



# Limited records and reputation bubbles <sup>☆</sup>

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## Abstract

This paper offers a tractable and fully rational model to study the economics of reputation in a dynamic market with limited record-keeping, i.e., a market in which new entrants observe only the last few periods of play of the long-run player instead of the full history of the market. We show that trust is gradually granted to the opportunistic long-run player despite the fact that his type is perfectly observed by the short-run opponents, and the perfectly informed short-run players ride and drive up “reputation bubbles” at the expense of their uninformed successors. We characterize equilibrium payoffs uniformly over time, which is useful for analyzing ongoing repeated relationships where the starting moments have passed.

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## 1. Introduction

Reputational concerns are an important dimension of informal incentives in dynamic markets, and reputation models play a central role in the economics of long-run relationships. Typically, our models assume that agents see the full history of past transactions. In reality, however,

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reputation inference is often based on limited information. For instance, in many countries, access to borrower's credit history is limited, insurance companies only observe the most recent driving records that are cleared after a fixed period of time, and the Better Business Bureau in the U.S. only reports customer complaints from the last 36 months.<sup>1</sup> In less formal markets, information is conveyed by word of mouth, and in many new and fast-growing markets, the lack of transparency over past transactions is often due to the relatively slow development of monitoring institutions.

In this paper we analyze a reputation model with limited record keeping capturing that common feature of markets. We show how reputation effects change qualitatively as a result of limited records by comparing our results to the existing reputation literature. Our results hopefully shed light on the role of record keeping and could be used in further research to answer questions about the optimal design of online reputation systems.

In our fully rational model there is a sequence of new agents who enter over time (one per period) and interact with a long-run player who can be either of a commitment or an opportunistic type. Under the assumption of limited records of the long-run player's past actions, we show that all equilibria are characterized by recurrences of "reputation bubbles", along which short-run players ride and drive up the reputation bubble by granting increasing amount of trust to the opportunistic long-run player under perfect knowledge of his type – which does not happen in existing reputation models. The strength of our characterization comes from the fact that it works for all large discount factors, and it is independent of the size of the finite records or the exact values of the prior belief, as long as the prior assigns positive probability, no matter how small, to a commitment type. We show that the long-run player must achieve a high reputation payoff if the records are long but finite. Our payoff bound is uniform across time and across all equilibria. A large literature initiated in [12] under complete records has devoted efforts to bound payoffs at the beginning of the game, whereas many applications we economists investigate feature ongoing repeated relationships where the starting moments have passed. Thus our payoff result could shed light on these ongoing relationships (see also [11]). Via an example, we show that with complete records there exist equilibria with a very low payoff for the long-run player in the long run. Hence, no such uniform reputational payoff property exists in games with complete records. In other words, the natural assumption of limited records yields a model of reputation with sharp behavioral and payoff predictions at any time of the game.

In our model in each period the long-run and short-run players act simultaneously (our model is equivalent to a sequential move game in which the short-run player moves first). The short-run player chooses an action  $y \in [0, 1]$  which indicates how much he trusts the long-run player who in turn decides how much to exploit the trust. The long-run player's action is  $x \in [0, 1]$ , where a lower action represents more exploitation. As usual in the reputation literature, we assume that the long-run player might be potentially a commitment type who always plays a fixed non-opportunistic action  $c > 0$ . A short-run player entering the game sees the long-run player's actions in the most recent  $K$  periods. For tractability, we assume that a short-run player does not know how many interactions have occurred before his entry, but has a Bayesian prior over the times at which he enters the game.

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<sup>1</sup> Limited records are also common in online markets. For instance, at [Elance.com](http://Elance.com), an online labor market, the default view of user feedback for contractors contains information from the last 12 months. Additionally, "star ratings" in online markets are often combined with a list of individual reviews ranked by date. If users do not read all reviews and are more likely to read recent ones than old ones (say, the first page), then our model of limited records applies to such markets as well.

We make several assumptions about the stage game payoffs to capture the reputational trade-off. The important ones are that the long-run player always has a static incentive to fully exploit the short-run player's trust by taking the lowest possible action,  $x = 0$ , and that the static incentive to exploit grows as the short-run player becomes more trusting. The model assumptions capture the following application: a long-run contractor/seller chooses the quality ( $x$ ) of a product, and a short-run consumer chooses the quantity/size of a project ( $y$ ). The larger the project, the more tempting it is for the contractor to shirk. Our results stem from the interaction of assuming records are finite and these payoff assumptions. The results are summarized as follows.

First, under finite records, belief over types no longer serves as a sufficient statistic of equilibrium strategies. We show that *all* stationary perfect Bayesian equilibria (PBE) depend on a simple statistic of the observed history. In any equilibrium after a history containing at least one action  $x \neq c$ , the opportunistic long-run player mixes between fully exploiting a short-run player's trust (playing  $x = 0$ ) and mimicking the non-opportunistic type. Once the history is "clean" (i.e., consistent with non-opportunistic play), the rational long-run player exploits his opponent's trust for sure. The short-run players choose a positive amount of trust in every period, even when the short-run players *know* that the long-run player is opportunistic. A central feature of our model is that even if the current short-run player knows the long-run player's type, and he knows that several future short-run players will know the long-run player's type, some short-run players in the remote future might be uncertain about the long-run player's type. Indeed, we show that in the *entire* reputation building process, the type of the long-run player is *known* to the short-run players. They "collude" with the opportunistic type because they understand that the opportunistic player has incentives to build up his reputation to exploit it even more in the future. We call this phenomenon "riding reputation bubbles". This logic is, in spirit, related to that of Abreu and Brunnermeier [1] who show that a lack of common knowledge of a financial bubble creates the incentives for the perfectly informed arbitrageurs to ride the bubble. In [1], a bubble grows exogenously and the burst of the bubble is once and for all. In contrast, in our model, the perfectly informed short-run players ride on the "reputation bubble" to take advantage of the reputation building incentives of the long-run player who will exploit future uninformed short-run players. Moreover, reputation bubbles endogenously grow and burst, and this process repeats itself. Riding reputation bubbles is a defining feature of our model. We return to this point in the related literature section.

Second, to investigate payoff limits in games with finite records, the order of taking limits ( $K$  and the discount factor  $\delta$ ) becomes relevant, while in games with unlimited records there is only one limit to take since  $K = \infty$ . We prove a payoff bound if the record length,  $K$ , is large enough (we characterize both orders). As the long-run player becomes infinitely patient, this bound converges to the payoffs when he can pre-commit. From the point of view of designing reputation systems that result implies that it is possible to sustain high payoffs with large but finite record keeping, even though the behavior changes a lot. Moreover, in contrast to [12], our bound applies at any time of the equilibrium play, not just at the beginning of the game, providing a stronger long-run prediction. While the uniformity of the payoff bound stems from the finite memory structure of the game, finite records pose additional challenge to the proof which we explain more carefully later. The classic technique from [12] is no longer sufficient to establish the bound – we need to utilize additional properties of the equilibrium behavior established in the first part of the paper. Indeed, our proof reveals a trade-off faced by the reputation carrier: if the record length is short, reputation building is not effective (e.g., reputation cannot be built in the extreme case of no record), while if the record length is long, rebuilding a clean record is too costly.

Our findings are relevant for understanding the functioning of reputational concerns in applications. In markets where complete and impartial record keeping is hard or even impossible, as is the case for example in developing economies or in grey markets, transactions are often conducted with limited records. Since a lack of good record keeping is often correlated with weak contractual enforcement or a complete lack of it, reputation effects are believed to be an important substitute for the formal legal system, as argued, for example, in [13]. In the same vein, existing literature focused on how reputational concerns can successfully restore formal contractual incentives in informal markets, and how long-run players can be motivated to behave non-opportunistically to “keep their reputation”. In contrast, we show that if record keeping is limited, then reputational concerns necessarily create incentives for repeatedly massaging one’s reputation, consistent with informal observations of these markets.

Strategic mortgage default offers an example of a market on which our results might shed some light. A strategic mortgage default occurs when a homeowner stops paying their mortgage even though they are still financially able to do so, and it is a widespread phenomenon. Los Angeles Times<sup>2</sup> reported the following surprising findings: (1) “strategic defaulters often go straight from perfect payment histories to no mortgage payments at all”, (2) “homeowners with large mortgage balances generally are more likely to pull the plug than those with lower balances”, and (3) “people with credit ratings in the two highest categories . . . are far more likely to default strategically than people in lower score categories”. These findings indicate a sophisticated reputation manipulation. These findings are closely related to our theoretical predictions. We find that (1) the strategic player is more likely to take the static dominant action when he achieves a clean record, (2) he cheats for sure when the stake is the highest, and (3) a partner is exploited with the highest probability when he trusts the most. Here one could interpret homeowners as long-run players facing different lenders, and the credit record shows whether there is a default. Indeed, the impact of a single default on credit scores is not permanent, which incentivizes strategic defaults.<sup>3</sup>

### 1.1. Related literature

The reputation formation mechanism in our model is different from that of the existing literature. Belief evolution has been the prevalent feature of reputation and career concern models since the classic work of [18,23], and [26]. See also a recent model of [15] that studies belief formation of boundedly rational agents. Players cannot ride the bubbles in the models of this large literature – once the uninformed players learn the type of their opponents, trust is lost forever. A recent body of work has been focusing on the sustainability of reputation incentives and the role of institutions. Mailath and Samuelson [21] study a model in which the strategic type takes actions to separate herself from, rather than mimic, the bad type. They investigate how ownership transactions affect the investment on reputation formation. Bar-Isaac [4] explores the institution of partnership and team production as a means of introducing persistent adverse selection. Tadelis [27] studies the separation of reputation carrier’s identity and entity and show how market of name trading could sustain reputation incentives. Mailath and Samuelson [22]

<sup>2</sup> “Homeowners who ‘strategically default’ on loans a growing problem” by Kenneth R. Harney on September 20, 2009.

<sup>3</sup> Unfortunately, the mortgage data are not sufficient to detect any cycles of reputation building, but they are suggestive that some borrowers considering future default try to first improve their credit ratings to secure larger loans, as our model predicts.

and Bar-Isaac and Tadelis [5] provide useful overviews of additional theoretical models and their applications.

The equilibrium patterns in our model are related to the ones described in papers with switching types (see for example [24] and [28]). In this literature, the exogenous, privately observed stochastic process that governs type changing is the driving force of equilibrium dynamics. This stochastic process replenishes short-run players' beliefs about long-run player's types from time to time. Consequently, beliefs serve as state variables and gradual forgiveness is granted precisely because belief improves. This is not the case in our model which features a dynamics of riding reputation bubbles. Gradual forgiveness is *not* a part of our story, because type is fixed and hence short-run players *know* their opponent's types as long as they see a non-commitment action in the past. As a result, equilibria in our game are *not* Markov with the belief as the state.

Technically, our paper contributes to the literature of repeated games with bounded memory by looking at a class of *incomplete information* games. We want to emphasize an important difference between our model and the existing literature: that in our model the long-run player has the ability to “clean up” history by unilaterally playing good actions for long enough regardless of the (on- and off-equilibrium) actions taken by his opponents. We get that property by assuming that the past actions of short-run players are not observable by current short-run players. Otherwise (as we illustrate in Section 3.3), the bootstrapping type of equilibrium would be re-introduced – as in the literature on repeated games with bounded memory under complete information, for example, [25,20], and [2,3]. We think this feature of bounded memory is realistic in applications.

Bounded memory or finite record can be viewed as a reduced form of costly monitoring. Liu [19] considers a product-choice game with explicit information acquisition cost. Costly endogenous monitoring in repeated games has been previously investigated in [6] and [16]. However, costly observation has a distinctive feature from our model. With observation costs, a phenomenon of “random auditing” will appear. Even though payoff characterization is intractable with costly information, [19] constructs a class of equilibria with reputation cycles, while we offer a *complete* equilibrium characterization for a rich class of games and identify the assumptions of the underlying games for which reputation dynamics ensued. Moreover, the phenomenon of riding a reputation bubble under the knowledge of the long-run player's type, which is central in our model, does *not* appear in a model of costly observation; that is, if the short-run player observes a cheating action of the long-run player, no trust can ever be granted. The reason is intuitive: if trust is granted in Liu's  $2 \times 2$  product-choice game even though short-run players find out a cheating action by paying a cost, then the short-run players should not pay the cost in the first place.

Besides studying markets that have exogenously limited records, our paper is also a step towards understanding the trade-offs in the design of record keeping. For example, what are the consequences of the public credit registers (PCRs) or the Better Business Bureau changing the time window of data reporting? In the credit history case, all PCRs in the European Union eventually “forget” transactions which occurred in the remote past. Jappelli and Pagano [14] argue that “this feature may reflect a concern to offer a ‘second chance’ to defaulting debtors, which may be justified not only on equity grounds but also for economic efficiency”. In this paper, instead of arguing when and whether limited record-keeping is optimal, we investigate the positive implications of limited records but clearly the model can be also used for normative questions. Relatedly, Ekmekci [10] considers the design of finite-state rating system to support a high payoff equilibrium in product-choice games.

The rest of the paper is organized as follows. We introduce the model in Section 2. Section 3 studies the behavioral implications of reputation and Section 4 studies the payoff implications. Section 5 concludes. Appendices A–D contain omitted proofs.

## 2. Model

### 2.1. Basic setup

There is one long-run player (player 1) who over time,  $t = 0, 1, \dots$ , plays with an infinite sequence of short-run players. We refer to a generic short-run player as player 2. A short-run player who arrives at time  $t$  plays one stage game with the long-run player and exits the game.

We consider the following stage game. In the stage game the players move simultaneously.<sup>4</sup> Player 2 chooses an amount of trust,  $y \in [0, 1]$  (denote the action space by  $Y$ ). Player 1 decides how much to honor player 2's trust by choosing  $x \in [0, 1]$  (denote the action space by  $X$ ), with  $1 - x$  being a measure of how much player 1 “abuses” player 2's trust. The stage game payoffs are  $g_1(x, y)$ ,  $g_2(x, y)$  for the two players respectively and they are both continuous. Player 1 maximizes the expected discounted payoff with a discount factor  $\delta \in (0, 1)$ . Each short-run player maximizes his stage game payoff.

We make the following natural assumptions on the stage-game payoffs. In the repeated-sale interpretation of our model, Assumption 1 says that in a one-shot interaction seller prefers to produce the lowest (cheapest) quality; Assumption 2 further says that seller's benefit from lowering the quality is higher when the buyer purchases a larger quantity; Assumption 3 says that buyer will purchase more if he anticipates a higher quality.

**Assumption 1** (*Myopic incentive of player 1*).  $g_1(x, y)$  is strictly decreasing in  $x$ . As a consequence,  $x = 0$  is a dominant strategy for player 1 in a single repetition of the game.<sup>5</sup>

**Assumption 2** (*Monotone incentive of player 1*).  $g_1(x, y) - g_1(x', y)$  is strictly increasing in  $y$  for any  $x < x'$ . In words, player 1 has stronger static incentives to abuse player 2 for a higher (trusting) action of player 2 or, to put it in the context of the product-choice game, it is more expensive for the firm to provide high quality when the consumers buy more.<sup>6</sup>

**Assumption 3** (*Myopic incentive of player 2*). For any (mixed) action by player 1, player 2 has a unique best response, denoted by  $y^*(x)$  for pure actions and  $y^*(v)$  for mixed actions  $v$ . Moreover,  $y^*(v)$  increases if  $v$  increases in the first-order stochastic dominance sense. We normalize the payoffs and remove strictly dominated strategies for player 2, so that player 2's best response to player 1's action 0 is  $y^*(0) = 0$ . This makes  $(0, 0)$  the unique Nash equilibrium of the stage game.

Observe that  $x = 0$  and  $y = 0$  are simply a normalization; for example, in the repeated sale model, we can interpret  $y = 0$  as the lowest demand in response to the worst quality, though the lowest demand could still be positive. Assumptions 1 and 2 are the essence of the reputation model we consider. Assumption 3 simplifies the analysis – player 2 simply reacts myopically

<sup>4</sup> All of our results hold in a sequential-move game where the short-run players move first.

<sup>5</sup> All the analysis allows also that  $g_1(x, y)$  is weakly decreasing in  $x$  for each  $y$ , and strictly decreasing when  $y > 0$ .

<sup>6</sup> See Appendix C for a further discussion of this assumption.

to his expectations about player 1's strategy; the pure action assumption can be justified by, for example, risk aversion.

## 2.2. Incomplete information

We introduce a behavioral type for player 1. Player 1 has two types. With probability  $\mu^* \in (0, 1)$  player 1 is a commitment type who always plays  $c \in (0, 1]$  in the repeated game. With probability  $1 - \mu^*$  player 1 is rational (or opportunistic) with the strategies and payoffs specified above. The type space is  $\Theta = \{r, c\}$ , where  $r$  stands for “rational type” and with some abuse of notation,  $c$  for “commitment type” who always plays action  $c$ . We emphasize that  $c$  is not necessarily the Stackelberg type.

We add the following two assumptions about the stage game payoffs to capture the trade-off of reputations in the presence of commitment types.

**Assumption 4** (*Reputation is valuable*).  $g_1(c, y^*(c)) > g_1(0, 0)$ . In words, player 1 prefers to pre-commit to  $c$  rather than play the static Nash outcome (recall  $y^*(c)$  is player 2's best response to  $c$ ).

**Assumption 5** (*Player 1 wants to be trusted*).  $g_1(x, y)$  is strictly increasing in  $y$ . Namely, player 1 prefers a trusting action of player 2.

Throughout, we assume a high-enough discount factor. Let  $\bar{\delta} = \frac{g_1(0, y^*(c)) - g_1(c, y^*(c))}{g_1(0, y^*(c)) - g_1(0, 0)}$ . **Assumptions 1 and 4** guarantee  $\bar{\delta} \in (0, 1)$ . We assume  $\delta > \bar{\delta}$ . The choice of  $\bar{\delta}$  is such that

$$(1 - \delta)g_1(0, y^*(c)) + \delta g_1(0, 0) < g_1(c, y^*(c)) \quad (1)$$

for each  $\delta > \bar{\delta}$ . Therefore, maintaining a reputation of commitment type (with an associated pay-off as on the right-hand side of (1)) is more desirable than cashing in on the reputation and losing the reputation forever (with a corresponding payoff as on the left-hand side of (1)). Consequently, for large  $\delta$ , player 1 has incentives to build up reputation. We shall see later that the exact equilibrium incentives are more subtle – since player 1 can regain his reputation, the incentive to maintain a reputation will unravel though not completely.

## 2.3. Finite records and stationary strategies

For tractability, in this model we introduce another restriction about the information sets of the short-run players. In particular, assume that a short-run player does not know the number of transactions before them – or the calendar time since the game starts. But as a Bayesian player, they share a common prior belief  $P$  over the periods in which they are likely to enter the game *before* entering the game; i.e., they believe that they will enter in period  $t \geq 0$  with probability  $P(t) \geq 0$ . They will update their belief based on any new information they received.

The main assumption in our analysis is about the information of the short-run players. Unlike most of the reputation literature, we assume that the short-run players observe only a finite and partial history of past play. In particular, we assume a short-run player, upon entering the game, observes only the actions chosen by player 1 in the previous  $K$  periods. As explained in the introduction, the unobservability of the short-run players' action enables the long-run player to clean up his history unilaterally. This is the critical feature of the applications we are trying to



capture in this model. This feature also distinguishes our model from the finite memory literature. See Section 3.3 for more detailed comparison.

Formally, denote the finite histories of player 1's play with length  $k = 1, \dots, K$  by  $H^k = X^k$ , and the initial history by the null set  $H^0 = \{\emptyset\}$ . The set of all finite histories observable to player 2 is  $H = \bigcup_{k=0}^K H^k$ . Hence,  $H$  is the collection of information sets in which player 2 can be.

Given Assumption 3, we shall write player 2's strategy as  $\sigma : H \rightarrow Y$ . Finally, we restrict analysis to (stationary) PBE in which player 1 also plays a strategy which depends only on the information set of player 2. That is, we look at PBE in which  $\pi : H \rightarrow \Delta(X)$ .<sup>7</sup>

Standard Bayesian updating implies that  $\pi$ ,  $P$  and  $\mu^*$  induce a posterior belief for player 2 over player 1's type space  $\Theta$ ,  $\mu(\theta|h)$ , for each  $h \in H$  that is reached in equilibrium with positive probability. For any off the equilibrium histories in which player 1 played at least one action different from  $c$  in the last  $K$  periods, we assume  $\mu(c|h) = 0$ .<sup>8</sup> Of course, for all histories reached on the equilibrium path that contain at least one action different from  $c$  in the last  $K$  periods, the Bayes' rule implies  $\mu(c|h) = 0$  as well. Following the literature, we call player 2's posterior belief  $\mu(c|h)$  player 1's reputation (as a committed player).

Note that despite player 2 observing only a finite history, the game has perfect recall. Therefore we can use this standard equilibrium notion in our analysis. Our goal is to characterize all stationary PBE:

**Definition 1.**  $(\pi, \sigma, \mu)$  is a stationary PBE if  $\pi$  and  $\sigma$  (which depend only on  $H$ ) are best responses to each other given  $\mu$  and on the equilibrium path  $\mu$  is consistent with Bayes' rule given  $\pi$  and priors  $P$ ,  $\mu^*$  (while off the path, if player 2 observes any action different from  $c$ , then  $\mu(c|h) = 0$ ).

We denote by  $V(h)$  the expected equilibrium payoff of player 1 (rescaled to average per-period payoffs) given a history  $h$ . For parts of the analysis it will be easiest to describe the equilibria in case  $P$  is the uniform improper prior – for  $h \in H^K$  the resulting beliefs are equal to the fraction of time type  $\theta$  reaches a given state on the equilibrium path. See Appendix C.1 for more discussion of beliefs.

### 3. Reputation effects on behavior

In this section we characterize the equilibrium behavior in all stationary PBE.

#### 3.1. Sufficient statistics of history

The set  $H = \bigcup_{k=0}^K H^k$  is a continuum, making it potentially difficult to describe the equilibrium strategies. In our model, it matters not only what short-run players know about the long-run

<sup>7</sup> By a standard argument, if player 2's strategy is measurable with respect to  $H$  then player 1 has a best response which is also  $H$ -measurable. Hence, for every equilibrium in which player 1's strategy is a more complicated function of the entire history – which for him includes all his past actions and all past actions of the short-run players – there exists a stationary equilibrium with the same payoff to player 1. Note also that the prior belief  $P$  over calendar times affect belief updating. Therefore, technically, the assumption of unobservable calendar time is not equivalent to assuming observable calendar times and stationary strategies of player 2.

<sup>8</sup> The only other off-equilibrium histories are such that  $P(t) = 0$ . For  $t < K$ , we assume that in that case player 2 updates using  $\pi$  and  $\sigma$ . For  $t > K$ , we assume that  $P(t)$  assigns positive probability to at least one such time.



player, but also what they know about what future short-run players would know. This reasoning suggests that beliefs about long-run player's type alone are insufficient to pin down all the strategic incentives. We decompose the set  $H$  into three subsets. The first subset consists of histories in which a non-commitment action has been played, and we denote this set by  $H_-$ . The second subset is a singleton – history  $(\underbrace{c, \dots, c}_K)$ , i.e., a history in which player 1 played  $c$  in the previous

$K$  periods. We call it the “clean history”. The residual is a finite set  $\{\emptyset, C^1, \dots, C^{K-1}\}$  where  $C^k$  is a history of length  $k$  and the commitment action has been played in all periods so far.

We can partition  $H_-$  further into a finite collection by counting how close the player is to a clean history. Formally,  $I(h)$  for a history  $h = (x^0, x^1, \dots, x^t) \in H_-$ ,  $0 \leq t < K$  is

$$I(h) = t - \max\{k: x^k \neq c\}.$$

Therefore,  $I(h)$  measures the number of commitment actions in  $h$  since the most recent non-commitment action. For convenience, let us define  $I(h) = K$  if  $h = (\underbrace{c, \dots, c}_K)$  and call  $I(h)$  the

commitment index of a history. A commitment action played on a history  $h$  with  $I(h) = k$  will increase this index by 1 (or the index remains at  $K$  if  $k = K$ ). A non-commitment action will reduce the index down to 0.

For a fixed  $K$ , we denote the set of indices as  $\mathbf{I} = \{0, 1, 2, \dots, K\}$ . We claim that  $I(h)$  contains all the strategically relevant information in any stationary PBE at history  $h \in H_-$ <sup>9</sup>:

**Proposition 1.** *In any stationary PBE, equilibrium strategies for histories in  $H_-$ , on and off the equilibrium path, depend only on  $I(h)$ . That is, if  $I(h) = I(h')$ , then  $\sigma(h) = \sigma(h')$  and  $\pi(h) = \pi(h')$ .*

This result greatly simplifies our analysis by allowing us to focus on a finite state space  $\mathbf{I} \cup \{\emptyset, C^1, \dots, C^{K-1}\}$ . We call  $\mathbf{I}$  the set of *regular indices for regular histories* and  $\{\emptyset, C^1, \dots, C^{K-1}\}$  the *special indices for special histories*. From now on write strategies  $\pi: \mathbf{I} \cup \{\emptyset, C^1, \dots, C^{K-1}\} \rightarrow \Delta(X)$  and  $\sigma: \mathbf{I} \cup \{\emptyset, C^1, \dots, C^{K-1}\} \rightarrow Y$ . We write  $\mu_k = \mu(c|h)$  for a regular history  $h$  with  $I(h) = k$ . By definition,  $\mu_K > 0$  and  $\mu_0 = \mu_1 = \dots = \mu_{K-1} = 0$ .

Since, as we established in Proposition 1, the continuation payoff of player 1 depends only on the commitment index of history, if he chooses to play an action different from  $c$ , his best response is to play 0 (his myopic best response; see Lemma 4 in Appendix A for a formal proof). In other words,  $\pi(k)$  assigns positive probability to at most two actions,  $\{0, c\}$ . Abusing (to simplify) notation a bit, for regular indices we write player 1's strategy as  $\beta_k$  where  $\beta_k$  is the probability player 1 assigns to action  $c$  (which means he assigns probability  $1 - \beta_k$  to action 0) and player 2's strategy as  $y_k \equiv \sigma(k)$ . To further simplify notation, we also write  $y^*(\beta_k)$  as player 2's best response when he believes that action  $c$  is played with probability  $\beta_k$ .

### 3.2. Reputation bubble

We now focus on the strategies in states in  $H^K$ , i.e., after histories  $h^t$  for  $t \geq K$ . We deal with the initial histories in Appendix B.3. We claim the following characterization of stationary PBE. In every equilibrium, when the index of the history is  $k < K$ , player 1 mixes between 0 and  $c$ .

<sup>9</sup> Even though we have assumed  $\delta > \bar{\delta}$ , this proposition holds for all  $\delta$ .

When he achieves the “clean history”, index  $K$ , he plays 0 for sure, exploiting his reputation. In every period player 2 trusts player 1 to some extent – and the equilibrium degree of trust is increasing in how close the history is to the clean history and it is maximized at the clean history.

The strength of our characterization comes from the fact that it works for all large discount factors and the qualitative prediction is independent of the exact values of  $K$  and  $\mu^*$  as long as  $\mu^*$  assigns positive probability, no matter how small, on the commitment type. Formally,

**Theorem 1.** *For any  $\delta > \bar{\delta}$ ,  $K > 0$ , and any prior  $\mu^* > 0$ , any stationary PBE takes the following form:*

1. *There exists a strictly increasing sequence  $\{\beta_k\}_{k=0}^{K-1} \subset (0, 1)$  such that, if the index of the history is  $k < K$ , player 1 plays  $c$  with probability  $\beta_k$  and 0 with probability  $1 - \beta_k$ ; player 2 plays  $y_k = y^*(\beta_k)$ .*
2. *If the index is  $K$ , player 1 plays action 0 (with probability 1); player 2 plays  $y^*(\mu_K)$ , where  $\mu_K > \beta_{K-1}$ .*

Part 1 of [Theorem 1](#) describes the behavior of riding the bubble – as  $k < K$ , player 2 knows player 1’s type, but trust is granted and is increasing in  $k$ . Part 2 describes the burst – player 1 pricks the bubble to exploit the maximal amount of trust. Technically, we see the equilibrium strategies critically depend on  $k$  though  $\mu_k = 0$  for all  $k < K$ . Hence belief over types is not a sufficient statistic.

The proof of [Theorem 1](#) is developed through a series of lemmas that show the following:

(1) When the history is clean (with index  $K$ ), the type  $r$  player 1 plays 0 for sure. If not, he would play  $c$  after every history because [Assumption 2](#) implies that, if he is not tempted to play 0 when player 2 plays  $y$ , he is not tempted for all  $y' < y$ . But, if he plays  $c$  after every history, then for all histories player 2 would play  $y^*(c)$  – there would be no intertemporal incentives. As a result, it would be a strict best response for player 1 to play 0, a contradiction. Similarly, for any regular history it cannot be the case that player 1 plays  $c$  with probability 1 – that would be the best period for him to deviate to 0!

(2) After a history with index  $k$ , player 1 either plays 0 with probability 1 or he plays  $c$  with probability strictly higher than in state  $k - 1$ . The reasoning is as follows. Suppose player 1 plays  $c$  with positive probability in states  $k - 1$  and  $k$ . The continuation payoff from playing 0 is the same in both cases but playing 0 in state  $k - 1$  delivers the extra myopic payoff sooner. For player 1 to be willing to play  $c$  in state  $k - 1$  it must be that he is rewarded next period with an even higher reward for playing 0, to compensate for discounting (this is captured by Eq. (2)).

(3) If  $\delta > \bar{\delta}$ , then playing 0 in every period is not an equilibrium. In fact, playing 0 for sure in any of the regular histories is not an equilibrium. The reason is that otherwise in state  $K$  player 2 would assign a very high probability to type  $c$  and play close to  $y^*(c)$ . If  $\delta$  is high enough then type  $r$  of player 1 prefers to mimic type  $c$  instead of deviating once and getting  $g_1(0, 0)$  afterwards.

(4) Combining steps 2 and 3,  $\beta_k \in (0, 1)$  for all  $k \in \{0, \dots, K - 1\}$ . Since player 1 is indifferent in states  $k < K$ , the incentive constraints require, for  $k \in \{0, \dots, K - 1\}$ ,

$$g_1(0, y_k) - g_1(c, y_k) = \delta[g_1(0, y_{k+1}) - g_1(0, y_0)], \quad (2)$$

where player 2’s best response requires  $y_k = y^*(\beta_k)$ . These equations almost pin down the equilibrium. The only remaining equation is for state  $K$  to pin down  $\mu_K$  and  $y_K$ . That in general depends on  $P(t)$ ,  $\mu^*$ , and the equilibrium strategies after the special histories (in the initial

$K - 1$  periods with no deviations from  $c$ ). Eq. (2) requires  $\{\beta_k\}_{k=0}^{K-1}$  to be strictly increasing and  $\mu_K > \beta_{K-1}$ . In Appendices A–D, we characterize equilibrium strategies for the initial periods in which player 2 knows the calendar time (he deduces it from the length of the history).

We emphasize that player 1's equilibrium strategies in the first  $K$  periods upon a (short) clean history are very different. Hence, if player 2 observes the calendar time, his belief updating upon observing  $K$  consecutive  $c$ 's when he enters at period  $K$  is different from that when he enters in future periods.

In the special case of the improper uniform prior we can characterize the equilibrium even further.

**Proposition 2.** Assume  $P$  is the “improper uniform prior”. The equilibrium strategies  $\beta_0, \beta_1, \dots, \beta_{K-1}$  and the posterior belief  $\mu_K$  are completely characterized by (2),

$$\mu_K = \frac{\mu^*(1 + \beta_0 + \beta_0\beta_1 + \dots + \beta_0\beta_1 \dots \beta_{K-1})}{\mu^*(1 + \beta_0 + \beta_0\beta_1 + \dots + \beta_0\beta_1 \dots \beta_{K-2}) + \beta_0\beta_1 \dots \beta_{K-1}}, \quad (3)$$

$$y_K = y^*(\mu_K). \quad (4)$$

To summarize, we have established in this section the behavioral predictions of our model with limited records. An interesting and new feature of the model is that, in the reputation building stage, player 2 plays trusting actions even though he knows that player 1 is the rational type. Moreover, player 2's trust grows over the indices. This relationship building is different from what we observe in models with changing types such as [24]. In Phelan's model, the short-run players never know for sure player 1's type. Their trust grows precisely because they increasingly believe that the long-run player's type is the commitment type. This is not the case in our model. The short-run players trust the long-run player not because they assign a high probability to the long-run player being the commitment type. They know for sure that they are playing against a rational type, but they know that with positive probability some future short-run players will not know what they know. They also understand the long-run player's incentive to cheat future short-run players. As a result, they ride the bubble and effectively exploit future short-run players.

We stress that the dynamic reputation massaging in our model could not be a part of equilibrium if the short-run players always see the complete history of player 1's actions. Why? If player 2 sees the full history of player 1's play, then there is no room for player 1's “manipulation” – player 2 can be “surprised” on the equilibrium path only once. To see this, note that if player 1 ever played an action different than  $c$  in the past, player 2 would know that he faces the rational type and would expect  $x = 0$  is coming. As a result, under complete records player 2 would not trust player 1 and would avoid being exploited (play  $y = 0$ ). But that would mean that player 1 should deviate to 0 one period before and the whole equilibrium would unravel. In contrast, with complete records and sufficiently high  $\delta$  it is easy to construct grim-trigger equilibria in which player 1 always plays  $c$ .

### 3.3. More on the role of limited records

Our assumption of limited records enables the long-run player to “clean up” his history, which is new in the reputation literature. This realistic and important feature is shared by many applications and as we discussed above, it radically changes the equilibrium behavior (and as we show in the next section, it leads to new, stronger payoff predictions as well).

The assumption of limited records has two components – player 2 observes only a finite history and he only observes the past actions of player 1. Both are important. Indeed, if the past actions of a short-run player were observed, the current player 2 could infer something about the behavior of the long-run player beyond the last  $K$  periods. This type of inference re-introduces the infinite memory into the model and it is possible to construct equilibria that in which the long-run player never deviates from  $c$  on the equilibrium path and once he does, the play  $(0, 0)$  ensues forever. The folk theorems obtained in [20] in complete information games are based on this bootstrapping logic.

Again, what distinguishes our model from the bounded memory literature is precisely a player's ability to unilaterally clean up history, regardless of the opponents' actions (both on and off the equilibrium path). This is particularly compelling when one long-run player plays against a sequence of different short-run players over time. The assumption that short-run players' actions are not observable to other short-run players is relevant in many real-life markets. For example, the Better Business Bureau does not reveal volume of transactions at a given retailer nor details on trading history; driving records show the driving faults but do not show the past insurance coverage or premium. In general, a designer of the record-keeping system may not want to reveal the data for privacy reasons (for example, a university may not want to reveal the grades of the students that wrote negative feedback for a course, eBay may not want to reveal the size of the transactions of buyers providing feedback, etc.). Finally, in some situations the long-run player may observe the short-run players' actions privately and is not able to reveal them credibly.

#### 4. Reputation effects on payoffs

In this section, we provide three results regarding the lower bounds of equilibrium payoffs. We first apply the classic argument of [12] to obtain a payoff bound at the beginning of the game. This bound is not ideal for games with limited records. We refine this argument to obtain a tighter bound uniformly for all histories. Finally, via an example, we show that uniform payoff bound is not possible in games with unlimited records.

##### 4.1. Payoff bounds

We now turn to the equilibrium payoffs. First using the argument of [12], we obtain the following bound:

**Proposition 3.** *For any  $\varepsilon > 0$  and  $\mu^* \in (0, 1)$ , there exists an integer  $K(\varepsilon, \mu^*) > 0$  (independent of  $\delta$ ) such that if  $K > K(\varepsilon, \mu^*)$  then in any stationary PBE player 1's payoff computed at period 0 is at least  $B(\delta, K) = (1 - \delta^{K(\varepsilon, \mu^*)})g_1(c, 0) + (\delta^{K(\varepsilon, \mu^*)} - \delta^K)g_1(c, y^*(c)) + \delta^K g_1(0, 0) - \varepsilon$ , which converges to  $(1 - \delta^{K(\varepsilon, \mu^*)})g_1(c, 0) + \delta^{K(\varepsilon, \mu^*)}g_1(c, y^*(c)) - \varepsilon$  as  $K \rightarrow \infty$ .*

The proof follows almost exactly the steps in [12]. For brevity we omit the proof, but provide the general idea (see Fig. 1 for an illustration). Player 2's best response is continuous.<sup>10</sup> For any  $\varepsilon > 0$ , there exists  $\eta(\varepsilon) > 0$  such that if player 2 believes that player 1 plays  $c$  with a probability more than  $1 - \eta(\varepsilon)$ , then player 2's best response will be very close to  $y^*(c)$ . If that happens, player 1's payoff from playing  $c$  will be  $\varepsilon$ -close to  $g_1(c, y^*(c))$ . Fudenberg and Levine [12]

<sup>10</sup> The bounded singleton valued upper semi-continuous best response correspondence must be continuous.

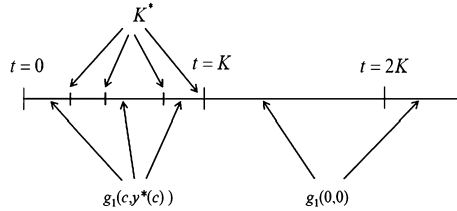


Fig. 1. Illustration of payoff bounds with finite record.

show that if player 1 plays  $c$  constantly, there exists  $K(\varepsilon, \mu^*)$  such that there are no more than  $K(\varepsilon, \mu^*)$  periods in which player 2 expects player 1 to play  $c$  with a probability less than  $1 - \eta(\varepsilon)$ .<sup>11</sup> The intuition is that if there are many periods in which player 2 expects player 1 to play  $c$  with a probability not close to 1, then after observing a sequence of  $c$  the belief player 2 assigns to the  $c$  type has to increase close to 1. The bound is then constructed by considering a deviation by player 1 to playing  $c > 0$  for the first  $K$  periods and then playing 0. (We cannot apply the argument of Fudenberg and Levine [12] to any periods with finite record – see the discussion below.) In the worst case scenario player 1 receives  $g_1(c, 0)$  in each of these periods, and for the remaining  $K - K(\varepsilon, \mu^*)$  periods, player 1 is guaranteed to receive a payoff of at least  $g(c, y^*(c)) - \varepsilon$ . After the  $K$  periods (when player 2 can no longer observe the calendar time), player 1's payoff is at least  $g_1(0, 0)$ . That yields the bound.

**Remark 1.** Proposition 3 is not satisfactory for two reasons. First, the bound comes strictly from the part of the game in which the information sets are exactly as in a standard game with complete records. Hence, it does not tell us much about the impact of the limited records. To learn about that one may be more interested in bounding the equilibrium payoffs at  $t \geq K$ . Second, the limit of this payoff bound for patient players depends crucially on the order of taking limits:  $\lim_{\delta \rightarrow 1} \lim_{K \rightarrow \infty} B(\delta, K) = g_1(c, y^*(c)) - \varepsilon$ , but  $\lim_{\delta \rightarrow 1} B(\delta, K) = g_1(0, 0) - \varepsilon$  for any  $K > K(\varepsilon, \mu^*)$ . Note that only the first limit matters in games with complete records where  $K = \infty$  by assumption, but with limited records the second limit seems to be of interest as well.

**Remark 2.** To address these two issues, we provide a tighter bound for PBE uniformly over time. One might think of applying the argument of [12] after any history to provide such a bound for any fixed  $K$ . But this will not work for the following reason. Fudenberg and Levine [12] show that there are at most  $K(\varepsilon, \mu^*)$  periods in which player 2 does not play close to  $y^*(c)$ , but it is not clear which those  $K(\varepsilon, \mu^*)$  periods are. In fact, no matter how large  $K$  is, the period  $t = K$  might be exactly one of the  $K(\varepsilon, \mu^*)$  periods in which player 2 does not play close to  $y^*(c)$ . But any period beyond  $K$  looks to player 2 just like period  $K$  if he sees a sequence of  $c$ 's! If player 2 plays very differently from  $y^*(c)$  after  $t = K$ , we cannot bound the payoffs uniformly using Fudenberg and Levine's technique. For example, if player 2 assigns a high probability that he enters at  $t = K$  and he expects that at  $t = K$  after playing  $c$  for  $K$  times player 1 plays 0, then after that history player 2 will play 0. Hence, a simple reasoning based on a deviation to play  $c$  always is not enough and we need to use the equilibrium characterization from the previous section to establish the second main result of this paper.

<sup>11</sup> We can take  $K(\varepsilon, \mu^*) = \frac{\ln \mu^*}{\ln \eta(\varepsilon)}$ .

Motivated by the discussion above, our approach is to show that the posterior belief on the commitment type,  $\mu_K$ , is *bounded from below* in any PBE and the lower bound is close to 1 when  $K$  is large enough (and hence player 2 must play an action close to  $y^*(c)$ ).

**Lemma 1.** *For any  $\eta \in (0, 1)$ , whenever  $K > \frac{\ln(\frac{\eta\mu^*}{1-\mu^*})}{\ln(1-\eta)} - 1$ , in any stationary PBE we have  $\mu_K > 1 - \eta$ .*

The general idea behind the proof is as follows. Suppose to the contrary that  $\mu_K \leq 1 - \eta$ . Then, as we have shown in [Theorem 1](#),  $\beta_k < \mu_K \leq (1 - \eta)$ , so from the history with index 0, the rational type of player 1 reaches the clean history (with index  $K$ ) with a probability at most  $\prod_{k=0}^{K-1} \beta_k < (1 - \eta)^K$ . But then, if the prior  $P(t)$  assigns all the probabilities to the tails (such as the improper uniform prior), for high enough  $K$ , player 2 would assign a very high probability to type  $c$  after seeing the clean history, a contradiction. The extra twist is that, if  $P(t)$  assigns positive probability to periods  $\{K, \dots, 2K - 1\}$ , the belief depends also on the strategies after the special initial histories (with indices  $\emptyset, C^1, \dots, C^{K-1}$ ), but we shall argue (in [Appendix B.3](#)) that  $\beta_k > \beta(C^k)$  and  $\beta_0 > \beta(\emptyset)$ , so the same bound applies.

This lemma leads to our new payoff bound.

**Theorem 2.** *For any  $\varepsilon > 0$  and  $\mu^* \in (0, 1)$ , there exists an integer  $K(\varepsilon, \mu^*)$  independent of the equilibrium and the discount factor such that player 1's payoff in any stationary PBE with limited records of length  $K > K(\varepsilon, \mu^*)$  at any history is bounded from below by*

$$(1 - \delta^K)g_1(c, 0) + \delta^K g_1(c, y^*(c)) - \varepsilon,$$

which converges to  $g_1(c, y^*(c)) - \varepsilon$  as  $\delta$  goes to 1.

**Proof.** From the continuity of player 2's best response and the continuity of player 1's payoff function, for any  $\varepsilon > 0$  there exists an  $\eta(\varepsilon) > 0$  such that if player 2 believes that player 1 plays  $c$  with probability at least  $1 - \eta(\varepsilon)$ , then player 1's payoff is  $\varepsilon$ -close to  $g_1(c, y^*(c))$  this period.

Consider player 1's deviation to always playing  $c$ . If  $\mu_K > 1 - \eta(\varepsilon)$ , the result is immediate: in the worst case, player 1 gets  $g_1(c, 0)$  in the first  $K$  periods, and stays in state  $K$  with a payoff of at least  $g_1(c, y^*(c)) - \varepsilon$ . If  $\mu_K \leq 1 - \eta(\varepsilon)$ , then take  $K(\varepsilon, \mu^*) = \ln(\frac{\eta(\varepsilon)\mu^*}{1-\mu^*}) / \ln(1 - \eta(\varepsilon)) - 1$ . The rest follows from [Lemma 1](#).  $\square$

It is easy to see that the payoff bound established in [Theorem 2](#) work for all PBE (see footnote 7). It is worth noting that the record length,  $K$ , plays a dual role on the equilibrium path. A larger  $K$  makes a clean history a convincing signal for the commitment type which increases reputation benefit, but it also makes it harder to clean a history and hence lowers the equilibrium payoffs. This reminds us the education signaling models with multiple equilibrium levels of education (see also [\[17\]](#) for a repeated signaling model with persistent private information and its connection to reputation formation).

Summing up this section, we have shown that even though with limited records the equilibrium behavior is quite different from that with complete records, if the records are long enough and player 1 is patient enough, he can still achieve payoffs close to  $g_1(c, y^*(c))$ . The difference from the game with complete records is that we are able to establish our payoff bound for any time in the game. We also want to emphasize that [Ekmekci et al. \[11\]](#) obtain a uniform payoff bound for reputation games under a very different mechanism. In their model, full history

is observed but the long-run player's type changes over time. They employ a powerful entropy technique to general payoff structures which include the model of [24] as a special case.

#### 4.2. Payoffs under complete records

To finish this section, we present an example that shows how, under complete records, it is impossible to have a reputational payoff bound which is uniform across all periods. Making use of a public randomization device, we construct an equilibrium in which players' behavioral strategies after each history depend only on reputation levels the history induces at the previous two periods. We consider the following payoff functions:

$$g_1(x, y) = (2 - x)y, \quad \text{and} \quad g_2(x, y) = -(x - y)^2.$$

Assume  $c = \frac{1}{3}$ . If player 2 plays  $\frac{1}{3}$  with probability  $p$  and 0 with probability  $1 - p$ , player 2's best response is  $\frac{p}{3}$  due to his quadratic payoff function. Player 1's Stackelberg action is 1 and his Stackelberg payoff is  $g_1(1, 1) = 1$ . We abuse some notation by writing  $\mu_t$  as the belief at the beginning of period  $t$ , with  $\mu_0 = \mu^*$ . We shall show the following is a perfect Bayesian equilibrium for  $\delta > \frac{1}{2}$ .

1. If the opportunistic type has reputation  $\mu_t \in (0, 1)$ , he plays  $x = c$  with probability  $\frac{(\mu_t)^{\frac{1}{2}} - \mu_t}{1 - \mu_t} \in (0, 1)$  and  $x = 0$  with the remaining probability.
2. If  $\mu_t > 0$  and  $\mu_{t+1} = 0$ , then a public randomization device is invoked at the beginning of period  $t + 1$  to determine the play of the continuation game:
  - (a) with probability  $1 - q(\mu_t)$ , the static Nash equilibrium  $(0, 0)$  is repeated indefinitely.
  - (b) with probability  $q(\mu_t)$ , the following trigger strategy profile is played:  $(1, 1)$  is played on the equilibrium path, and a player 1's deviation triggers the repetition of static Nash equilibrium  $(0, 0)$ .
3. Deviations of the short-run player 2 are ignored.

For the trigger strategy profile in part 2(b) to form an equilibrium, it suffices that  $\delta > \frac{1}{2}$ , where  $\delta = \frac{1}{2}$  is the unique solution for  $(1 - \delta)g_1(0, 1) + \delta g_1(0, 0) = g_1(1, 1)$ .

We now determine  $q(\mu_t)$  and verify that parts 1–3 indeed form a PBE. Suppose  $\mu_t \in (0, 1)$ . Then the posterior belief upon  $c$  is observed is

$$\mu_{t+1} = \frac{\mu_t}{\mu_t + (1 - \mu_t) \frac{(\mu_t)^{\frac{1}{2}} - \mu_t}{1 - \mu_t}} = (\mu_t)^{\frac{1}{2}}.$$

According to the candidate equilibrium prescribed by part 1, the total probability that  $c$  is played at period  $t$  by both types of player 1 is  $(\mu_t)^{\frac{1}{2}}$ . Player 2's best response to this belief is  $\frac{1}{3}(\mu_t)^{\frac{1}{2}}$ . Player 1's expected normalized discounted payoff from playing 0 is

$$\begin{aligned} & (1 - \delta)g_1\left(0, \frac{1}{3}(\mu_t)^{\frac{1}{2}}\right) + \delta[(1 - q(\mu_t))g_1(0, 0) + q(\mu_t)g_1(1, 1)] \\ &= \frac{2}{3}(1 - \delta)(\mu_t)^{\frac{1}{2}} + \delta q(\mu_t). \end{aligned}$$

If player 1 plays  $c = \frac{1}{3}$  this period and plays  $c$  thereafter, his normalized discounted continuation payoff is



$$(1 - \delta) \sum_{s=t}^{\infty} \delta^{s-t} g_1 \left( \frac{1}{3}, \frac{1}{3} (\mu_s)^{\frac{1}{2}} \right) = \frac{5}{9} (1 - \delta) \sum_{s=t}^{\infty} \delta^{s-t} (\mu_s)^{\frac{1}{2}}.$$

For player 1 to randomize at  $t$  as prescribed by part 1, the following indifference condition needs to hold:

$$(1 - \delta) \frac{2}{3} (\mu_t)^{\frac{1}{2}} + \delta q(\mu_t) = \frac{5}{9} (1 - \delta) \sum_{s=t}^{\infty} \delta^{s-t} (\mu_s)^{\frac{1}{2}}.$$

Therefore,

$$q(\mu_t) = \frac{1 - \delta}{\delta} \left( \frac{5}{9} \sum_{s=t+1}^{\infty} \delta^{s-t} (\mu_s)^{\frac{1}{2}} - \frac{1}{9} (\mu_t)^{\frac{1}{2}} \right). \quad (5)$$

Since  $\mu_s > \mu_t$  for  $s > t$  and  $\delta > \frac{1}{2}$ ,

$$\begin{aligned} q(\mu_t) &> \frac{1 - \delta}{\delta} \left( \frac{5}{9} (\mu_t)^{\frac{1}{2}} \sum_{s=t+1}^{\infty} \delta^{s-t} - \frac{1}{9} (\mu_t)^{\frac{1}{2}} \right) \\ &= \frac{1 - \delta}{\delta} (\mu_t)^{\frac{1}{2}} \left( \frac{5}{9} \frac{\delta}{1 - \delta} - \frac{1}{9} \right) > \frac{4}{9} (\mu_t)^{\frac{1}{2}} > \frac{4}{9} (\mu^*)^{\frac{1}{2}}. \end{aligned}$$

Since  $\mu_s < 1$ , we have

$$q(\mu_t) < \frac{5}{9} \frac{1 - \delta}{\delta} \sum_{s=t+1}^{\infty} \delta^{s-t} = \frac{5}{9}.$$

Therefore,  $q(\mu_t) \in (\frac{4}{9} (\mu^*)^{\frac{1}{2}}, \frac{5}{9})$  given by (5) is a well-defined probability.

To summarize, in the constructed equilibrium the opportunistic player 1 randomizes between  $c$  and 0 before revealing his type and the sequence of reputation levels before revelation is  $\{(\mu^*)^{\frac{1}{2t}}\}$ . With a positive probability, player 1 reveals his type and then either the static Nash equilibrium is played with a continuation payoff of 0 or the Stackelberg equilibrium is played with continuation payoff of 1.

## 5. Conclusion

In this paper, we have studied the impact of limited records on reputation effects. We have shown that limited records dramatically change the equilibrium behavior and lead to long-run predictions of reputation that are not possible under complete records.

An interesting direction for future research is to consider games with imperfect monitoring. In our model, the actions of player 1 are observed without noise. Yet, in some applications, it is natural to ask what happens if monitoring is imperfect. We have done some preliminary work which shows that, in the model with a commitment type, strategies will depend on the history in a more complex way – it will no longer be sufficient to keep track only of the commitment index because even a history containing some non-commitment actions would be consistent with player 1 being the  $c$  type. Nevertheless, we conjecture that, for sufficiently low noise in monitoring, there would exist equilibria similar to the ones described in this paper (we have constructed examples for  $K = 1$ ). Cripps et al. [9] have shown that reputation effects are short-run in imperfect monitoring models, and, under fairly reasonable assumptions, “anything goes” with regard

to long-run equilibrium behavior and payoffs. Ekmekci [10] showed that, if histories are restricted to a rating system, there exists an equilibrium with high continuation payoffs after all histories. We conjecture that, in our games with limited records, all equilibria would have high payoffs after all histories, for sufficiently precise monitoring. Yet, it remains to be discovered what other new, dynamic effects limited records would introduce in games with imperfect monitoring. One might also take a mechanism design approach to study limited records. These richer models, though beyond the scope of the current work, might help to understand the reputation phenomenon observed in online auctions; see, e.g., [7].

## Appendix A. Proof of Proposition 1: Sufficient statistics

In this appendix we prove Proposition 1:

**Proposition 1.** *In any stationary PBE, equilibrium strategies for histories in  $H_-$ , on and off the equilibrium path, depend only on  $I(h)$ . That is, if  $I(h) = I(h')$ , then  $\sigma(h) = \sigma(h')$  and  $\pi(h) = \pi(h')$ .*

Our proof utilizes the following auxiliary relation between histories, which we term “ $n$ -similarity”. In words, if  $h$  and  $h'$  are  $n$ -similar, then  $h$  and  $h'$  share the same most recent  $n - 1$  elements and their  $n$ th recent entries are different from the commitment action  $c$ . Let  $L(h)$  be the length of the history  $h$ . Formally,

**Definition 2.** For any  $h, h' \in H_-$ , we say that  $h$  and  $h'$  are  $n$ -similar, for  $1 \leq n \leq \min\{L(h), L(h')\}$ , denoted by  $h \sim^n h'$ , if

- 1)  $\tau^{n-1}(h) = \tau^{n-1}(h')$  and
- 2)  $\tau^n(h) \neq (\tau^{n-1}(h), c)$  and  $\tau^n(h') \neq (\tau^{n-1}(h'), c)$ .

For example, take  $K = 5$ .

- 1)  $h = (0, 0, c, c, c)$  and  $h' = (0, c, c, c, c)$  are not  $n$ -similar for any  $n$  (even though last 3 entries are the same, since the 4th most recent entry in  $h'$  is  $c$  they are not 4-similar).
- 2)  $h = (0, 0, c, c, c)$  and  $h' = (c, 0, c, c, c)$  are 4-similar.
- 3)  $h = (0, 0, c, 0, 0)$  and  $h' = (0, 0, 0, 0, 0)$  are 1-similar and 2-similar.
- 4)  $h = (0, c)$  and  $h' = (c, c, 0, 0, c)$  are 1-similar.

We use the following relationship between the commitment index of the histories and their  $n$ -similarity.

**Lemma 2.** *For any  $h, h' \in H_-$ , if  $I(h) = I(h')$ , then either  $h = h'$  or  $h \sim^n h'$ , for some  $n \in \{1, 2, \dots, K\}$ .*

**Proof.** Suppose  $I(h) = I(h') = i$  and  $h \neq h'$ . This implies that  $i < \min\{L(h), L(h')\} \leq K$  and that there exist some  $x, x' \neq c$  such that  $\tau^{i+1}(h) = (x, \underbrace{c, c, \dots, c}_i)$  and  $\tau^{i+1}(h') = (x', \underbrace{c, c, \dots, c}_i)$ . Therefore,  $h \sim^{i+1} h'$ .  $\square$

Our next (main) step is to show that if two regular histories have the same index, then player 2 plays the same strategy after seeing these two strategies, and player 1 payoffs in these two states are the same.

**Lemma 3.** *If  $I(h) = I(h')$ , then  $\sigma(h) = \sigma(h')$  and  $V(h) = V(h')$ .*

**Proof.** By Lemma 2, we only need to show that if  $h \sim^n h'$  for some  $n \in \{1, 2, \dots, K\}$ , then  $\sigma(h) = \sigma(h')$  and  $V(h) = V(h')$ . We do so by induction. Consider any stationary PBE,  $(\pi, \sigma, \mu)$ .

*Step 1:* If  $h \sim^K h'$ , then  $\sigma(h) = \sigma(h')$  and  $V(h) = V(h')$ .

Suppose to the contrary that  $\sigma(h) \neq \sigma(h')$ , and, without loss of generality, take  $\sigma(h) > \sigma(h') \geq 0$ . Since player 1's type is known to player 2 after each of the two histories (if two histories are  $n$ -similar, they both have to contain at least one action different from  $c$ ) there exists  $d > 0$  such that  $\pi(h)([d, 1]) > 0$  (otherwise, we would have player 1 play 0 for sure and  $\sigma(h) = 0$ ). Let  $D$  be the set of such  $d$ .

Next, it must be that  $\pi(h')([0, d)) > 0$  for some  $d \in D$ . Suppose, to the contrary that  $\pi(h')([d, 1]) = 1$  for any  $d \in D$ , then  $\pi(h')$  first order dominates  $\pi(h)$ . This would violate Assumption 3 and the assumption  $\sigma(h) > \sigma(h')$ .

Therefore, there exists  $d > 0$  such that

$$\pi(h)([d, 1]) > 0, \quad \pi(h')([0, d)) < 1.$$

Take  $d^* \in [d, 1]$  from the support of  $\pi(h)$  and  $d_* \in [0, d)$  from the support of  $\pi(h')$ . The IC constraint for player 1 in state  $h$  implies

$$(1 - \delta)(g_1(d^*, \sigma(h)) - g_1(d_*, \sigma(h))) \geq \delta(V((h, d_*) - V((h, d^*))), \quad (\text{A.1})$$

where  $h, d^*$  is the history obtained by appending  $d^*$  to  $h$  and then dropping the oldest entry of  $h$ , and  $V((h, d^*))$  indicates the continuation payoff at this history.

Similarly, the incentive constraint in state  $h'$  implies that

$$(1 - \delta)(g_1(d^*, \sigma(h')) - g_1(d_*, \sigma(h'))) \leq \delta(V((h', d_*)) - V((h', d^*))). \quad (\text{A.2})$$

Note that  $(h', d_*) = (h, d_*)$  and  $(h', d^*) = (h, d^*)$  because  $h \sim^K h'$  (since the two histories are  $K$ -similar, after appending the same action today and dropping the latest action, they become identical). Combining (A.1) and (A.2), we get

$$g_1(d^*, \sigma(h')) - g_1(d_*, \sigma(h')) \leq g_1(d^*, \sigma(h)) - g_1(d_*, \sigma(h))$$

which contradicts Assumption 2.

Therefore,  $\sigma(h) = \sigma(h')$  if  $h \sim^K h'$ . It follows immediately that  $V(h) = V(h')$ .

*Step 2:* Assume, for any  $k \geq 2$ , that if  $h \sim^k h'$ , then  $V(h) = V(h')$  and  $\sigma(h) = \sigma(h')$ . We claim this implies that the same is true for  $k - 1$ .

Suppose  $h \sim^{k-1} h'$ , but  $\sigma(h) \neq \sigma(h')$ . Then, either  $\sigma(h) > \sigma(h')$ , or  $\sigma(h') > \sigma(h)$ . Assume the former, without loss of generality. Following the argument in Step 1, we obtain (A.1) and (A.2). But since  $(h', d_*) \sim^k (h, d_*)$ , and  $(h', d^*) \sim^k (h, d^*)$ , we again derive a contradiction with Assumption 2 using the induction assumption.  $\square$

As we argued in the text, in any equilibrium, player 1 plays only either  $c$  or 0 after any history (not only after regular histories)

**Lemma 4.** *In any equilibrium after any history, player 1 plays with positive probability only 0 or  $c$ .*

**Proof.** Assume that player 1 plays a strategy  $\pi(h)$  that puts positive probability on  $(0, c) \cup (c, 1]$ . Then, player 2's best response is  $\sigma(h) > 0$  by [Assumption 3](#). Now consider player 1's intertemporal incentive at  $h$ . For any  $x \in (0, c) \cup (c, 1]$  in the support of  $\pi(h)$ ,

$$\begin{aligned} V(h) &= (1 - \delta)g_1(x, \sigma(h)) + \delta V((h, x)) \\ &< (1 - \delta)g_1(0, \sigma(h)) + \delta V((h, x)) \\ &= (1 - \delta)g_1(0, \sigma(h)) + \delta V((h, 0)), \end{aligned}$$

where the last equality follows from [Lemma 3](#) because  $I((h, x)) = I((h, 0))$ . Therefore,  $x$  is not a best response.  $\square$

Finally, we can pin down the strategy of player 1.

**Lemma 5.**  $\pi(h) = \pi(h')$  if  $I(h) = I(h')$ .

**Proof.** By [Lemma 3](#), if  $I(h) = I(h')$  then  $\sigma(h) = \sigma(h')$ . By [Lemma 4](#), player 1 only plays 0 and  $c$ . If  $\pi(h) \neq \pi(h')$  then  $\pi(h)$  and  $\pi(h')$  can be ranked according to the first-order stochastic dominance. Therefore, when  $I(h) < K$ ,  $\sigma(h) \neq \sigma(h')$ , a contradiction.  $\square$

This finishes the proof of [Proposition 1](#).

## Appendix B. Proof of results on behavioral predictions

### B.1. Proof of [Theorem 1](#)

In this appendix we prove the general characterization of [Theorem 1](#) (the structure of all stationary PBE with sufficiently patient players).

Since [Proposition 1](#) established that equilibrium strategies depend only on the index of the history, we now write the continuation payoffs  $V(i)$  as a function of the index alone. Recall that we now denote by  $y^*(\beta)$  player 2's best response if he expects player 1 will play  $c$  with probability  $\beta$  and 0 with probability  $(1 - \beta)$ , and we denote by  $y_k$  the equilibrium strategy of player 2 as a function of the history index  $k$ .

Our first lemma and corollary establish that in state  $K$ , when player 2 observes a clean history consisting of  $K$  observations of  $c$ , the rational type of player 1 plays 0 for sure.

**Lemma 6.** *If player 1 weakly prefers action  $c$  in state  $K$ , then, in each state  $i = 0, 1, \dots, K - 1$ , we have (1) player 1 weakly prefers action  $c$ , (2)  $y_i = y_K$ , and (3)  $V(i) = V(K)$ .*

**Proof.** Since player 1 weakly prefers action  $c$  in state  $K$ ,  $V(K) = g_1(c, y_K)$ . Assume, for induction, that, for  $i = k + 1, \dots, K$ , the three properties hold. Consider  $i = k$ .

*Step 1:* We first show that  $y_k \leq y_{k+1} = y_K$ .

Suppose to the contrary that  $y_k > y_{k+1}$ . Let  $r_k(x)$  be player 1's payoff from playing  $x$  in state  $k$  once and then returning to the equilibrium strategy. These payoffs are

$$r_k(0) = (1 - \delta)g_1(0, y_k) + \delta V(0), \quad r_k(c) = (1 - \delta)g_1(c, y_k) + \delta V(K)$$

and, analogously,

$$r_{k+1}(0) = (1 - \delta)g_1(0, y_{k+1}) + \delta V(0), \quad r_{k+1}(c) = (1 - \delta)g_1(c, y_{k+1}) + \delta V(K).$$

Note that  $y_k > 0$  by the assumption  $y_k > y_{k+1}$ . Thus  $r_k(0) - r_k(c) \leq 0$  and

$$\begin{aligned} & [r_{k+1}(0) - r_{k+1}(c)] - \underbrace{[r_k(0) - r_k(c)]}_{\leq 0} \\ &= (1 - \delta)\{[g_1(0, y_{k+1}) - g_1(c, y_{k+1})] - [g_1(0, y_k) - g_1(c, y_k)]\} < 0, \end{aligned}$$

where the inequality follows from [Assumption 2](#) and  $y_k > y_{k+1}$ . This requires  $r_{k+1}(0) - r_{k+1}(c) < 0$ .

But then  $\beta_{k+1} = 1$  which contradicts  $y_k > y_{k+1}$  (because  $y_k$  is never higher than  $y^*(c) = y^*(1) = y_{k+1}$ ).

*Step 2:* We finish the induction.

We either have  $\beta_k = 1$  or  $\beta_k < 1$ . If  $\beta_k = 1$ , then  $y_k = y_{k+1}$  because  $\beta_k = 1$  implies  $y_k = y^*(c)$ . Step 1 implies  $y_{k+1} \geq y_k$  and  $y_j \leq y^*(c)$  for any  $j$ . Properties (1) and (3) hold in this case as well.

Now, suppose  $\beta_k < 1$  and  $y_k < y_{k+1}$ , then from calculations in Step 1,

$$[r_{k+1}(0) - r_{k+1}(c)] - [r_k(0) - r_k(c)] > 0. \quad (\text{B.1})$$

But,  $\beta_k < 1$  requires  $r_k(0) \geq r_k(c)$  (0 is a best response in state  $k$ ). To satisfy inequality (B.1) we require  $r_{k+1}(0) > r_{k+1}(c)$ , i.e., player 1 strictly prefers action 0 in state  $k + 1$ , contradicting the induction hypothesis.

Finishing up, we have shown  $y_k = y_{k+1}$ . Using calculations in step 1, it yields  $r_k(0) - r_k(c) = r_{k+1}(0) - r_{k+1}(c)$ . In turn, this implies that player 1 weakly prefers action  $c$  in state  $k$  and obviously property (3) holds as well.  $\square$

**Corollary 1.** *Player 1 strictly prefers action 0 in state  $K$  and  $\beta_K = 0$ .*

**Proof.** If player 1 weakly prefers action  $c$ , then player 2's strategies do not depend on the history of the game, according to the previous lemma (and since  $\mu_K \geq \mu^*$ ,  $y > 0$  in every period). As a result, player 1 would strictly prefer action 0, a contradiction.  $\square$

The next step is to show that  $y_k$  is 0 or  $y_k - y_{k-1} > 0$  and analogously the payoffs are weakly increasing in  $k$ .

**Lemma 7.** *Player 2's strategy  $\{y_i\}_{i=0}^K$  and player 1's payoff  $\{V(i)\}_{i=0}^K$  are weakly increasing in  $i$ . Moreover, if for some  $j < K$ ,  $y_{j+1} > 0$ , then they are strictly increasing for all  $i \geq j$ . As a result, for any  $i < K$ ,  $y_i < y^*(c)$  and  $\beta_i < 1$ .*

**Proof.** We first show that  $y_{K-1} < y_K$ .

There are two possibilities:  $y_{K-1} = 0$  and  $y_{K-1} > 0$ . In the first case, immediately  $y_{K-1} < y_K$  since  $y_K \geq y^*(\mu^*)$ . Also, immediately  $V(K) > V(K-1)$  (since  $y_{K-1} = 0 \Leftrightarrow \beta_{K-1} = 0$  and we have  $y_K > 0$  and  $\beta_K = 0$ ).

Now consider  $y_{K-1} > 0 \Leftrightarrow \beta_{K-1} > 0$ .

Since player 1 strictly prefers action 0 in state  $K$  and weakly prefers  $c$  in state  $K-1$ , we have

$$(1-\delta)g_1(0, y_K) + \delta V(0) > (1-\delta)g_1(c, y_K) + \delta V(K),$$

$$(1-\delta)g_1(c, y_{K-1}) + \delta V(K) \geq (1-\delta)g_1(0, y_{K-1}) + \delta V(0).$$

Summing up the two inequalities, we have  $g_1(0, y_K) - g_1(c, y_K) > g_1(0, y_{K-1}) - g_1(c, y_{K-1})$ . By [Assumption 2](#), we have  $y_K > y_{K-1}$ , as claimed. Moreover,

$$V(K) > (1-\delta)g_1(c, y_K) + \delta V(K) \geq (1-\delta)g_1(c, y_{K-1}) + \delta V(K) = V(K-1),$$

which establishes the claim for  $K$  and  $K-1$ .

Now consider any  $i$  and suppose  $y_i = 0$  for some  $i$ . Let  $i^* < K$  be the highest such  $i$ . We claim that for all  $i < i^*$ ,  $y_i$  is zero as well. Suppose not, then there are two states,  $i$  and  $i+1$ , such that  $\beta_i > 0$ ,  $\beta_{i+1} = 0$ , and  $i+1 < K$ . Since  $\beta_{i+1} = 0$ ,  $y_{i+1} = 0$ , and hence,  $V(i+1) = V(0)$  and the incentive compatibility constraint is

$$(1-\delta)g_1(c, y_i) + \delta V(i+1) \geq (1-\delta)g_1(0, y_i) + \delta V(0),$$

which cannot be satisfied (if player 1 plays 0 for sure next period and this period  $y > 0$ , he prefers to speed up playing 0).

Now suppose that  $y_i > 0$  and  $y_i - y_{i-1}$  is not strictly positive. Let  $i^*$  be the largest state in which  $y_{i^*-1} \geq y_{i^*} > 0$ . By the result for  $K$  and  $K-1$ , we know that  $1 \leq i^* < K$ . It must also be the case that  $\beta_{i^*} < 1$ . Otherwise,  $y_{i^*} \geq y_{i^*+1}$ , contradicting the definition of  $i^*$ . Since  $\beta_{i^*} > 0$  and  $\beta_{i^*-1} > 0$ ,

$$\begin{aligned} V(i^*-1) &= (1-\delta)[g_1(c, y_{i^*-1}) + \delta g_1(c, y_{i^*}) + \dots + \delta^{K-i^*-1} g_1(c, y_{K-2})] \\ &\quad + \delta^{K-i^*} V(K-1) \\ &\geq (1-\delta)g_1(0, y_{i^*-1}) + \delta V(0) \end{aligned}$$

and

$$\begin{aligned} V(i^*) &= (1-\delta)[g_1(c, y_{i^*}) + \delta g_1(c, y_{i^*+1}) + \dots + \delta^{K-i^*-1} g_1(c, y_{K-1})] + \delta^{K-i^*} V(K) \\ &= (1-\delta)g_1(0, y_{i^*}) + \delta V(0). \end{aligned}$$

The last equality follows because  $0 < \beta_{i^*} < 1$ .

Subtracting the second expression from the first, and rearranging terms, we have,

$$\begin{aligned} \sum_{k=i^*+1}^{K-1} \delta^{k-i^*} [g_1(c, y_{k-1}) - g_1(c, y_k)] + \delta^{K-i^*} [V(K-1) - V(K)] \\ \geq [g_1(0, y_{i^*-1}) - g_1(0, y_{i^*})] - [g_1(c, y_{i^*-1}) - g_1(c, y_{i^*})]. \end{aligned}$$

For each  $k > i^*$ , we have  $y_{k-1} \leq y_k$ , and, hence,  $g_1(c, y_{k-1}) - g_1(c, y_k) \leq 0$ . Furthermore,  $V(K-1) < V(K)$  as we have shown above. Therefore, from the above inequality,

$$g_1(0, y_{i^*-1}) - g_1(c, y_{i^*-1}) < g_1(0, y_{i^*}) - g_1(c, y_{i^*}).$$

By [Assumption 2](#),  $y_{i^*-1} < y_{i^*}$ , contradicting the definition of  $i^*$ . So it must be that  $y_i > y_{i-1}$  whenever  $y_i > 0$ .

When  $y_i$  is strictly increasing,  $V(i) = (1-\delta)g_1(0, y_i) + \delta V(0)$  is also strictly increasing. Since  $y_K \leq y^*(c)$ , it must be that for all  $k$ ,  $y_k < y^*(c)$  and hence  $\beta_k < 1$ .  $\square$

So, we have established that in any stationary PBE, in state  $K$  player 1 plays 0, and in all other states he plays 0 with positive probability. The next lemma shows that he must also play  $c$  with positive probability in all states  $i < K$ .

**Lemma 8.** *If  $\delta > \bar{\delta}$ , then  $\beta_i > 0$  for each  $i < K$ . In words, player 1 plays the commitment action  $c$  with positive probability if  $i < K$ .*

**Proof.** Assume to the contrary that the statement is not true, i.e., player 1 plays 0 with probability 1 in some states lower than  $K$ . Let  $i^*$  be the smallest such state at which player 1 plays 0 with probability 1.

We first show  $i^* = 0$ . Suppose instead  $i^* > 0$ . By definition,  $i^* < K$ . Then player 2 plays 0 at  $i^*$  because his belief is  $\mu_{i^*} = 0$ . Player 1 payoffs at states  $i = 0, 1, \dots, i^* - 1$  and  $i^*$  are

$$\begin{aligned} V(i) &= (1 - \delta)g_1(c, y_i) + \delta V(i + 1), \quad i = 0, 1, \dots, i^* - 1, \\ V(i^*) &= (1 - \delta)g_1(0, 0) + \delta V(0). \end{aligned}$$

Note that

$$\begin{aligned} V(i^* - 1) &= (1 - \delta)g_1(c, y_{i^*-1}) + \delta V(i^*) \\ &< (1 - \delta)g_1(0, y_{i^*-1}) + (1 - \delta)\delta g_1(0, 0) + \delta^2 V(0) \\ &< (1 - \delta)g_1(0, y_{i^*-1}) + (1 - \delta)\delta g_1(0, y_0) + \delta^2 V(0) \\ &\leq (1 - \delta)g_1(0, y_{i^*-1}) + \delta V(0), \end{aligned}$$

where the first and second inequalities follow from the monotonicity of  $g_1$  given  $y_i > 0$ , and the third inequality follows from the incentive constraints in state 0. Therefore, we conclude that player 1 should play action 0 instead of action  $c$  in state  $i^* - 1$ , contradicting  $i^* > 0$ . Therefore,  $i^* = 0$ .

Second, suppose that  $i^* = 0$ , which means  $\beta_0 = 0$  and implies  $y_0 = 0$ . Therefore,  $V(0) = g_1(0, 0)$ .

Case 1: Suppose  $y_K = y^*(c)$ .

Note that if  $\delta > \frac{g_1(0, y^*(c)) - g_1(c, y^*(c))}{g_1(0, y^*(c)) - g_1(0, 0)} = \bar{\delta}$ , then

$$(1 - \delta)g_1(0, y^*(c)) + \delta g_1(0, 0) < g_1(c, y^*(c)).$$

The left-hand side is player 1's payoff if he plays 0 in state  $K$ , and the right-hand side is his payoff if he plays  $c$  forever. Therefore, player 1 has a unique best response at state  $K$  – to play  $c$ .

Case 2: Suppose  $y_K < y^*(c)$ .

This implies that there exists  $t$  such that  $P(t) > 0$ , player 1 plays  $c$  in periods  $(t - 1, \dots, t - K)$  with positive probability, and at  $t$ , if the index is  $K$ , player 1 plays 0 with positive probability. If  $\beta_0 = 0$ , as we assumed, that can happen only if player 1 plays  $c$  with positive probabilities in periods  $0, \dots, K - 1$  and then plays 0 with positive probability.

Consider history  $h_c^n = (c, c, \dots, c) \in H^n$ , for  $n < K$ , which has index  $C^n$ . This is a history in period  $t = n < K$  in which player 1 has played  $c$  in all the periods so far. We claim that if  $\beta_0 = 0$ , then in period 0 player 1 plays 0 for sure, and, hence, we have a contradiction – a history with index  $K$  is never reached by a type  $r$  player 1.

We show this claim by induction. First, consider a history with index  $C^{K-1}$ . Since player 1 plays  $c$  with positive probability after that history, his IC constraint is



$$(1 - \delta)g_1(c, \sigma(C^{K-1})) + \delta V(K) \geq (1 - \delta)g_1(0, \sigma(C^{K-1})) + \delta V(0).$$

We combine it with the IC constraint in state  $K - 1$  (which is a history for  $t > K$ ). Since, as we have shown above, in state  $k < K$  player 1 plays 0 with positive probability,

$$(1 - \delta)g_1(0, y_{K-1}) + \delta V(0) \geq (1 - \delta)g_1(c, y_{K-1}) + \delta V(K).$$

Adding these two IC constraints together, and re-arranging terms, we obtain

$$g_1(0, y_{K-1}) - g_1(c, y_{K-1}) \geq g_1(0, \sigma(C^{K-1})) - g_1(c, \sigma(C^{K-1})).$$

Using [Assumption 2](#), this implies

$$y_{K-1} \geq \sigma(C^{K-1}), \quad V(K-1) \geq V(C^{K-1}).$$

We proceed with induction. Suppose that for all  $\{n, n+1, \dots, K-1\}$  we have

$$y_n \geq \sigma(C^n), \quad V(n) \geq V(C^n).$$

Then the IC constraints in states  $C^{n-1}$  and  $k = n - 1$  are (recall that  $\beta_n < 1$  and we supposed  $\beta_{C^n} > 0$ )

$$(1 - \delta)g_1(c, \sigma(C^{n-1})) + \delta V(C^n) \geq (1 - \delta)g_1(0, \sigma(C^{n-1})) + \delta V(0),$$

$$(1 - \delta)g_1(0, y_{n-1}) + \delta V(0) \geq (1 - \delta)g_1(c, y_{n-1}) + \delta V(n),$$

where we used the result that even for  $t \in \{1, \dots, K-1\}$ , if player 1 plays any action other than  $c$ , the continuation equilibrium depends only on the commitment index of the history, which after the first non- $c$  action, is 0.

Adding these two IC constraints together, and rearranging terms, we obtain

$$g_1(0, y_{n-1}) - g_1(c, y_{n-1}) \geq g_1(0, \sigma(C^{n-1})) - g_1(c, \sigma(C^{n-1})),$$

and, again by [Assumption 2](#), this implies that

$$y_{n-1} \geq \sigma(C^{n-1}), \quad V(n-1) \geq V(C^{n-1}).$$

Going all the way to  $n = 1$ , the next iteration establishes a bound on the strategy of player 1 at the beginning of the game ( $t = 0$ ): he plays  $c$  with probability at most  $\beta_0$ . But that leads to a contradiction; we cannot have  $\beta_0 = 0$  and  $y_K < y^*(c)$ , since then the only type ever reaching history with index  $K$  would be the commitment type, which would imply  $\mu_K = 1$  and  $y_K = y^*(c)$ .  $\square$

**Corollary 2.**  $\{y_i\}_{i=0}^K$  and  $\{\beta_i\}_{i=0}^{K-1}$  are strictly increasing and  $0 < \beta_i < 1$ ,  $i = 0, 1, \dots, K-1$ .

**Proof.** We have shown that  $\beta_i > 0$  for all  $i \in \{0, \dots, K-1\}$ . Hence  $y_i > 0$  and, by [Lemma 7](#),  $\{y_i\}_{i=0}^K$  is a strictly increasing sequence. Since  $\{y_i\}_{i=0}^K$  is strictly increasing,  $\{\beta_i\}_{i=0}^{K-1}$  must be strictly increasing. Since  $y_{K-1} < y_K$ , we have  $\beta_{K-1} < 1$ .  $\beta_i > 0$  is shown in [Lemma 8](#).  $\square$

**Corollary 3.**  $\{V(i)\}_{i=0}^K$  is strictly increasing.

**Proof.** We have already shown that  $V(K-1) < V(K)$  in [Lemma 7](#). For each  $i = 0, 1, \dots, K-1$ , we have  $0 < \beta_i < 1$  and hence

$$V(i) = (1 - \delta)g_1(0, y_i) + \delta V(0).$$

Since  $y_i$  is strictly increasing,  $V(i)$  is strictly increasing.  $\square$

### B.2. Proof of Proposition 2

This result follows from Theorem 1. In state  $k \leq K - 1$ , player 1 is indifferent between *playing 0 immediately, returning to state 0, and staying there by playing 0 repeatedly; or playing  $c$  once, moving to state  $k + 1$ , playing 0 there, and returning to state 0*. The indifference condition is

$$(1 - \delta)g_1(0, y_k) + \delta g_1(0, y_0) = (1 - \delta)g_1(c, y_k) + \delta(1 - \delta)g_1(0, y_{k+1}) + \delta^2 g_1(0, y_0),$$

which simplifies to Eq. (2) in the main text. Eq. (3) in the main text is immediate from the Bayes' rule.

### B.3. Initial $K$ periods

In the initial  $K$  periods of the game, player 2 knows the calendar time simply from the length of the history. As a result, he (and hence player 1) can play differently in these initial periods than later in the game. We have already established in Proposition 1 that, if any action  $x \neq c$  is observed in the initial periods, the equilibrium depends only on the commitment index of the history.

So, to finish characterization of the equilibrium, we need to pin down  $\sigma(C^k)$  and  $\beta(C^k)$  for  $k \in \{0, \dots, K - 1\}$  (with  $C^0 \equiv \emptyset$  being the empty history). First of all, a stationary PBE always exists. It is described by a finite collection of  $\beta_0, \dots, \beta_K, \beta(C^0), \dots, \beta(C^{K-1})$ ,  $\mu_K$ , and  $\{\mu(c|C^k)\}_{k=1}^{K-1}$ , together with the corresponding unique player 2 best responses – since the payoffs are continuous and the action space is compact, standard arguments guarantee existence. For a general  $P(t)$ , equilibrium strategies  $\sigma(C^k)$  and  $\beta(C^k)$  can be computed as a fixed point of the best-response conditions and Bayes' rule equations, noting that they depend on  $\mu_K$  via  $V(K)$  and, in turn, if  $P(t) > 0$  for  $t \in \{K, 2K - 1\}$ ,  $\mu_K$  depends on  $\beta(C^k)$ .

Finally, in the end of proof of Theorem 1 we have also established that for all  $k \in \{0, \dots, K - 1\}$ , if in equilibrium  $\beta(C^k) > 0$ ,

$$y_k \geq \sigma(C^k), \quad V(k) \geq V(C^{k-1}),$$

and, if  $\beta(C^k) > 0$ , then  $\sigma(C^{k+1}) > \sigma(C^k)$ . The inequality  $y_k \geq \sigma(C^k)$  additionally implies that

$$\beta_k > \beta(C^K),$$

since at history  $C^k$  player 2 believes that player 1 will play  $c$  with probability at least  $\mu^* + (1 - \mu^*)\beta(C^K)$ , and his response is weakly lower than that to  $\beta_k$ . Note that this last bound holds even if  $\beta(C^k) = 0$ , which can be the case for some  $\mu^*$ .

## Appendix C. Proof of payoff results

### C.1. Derivation of belief updating

Let  $\Omega = \Theta \times X^\infty$  be the set of all possible outcomes. For example, a state  $(\theta, x^0, x^1, \dots) \in \Omega$  describes a situation where player 1's type is  $\theta$ , and player 1's period  $t \geq 0$  play is  $x^t$ . We consider the  $\sigma$ -algebra on  $\Omega$  generated by  $(t + 1)$ -dimensional calendar sets  $\{\theta\} \times B_0 \times B_1 \times \dots \times B_{t-1}$ , where  $\theta \in \Theta$  and  $\{B_t\}$  is a sequence of Borel measurable subsets of  $X$ . For each history  $h = (x^0, x^1, \dots, x^{t-1}) \in H^t$ , we identify  $h$  with the subset  $\{(\hat{x}^0, \hat{x}^1, \dots) \in X^\infty : \hat{x}^0 = x^0, \hat{x}^1 = x^1, \dots, \hat{x}^{t-1} = x^{t-1}\}$ . Denote this set by  $(\text{Proj})^{-1}(h)$ .

*Step 1.* We first define the probability measure  $Q$  on  $\Omega$  induced by  $\pi : H \rightarrow \Delta X$  and  $\mu^*$ .

The rational player 1's strategy  $\pi : H \rightarrow \Delta(X)$  induces a probability measure  $\Pi^t$  over measurable rectangles  $B_0 \times B_1 \times \cdots \times B_t$  of the  $(t+1)$ -dimensional space  $X^{t+1}$ ,  $t \geq 0$ , inductively as follows.

$$\begin{aligned}\Pi^0(B_0) &= \pi(\emptyset)(B_0), \\ \Pi^t(B_0 \times B_1 \times \cdots \times B_t) &= \int_{B_0 \times B_1 \times \cdots \times B_{t-1}} \pi(x^0, x^1, \dots, x^{t-1})(B_t) d\Pi^{t-1}.\end{aligned}$$

By Kolmogorov Extension Theorem, the sequence of measures  $\{\Pi^t\}$  extends uniquely to a probability measure  $\Pi$  over  $X^\infty$ . Then  $\Pi$  and  $\mu^*$  induce  $Q \in \Delta(\Omega)$  as follows. For any measurable set  $A \subset X^\infty$ .

$$Q(\{c\} \times A) = \begin{cases} \mu^*, & \text{if } (c, c, \dots) \in A, \\ 0, & \text{otherwise,} \end{cases} \quad Q(\{r\} \times A) = (1 - \mu^*)\Pi(A).$$

In words,  $Q$  is an outside observer's belief about the outcomes if player 1 plays  $\pi$ .

*Step 2.* We derive player 2's ex ante belief,  $\mu$ , over  $\Theta \times \tau^K(H)$  before entering the game.

Formally,  $\mu$  is induced by  $Q$  and  $P$  as follows. For each  $\theta \in \Theta$ , if  $A \subset \bigcup_{t=0}^{K-1} H^t \subset \tau^K(H)$ , then

$$\mu(\{\theta\} \times A) = P(t)Q(\{\theta\} \times (\text{Proj})^{-1}(A)),$$

and if  $A \subset H^K \subset \tau^K(H)$ , then

$$\mu(\{\theta\} \times A) = \lim_{n \rightarrow \infty} \sum_{t=K}^{K+n} P(t)Q(\{\theta\} \times X^{t-K} \times (\text{Proj})^{-1}(\tau^K(A))).$$

*Step 3.* After player 2 enters the game and observes a truncated history,  $h \in \tau^K(H) = \bigcup_{t=0}^K H^t$ , he updates his belief on the type of player 1. Let's denote this posterior belief by  $\mu(\theta|h)$ . We take  $\mu(\theta|h)$  as a version of the conditional probability  $\mu(\{\theta\} \times \tau^K(H) | \Theta \times \{h\})$  such that  $\mu(c|h) = 0$  if  $\mu((c, h)) = 0$ .

**Remark 3.** In parts of the paper we use the special case of  $P$  being the improper uniform prior. In computing beliefs in this case, only Step 2 needs to be refined. For each  $\theta \in \Theta$ , if  $A \subset \bigcup_{t=0}^{K-1} H^t$ , then

$$\mu(\{\theta\} \times A) = \frac{1}{K}Q(\{\theta\} \times (\text{Proj})^{-1}(A)).$$

For  $A \subset H^K \subset \tau^K(H)$ , define

$$\mu(\{\theta\} \times A) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=K}^{K+n} Q(\{\theta\} \times X^{t-K} \times (\text{Proj})^{-1}(\tau^K(A)))$$

if the limit exists (and  $\mu$  is arbitrary otherwise).<sup>12</sup>

<sup>12</sup> The existence is not guaranteed for an arbitrary strategy  $\pi$ . Alternatively, we can define  $\mu$  from the ergodic limit of the Markov process induced by  $\Pi$  on  $H^K$  with the initial measure  $\Pi^{K-1}$ . The ergodic limit need not exist because  $H^K$  is a continuum. However, we show that for equilibrium  $\pi$  the limit exists and coincides with the unique invariant measure.

### C.2. Proof of Lemma 1

**Proof of Lemma 1.** Assume instead  $\mu_K \leq 1 - \eta$ . In Theorem 1 we have established that in any stationary PBE,  $\beta_k < \mu_K$  and  $\beta(C^k) < \beta_k$  for the initial histories (see Appendix B.3).

Hence, the probability of observing a history with index  $K$  in period  $t \geq K$ , conditional on player 1 being the  $r$  type, is bounded above by

$$\prod_{k=0}^{K-1} \beta_k < \mu_K^K \leq (1 - \eta)^K.$$

We now use the Bayes' rule to show that this contradicts  $\mu_K \leq 1 - \eta$ , for high enough  $K$ . Write  $C^K$  as the history  $(c, c, \dots, c) \in H^K$  (the history with index  $K$ ). Using the notation in the derivation of  $\mu(\theta|h)$  in Appendix C.1 below,  $\Pi^t((\text{Proj})^{-1}(C^K)) < (1 - \eta)^K$  for  $t \geq K$  by our analysis above. Therefore,

$$\begin{aligned} Q(\{c\} \times X^{t-K} \times (\text{Proj})^{-1}(C^K)) &= \mu^*, \\ Q(\{r\} \times X^{t-K} \times (\text{Proj})^{-1}(C^K)) &< (1 - \mu^*)(1 - \eta)^K. \end{aligned}$$

The joint probability of player 1 being the  $c$  type and player 2 observing the  $C^K$  history, denoted  $\mu(c, C^K)$ , is

$$\mu(c, C^K) = \mu^* \sum_{t=K}^{\infty} P(t).$$

We also get

$$\mu(r|C^K) = \sum_{t=K}^{\infty} P(t) Q(\{r\} \times X^{t-K} \times (\text{Proj})^{-1}(C^K)) \leq (1 - \mu^*) \sum_{t=K}^{\infty} P(t) (1 - \eta)^K.$$

Hence,

$$\begin{aligned} \mu_K = \mu(c|C^K) &\geq \frac{\mu^* \sum_{t=K}^{\infty} P(t)}{\mu^* \sum_{t=K}^{\infty} P(t) + (1 - \mu^*) \sum_{t=K}^{\infty} P(t) (1 - \eta)^K} \\ &= \frac{\mu^*}{\mu^* + (1 - \mu^*)(1 - \eta)^K}. \end{aligned}$$

Therefore,  $\mu_K > 1 - \eta$  whenever  $K > \frac{\ln(\frac{\eta\mu^*}{1-\mu^*})}{\ln(1-\eta)} - 1$ , a contradiction.  $\square$

### Appendix D. More on payoff assumptions

In the game we study, player 1 has stronger static incentives to abuse player 2 for a higher (trusting) action of player 2. This property is captured by Assumption 2, the long-run player's payoff function is strictly submodular in the profile of pure strategies. This natural property of reputation games plays a crucial role in shaping reputation incentives and equilibrium dynamics. We now show that the flip side of Assumption 2 leads to very different equilibrium predictions.<sup>13</sup>

<sup>13</sup> In repeated prisoner's dilemma game, Cole and Kocherlakota [8] observe that supermodularity determines whether a folk theorem can be obtained via symmetric strategies.

We show that if [Assumption 2](#) is violated then “full cooperation” can be achieved in equilibrium. The strength of this result is that it does not depend on the prior belief  $\mu^*$  and record size  $K$ . It works even when  $\mu^* = 0$  and  $K = 1$ . Let  $\hat{\delta} = \frac{g_1(0,0) - g_1(c,0)}{g_1(c,y^*(c)) - g_1(c,0)}$ . From [Assumptions 1](#) and [4](#), we know  $\hat{\delta} \in (0, 1)$ .

**Assumption 2\*.**  $g_1(x, y) - g_1(x', y)$  is weakly decreasing in  $y$  for any  $x < x'$ .

**Proposition 4.** Suppose [Assumption 2](#) is replaced by [Assumption 2\\*](#) and other assumptions remain unchanged. Then for any  $\delta > \hat{\delta}$  and any  $K \geq 1$  there is an equilibrium in which  $(c, y^*(c))$  is played in every period on the equilibrium path.

**Proof.** We construct the equilibrium with two phases. The play starts with the cooperation phase (C) in which  $(c, y^*(c))$  is played. Player 2’s deviation is ignored. If player 1 deviates, the deviation phase (D) is played: player 1 plays  $x = c$  with probability  $p \in (0, 1)$  and  $x = 0$  with probability  $1 - p$ ; player 2 plays his static best response which we simply write as  $y^*(p)$ ; the play returns to phase (C) only after  $x = c$  for one period and remains at phase (D) otherwise. This strategy profile only utilizes one period memory and, hence, is well-defined for any  $K \geq 1$ .

We shall choose the values of  $p$  and  $y^*(p)$  appropriately to make the strategy profile an equilibrium. For player 1 to be indifferent in phase (D), we require  $y^*(p)$  be a solution of

$$g_1(0, y) = (1 - \delta)g_1(c, y) + \delta g_1(c, y^*(c)). \quad (\text{D.1})$$

We claim that when  $\delta > \hat{\delta}$ ,  $y = y^*(p)$  is indeed a solution to Eq. (D.1) and  $0 < y^*(p) < y^*(c)$ . The argument is as follows. By [Assumption 1](#),

$$g_1(0, y^*(c)) > (1 - \delta)g_1(c, y^*(c)) + \delta g_1(c, y^*(c)),$$

and when  $\delta > \hat{\delta}$ ,

$$g_1(0, 0) < (1 - \delta)g_1(c, 0) + \delta g_1(c, y^*(c)).$$

The claim follows directly from the intermediate value theorem and continuity of  $g_1$ . We now pick  $p \in (0, 1)$  such that  $y^*(p)$  is a best response. The existence of  $p$  follows from the intermediate value theorem and the continuity of player 2’s best response.

In phase (C), the one-step deviation condition for player 1 is

$$g_1(c, y^*(c)) \geq (1 - \delta)g_1(0, y^*(c)) + \delta g_1(0, y^*(p)). \quad (\text{D.2})$$

In view of Eq. (D.1), condition (D.2) can be rewritten as

$$g_1(0, y^*(p)) - g_1(c, y^*(p)) \geq g_1(0, y^*(c)) - g_1(c, y^*(c)),$$

which is guaranteed to hold by [Assumption 2\\*](#) and the fact that  $y^*(p) < y^*(c)$ .  $\square$

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