinatoric aspect, namely which coalitions are least cost for each proposer, and a numerical aspect. Unlike linear programming or Nash equilibrium for two person games, even once the combinatoric aspect has been solved, the numerical aspect is nonlinear, so there is little hope of finding an algorithm based on linear pivoting such as the simplex algorithm for linear programming, the Lemke-Howson algorithm (Lemke and Howson (1964)) for two player games, or the Lemke (1965) algorithm for linear complementary problems. Another potential approach to computation is based on homotopy: starting with recognition probabilities for which the stationary subgame perfect equilibrium payoff vector is known, follow a path of (recognition probability vector, payoff vector) pairs until one reaches the recognition probabilities of interest. Due to the nonlinearity, this approach is likely to be difficult to analyze theoretically, but may well be practical in many applications.

This problem is interesting from the point of view of computer science, which has an active line of literature concerned with the complexity of computing fixed points, with special emphasis on the computation of Nash equilibria. (See Etessami and Yannakis (2007) and Papadimitriou (2007).) Computation of the vector of continuation values is a special problem in this class because it is nonlinear and a unique solution is guaranteed, but our proof of uniqueness does not supply an algorithm. Recall that **P** is the class of decision problems that can be decided by a Turing machine whose running time is bounded by a polynomial function of the size of the input, and **NP** is the class of decision problems such that a positive answer has a "witness" that can be verified in polynomial time. For example, if a graph has a clique of size k (that is, k vertices such that any two are the endpoints of an edge) then such a clique is a witness for that fact. Similarly, **coNP** is the class of decision problems for which a negative answer has an easily verified witness; obviously it is the class of decision problems whose negations are in **NP**. It is not known whether the intersection of **NP** and **coNP** contains problems that are not in **P** (whether **NP** contains problems that are not in **P** is currently one of the most important open problems in mathematics) but there are currently very few problems in the intersection of **NP** and **coNP** for which no polynomial time algorithm is known. The best known examples are related to objects whose unique existence is guaranteed, specifically factoring of integers and equilibrium payoffs of "simple" zero sum stochastic games (cf. Johnson (2007)). Decision problems related to v are perhaps a source of such problems, but because the components of v need not be rational, it is not clear that such problems can be placed in **NP**.

There are many interesting questions concerning the vector v, which can be investigated either theoretically or computationally. Eraslan (2002) shows that in the case of k-majority rule, if the recognition probabilities are all the same, then the costs of players, as potential members of a coalition, are ordered in the same way as the discount factors. That is, if $\delta_i \leq \delta_j$, then $\delta_i v_i \leq \delta_j v_j$. She also shows that if all players have the same discount factor, then the costs of players are ordered in the same way as the recognition probabilities, i.e., $p_i \leq p_j$, then $\delta_i v_i \leq \delta_j v_j$. It is natural to ask whether these results generalize to coalition structures that are symmetric in the sense that for any i and j there is a permutation of N that maps i to j and preserves (in the natural sense) the structure of the sets of minimal winning coalitions. There are a host of additional questions concerning monotonicity of v_i as an agent i becomes more or less powerful due to changes in the structure of the sets of minimal winning coalitions.

Our results permit the definition of a new power index. The Shapley value (Shapley (1953), Shapley and Shubik (1954)) is a function that assigns a vector of payoffs to each TU game, and the Shapley-Shubik (Shapley and Shubik (1954)) power index is the application of the Shapley value to simple games. The power indices of Banzhaf (1965, 1968) Deegan and Packel (1978) and Johnston (1978) are functions with the same domain and range: each assigns a vector of individual "powers" to each simple game. In our framework there is a simple game (the system of winning coalitions) and other parameters, namely the recognition probabilities and the discount factors. To obtain a power index comparable to those mentioned in Section 2 one may take the limit of our vector of equilibrium continuation payoffs, for the case of symmetric recognition probabilities, as the common discount factor goes to one. The power index obtained in this way has clear noncooperative foundations, in line with the Nash program. These are certainly open to question in some applications, and can be compared with other noncooperative foundations for cooperative solutions (e.g., Gul (1989)).

Many interesting topics concern generalizations or variations of the model. It would be desirable to extend this paper's model to allow for different proposer-coalition pairs to generate pies of different size. Among other things, one could then investigate the hypothesis that the coalitions that form are the most productive. Whether our uniqueness result extends to such a model is an open question. There seems to be considerable scope for additional work on alternative bargaining protocols such as Chatterjee et al. (1993), Ray (2007), and Kawamori (2008). Another possibility studied by Montero (2006) is to search for vectors of recognition probabilities that are *self-confirming* in the sense that they coincide with the resulting vector of continuation values. As we mentioned in the introduction, she shows that points in the nucleus of a proper simple game have this property. Comparison of the properties of the various power indices seems like an interesting direction for theoretical investigation.

Appendix

This appendix gives a precise description of the bargaining game and shows that the expected payoff vectors generated by its stationary subgame perfect equilibria satisfy the characterization given by the definition of reduced equilibrium. This involves a certain amount of advanced measure theory, but, taking these tools as given, the work is straightforward and unsurprising. Since, quite understandably, many papers in this area do not fill in these details, we hope this appendix may be useful as a reference or model for other authors.

We describe the progress of the game within a single active period in terms of the space of *within-period histories*

$$H := H^0 \cup H^1 \cup \ldots \cup H^n.$$

Here H^0 is the set Σ_n of permutations of $\{1, \ldots, n\}$. There is a given probability distribution q on Σ_n. If $\sigma \in \Sigma_n$ is realized, then $\sigma(1)$ is the proposer, and the other agents vote on her proposal sequentially in the order $\sigma(2), \ldots, \sigma(n)$. Therefore $p_i/(1-p_0) = \sum_{\sigma(1)=i} q(\sigma)$ for all i. (One consequence of our analysis here is that other aspects of q have no influence on equilibrium outcomes.) After σ is realized, $\sigma(1)$ chooses a proposal $x \in \Pi$, so $H_1 = \Sigma_n \times \Pi$. The agents then vote in order, with each seeing the proposal and the earlier agent's votes, then choosing an element of $\{\text{Yes}, \text{No}\}$, so that for each $k = 1, \ldots, n$,

$$H^k := \Sigma_n \times \Pi \times \{\text{Yes}, \text{No}\}^{k-1}.$$

For $k = 0, \ldots, n-1$ and $i = 1, \ldots, n$ let H_i^k be the set of within-period histories in H^k at which agent i chooses. Thus $H_i^0 = \{\, \sigma \in H^0 : \sigma(1) = i \,\}$, and for $k = 2, \ldots, n$ we have

$$H_i^k = \{\, (\sigma, x, b_{\sigma(2)}, \ldots, b_{\sigma(k)}) \in H^k : \sigma(k+1) = i \,\}.$$

The measure-theoretic description of behavior strategies involves transition probabilities. In general, if $(\Omega, \mathcal{A})$ is a measurable space, let $\mathcal{P}(\Omega)$ be the set of probability measures on Ω. If $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are measurable spaces, a function $P_{12} : \Omega_1 \to \mathcal{P}(\Omega_2)$ is a *transition probability* if

$$P_{12}(\cdot)(A_2) : \Omega_1 \to [0, 1]$$

is measurable for all $A_2 \in \mathcal{A}_2$. Let $\mathcal{P}(\Omega_1, \Omega_2)$ be the set of transition probabilities from Ω_1 to Ω_2.

A (stationary) *proposer strategy* for agent $i = 1, \ldots, n$ is a transition probability

$$\pi_i \in \mathcal{P}(H_i^0, \Pi).$$

Fix proposer strategies $\pi_1, \ldots, \pi_n$ for the various agents. A (stationary) *responder strategy* for agent i is a transition probability

$$\rho_i \in \mathcal{P}(H_i^1 \cup \ldots \cup H_i^{n-1}, \{\text{Yes}, \text{No}\}).$$

Fix responder strategies $\rho_1, \ldots, \rho_n$.

Let

$$\pi \in \mathcal{P}(H^1, \Pi)$$

be the transition probability that agrees with π_i on each H_i^0. Abusing notation, we also use the symbol π to denote the profile $(\pi_1, \ldots, \pi_n)$; the correct interpretation will always be clear from context. Let $\rho := (\rho_1, \ldots, \rho_n)$, and for $h = 1, \ldots, n-1$ let

$$\rho^h \in \mathcal{P}(H^h, \{\text{Yes}, \text{No}\})$$

be the transition probability that agrees with ρ_i on each H_i^h.

We now need to define and characterize the measures on $H^0, \ldots, H^n$ induced by (π, ρ). The following result generalizes Fubini's theorem and is a fundamental result for the theory of Markov chains.

Lemma A.1. *For any $\lambda \in \mathcal{P}(\Omega_1)$ and $P_{12} \in \mathcal{P}(\Omega_1, \Omega_2)$ there is a unique probability measure $\lambda \otimes P_{12} \in \mathcal{P}(\Omega_1 \times \Omega_2)$ satisfying*

$$(\lambda \otimes P_{12})(A_1 \times A_2) = \int_{A_1} P_{12}(\cdot)(A_2)\, d\lambda$$

for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. If $X : \Omega_1 \times \Omega_2 \to \mathbb{R}_+$ is integrable, then

$$\int_{\Omega_2} X(\omega_1, \omega_2)\, P_{12}(\omega_1)(d\omega_2)$$

is a measurable function of ω_1, and

$$\int_{\Omega_1 \times \Omega_2} X\, d(\lambda \otimes P_{12}) = \int_{\Omega_1} \Big[\int_{\Omega_2} X(\omega_1, \omega_2)\, P_{12}(\omega_1)(d\omega_2) \Big]\, \lambda(d\omega_1).$$

Proof. E.g., Proposition III.2.1 of Neveu (1965). □

The measure on H^n induced by q and (π, ρ) is now seen to be

$$q \otimes \pi \otimes \rho^1 \otimes \cdots \otimes \rho^{n-1}.$$

Let $\mathcal{O}$ be the disjoint union of Π and $\{\text{No}\}$, endowed with the obvious σ-algebra, and define the *outcome function* $o : H^n \to \mathcal{O}$ by setting

$$o(\sigma, x, b_{\sigma(2)}, \dots, b_{\sigma(n)}) = \begin{cases} x, & \{i\} \cup \{\, j : b_j = \text{Yes}\,\} \in \mathcal{S}_{\sigma(1)}, \\ \text{No}, & \text{otherwise.} \end{cases}$$

The measure on $\mathcal{O}$ induced by (π, ρ) is

$$(q \otimes \pi \otimes \rho^1 \otimes \cdots \otimes \rho^{n-1}) \circ o^{-1}.$$

In order to define subgame perfection we need to also define the measures on $\mathcal{O}$ induced by beginning with a particular partial history and continuing according to (π, ρ). For $h \in H$ we define

$$\kappa(h, \pi, \rho) := \begin{cases} (\delta_h \otimes \pi \otimes \rho^1 \otimes \cdots \otimes \rho^{n-1}) \circ o^{-1}, & h \in H^0, \\ (\delta_h \otimes \rho^j \otimes \cdots \otimes \rho^{n-1}) \circ o^{-1}, & h \in H^j,\ j = 1, \dots, n-1, \\ \delta_h \circ o^{-1}, & h \in H^n. \end{cases}$$

For $h \in H$ and $v \in V$ we define $\tau(h, \pi, \rho, v) \in [0,1]^n$ by setting

$$\tau_i(h, \pi, \rho, v) = \kappa(h, \pi, \rho)(\{\text{No}\}) \cdot \delta_i v_i + \int_\Pi x_i \, \kappa(h, \pi, \rho)(dx).$$

To simplify notation we will usually write $\tau(\pi, \rho, v)$ in place of $\tau(A, \pi, \rho, v)$. We say that (π, ρ) is a *within period subgame perfect v-equilibrium* if

$$\tau_i(h, \pi, \rho, v) \geq \tau_i(h, (\pi_1, \dots, \tilde{\pi}_i, \dots, \pi_n), (\rho_1, \dots, \tilde{\rho}_i, \dots, \rho_n), v)$$

for all $h \in H$, all $i = 1, \dots, n$, and all proposer strategies $\tilde{\pi}_i$ and responder strategies $\tilde{\rho}_i$ for i.

Since it would be tedious and serve little purpose, we will not give a formal definition of stationary subgame perfect equilibrium. Informally, a stationary subgame perfect equilibrium is a pair (π, ρ) such that after any history of the larger game, playing according to these strategies is optimal. Stationarity implies that there is a well defined vector v of expected payoffs prior to the beginning of the game, and that this vector satisfies the fixed point characterization above: each player i's expected payoff consists of the probability that the game ends in the current period, times the expected payoff conditional on this event, plus the probability that it does not end times $\delta_i v_i$. Subgame perfection clearly implies that the the pair (π, ρ) is a within period subgame perfect v-equilibrium. On the other hand, (π, ρ) is a stationary subgame perfect equilibrium whenever it is a within period subgame perfect v-equilibrium and v is also the vector of expected payoffs when behavior in the first period is governed by (π, ρ) and the value of failure to reach agreement is given by the numbers $\delta_i v_i$.

For the remainder of the Appendix we work with a fixed $v \in V$ and a within period subgame perfect v-equilibrium (π, ρ). Consider a particular $k = 2, \ldots, n$ and $h = (\sigma, x, b_{\sigma(2)}, \ldots, b_{\sigma(k-1)}) \in H^k$ let

$$C(h) := \{\sigma(1)\} \cup \{\, \sigma(j) : 2 \leq j \leq k-1 \text{ and } b_{\sigma(j)} = \text{Yes}\,\}$$

be the set of agents who have voted to approve the proposal, let

$$D(h) := \{\, \sigma(j) : j = k, \ldots, n \text{ and } x_{\sigma(j)} > \delta_{\sigma(j)} v_{\sigma(j)}\,\}$$

be the set of agents in $\{\sigma(k), \ldots, \sigma(n)\}$ who prefer x to their disagreement payoff, and let

$$\overline{D}(h) := \{\, \sigma(j) : j = k, \ldots, n \text{ and } x_{\sigma(j)} \geq \delta_{\sigma(j)} v_{\sigma(j)}\,\}$$

be the set of agents in $\{\sigma(k), \ldots, \sigma(n)\}$ who do not prefer their disagreement payoff to x.

From this point forward we assume that supersets of winning coalitions are winning. That is, for each i and $S_i \in \mathcal{S}_i$, if $S_i \subset T_i \subset N$, then $T_i \in \mathcal{S}_i$.

Lemma A.2. *If $C(h) \cup D(h) \in \mathcal{S}_{\sigma(1)}$, then the proposal will be implemented with probability one.*

Proof. The claim is certainly correct if $k = n$, since then the last agent will vote according to her interest if her vote makes a difference. Suppose it is true with k replaced with $k+1$. If $\sigma(k) \notin D(h)$, then passage of the proposal is certain, by virtue of the induction hypothesis, regardless of how $\sigma(k)$ votes. (In particular, $\sigma(k)$ cannot defeat the proposal by voting in favor of it.) If $\sigma(k) \in D(h)$, then $\sigma(k)$ can insure passage of the proposal by voting in favor, and in a subgame perfect equilibrium will not vote against unless passage is also guaranteed in that event. $\square$

Lemma A.3. *If $C(h) \cup \overline{D}(h) \notin \mathcal{S}_{\sigma(1)}$, then the probability of implementing the proposal is zero.*

Proof. The claim is certainly correct if $k = n$, since then the last agent will vote according to her interest if her vote makes a difference. Suppose it is true with k replaced with $k+1$. If $\sigma(k) \in \overline{D}(h)$, then regardless of how $\sigma(k)$ votes the proposal will certainly not be implemented, and if $\sigma(k) \notin \overline{D}(h)$, then $\sigma(k)$ can insure rejection by voting against, and will only vote in favor if that also results in rejection with probability one. $\square$

For $i = 1, \ldots, n$ define

$$\mathbf{w}_i(v) = 1 - \min_{S_{ik} \in \mathcal{S}_i} \sum_{j \neq i} \delta_j v_j.$$

For each i let

$$\mathcal{S}_i^*(v) := \operatorname{argmin}_{S_{ik} \in \mathcal{S}_i} \sum_{j \in S_{ik}} \delta_j v_j$$

be the set of minimum cost coalitions. For each $\sigma \in \Sigma_n$ and $k = 1, \ldots, K_{\sigma(1)}$ let $\eta^*_{\sigma(k)}$ be the probability that $\pi_{\sigma(1)}(A, \sigma)$ assigns to the allocation in which each member j of $S_{\sigma(1)k}$ receives $\delta_j v_j$, $\sigma(1)$ receives $1 - \sum_{j \in S_{\sigma(1)k}} \delta_j v_j$, and all other agents receive 0.

Proposition A.3. *For each* σ

$$\sum_{S_{\sigma(1)k} \in \mathcal{S}^*_{\sigma(1)}(v)} \eta^*_\sigma(S_{\sigma(1)k}) = 1,$$

and the proposal is accepted with probability one.

Proof. Lemma A.2 implies that $\sigma(1)$ can insure the implementation of any proposal x such that $\{ j \neq \sigma(1) : x_j > \delta_j v_j \} \in \mathcal{S}_{\sigma(1)}$. Lemma A.3 implies that $\sigma(1)$ cannot hope to implement a proposal x with $\{ j \neq \sigma(1) : x_j \geq \delta_j v_j \} \notin \mathcal{S}_{\sigma(1)}$. Combining these two results, we find that $\sigma(1)$ has expected utility $\mathbf{w}_{\sigma(1)}(v)$, and that this expected utility can be achieved only if a proposal is implemented with probability one. $\square$

At this point we have shown that every vector of expected payoffs resulting from a stationary subgame perfect equilibrium is the first component of a reduced equilibria. We conclude by proving the converse, which also establishes that our theory is not vacuous because stationary subgame perfect equilibria actually exist.

Proposition A.4. *Consider* $v \in V$ *and* (π, ρ) *such that:*

(i) for each $\sigma \in \Sigma_n$, $\pi_{\sigma(1)}$ assigns all probability to proposals such that, for some $S_{\sigma(1),k} \in \mathcal{S}_i^*(v)$, each member j of $S_{\sigma(1)k}$ receives $\delta_j v_j$, $\sigma(1)$ receives $1 - \sum_{j \in S_{\sigma(1)k}} \delta_j v_j$, and all other agents receive 0;

(ii) for each i, in response to a proposal x with $x_i \geq \delta_i v_i$, ρ_i assigns all probability to voting to accept, and in response to a proposal x with $x_i < \delta_i v_i$, ρ_i assigns all probability to voting to reject.

Then (π, ρ) is a within period subgame perfect x-equilibrium.

Proof. Since, under ρ, a responder's vote has no effect on the votes of subsequent voters, it is clear there is no improving deviation from ρ. Given that responders are playing ρ, the proposer $\sigma(1)$ is achieving the expected payoff $\mathbf{w}_{\sigma(1)}(v)$, and Lemma A.3 implies that there is no way to do better than this. $\square$

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