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Optimal choice of health and retirement in a life-cycle model

(very preliminary)

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Abstract

We examine within a life-cycle set-up the simultaneous choice of health care and retirement (together with consumption), when health care contributes to both a reduction in mortality and in morbidity. Health tends to impact on retirement via morbidity, determining the disutility of work, and through longevity, determining the need to accumulate retirement wealth. In contrast, the age of retirement drives health through changes in the value of survival and the value of morbidity reductions. We compare the allocation for a first-best setting where the individual can transfer wealth freely within a perfect annuity market with a second-best set-up with an incentive incompatible (yet actuarially fair) retirement scheme and an imperfect annuity market and show how the inefficiencies shape the health-retirement nexus. Numerical analysis illustrates the workings of our model.

Keywords: annuities, demand for health, moral hazard, life-cycle-model, optimal control, retirement, value of life.

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1 Introduction

It is well known that population ageing has significant repercussions for the funding of retirement pensions. Increasing longevity implies that more pension funding needs to be accumulated for a given age of retirement. At the same time and perhaps more importantly, declining fertility rates imply for pay-as-you-go (PAYG) funding a widening gap at aggregate level between contributions and benefits. Reform proposals range from parametric adjustments in the pension scheme to a shift from PAYG to a funded pension system (see e.g. Disney 2000), but they typically include as a corner stone an increase in the retirement age.

This brings into focus the question of health: On the one hand, if individuals are expected to work longer productively, this requires they are still in shape to do so. On the other hand, significant changes in the length of the working life may alter the individual's behaviour towards their own health. More generally, the increase in longevity as one of the factors causing the need to readjust the retirement scheme in and of itself is determined by the individual's health and its health-related behaviour. The link between retirement and health at individual level has received considerable attention in the empirical literature. However, this is not reflected in theoretical work. While there are first efforts at gaining theoretical insights (Bloom et al. 2007, Galama et al. 2008) into the relationship between health and retirement, to date this is still underexplored, and even less is known so far on the relationship between pension funding (rather than retirement itself) and health.

The relationship between health and retirement arises through (at least) four channels: (i) the positive relationship between health and longevity; (ii) the positive impact of health on productivity; (iii) the impact of work on health; and (iv) the impact of work-related income (i.e. earnings) on health. We briefly consider these effects in turn: (i) By driving increases in longevity, improvements in health trigger a need for accumulating greater retirement wealth and thus a need for an increase in the pension age. (ii) By raising productivity (and/or lowering the disutility of work) health improvements allow the individual to earn a higher income over a fixed length of working life. In and of itself the wealth effect may allow the individual to reduce its retirement age, offsetting the impact of greater longevity. However, improvements in productivity also imply a substitution effect, which would call for an increase in retirement, leaving the overall effect ambiguous. (iii) The supply of labour is likely to have a direct impact on health, the direction of which depends on the working environment. If work is physically demanding, stressful and/or dangerous, the impact of labour supply on health is negative. Thus, (an early) retirement would lead to a reduced depreciation of health (possibly but perhaps less likely even to improvements in health). On the other hand, if the supply of labour allows an individual to 'stay in shape', the impact of retirement on health may, in fact, be negative. (iv) Increases in life-time labour income coming with a later retirement allow the individual to consume greater amounts of health care, thus leading to improved health.

A large body of empirical literature has been devoted to disentangling the various strands of relationships between health and retirement. Three issues have attracted particular focus: (i) The relative importance of health versus financial incentives as determinants of retirement decisions. As of yet, the question as to what is the main driver remains unresolved with evidence pointing both at financial incentives (e.g. Bazzoli 1985; French 2005) and at health (Dwyer and Mitchell 1999; McGarry 2004) as the prime driver. Lumsdaine and Mitchell (1999) conclude in a review of the literature that financial incentives can explain about half of the observed variation of retirement rates in the US. The role of financial incentives may well depend on the particular route to retirement as, indeed, on the individual's health state in and of itself. Kerkhofs and Lindeboom (1999) find that financial incentives rather than health explain the selection into early retirement schemes but health rather than financial concerns explains the selection into disability insurance or unemployment insurance as a pathway to retirement. Banks et al. (2007) find that financial incentives are only important determinants of retirement for individuals who are in good health, whereas poor health is a good predictor of early retirement. (ii) The role of a gradual deterioration of health as opposed to health shocks as explanators of retirement: Methodologically, health shocks offer a convenient way of overcoming the problem of endogeneity bias caused by the correlation of measures of health and unobserved heterogeneity (Disney et al. 2006). They are good predictors of the onset of disability and thus of (early) retirement (Lindeboom et al. 2006), but as they are rare events, the more gradual deterioration of health must play what is perhaps the more important role in explaining retirement. (iii) The reverse impact of retirement on health: Again the evidence is inconclusive (see e.g. the literature review Dave et al. 2006).

A positive link between longevity and the number of life-years spent in good health is empirically well established by now (for a discussion and references see Bloom et al. 2007). According to the 'compression of morbidity' hypothesis the period spent in ill health towards the end of life is compressed (either in the absolute number of life-years or at least in proportion to total life-time). The health improvements over the life course underlying the increase in longevity should also imply that individuals enjoy greater productivity (or lower disability from labour supply). Bloom et al. (2007) employ this relationship within a life-cycle model with endogenous retirement to examine how increasing longevity bears on the optimal timing of retirement. They show that the income effect related to higher productivity dominates the substitution effect so that individuals respond to increases in longevity by demanding both more leisure and more consumption. Hence, while individuals tend to increase the length of their working lives (in order to accumulate greater retirement wealth) they do so less than proportionately, implying that a greater share of the lifetime gained is spent in retirement. Bloom et al. (2007) consider a set-up without public pension scheme. Individuals are able to spread their wealth over their life-course by transacting on a perfect annuities market. While the conclusions thus emerge for a first-best laissez-faire economy, they suggest the following adjustments of public pension schemes in response to increases in longevity: On the one hand, the retirement age should be increased (but less than proportionately), a recommendation broadly in line with reform proposals; on the other hand, benefit rates should be increased rather than reduced (in order to allow a better spread of life-time consumption), a recommendation which stands in contrast to most reform proposals.[1] This notwithstanding, their model does not explicitly analyse the role of pension funding for retirement. Furthermore, it assumes that longevity and, by implication, health are entirely exogenous and cannot be influenced by individual behaviour.

The endogeneity of health is addressed by Galama et al. (2008) who extend the Grossman (1972) model of life-cycle demand for health to allow individual choices of both retirement and health expenditure (besides consumption). Health expenditure contributes towards greater productivity and provides a direct benefit of consumption but does not raise longevity. Indeed, the life span is exogenous and fixed throughout the analysis. Galama et al. (2008) derive a number of different patterns of health, health expenditure, and consumption depending on the (exogenous) onset of retirement. At the point of retirement, health loses its 'productive' value, implying that the desired level of the health stock decreases discontinuously. Hence, according to one scenario, individuals may begin to invest in health at some point prior to retirement in order to maintain their stock of health up to the point of retiring. Post-retirement they forego health investments up to the point where health has depreciated to the lower optimal level at which (re-)investments become necessary in order to maintain the consumption value of health. In a second stage of analysis the authors solve numerically for the optimal retirement age. These simulations show for instance, that workers who earn a higher base-line wage tend to (re-)invest earlier in their health and tend to retire later. Thus, in their model, and in contrast to Bloom et al. (2007), the substitution effect in the consumption-leisure trade-off is dominant. In contrast, higher levels of initial health induce earlier retirement by a pure income effect, whereas, surprisingly perhaps, a greater deterioration of health induces later retirement (a negative income effect). Finally, Galama et al. (2008) study the effects of a retirement scheme consisting of an exogenous base-pension and the individual's contributions to a pension plan.[2] Pension funds are accumulated from an earnings-tax, and adjusted for the age of retirement. The authors find that the retirement age decreases both in the base-pension (a pure income effect) and in the tax rate (the substitution effect dominating the income effect).

Noting that the papers by Bloom et al. (2007) and by Galama et al. (2008) are complementary to each other but that each of them is omitting an important aspect of the health-retirement-longevity nexus, we propose a model allowing for an analysis that embraces the longevity and productivity effects of health (similar to Bloom et al. 2007) but at the same time endogenises the individual's choice of health care (similar to Galama et al. 2008). Thus, we obtain a richer model that captures some of the feedback of longevity on retirement (as in Bloom et al. 2007) but, at the same time, allows us to analyse how retirement age and, indeed, the underlying social security scheme affect the choice of health care. This latter aspect relates our work to Davies and Kuhn (1992) and Philipson and

[1] Of course, the need for lower benefits (cum greater contribution rates) may not arise from the increase in longevity but rather from the reduction in fertility rates. As Bloom et al. (2007) only focus on the increase in longevity - and only at individual level - the fertility related component of population ageing is missing from their model.

[2] Note that the model is not closed as base-pensions do not have to be financed by the individual.

Becker (1998) who study the moral hazard effect of social security where a guaranteed pension stream induces individuals to invest excessively in longevity.[3] In contrast to these papers, however, we take additional account of retirement-related moral hazard, where an incentive incompatible pension scheme typically induces early retirement (for an overview see e.g. Fenge and Pestieau 2005: chapter 3). Our analysis proceeds by examining a first-best allocation where the individual can transfer wealth freely through perfect annuities, the return of which adjusts to individual health investments and comparing it in turn to the second-best allocation in the presence of an incentive incompatible (yet actuarially fair) retirement scheme and/or an imperfect annuity market, where the return is exogenous from the individual's perspective.

We show inter alia that whereas (pre-retirement) health investments and retirement age are complementary in the first-best this need not be the case in the second-best. Furthermore, we show that the health-related moral hazard effects depend crucially on both the pension scheme and the degree of annuity imperfection. For a given retirement age the pension system tends to increase the incentive to invest in health over the whole life-course corresponding to the longevity-related moral hazard. However, this is not the case in the presence of an imperfect annuity market (as is realistic). Moreover, to the extent that the pension system crowds out annuity wealth, this reduces the incentive to invest in excess longevity and may even turn around the incentive if the pension scheme leads to the accumulation of private debt. We then study the impact of health investments on the age of retirement, which, in the presence of imperfections, is quite varied. In our analysis we rely on the value of health, an extension, to include morbidity, of the familiar value of life, which allows us to trace out the impact of the retirement scheme and the annuity market on the demand for health care. Lack of tractability constrains our analytics to partial impacts. Although these are insightful, we complement our analysis by numerical simulations, which help to identify life-cycle-related impacts of the pension scheme and inefficiencies in the annuity market.

The remainder of the paper is organised as follows. Section 2 deals with the first-best allocation, starting with a presentation of the model (section 2.1) followed by a technical derivation of the first-order conditions (section 2.2). Section 2.3 then proceeds to derive the value of health and uses it to interpret the first-order condition and examine the relationship between health and retirement. The section concludes with a presentation of the dynamics of consumption and health care (section 2.4). Section 3 deals with the second-best allocation, introducing the incentive incompatible pension scheme and our notion of imperfect annuities (section 3.1). After a technical derivation of the first-order conditions in section 3.2, we proceed again to derive the value of health and use it in interpreting the first-order conditions as well as the inter-relationship between health and retirement. We further show how it is shaped by the inefficiencies (section 3.3). Section 3.4 presents the

[3] Cremer et al. (2006) analyse a different form of 'moral hazard' regarding the individual choice of health care: By improving health, individuals are able to diminish their disutility of working. Cremer et al. (2006) then study how retirement policies can be used to affect redistribution towards individuals with poorer health endowments. Our model embraces both elements of moral hazard: By investing in health, individuals affect both longevity and disutility from work.

dynamics of consumption and health. Section 4 provides a number of numerical results and section 5 concludes.

2 First-best allocation

2.1 The model

Consider an individual who chooses age-specific consumption, $c(t)$, health care, $h(t)$, and the age of retirement, τ, to maximise lifetime utility. The life-course falls into two distinct periods, separated by τ: working life and time in retirement. Similar to Bloom et al. (2007) we assume that in each period the individual enjoys utility from consumption $u(c(t))$. During her working life the individual suffers a disutility of $\nu\left(S(t)\right)$, where the age-specific survival probability $S(t)$ proxies for the current state of the individual's health. The period utility from consumption is concave: $u_c > 0$, $u_{cc} \leq 0$. In addition, we assume $u(0) \geq 0$ and $u_c(0) = +\infty$.

The survival probability evolves according to the age specific mortality rate, $\mu(t, h(t))$, which depends on the current age and the health investment. The corresponding state equation is:

$$\dot{S}(t) = -\mu(t, h(t))S(t), \quad S(t_0) = 1, \tag{1}$$

where t_0 indicates the birth date. We assume that the mortality rate $\mu(t, h(t))$ satisfies

$$\begin{aligned} \mu(t, h(t)) &\in (0, \tilde{\mu}(t)] \ (\forall t); \quad \mu(t, 0) = \tilde{\mu}(t),\ \mu(t, \infty) \geq 0 \ (\forall t) \\ \mu_h(\cdot) &< 0,\ \mu_{hh}(\cdot) > 0; \quad \mu_h(t, 0) = -\infty,\ \mu_h(t, \infty) = 0 \ (\forall t), \end{aligned}$$

where $\tilde{\mu}(t)$ is the "natural" mortality rate resulting without any health care. Hence, by purchasing health care and lowering the instantaneous mortality rate an individual can improve its survival prospects but only so at diminishing returns. At this point we note that we can (re-)interpret $S(t)$ as the individual's age-specific stock of health, where (1) then describes the development of the stock of health over the life-course. Similar to the models by e.g. Grossman (1972), Ehrlich and Chuma (1990) or Galama et al. (2008) the health stock is then subject to depreciation. Health care allows to reduce the depreciation but (in contrast to the models a la Grossman 1972) not to reverse it. We realise that the assumption of a one-to-one relationship between health stock and survivorship and the notion that health cannot be recuperated are probably overly simplistic. In a more realistic setting one would expect mortality rates to depend on a stock of health subject to depreciation and health investments. Furthermore, one may envisage improvements in the health stock after periods of illness. Nevertheless, we believe that in addition to being expedient in terms of modelling economy (one state instead of two) our approach has the conceptual advantage that it does not give rise to a positive relationship between the stock of health and expenditure as in Grossman (1972), which is at odds with empirical evidence. In our model the demand for health depends on survival prospects and, thus, on the stock of health, but it is importantly shaped by the impact of health on mortality (=depreciation),

$\mu_h(\cdot)$. Indeed, as we will show below (see sections 2.4, 3.4 and in particular section 4) our model is compatible with an inversely U-shaped age-profile of health expenditure despite an ongoing reduction of the health stock.[4] Regarding the irreversibility of health decline, we should stress that our model is written from an ex-ante rather than an ex-post perspective. We do not have in mind the treatment of certain specific illnesses, leading to a recovery and, thus, to an increase in the health stock, but rather more broadly the incentives to shape the long-run development of health and mortality. Here, the long-run depreciation of health is documented not only by declining survivorship (in fact, typically at increasing rates) but also by the gradual accumulation of health deficits (for an overview see Strulik 2010: section 6).

We assume that the stock of health $S(t)$ at age t diminishes the disutility of work $\nu(S(t))$ at this age but at a decreasing rate, i.e. $\nu_S < 0$, $\nu_{SS} \geq 0$. Note that the positive relationship between the stock of health and survival is broadly in line with the compression of morbidity hypothesis. Bloom et al. (2007) employ this argument when assuming that disutility of work increases with age and falls with the life expectancy at birth. Our model captures both aspect - poorer health with the progression of age and the compression of morbidity coming along with an increase in life expectancy - with one and the same variable. Furthermore, in our model the health is subject to individual choice, whereas in Bloom et al. (2007) it is entirely exogenous.

Consumption, health and savings are financed out of earnings $w(t)$ during the first (working) phase of life. We assume that earnings bounded and exogenously given. In order to sustain consumption and health during the second (retirement) phase when no earnings accrue, the individual invests in annuities. We follow Yaari (1965) in considering a set-up in which individuals can fully annuitise their wealth by trading actuarial notes at the gross interest $r + \mu(t, h(t))$. It is important to note at this point that from a first-best perspective the return on annuities should be fully responsive to the individual's health investments. Although most real world markets for annuities or life-insurance are likely to fail this property, we do maintain this assumption for the purpose of establishing here a true first-best solution and compare this to an imperfect 'real-world' annuity market (see section 3.1 below).

We therefore end up with the following two stage budget equation where we assume that the individual starts and ends with zero assets.

$$\begin{aligned}
\dot{A}(t) &= w(t) - c(t) - h(t) + (r + \mu(t, h(t)))A(t), \quad A(t_0) = 0 \quad \text{for } t < \tau \\
\dot{A}(t) &= -c(t) - h(t) + (r + \mu(t, h(t)))A(t), \quad A(T) = 0 \quad \text{for } t > \tau.
\end{aligned}$$

Here, T denotes a point in time at which the individual has certainly perished. Applying a discount rate ρ, we can formulate the following dynamic optimisation problem with state variables $A(t), S(t)$ and control variables $c(t), h(t), \tau$:

[4] Galama et al. (2008) provide a different rationale as to why health expenditure falls with the stock of health. In their model, following Wolfe (1985), an individual does not demand any health care up to the point that natural depreciation has eroded a stock of health, which initially is large, to the level at which positive health investments become profitable.

$$\max_{c(t),h(t),\tau} \int_{t_0}^{\tau} e^{-\rho t} S(t)(u(c(t)) - \nu(S(t)))dt + \int_{\tau}^{T} e^{-\rho t} S(t)u(c(t))dt \tag{2}$$

s.t.

$$\begin{aligned}
\dot{A}(t) &= w(t) - c(t) - h(t) + (r + \mu(t, h(t)))A(t), \quad A(t_0) = 0 \quad \text{for } t < \tau \\
\dot{A}(t) &= -c(t) - h(t) + (r + \mu(t, h(t)))A(t), \quad A(T) = 0 \quad \text{for } t > \tau \\
\dot{S}(t) &= -\mu(t, h(t))S(t), \quad S(t_0) = 1
\end{aligned}$$

Here, the first-integral of the objective function denotes the expected present value of the utility stream over the working life, while the second integral denotes the expected present value of the utility stream during retirement. The constraints are given by the (two part) budget equations, the movement of survivorship/health capital as well as the initial and end point conditions.

2.2 Optimality conditions (technical)

To solve the problem in (2) we apply the optimality conditions for two-stage optimal control problems as outlined in Grass et al. (2008, pp. 386). Using the current value Hamiltonians for the first and second periods

$$\begin{aligned}
\mathcal{H}^1 &= Su(c) - S\nu(S) - \lambda_S^1 \mu(h)S + \lambda_A^1(w - c - h + (r + \mu(h))A) \\
\mathcal{H}^2 &= Su(c) - \lambda_S^2 \mu(h)S + \lambda_A^2(-c - h + (r + \mu(h))A)
\end{aligned}$$

we obtain the following set of first order conditions for period $i = 1, 2$ (at an inner optimum)

$$\mathcal{H}_c^i = Su_c(c) - \lambda_A^i \overset{!}{=} 0, \tag{3}$$

$$\mathcal{H}_h^i = \mu_h(h)(A\lambda_A^i - S\lambda_S^i) - \lambda_A^i \overset{!}{=} 0, \tag{4}$$

which determine optimal consumption and health investments, respectively. For the adjoint equations $i = 1, 2$ we obtain

$$\begin{aligned}
\dot{\lambda}_S^1 &= (\rho + \mu(h))\lambda_S^1 - (u(c) - \nu(S) - S\nu_S(S)), \\
\dot{\lambda}_S^2 &= (\rho + \mu(h))\lambda_S^2 - u(c), \\
\dot{\lambda}_A^i &= (\rho - r - \mu(h))\lambda_A^i.
\end{aligned}$$

To account for the switching instant at the age of retirement, the following matching conditions for the adjoint variables

$$\begin{aligned}
\lambda_A^1(\tau) &= \lambda_A^2(\tau) =: \lambda_A^\tau \\
\lambda_S^1(\tau) &= \lambda_S^2(\tau) =: \lambda_S^\tau
\end{aligned}$$

and the Hamiltonian ($\mathcal{H}^1(\tau) = \mathcal{H}^2(\tau)$)[5]

$$\frac{\nu(S(\tau))}{u_c(c(\tau))} = w(\tau) \tag{5}$$

must hold. The latter condition determines the optimal age of retirement. The following Lemma establishes a set of sufficient conditions for the existence of a unique (and interior) age of retirement.

Lemma 1 *An interior solution to (10) exists if*

E1) $\frac{\nu(1)}{u_c(c(t_0))} < w(t_0)$

E2) $\lim_{t\to T} \nu(S(t)) = +\infty$ *or* $\frac{\nu(S(T))}{u_c(c(T))} > w(T)$

are satisfied. The resulting solution $\tau \in (t_0, T)$ *is unique if*

U1) $\rho \geq r$

U2) $w'(t) \leq 0$ *or* $w'(t) > 0$, $w''(t) \leq 0$

Proof: From the matching conditions and the necessary first order condition consumption and health expenditures are continuous at τ. The first order conditions also guarantee that both controls are continuous over the whole planning horizon. According to E1 the left hand side of the matching condition (5) is strictly greater than the right hand side at the beginning of the planning horizon. E2 implies the opposite at the end of the planning horizon. The continuity of both sides guarantees the existence of at least one $\tau \in (t_0, T)$ where both sides are equal.

U1 implies that consumption is non-decreasing over time. Thus the marginal utility $u_c(c)$ does not increase over time. The disutility of work increases over time as the survival probability is strictly decreasing according to (1). Thus the left hand side of the matching condition is increasing and convex over time. U2 implies that the wage (representing the right hand side of the matching condition) is concave over time. Putting this together the solution τ has to be unique.[6] $\square$

Before turning to an interpretation of the optimality conditions, we introduce the value of health as a convenient measure for our further analysis.

[5]The general matching conditions are more complicated as they allow for switching costs and also the fact that the first and/or second period objective might depend on the switching time (cf. Grass et al. 2008, p. 387).

[6]We should stress that the conditions are sufficient. In particular, this applies to U1, a condition that is overly restrictive.

2.3 Value of Health

We can calculate the value of health as the willingness to pay for a small reduction of the mortality rate - or equivalently the depreciation rate on the health stock - at age t. Conceptually, the value of health is identical to the value of life, as was first developed in a formal manner by Shepard and Zeckhauser (1984) (see also Rosen 1988; and Johansson 2002, Murphy and Topel 2006). Denoting by V the value function corresponding to the optimisation problem in (2), we define the value of health (VOH) as

$$\psi^i(t) := -\frac{\partial V/\partial \mu}{\partial V/\partial A} = \frac{\lambda_S^i S - \lambda_A^i A}{\lambda_A^i} = \frac{\lambda_S^i S}{\lambda_A^i} - A = \frac{\lambda_S^i}{u_c} - A. \tag{6}$$

where the last equality follows from the first-order condition (3). Integrating the adjoint equation $\dot{\lambda}_S^i$ and the budget equation, and substituting λ_S^i and A, we obtain the following expression for the VOH[7]

$$\begin{aligned}
\psi^1(t) \quad : &= \psi(t \leq \tau) = \int_t^\tau e^{-r(s-t)} \frac{S(s)}{S(t)} \frac{u\left(c\left(s\right)\right) - v\left(S\left(s\right)\right)}{u_c\left(c\left(s\right)\right)} ds \\
&+ \int_\tau^T e^{-r(s-t)} \frac{S(s)}{S(t)} \frac{u\left(c\left(s\right)\right)}{u_c\left(c\left(s\right)\right)} ds + H\left(t\right) - E\left(t\right) \\
&- \int_t^\tau e^{-r(s-t)} \frac{S(s)}{S(t)} \frac{S(s)\nu_S}{u_c\left(c\left(s\right)\right)} ds \qquad (7) \\
\psi^2(t) \quad : &= \psi(t \geq \tau) = \int_t^T e^{-r(s-t)} \frac{S(s)}{S(t)} \frac{u\left(c\left(s\right)\right)}{u_c\left(c\left(s\right)\right)} ds - E\left(t\right), \qquad (8)
\end{aligned}$$

where

$$\begin{aligned}
H\left(t\right) \quad : &= \int_t^\tau e^{-r(s-t)} \frac{S(s)}{S(t)} w\left(s\right) ds \qquad (9) \\
E\left(t\right) \quad : &= \int_t^T e^{-r(s-t)} \frac{S(s)}{S(t)} \left(c\left(s\right) + h\left(s\right)\right) ds \qquad (10)
\end{aligned}$$

denote, respectively, human wealth, i.e. income over the remaining life course, and expenditure over the remaining life course. The VOH during the working phase can then be decomposed into five components corresponding to the terms on the RHS of (7): (i) The gross value attached to the remaining working life, consisting of the (discounted) gross value of each year of working life $\frac{u-\nu}{u_c}$ summed over the individual's remaining working life. (ii) The gross value attached to the retirement phase, amounting to the (discounted) gross consumer surplus $\frac{u}{u_c}$ aggregated over the remaining life-course. (iii) Human wealth $H\left(t\right)$, and (iv) expenditure over the remaining life-course, $E\left(t\right)$. (v) The aggregate value of reductions in the disutility of work, $-\frac{S\nu_S}{u_c} > 0$, resulting from health improvements over

[7] It is easy to check that $\psi^1(\tau) = \psi^2(\tau)$. Thus, the VOH also satisfies a matching condition at the switching point.

the remaining working life. Note that (i)-(iv) correspond to the value of survival, as is typically embraced by the value of a statistical life as in the literature referenced earlier.[8] Obviously, the value of survival refers to reductions in mortality. In contrast, (v) can be understood as the value assigned to reductions in morbidity. Equation (8) denotes the VOH after entry into retirement, where the disutility of work, the value of work-related improvements in health and human wealth are no longer relevant.

Noting that $\lambda_A^i(t) = \lambda_A^i(s)\, e^{-(\rho-r)(s-t)}\frac{S(t)}{S(s)}$, combining (3) and (4) and using (6) we can express the optimality conditions for $\{c^*(t), h^*(t), \tau^*\}$ as

$$\frac{u_c(c^*(t))}{u_c(c^*(s))e^{-\rho(s-t)}} = e^{r(s-t)}, \tag{11}$$

$$-\frac{1}{\mu_h(h^*(t))} = \psi^i(t), \tag{12}$$

$$\frac{\nu(S(\tau^*))}{u_c(c^*(\tau^*))} = w(\tau^*). \tag{13}$$

The distribution of consumption over the life-cycle is determined by the familiar Euler equation (11), equating the marginal rate of intertemporal substitution with the compounded interest. Health investments are chosen such that the effective cost of reducing mortality by on unit equals the VOH. Note that while the distribution of consumption is chosen irrespective of whether the individual is in its working life or retired, this is not the case for health investments. Finally, given that the conditions in Lemma 1 are satisfied, retirement occurs at the point where the monetary disutility of working just equals the earnings. Indeed, as Lemma 1 shows, if the wage rate exceeds (falls short of) the monetary disutility it is optimal to remain in employment (to retire), a familiar optimality condition for retirement (see e.g. Bloom et al. 2007).

We conclude this section by examining the interrelationship between health and retirement. Considering the partial effect of the retirement age on health, i.e. for a given life-cycle allocation $\{c^*(t), h^*(t)\}$, we find from (12) that

$$\frac{\partial h^*(t)}{\partial \tau} = \frac{\mu_{hh}}{\mu_h^2}\frac{\partial \psi^i(t)}{\partial \tau},$$

where

$$\begin{aligned}
\frac{\partial \psi^1(t)}{\partial \tau} &= e^{-r(\tau-t)}\frac{S(\tau)}{S(t)}\left[\frac{-v(S(\tau))}{u_c(c(\tau))} - \frac{S(\tau)\nu_S}{u_c(c(\tau))}\right] + \frac{\partial H(t)}{\partial \tau} \\
&= e^{-r(\tau-t)}\frac{S(\tau)}{S(t)}\left[\frac{-v(S(\tau))}{u_c(c(\tau))} + w(\tau) - \frac{S(\tau)\nu_S}{u_c(c(\tau))}\right], \\
\frac{\partial \psi^2(t)}{\partial \tau} &= 0.
\end{aligned}$$

Thus, an increase in retirement age has a direct impact on the VOH only during the working life. Unsurprisingly, there is no direct impact during the retirement phase, as the

[8]Typically this literature does not account for retirement.

VOH is forward looking while the change of the retirement age is a thing of the past. The VOH during the working phase increases (falls) in retirement age if and only if the value of a morbidity reduction exceeds the difference between the (monetary) disutility of work and the wage rate. Indeed, for an optimal choice of retirement we then obtain

$$\frac{\partial \psi^1(t)}{\partial \tau} = -e^{-r(\tau^*-t)} \frac{S(\tau^*)}{S(t)} \frac{S(\tau^*)\nu_S}{u_c\left(c^*\left(\tau^*\right)\right)} > 0,$$

implying that retirement is complementary to health. A postponement of retirement (in the vicinity of the optimum) implies an additional benefit to health investments as these help to lower the disutility of working through an additional year, say.

Considering the partial effect of health on retirement, we find from (13) that

$$\frac{\partial \tau^*}{\partial h(t)} |t \leq \tau^* = \frac{v_S}{w' - v_S/u_c \dot{S}\left(\tau^*\right)} \frac{\partial S\left(\tau^*\right)}{\partial h(t)} = \frac{-v_S \mu_h(h(t)) S\left(\tau^*\right)}{w' - v_S/u_c \dot{S}\left(\tau^*\right)} > 0.$$

By improving the health stock at the point of retirement, health investments during the working age contribute to a lower disutility of work and, thus, provide an incentive for a postponement of retirement. We can therefore summarise

Proposition 1 *Pre-retirement health investments and the retirement age are complements in a partial sense.*

The complementarity between health during the working life and the length of the working life is intuitive. Nevertheless, we need to caution that the partial view neglects any effects that are transmitted through concomitant changes in the life-cycle paths of consumption and health, $\{c\left(t\right), h\left(t\right)\}$. To illustrate the importance of these effects, we consider two examples: (i) The impact of retirement age on health investments during the retirement phase. While the partial impact is zero, the full impact is likely to be positive: The increase in human wealth and, thus, in life-time income that comes with a postponement of retirement allows a greater scope for consumption and health investments. In as far as these are spread over the life-cycle, they also raise the VOH during the retirement phase and, thereby, imply a complementarity between retirement age and post-retirement health investments. (ii) Pre-retirement health investments have to be financed by way of a reduction in consumption (at the point of retirement). Note that this embraces both the direct expenditure $h\left(t\right)$ and an indirect cost as for an increase in longevity, additional retirement consumption needs to be accommodated. The decline in $c\left(\tau\right)$ then leads to a lower value of the disutility of work $\frac{v(S(\tau))}{u_c(c(\tau))}$, pushing further towards an increase in the retirement age. While these considerations tend to support the notion that health and retirement age are complementary, we cannot rule out circumstances in which the full effects may, indeed, reverse the relationship. This not withstanding our numerical analysis in section 4 supports the notion of complementarity.

2.4 Dynamics of consumption and health investments

From the first order condition (11) we can derive the time path of consumption (omitting time indices)

$$\dot{c}^* = \frac{u_c(c^*)}{u_{cc}(c^*)}(\rho - r),$$

which is the dynamic representation of the Euler equation. As is common, consumption increases (decreases) over the life-course if and only if the discount rate falls short of (exceeds) the interest rate. From (12) we obtain the time path of health investments

$$\begin{aligned}
\text{for } t \ \le \ \tau^*: \quad \dot{h}^* &= -\frac{\mu_{ht}(h^*)}{\mu_{hh}(h^*)} - \frac{\mu_h(h^*)}{\mu_{hh}(h^*)} \left[\begin{array}{c} r + \mu(h^*) \\ -\frac{1}{\psi^1(t)} \left(\frac{u-\nu}{u_c} + w - c^* - h^* - \frac{\nu_S S}{u_c} \right) \end{array} \right], \\
\text{for } t \ > \ \tau^*: \quad \dot{h}^* &= -\frac{\mu_{ht}(h^*)}{\mu_{hh}(h^*)} - \frac{\mu_h(h^*)}{\mu_{hh}(h^*)} \left[r + \mu(h^*) - \frac{1}{\psi^2(t)} \left(\frac{u}{u_c} - c^* - h^* \right) \right].
\end{aligned}$$

The evolution of health investments is determined by two forces: (i) The change in the marginal effectiveness of health investments with the progress of age, corresponding to the first term on the RHS of both branches of the path. Recalling that $\mu_{hh} > 0$, i.e. decreasing returns to health care at any given age, it follows that health investments tend to increase directly with age as long as the marginal effectiveness of health care increases with age, $\mu_{ht} < 0$. Typically, this tends to be true for ages up to the 60s and 70s with health expenditure having little impact at young ages before the onset of life-threatening conditions. For the highest ages health expenditure is likely to become less effective in combatting mortality so that $\mu_{ht} > 0$. (ii) The change over the life-course in the VOH. Indeed, it can be verified that the second term on the RHS equals the rate of change in the respective VOH $\frac{\dot{\psi}^i(t)}{\psi^i(t)}$. On the one hand, the VOH increases with the effective interest rate $r + \mu$; on the other hand, it falls as with each passing life year the value of this year is written off.[9] During the working life ($t \le \tau^*$), this includes the gross value of a life year, the net savings, and the value of morbidity declines, $\frac{u-\nu}{u_c} + w - c^* - h^* - \frac{\nu_S S}{u_c}$. For the retirement phase ($t > \tau^*$) the value of a life years passed excludes the disutility of working, earnings and the value of morbidity declines, $\frac{-\nu}{u_c} + w - \frac{\nu_S S}{u_c}$. Recalling that at the point of optimal retirement, τ^*, we have $\frac{-\nu}{u_c} + w = 0$, it follows that at this point the depreciation of the VOH falls discontinuously by an amount $-\frac{\nu_S S}{u_c} > 0$, corresponding to the reduction in morbidity, which is no longer valued during retirement. Thus, with the onset of retirement health investments may suddenly increase at a larger pace; decline at a smaller pace; or indeed relapse from a decline back into an increase (see Figures 2 and 6 in section 4).

[9] Recall here that $-\frac{\mu_h(h)}{\mu_{hh}(h)} > 0$.

3 The allocation with imperfect pensions and annuities

3.1 The model with imperfections

We now apply our model to study the impact on health and retirement of two distinct imperfections. (i) An imperfect pension scheme; and (ii) an imperfect annuity market. We discuss the precise nature of these imperfections in turn.

Pensions: We consider a pension scheme that is (a) fully funded (i.e. there are no transfers between different cohorts); (b) actuarially neutral; and (c) incentive incompatible. Specifically, we assume that workers pay a constant fraction of wages $\alpha \in (0,1)$ when working and receive an annual pension benefit p during their retirement. Actuarial neutrality then implies that the discounted value of the expected stream of contributions equals the discounted value of expected pension payments.

$$\alpha \int_{t_0}^{\tau} w(t)S(t)e^{r(\tau-t)}dt = p\int_{\tau}^{T} S(t)e^{-r(t-\tau)}dt.$$

Generally, the pension scheme can be structured in a number of different ways to guarantee actuarial neutrality. Both, contribution rates and pension benefits could be linked to the retirement age, τ, and survival schedule $S :=_{t\in[t_0,T]} \{S(t)\}$. A defined contribution scheme is then a scheme, where for a contribution rate, α, the pension benefit is given by a function

$$p(\alpha,\tau,S) = \frac{\alpha\int_{t_0}^{\tau} w(t)S(t)e^{r(\tau-t)}dt}{\int_{\tau}^{T} S(t)e^{-r(t-\tau)}dt},$$

depending (inter alia) on the contribution rate, the retirement age and the survival function.[10] In the following, we consider a scheme that is incentive incompatible in the sense that from the individual's perspective the pension benefit constitutes a fixed payment regardless of the retirement age that is chosen ex-post and regardless of the individual's survival schedule. From an ex-ante perspective, actuarial neutrality then requires that $p(\alpha,\overline{\tau},\overline{S})$ is calculated on the basis of the expected retirement age $\overline{\tau}$ and the expected survival schedule $\overline{S}$. Thus, we write

$$\overline{p} := p(\alpha,\overline{\tau},\overline{S}) = \frac{\alpha\int_{t_0}^{\overline{\tau}} w(t)\overline{S}(t)e^{r(\overline{\tau}-t)}dt}{\int_{\overline{\tau}}^{T} \overline{S}(t)e^{-r(t-\overline{\tau})}dt}, \tag{14}$$

with $\bar{S}(t) = \exp\left[-\int_{t_0}^{t}\bar{\mu}(s)ds\right]$. All variables with a 'bar' are then to be understood as exogenous from the individual's ex-post point of view. Nevertheless, from an ex-ante perspective, and given rational expectations, the benefit is calculated on the basis of $\overline{\tau} = \tau$ and $\overline{S}(t) = S(t)$, or equivalently $\bar{\mu}(t) = \mu(t, h(t))$. Such a pension scheme gives rise to two

[10] In Appendix A we derive the function $p(\alpha,\tau,S)$ from a dynamic formulation of the pension scheme based on the accumulation / decumulation of pension wealth if this is invested at the interest rate $r+\mu$.

forms of moral hazard. First, as the pension benefit does not depend on actual retirement, this tends to induce a sub-optimal (indeed a premature) pension entry. Retirement-related moral hazard is well-recognised both in the academic literature and in the policy debate (for a recent review see e.g. Fenge and Pestieau 2005: chapter 3). In principle, it can be tackled by appropriate adjustments to the retirement scheme, involving discounts for early retirement (see e.g. Börsch-Supan 2000, Breyer and Kifmann 2002, Berkel and Börsch-Supan 2004 for a PAYG scheme). Second, health-related moral hazard arises as individuals are able to influence their mortality through health investments but this cannot be controlled for by health-state contingent variations in the pension payment. With a guaranteed pension stream, the individual has an incentive to invest excessively into survival. Arguably, it is difficult to obtain a good - not to mention a verifiable - measure of $S(t)$, if only for the reason that it is constantly changing over the life course. Therefore, health-related moral hazard is much more difficult to control than retirement-related moral hazard.

Annuities: Individuals can also accumulate private wealth (or debt) by investing in annuities. We assume that in contrast to e.g. Hurd (1989) and Leung (1994, 2007) full availability of annuities to the individual; and in contrast to e.g. Brunner and Pech (2008) actuarially fair pricing of annuities. This notwithstanding, annuity prices are typically based on expected or average mortality $\bar{\mu}(t)$, as observed in life-tables, and not on the individual's current mortality $\mu(t, h(t))$. The individual will then take the return on annuities $r + \bar{\mu}(t)$ as given when choosing the level of health care. As Davies and Kuhn (1992) and Philipson and Becker (1998) show, this implies health-related moral hazard, where individuals tend to invest too much into longevity.

In order to study the implications of such an imperfection in the annuity market and in order to separate it out from the effects of imperfections in the pension system, we study a hypothetical annuity market, paying a return of $r + \theta\bar{\mu}(t) + (1-\theta)\mu(t, h(t))$ with $\theta \in [0, 1]$. The polar cases, $\theta = 0$ and $\theta = 1$, respectively, then correspond to a perfect and (fully) imperfect annuity market. This not withstanding, the annuity market is actuarially fair, implying that the market equilibrium obeys $\bar{\mu}(t) = \mu(t, h(t))$.

In the following, we will examine in tandem the effects of an imperfect pension system and an imperfect annuity market on individual life-cycle choices regarding health, retirement and consumption. In order to do so we establish the solution to the following dynamic optimisation problem with state variables $A(t), S(t)$ and control variables $c(t), h(t), \tau$:

$$\max_{c(t),h(t),\tau} \int_{t_0}^{\tau} e^{-\rho t} S(t)(u(c(t)) - \nu(S(t)))dt + \int_{\tau}^{T} e^{-\rho t} S(t) u(c(t))dt \tag{15}$$

s.t.

$$\begin{aligned}
\dot{A}(t) &= (1-\alpha)w(t) - c(t) - h(t) + (r + \theta\bar{\mu} + (1-\theta)\mu)A(t), \quad A(t_0) = 0 \quad \text{for } t \leq \tau \\
\dot{A}(t) &= \bar{p} - c(t) - h(t) + (r + \theta\bar{\mu} + (1-\theta)\mu)A(t), \quad A(T) = 0 \quad \text{for } t \geq \tau \\
\dot{S}(t) &= -\mu(t, h(t))S(t), \quad S(t_0) = 1
\end{aligned}$$

The imperfections bear on the asset dynamics. During the working stage of life the individual collects only net earnings $(1-\alpha)w(t)$, whereas during its retirement it receives a benefit $\bar{p}$, as given by (14). Furthermore, during both stages of life the individual faces a return to annuities of $r + \theta\bar{\mu} + (1-\theta)\mu$. All else remains as in the first-best problem (2).

Before proceeding with the analysis we should stress the following point. Our analysis is neither about the optimal design of a pension system nor about the question of whether or not a public pension system improves the allocation. Indeed, in the context of our model it would always be optimal to abolish the pension scheme, i.e. to set $\alpha \equiv 0$, even with a (fully) imperfect annuity market. The reason is that the moral hazard incentive with imperfect annuities is also underlying the public pension scheme. But then the moral hazard incentive with respect to retirement will necessarily generate a worse outcome for the public pension scheme. Thus, our analysis is strictly positive in the sense of examining the effects of these imperfections on individual behaviour.

3.2 Optimality conditions (technical)

To solve the problem in (15), again we apply the optimality conditions of two-stage optimal control models. From the current value Hamiltonians for the first and second periods

$$\begin{aligned}
\mathcal{H}^1 &= Su(c) - S\nu(S) - \lambda_S^1\mu(h)S + \lambda_A^1((1-\alpha)w - c - h + (r + \theta\bar{\mu} + (1-\theta)\mu)A) \\
\mathcal{H}^2 &= Su(c) - \lambda_S^2\mu(h)S + \lambda_A^2(\bar{p} - c - h + (r + \theta\bar{\mu} + (1-\theta)\mu)A)
\end{aligned}$$

we obtain the following set of first order conditions for period $i = 1, 2$ (for an inner optimum)

$$\begin{aligned}
\mathcal{H}_c^i &= Su_c(c) - \lambda_A^i \stackrel{!}{=} 0, && (16) \\
\mathcal{H}_h^i &= -\mu_h\big(S\lambda_S^i - A\lambda_A^i(1-\theta)\big) - \lambda_A^i \stackrel{!}{=} 0. && (17)
\end{aligned}$$

For the adjoint equations $i = 1, 2$ we obtain

$$\begin{aligned}
\dot{\lambda}_S^1 &= (\rho + \mu)\lambda_S^1 - (u(c) - \nu(S) - S\nu_S(S)), \\
\dot{\lambda}_S^2 &= (\rho + \mu)\lambda_S^2 - u(c), \\
\dot{\lambda}_A^i &= (\rho - r - \theta\bar{\mu} - (1-\theta)\mu)\lambda_A^i.
\end{aligned}$$

The following matching conditions for the adjoint variables account for the age of retirement:

$$\begin{aligned}
\lambda_A^1(\tau) &= \lambda_A^2(\tau) =: \lambda_A^\tau, \\
\lambda_S^1(\tau) &= \lambda_S^2(\tau) =: \lambda_S^\tau,
\end{aligned}$$

and $\mathcal{H}^1(\tau) = \mathcal{H}^2(\tau)$, which implies

$$\frac{\nu(S)}{u_c(c)} = (1-\alpha)w - \bar{p}. \tag{18}$$

The following Lemma establishes a set of sufficient conditions for the existence of a unique (and interior) age of retirement.

Lemma 2 *An interior solution to (18) if condition*

E1') $\frac{\nu(1)}{u_c(c(t_0))} < (1-\alpha)w(t_0)$

is satisfied. The resulting solution $\tau \in (t_0, T)$ *is unique if U1 and U2 are satisfied.*

Proof: In analogy to the problem in (2) consumption and health expenditures are continuous over the planning horizon. For the proof of existence we follow the same argument as in Lemma 1 observing that in equilibrium with $\overline{\tau} = \tau$ and setting $\tau = 0$ we have from (14) that $\overline{p} = p(\alpha, 0, \overline{S}) = 0$. Condition E2 is not necessary in this model, as it follows from (14) that $\lim_{\overline{\tau}=\tau \to T} \overline{p} = p(\alpha, \overline{\tau}, \overline{S}) = +\infty$.

Since $\frac{d\overline{p}}{d\tau}|_{\overline{\tau}=\tau} = p_{\overline{\tau}}(\alpha, \tau, \overline{S}) > 0$ it follows in analogy to Lemma 1 that U1 and U2 are sufficient for uniqueness. □

Again, we will provide interpretations for the first order conditions after having reported the value of health for this case.

3.3 Value of Health

For the problem in (15) we can derive the value of health as

$$\widehat{\psi}^i = \frac{\lambda_S^i S - (1-\theta)\lambda_A^i A}{\lambda_A^i} = \frac{\lambda_S^i}{u_c} - (1-\theta)A. \tag{19}$$

Again, integrating out and observing that in equilibrium we have $\bar{S}(s) = S(s)$ we obtain

$$\begin{aligned}
\widehat{\psi}^1(t) \; : \; &= \widehat{\psi}(t \leq \tau) = \int_t^\tau e^{-r(s-t)} \frac{S(s)}{S(t)} \frac{u(c(s)) - \nu(S(s))}{u_c(c(s))} ds \\
&+ \int_\tau^T e^{-r(s-t)} \frac{S(s)}{S(t)} \frac{u(c(s))}{u_c(c(s))} ds + (1-\theta)\left[\widehat{H}^1(t) - E(t)\right] \\
&- \int_t^\tau e^{-r(s-t)} \frac{S(s)}{S(t)} \frac{S(s)\nu_S}{u_c} ds \qquad (20) \\
\widehat{\psi}^2(t) \; : \; &= \widehat{\psi}(t \geq \tau) = \int_t^T e^{-r(s-t)} \frac{S(s)}{S(t)} \frac{u(c(s))}{u_c(c(s))} ds, \\
&+ (1-\theta)\left[\widehat{H}^2(t) - E(t)\right] \qquad (21)
\end{aligned}$$

where $E(t)$ as defined in (10) above, and where

$$\begin{aligned}
\widehat{H}^1(t) \; : \; &= \widehat{H}(t \leq \tau) = (1-\alpha)\int_t^\tau e^{-r(s-t)} \frac{S(s)}{S(t)} w(s)\, ds \\
&+ \bar{p} \int_\tau^T e^{-r(s-t)} \frac{S(s)}{S(t)} ds \qquad (22) \\
\widehat{H}^2(t) \; : \; &= \widehat{H}(t \geq \tau) = \bar{p} \int_t^T e^{-r(s-t)} \frac{S(s)}{S(t)} ds \qquad (23)
\end{aligned}$$

with $\bar{p}$ as defined in (14), denote human wealth pre-retirement, $H^1(t)$, and post-retirement, $H^2(t)$, respectively. As before, the VOH during the working period (20) can be decomposed into five components corresponding to (i) the gross value attached to the remaining working life, (ii) the gross value attached to the retirement phase, (iii) human wealth, $H^1(t)$, (iv) remaining life-time expenditure, $E(t)$, and (v) the aggregate value of morbidity reductions. Note, however, that an imperfect annuity market $\theta > 0$ implies a lower weighting of human wealth and remaining life-time expenditure. Indeed, if annuity prices do not adjust to individual health investments, i.e. for $\theta = 1$, human wealth and remaining life-time expenditure do not enter the VOH at all. Equation (21)) denotes the VOH after entry into retirement. Here, the presence of pension benefits $\bar{p} > 0$ implies that, in contrast, to the first-best setting there continues to be a positive level of human wealth. However, again an imperfect annuity market implies a lower weight on both human wealth and remaining life-time expenditure.

Before proceeding with an examination of the optimal allocation, we provide a characterisation of human wealth in the presence of an equilibrium pension system.

Lemma 3: *In equilibrium, where $\overline{\tau} = \tau$ and $\overline{S}(t) = S(t)$, it is true that*

$$\frac{\partial \widehat{H}^1(t)}{\partial \tau} + \frac{\partial \widehat{H}^1(t)}{\partial \overline{\tau}} = e^{-r(\tau-t)}\frac{S(\tau)}{S(t)}w(\tau) > 0 \tag{24}$$

$$\frac{\partial \widehat{H}^1(t)}{\partial \alpha} = \int_{t_0}^{t} e^{-r(s-t)}\frac{S(s)}{S(t)}w(s)\,ds > 0. \tag{25}$$

$$\frac{\partial \widehat{H}^2(t)}{\partial \tau} + \frac{\partial \widehat{H}^2(t)}{\partial \overline{\tau}} = \frac{\partial \widehat{H}^2(t)}{\partial \overline{\tau}} = \frac{\left(\alpha w(\overline{\tau}) + \bar{p}\right)\int_t^T e^{-r(s-t)}\frac{S(s)}{S(t)}ds}{\int_\tau^T \frac{S(s)}{S(\tau)}e^{-r(s-\tau)}ds} > 0 \tag{26}$$

$$\frac{\partial \widehat{H}^2(t)}{\partial \alpha} = \int_{t_0}^{\tau} w(s)\frac{S(s)}{S(t)}e^{r(t-\widehat{s})}d\widehat{s} > 0 \tag{27}$$

Proof: From (22) and (23) we obtain the derivatives

$$\begin{aligned}
\frac{\partial \widehat{H}^1(t)}{\partial \tau} &= e^{-r(\tau-t)}\frac{S(\tau)}{S(t)}\left[(1-\alpha)w(\tau)-\bar{p}\right] \\
\frac{\partial \widehat{H}^1(t)}{\partial \overline{\tau}} &= \bar{p}_{\overline{\tau}}\int_{\tau}^{T} e^{-r(s-t)}\frac{S(s)}{S(t)}ds \\
&= \frac{\alpha w(\overline{\tau})+\bar{p}}{\int_{\overline{\tau}}^{T}\frac{\overline{S}(\widehat{s})}{\overline{S}(\overline{\tau})}e^{-r(\widehat{s}-\overline{\tau})}d\widehat{s}}\int_{\tau}^{T} e^{-r(s-t)}\frac{S(s)}{S(t)}ds>0 && (28)\\
\frac{\partial \widehat{H}^1(t)}{\partial \alpha} &= -\int_{t}^{\tau} e^{-r(s-t)}\frac{S(s)}{S(t)}w(s)\,ds+\bar{p}_{\alpha}\int_{\tau}^{T} e^{-r(s-t)}\frac{S(s)}{S(t)}ds \\
&= -\int_{t}^{\tau} e^{-r(s-t)}\frac{S(s)}{S(t)}w(s)\,ds \\
&\quad +\frac{\int_{t_0}^{\overline{\tau}} w(\widehat{s})\overline{S}(\widehat{s})e^{r(\overline{\tau}-\widehat{s})}d\widehat{s}}{\int_{\overline{\tau}}^{T}\overline{S}(\widehat{s})e^{-r(\widehat{s}-\overline{\tau})}d\widehat{s}}\int_{\tau}^{T} e^{-r(s-t)}\frac{S(s)}{S(t)}ds && (29)\\
\frac{\partial \widehat{H}^2(t)}{\partial \tau} &= 0 \\
\frac{\partial \widehat{H}^2(t)}{\partial \overline{\tau}} &= \bar{p}_{\overline{\tau}}\int_{t}^{T} e^{-r(s-t)}\frac{S(s)}{S(t)}ds \\
&= \frac{\alpha w(\overline{\tau})+\bar{p}}{\int_{\overline{\tau}}^{T}\frac{\overline{S}(\widehat{s})}{\overline{S}(\overline{\tau})}e^{-r(\widehat{s}-\overline{\tau})}d\widehat{s}}\int_{t}^{T} e^{-r(s-t)}\frac{S(s)}{S(t)}ds>0 && (30)\\
\frac{\partial \widehat{H}^2(t)}{\partial \alpha} &= \bar{p}_{\alpha}\int_{\tau}^{T} e^{-r(s-t)}\frac{S(s)}{S(t)}ds \\
&= \frac{\int_{t_0}^{\overline{\tau}} w(\widehat{s})\overline{S}(\widehat{s})e^{r(\overline{\tau}-\widehat{s})}d\widehat{s}}{\int_{\overline{\tau}}^{T}\overline{S}(\widehat{s})e^{-r(\widehat{s}-\overline{\tau})}d\widehat{s}}\int_{\tau}^{T} e^{-r(s-t)}\frac{S(s)}{S(t)}ds>0
\end{aligned}$$

Using $\overline{\tau}=\tau$ and $\overline{S}(t)=S(t)$ we can write $\int_{\overline{\tau}}^{T}\frac{\overline{S}(\widehat{s})}{\overline{S}(\overline{\tau})}e^{-r(\widehat{s}-\overline{\tau})}d\widehat{s}=\int_{\tau}^{T}\frac{S(\widehat{s})}{S(\tau)}e^{-r(\widehat{s}-\tau)}d\widehat{s}$. Noting that s and $\widehat{s}$ are integrated over the same interval $[\tau,T]$ we then obtain

$$\int_{\tau}^{T}\frac{S(\widehat{s})}{S(\tau)}e^{-r(\widehat{s}-\tau)}d\widehat{s}=\frac{S(t)}{S(\tau)}e^{r(\tau-t)}\int_{\tau}^{T} e^{-r(s-t)}\frac{S(s)}{S(t)}ds. \qquad (31)$$

Using this relationship in (28) and setting $\overline{\tau}=\tau$ and $\overline{S}(\cdot)=S(\cdot)$ we obtain $\frac{\partial \widehat{H}^1(t)}{\partial \overline{\tau}}=[\alpha w(\tau)+\bar{p}]\frac{S(\tau)}{S(t)}e^{-r(\tau-t)}$. Substituting this into the aggregate effect $\frac{\partial \widehat{H}^1(t)}{\partial \tau}+\frac{\partial \widehat{H}^1(t)}{\partial \overline{\tau}}\frac{\partial \overline{\tau}}{\partial \tau}=\frac{\partial \widehat{H}^1(t)}{\partial \tau}+\frac{\partial \widehat{H}^1(t)}{\partial \overline{\tau}}$ gives the expression in (24). Similarly, inserting (31) into (29) and setting

$\overline{\tau} = \tau$ and $\overline{S}(\cdot) = S(\cdot)$ we obtain

$$\begin{aligned}
\frac{\partial \widehat{H}^1(t)}{\partial \alpha} &= -\int_t^\tau e^{-r(s-t)}\frac{S(s)}{S(t)}w(s)\,ds + \frac{e^{-r(\tau-t)}}{S(t)}\int_{t_0}^\tau w(\widehat{s})S(\widehat{s})e^{r(\tau-\widehat{s})}d\widehat{s} \\
&= -\int_t^\tau e^{-r(s-t)}\frac{S(s)}{S(t)}w(s)\,ds + \int_{t_0}^\tau w(\widehat{s})\frac{S(\widehat{s})}{S(t)}e^{r(t-\widehat{s})}d\widehat{s} \\
&= -\int_t^\tau e^{-r(s-t)}\frac{S(s)}{S(t)}w(s)\,ds \\
&\quad + \int_{t_0}^t w(\widehat{s})\frac{S(\widehat{s})}{S(t)}e^{r(t-\widehat{s})}d\widehat{s} + \int_t^\tau w(\widehat{s})\frac{S(\widehat{s})}{S(t)}e^{-r(\widehat{s}-t)}d\widehat{s} \\
&= \int_{t_0}^t w(\widehat{s})\frac{S(\widehat{s})}{S(t)}e^{r(t-\widehat{s})}d\widehat{s}.
\end{aligned}$$

where the last line follows when noting that s and $\widehat{s}$ are integrated over the same interval $[t, \tau]$. The expression in (26) and (27) follow when setting $\overline{\tau} = \tau$ and $\overline{S}(\cdot) = S(\cdot)$ and cancelling terms as is appropriate. □

When allowing the retirement system to adjust to the behaviour of a representative individual, we obtain the expected result that pre-retirement human wealth, $\widehat{H}^1(t)$, increases with the retirement age precisely by an amount equal to the (expected) earnings during the additional period of employment. More surprisingly perhaps, we find that pre-retirement human wealth also increases with the tax-rate on labour income α, or more generally, in the size of the pension scheme. Indeed, this result is not immediately obvious, as there are offsetting effects of the tax rate on pre-retirement human wealth. On the one hand, human wealth increases for the retirement phase due to a higher pension benefit. On the other hand, however, taxation at a higher rate lowers net earnings over the working life. Indeed, as the post-retirement component of $\widehat{H}^1(t)$ is discounted more heavily, on may be led to believe that the impact of the pension scheme may be similar to a tax and, thereby, reduce human wealth. As the Lemma shows, this is not the case for the funded system we are considering here. Given that in equilibrium pension funds attract the same rate of return as annuities, the individual is indifferent as to whether to engage in private savings or in forced savings through the pension scheme, the well-known neutrality of funded pensions. As far as the individual's human wealth at time $t \leq \tau$ is concerned, this applies to the (future) earnings path over the interval $[t, \tau]$. However, while annuities are fully tradeable, the individual has no direct access to the pension fund. Indeed, the pension scheme withdraws earnings at rate α over the working age and, thereby, reduces the individual's current assets. In exchange the contributions paid establish an entitlement from the individual's perspective, which is accessible, however, only after retirement. As its conversion into financial wealth is deferred into the future, the past accumulation $\alpha \int_{t_0}^t w(s)\frac{S(s)}{S(t)}e^{r(t-s)}ds$ then continues to be counted towards the individual's human wealth $\widehat{H}^1(t)$. Notably the impact of the pension scheme on human wealth is then strongest at the point of retirement, τ.

While there is no direct impact of the pension age on post retirement human wealth, an indirect effect arises through the (ex-ante) adjustment of the pension benefit, $\bar{p}$. An increase in the expected retirement age $\overline{\tau}$ contributes towards a higher pension benefit and, thus, towards post-retirement human wealth, $\widehat{H}^2(t)$.[11] Finally, an increase in the rate of the earnings tax α unambiguously raises post-retirement human wealth through an increase in the pension benefit.

Noting that $\lambda_A^i(t) = \lambda_A^i(s)\, e^{-(\rho-r)(s-t)} \frac{S(t)}{S(s)}$, combining (16) and (17) and using (19) we can now express the optimality conditions for $\left\{\widehat{c}(t), \widehat{h}(t), \widehat{\tau}\right\}$ as

$$\frac{u_c(\widehat{c}(t))}{u_c(\widehat{c}(s))e^{-\rho(s-t)}} = e^{r(s-t)} \tag{32}$$

$$-\frac{1}{\mu_h(\widehat{h}(t))} = \widehat{\psi}^i(t), \tag{33}$$

$$\frac{\nu(S(\widehat{\tau}))}{u_c(\widehat{c}(\widehat{\tau}))} = (1-\alpha)\, w(\widehat{\tau}) - p(\alpha, \widehat{\tau}, S)\,. \tag{34}$$

Again the distribution of consumption over the life-cycle is determined by the Euler equation. Similarly, health investments continue to be chosen such that the effective cost of reducing mortality by on unit equals the VOH. Differences in health behaviour in the presence of imperfections then have to be traced back to differences in the VOH. Finally, given that the conditions in Lemma 2 are satisfied, retirement occurs at the point where the monetary disutility of working just equals gross earnings net of the loss in social security wealth, $\alpha w(\widehat{\tau}) + p(\alpha, \widehat{\tau}, S)$ (see e.g. Fenge and Pestieau 2005, chapter 4).[12]

We can now examine the inter-relationship between health and retirement in the presence of imperfections as well as the effect on health and retirement arising from these imperfections. Again, we derive here the partial effects, assuming a given life-cycle allocation $\left\{\widehat{c}(t), \widehat{h}(t)\right\}$, deferring an examination of the full effects to the numerical analysis in section 4. We focus directly on the equilibrium effects, where $\overline{\tau} = \tau$ and $\overline{S}(t) = S(t)$. From (33) we find that health investments respond to a change in $x \in \{\tau, \alpha, \theta\}$ according to

$$\frac{\partial \widehat{h}(t)}{\partial x} = \frac{\mu_{hh}}{\mu_h^2} \frac{\partial \widehat{\psi}^i(t)}{\partial x}.$$

[11] Note that the annual pension benefit increases by an amount $\bar{p}_{\overline{\tau}} = \frac{\alpha w(\overline{\tau}) + \bar{p}}{\int_{\overline{\tau}}^T \frac{\overline{S}(\widehat{s})}{\overline{S}(\overline{\tau})} e^{-r(\widehat{s}-\overline{\tau})} d\widehat{s}}$. If retirement is postponed by a year, say, this allows to accumulate additional funds amounting to $\alpha w(\overline{\tau})$ and at the same time to save the annual retirement benefit $\bar{p}$. The increase in the total pension budget $\alpha w(\overline{\tau}) + \bar{p}$ is then spread over the (expected) remaining life-span $\int_{\overline{\tau}}^T \frac{\overline{S}(\widehat{s})}{\overline{S}(\overline{\tau})} e^{-r(\widehat{s}-\overline{\tau})} d\widehat{s}$.

[12] Hence, there is an implicit social secuirty tax $\alpha + \frac{p(\alpha, \widehat{\tau}, S)}{w(\widehat{\tau})}$.

and are therefore driven by changes in the VOH, where[13]

$$
\begin{aligned}
\frac{\partial\widehat{\psi}^1(t)}{\partial\tau} &= e^{-r(\tau-t)}\frac{S(\tau)}{S(t)}\left[\frac{-v\left(S\left(\tau\right)\right)}{u_c\left(c\left(\tau\right)\right)}-\frac{S(\tau)\nu_S}{u_c\left(c\left(\tau\right)\right)}\right] \\
&\quad +(1-\theta)\left(\frac{\partial H^1\left(t\right)}{\partial\tau}+\frac{\partial H^1\left(t\right)}{\partial\overline{\tau}}\right) \\
&= e^{-r(\tau-t)}\frac{S(\tau)}{S(t)}\left[\frac{-v\left(S\left(\tau\right)\right)}{u_c\left(c\left(\tau\right)\right)}+(1-\theta)w\left(\tau\right)-\frac{S(\tau)\nu_S}{u_c\left(c\left(\tau\right)\right)}\right], \qquad (35)\\
\frac{\partial\widehat{\psi}^2(t)}{\partial\tau} &= (1-\theta)\left(\frac{\partial H^2\left(t\right)}{\partial\tau}+\frac{\partial H^2\left(t\right)}{\partial\overline{\tau}}\right)=(1-\theta)\frac{\partial H^2\left(t\right)}{\partial\overline{\tau}}>0, \\
\frac{\partial\widehat{\psi}^i(t)}{\partial\alpha} &= (1-\theta)\frac{\partial H^i\left(t\right)}{\partial\alpha}>0, \quad i=1,2 \\
\frac{\partial\widehat{\psi}^i(t)}{\partial\theta} &= -\left(H^i\left(t\right)-E\left(t\right)\right)=A\left(t\right)\geq 0 \Longleftrightarrow A\left(t\right)\geq 0.
\end{aligned}
$$

Thus, an increase in retirement age has a direct impact on the VOH both before and after retirement. As a higher retirement age implies a larger equilibrium pension, this raises the value for surviving after retirement and, therefore, tends to increase post-retirement health investments. Surprisingly perhaps, the effect of the retirement age on the VOH pre-retirement now turns out to be ambiguous even when considering the impact at the optimal retirement age, $\widehat{\tau}$. Noting that for this case

$$
\frac{\partial\psi^1(t)}{\partial\tau}=e^{-r(\widehat{\tau}-t)}\frac{S(\widehat{\tau})}{S(t)}\left[p\left(\alpha,\widehat{\tau},S\right)-\theta w\left(\tau\right)-\frac{S(\widehat{\tau})\nu_S}{u_c\left(\widehat{c}\left(\widehat{\tau}\right)\right)}\right],
$$

we see that the impact depends both on the size of the pension scheme, as measured by α and on the degree of imperfection in the annuity market, θ. In a first-best, where $\alpha=\theta=0$ and $p\left(0,\widehat{\tau},S\right)=0$ we obtain a complementary relationship due to the benefit from lowering morbidity. The presence of a pension scheme reinforces the complementarity by generating additional value of survival through a greater pension benefit. An offsetting incentive arises from an imperfect annuity market. Recall that for annuity prices that do not respond to individual health, i.e. for $\theta=1$, human wealth does not count towards the VOH. Setting $\theta=1$ in (35) it then follows that the impact of retirement on the VOH balances the disutility from having to work for an additional period against the improvement in morbidity. As is readily verified the overall effect is negative if and only if the elasticity of disutility with respect to the health stock $\eta\left(v,S\right):=-\nu_S S/v$ lies below 1. Whether or not this is satisfied is an empirical question but the finding suggests that for those professions for which the impact of health on the disutility of work is small, an increase in the retirement age tends to reduce the incentive to invest in health.

Interestingly, the introduction or strengthening of a public pension scheme tends to increase the VOH and, thus, the incentive to invest in health, the reason being the increase

[13] Note that $\frac{\partial\widehat{\psi}^i(t)}{\partial\theta}=A\left(t\right)$ follows immediately from (19). The second equality can be verified after appropriate manipulations when using (22) and (23) with (10).

in human wealth over the whole life-cycle. From the discussion surrounding Lemma 3 we can infer the intuition. We have argued there that the pension scheme tends to transfer wealth from the pre- into the post-retirement period. In order to lay claim to this wealth the individual needs to ensure its survival up to the retirement age and beyond. Hence, the increase in the VOH with a resulting tendency towards higher health investments.[14] We should note, however, that this effect arises only if the annuity market responds to individual health. If this is not the case, i.e. for $\theta = 1$, a direct impact of the pension scheme on health investments is absent.

Finally, we find that the VOH increases (decreases) in the degree of annuity 'imperfection', θ, if the individual holds positive (negative) assets. This corresponds to the moral hazard with regard to longevity as discussed by Davies and Kuhn (1992) and Philipson and Becker (1998). An individual which holds annuity wealth will typically invest too much into longevity as it stands to benefit from the annuity over an extended life-time while receiving a constant return on the annuity. Naturally, in equilibrium the resulting excessive annuity is reflected in lower rates or return, implying that individuals end up with a less than optimal level of consumption. Note, however, that this result is reversed whenever the individuals is taking up debt. Indeed, for wage profiles which increase during the early part of working life (or even throughout their full working life), individuals are prone to incur a debt over the initial stages of life. In this case, annuity market imperfections tend to stifle the incentive to invest in health. This tendency is exacerbated in the presence of a pension scheme. Noting that financial wealth is given by

$$A(t) = \int_{t_0}^{t} \left[(1-\alpha)\, w(s) - c(s) - h(s)\right] \frac{S(s)}{S(t)} e^{r(t-s)} ds$$

it is evident that $\frac{\partial A(t)}{\partial \alpha} < 0$ and, indeed,

$$A(t) < 0 \Leftrightarrow \alpha > \overline{\alpha}(t) \in (0,1) := \frac{\int_{t_0}^{t} \left[w(s) - c(s) - h(s)\right] \frac{S(s)}{S(t)} e^{r(t-s)} ds}{\int_{t_0}^{t} w(s) \frac{S(s)}{S(t)} e^{r(t-s)} ds}.$$

Consider, the extreme example, where for $\alpha = 1$ all of the individual earnings are confiscated. In order to finance consumption individuals would need to run up negative assets over the full course of their working life, borrowing against their (huge) pension claims. In this case, clearly we would have $A(t) < 0$ for all $t \in (t_0, \tau]$, in fact even for some $t > \tau$, leading to a strong reduction in the incentive to invest in health.

We summarise the key results.

Proposition 2 *Consider the impact of $\{\tau, \alpha, \theta\}$ on health in a partial sense. (i) Under scenario 1 (individualised annuity pricing) health investments $\widehat{h}(t)$ tend to increase both in the age of retirement, τ, and in the extent of the pension scheme, α. (ii) Under*

[14] Indeed, the increase in VOH through the pension scheme is strongest at the point of retirement. Death at the point of retirement is particularly 'painful' as in this case the maximal social security wealth would be lost.

scenario 2 (average annuity pricing) pre-retirement health investments $\widehat{h}(t)$ tend to increase (decrease) in the retirement age if and only if the disutility of labour is elastic, $\eta(v,S)>1$ (inelastic, $\eta(v,S)<1$), whereas post-retirement health investments are unaffected. The extent of the pension scheme has no effect. (iii) Average as opposed to individualised annuity pricing tends to inflate (depress) health investments $\widehat{h}(t)$ if and only if the individual is in credit (debt). (iv) In the presence of imperfect annuity pricing and a large pension scheme with $\alpha>\overline{\alpha}(t)$, health investments are depressed below their first-best and longevity is too short.

In reality, a pension scheme $\alpha>0$ is likely to coexist with an imperfect annuity market $\theta=1$. The analysis in Philipson and Becker (1998) suggests that both, pension claims, which typically are unresponsive to individual longevity, and imperfect private annuities lead to moral hazard behaviour towards excessive longevity. Therefore, to a degree one would expect the two effects to reinforce each other, leading to sizeable inefficiency. Our analysis suggests that this is not the case for two reasons. On the one hand, an imperfect annuity market eradicates health-related moral hazard from the pension scheme. On the other hand, to the extent that the pension scheme reduces private financial wealth or even induces private debt, it eliminates or even reverses health-related moral hazard from the annuity market.

The partial determinants of the choice of retirement $\widehat{\tau}$ can be summarised as follows.

Proposition 3 *The partial effects of $\{h(t),\alpha,\theta\}$ on the age of retirement $\widehat{\tau}$ are given by*

$$\begin{aligned}\frac{\partial\widehat{\tau}}{\partial h(t)}\,|t\in[t_0,\widehat{\tau}] &\geq 0\Leftrightarrow\frac{\alpha\int_t^{\tau}w(s)S(s)e^{r(\tau-s)}ds}{\int_{\tau}^{T}S(s)e^{-r(s-\tau)}ds}+\frac{v_S S(\widehat{\tau})}{u_c\left(c(\widehat{\tau})\right)}<0;\\ \frac{\partial\widehat{\tau}}{\partial h(t)}\,|t\in(\widehat{\tau},T] &> 0;\quad\frac{\partial\widehat{\tau}}{\partial\alpha}<0;\quad\frac{\partial\widehat{\tau}}{\partial\theta}=0.\end{aligned}$$

Proof: Rewrite the first-order condition (34) to

$$\Omega\left(\widehat{\tau},h(t),\alpha,\theta\right)=(1-\alpha)\,w\left(\widehat{\tau}\right)-p\left(\alpha,\widehat{\tau},S\right)-\frac{\nu(S\left(\widehat{\tau}\right))}{u_c(\widehat{c}\left(\widehat{\tau}\right))}=0$$

and consider the derivatives of the function $\Omega\left(\tau, h(t), \alpha, \theta\right)$,

$$
\begin{aligned}
\Omega_{\tau} &= (1-\alpha)\, w'\left(\widehat{\tau}\right) - p_{\tau}\left(\alpha, \widehat{\tau}, S\right) - \frac{\nu_S}{u_c(\widehat{c}\left(\widehat{\tau}\right))}\dot{S} < 0 \\
\Omega_{h(t)}\left|t \in [t_0, \widehat{\tau}]\right. &= -p_{h(t)}\left(\alpha, \widehat{\tau}, S\right)\left|t \in [t_0, \widehat{\tau}]\right. + \frac{\nu_S \mu_h\left(t\right) S\left(\tau\right)}{u_c(\widehat{c}\left(\widehat{\tau}\right))} \\
&= \mu_h\left(t\right)\left[\frac{\alpha \int_t^{\tau} w(s)S(s)e^{r(\tau-s)}ds}{\int_{\tau}^{T} S(s)e^{-r(s-\tau)}ds} + \frac{\upsilon_S S(\widehat{\tau})}{u_c\left(c(\widehat{\tau})\right)}\right] \\
\Omega_{h(t)}\left|t \in (\widehat{\tau}, T]\right. &= -p_{h(t)}\left(\alpha, \widehat{\tau}, S\right)\left|t \in (\widehat{\tau}, T]\right. \\
&= \mu_h\left(t\right) p\left(\alpha, \widehat{\tau}, S\right) \frac{\int_t^{T} S(s)e^{-r(s-t)}ds}{\int_{\tau}^{T} S(s)e^{-r(s-\tau)}ds} < 0 \\
\Omega_{\alpha} &= -w\left(\widehat{\tau}\right) - p_{\alpha}\left(\alpha, \widehat{\tau}, S\right) \\
&= -w\left(\widehat{\tau}\right) - \frac{\int_{t_0}^{\tau} w(t)S(t)e^{r(\tau-t)}dt}{\int_{\tau}^{T} S(t)e^{-r(t-\tau)}dt} < 0.
\end{aligned}
$$

We then obtain for $x \in \{h(t), \alpha, \theta\}$ the derivatives $\frac{\partial \widehat{\tau}}{\partial x} = -\frac{\Omega_x}{\Omega_\tau}$ and, since $\Omega_\tau < 0$, it follows that $sgn\frac{\partial \widehat{\tau}}{\partial x} = sgn\Omega_x$. The results stated in the Proposition then follow immediately.□

Similar to the impact of retirement on health, we now find that the impact of health on retirement in the presence of a pension scheme is no longer clear-cut. With regard to health investments pre-retirement we find an offsetting effect. On the one hand, by contributing to the stock of health they continue to lower the disutility of work at the point of retirement and, thereby, provide an incentive to retire later. On the other hand, to the extent that improved pre-retirement survival increases the expected pension wealth accumulated at the point of retirement, this generates a higher pension benefit. A higher pension benefit, however, induces an incentive to retire earlier. In contrast to the first-best setting we now find that post-retirement health investments lead to a postponement of retirement. This is because the resulting increase in longevity leads to a reduction in the equilibrium pension benefit and, thus, provides an incentive for later retirement. Unsurprisingly, the introduction of the retirement scheme or an increase in its strength triggers earlier retirement for reasons of moral hazard. Finally, the nature of the annuity market has no direct impact on the retirement incentive.

Combining the partial effects of α and θ on health and retirement, while taking account of the complementarity, generally leaves an inconclusive picture. Even when disregarding the effects through changes in the full $\left\{\widehat{c}\left(t\right), \widehat{h}\left(t\right)\right\}$ schedules, not much can be said. Perhaps the clearest picture emerges for the case of an imperfect capital market $\theta = 1$. Recall that in this case, the pension system has no direct impact on health $\frac{\partial \widehat{h}(t)}{\partial \alpha} = 0$ and a priori leads to a direct reduction in the retirement age, $\frac{\partial \widehat{\tau}}{\partial \alpha} < 0$. Furthermore, the age of retirement has no direct impact on post-retirement health investments, $\frac{\partial \widehat{h}(t)}{\partial \tau}\left|t > \widehat{\tau} = 0\right.$ and an ambiguous impact on pre-retirement health investments $\frac{\partial \widehat{h}(t)}{\partial \tau}\left|t \leq \widehat{\tau} \lesseqgtr 0\right.$ depending on the elasticity of disutility of work. Similarly, pre-retirement health investments have an ambiguous impact on retirement $\frac{\partial \widehat{\tau}}{\partial \widehat{h}(t)} \left| t \leq \widehat{\tau} \lesseqgtr 0\right.$. Even here a variety of sub-cases is possible. (i) Suppose $\frac{\partial \widehat{h}(t)}{\partial \tau} \left| t \leq \widehat{\tau} > 0\right.$ as is the case if $\eta\left(v, S\right) > 1$ and $\frac{\partial \widehat{\tau}}{\partial \widehat{h}(t)} \left| t \leq \widehat{\tau} > 0\right.$ as is the case if α is not too large. With pre-retirement health and the age of retirement continuing to be complements, the introduction of a pension scheme would lead to a reduction in both health investments and the age of retirement. (ii) Suppose, in contrast, $\frac{\partial \widehat{h}(t)}{\partial \tau} \left| t \leq \widehat{\tau} < 0\right.$ as is the case if $\eta\left(v, S\right) < 1$ and $\frac{\partial \widehat{\tau}}{\partial \widehat{h}(t)} \left| t \leq \widehat{\tau} > 0\right.$. In this case, the introduction of a pension scheme tends to reduce the retirement age but at the same time tends to increase pre-retirement health investments. Other combinations obviously give rise to ambiguous outcomes.

All of this said, we should stress once again, that these results are only partial and do not take into account the shift in the life-cycle schedules of health-investments and consumption. In order to assess the full impact of changes in α and θ one would need to perform a full-blown comparative dynamic analysis. Instead of engaging in this, we will present in section 4 a range of numerical results highlighting the effects of imperfections on health investment, retirement and consumption.

3.4 Dynamics of consumption and health investment

Before resorting to the numerical analysis we briefly consider the dynamics for the model with imperfections. From the first order condition (32) we can derive the time path of consumption for both periods:

$$\dot{\widehat{c}} = \frac{u_c(\widehat{c})}{u_{cc}(\widehat{c})}\left[\rho - r - \theta\left(\bar{\mu} - \mu\right)\right] = \frac{u_c(\widehat{c})}{u_{cc}(\widehat{c})}(\rho - r),$$

where the second equality follows from the fact that in equilibrium $\mu = \bar{\mu}$ holds. Thus, the consumption path (but not the level of consumption) remains unaffected by both the retirement scheme and/or imperfections in the annuity market. From the first-order condition (33) we obtain the age profile of health investments (assuming $\mu = \bar{\mu}$ straightaway)

$$\text{for } t \leq \tau\text{:} \quad \dot{\widehat{h}} = -\frac{\mu_{ht}(\widehat{h})}{\mu_{hh}(\widehat{h})} - \frac{\mu_h(\widehat{h})}{\mu_{hh}(\widehat{h})}\left\langle \begin{array}{c} r + \mu(\widehat{h}) \\ -\frac{1}{\psi^1(t)}\left\{\begin{array}{c} \frac{u-\nu}{u_c} - \frac{\nu_S S}{u_c} \\ +(1-\theta)\left[(1-\alpha)\,w - \widehat{c} - \widehat{h}\right] \end{array}\right\} \end{array}\right\rangle$$

$$\text{for } t > \tau\text{:} \quad \dot{\widehat{h}} = -\frac{\mu_{ht}(\widehat{h})}{\mu_{hh}(\widehat{h})} - \frac{\mu_h(\widehat{h})}{\mu_{hh}(\widehat{h})}\left\langle \begin{array}{c} r + \mu(\widehat{h}) \\ -\frac{1}{\psi^2(t)}\left\{\frac{u}{u_c} + (1-\theta)\left(\overline{p} - \widehat{c} - \widehat{h}\right)\right\} \end{array}\right\rangle.$$

Again the evolution of health investments is determined by two forces: the change in the marginal effectiveness of health investments with the progress of age and the evolution over the life-course of the VOH. We note that the imperfections have a bearing on the depreciation of the VOH; to what extent and in what direction, however, is difficult to determine without further analysis. For instance, while a pension scheme with $\alpha > 0$ and $\overline{p} > 0$ tends to lower (raise) the value of a life year during the working (retirement) phase, corresponding to the terms in bracelets, it also changes the levels of the VOH in the respective numerators. Note that for the retirement phase ($t > \tau$) the value of life years passed no longer includes net earnings, the disutility of working and the value of a change in morbidity; instead it now includes the pension payment. Thus, at the point of retirement the value of a life year written off changes by an amount $-\left\{\frac{-\nu}{u_c} - \frac{\nu_S S}{u_c} + (1-\theta)\left[(1-\alpha)\,w - \overline{p}\right]\right\}$. At the point of optimal retirement, $\widehat{\tau}$, the change is given by $\frac{\nu_S S}{u_c} + \theta\left[(1-\alpha)\,w - \overline{p}\right] = \frac{\nu_S S}{u_c} + \theta\frac{\nu}{u_c}$. For a perfect annuity market ($\theta = 0$), this implies that similar to the first-best the reduction in the depreciation of the VOH tends to boost health investments immediately after retirement or, at least, to delay their decline. For an imperfect annuity market ($\theta > 0$) this effect is stifled. Indeed, for $\theta = 1$, the VOH may be depreciated at a higher rate if the elasticity of disutility of labour with respect to health satisfies $\eta\,(v, S) < 1$. In this case health investments tend to increase by less, or indeed, tend to decrease by more from the onset of retirement.

4 Numerical Results

In the following we apply numerical simulations to gain insight into the inefficiencies consumption, health investment and retirement generated by an imperfect pension scheme and/or annuity market. Throughout our numerical analysis we apply the following functional specifications. Per period utility is specified as

$$u(c(t)) = b + \frac{c(t)^{1-\sigma}}{1-\sigma}$$

where $b = 5$ and $\sigma = 2.5$. We assume that individual earnings $w(t)$ are constant over the life-cycle, and more specifically, that $w(t) = 1$. The maximum life-span T is set equal to 110. Mortality data have been taken from the human mortality data base (HMD) for the years 1990-2000. We model the mortality rate according to the proportional hazard model (see Kalbfleisch and Prentice 1980)

$$\mu(t, h(t)) = \tilde{\mu}(t)\phi(t, h(t)),$$

where $\tilde{\mu}(t)$ denotes the base mortality rate (effective in the absence of any health care) and $\phi(t, h(t))$ describes the impact of health care. While there is little evidence to guide our choice of the function $\phi(\cdot)$, it strikes us as reasonable to assume the following properties: $\phi_h < 0, \phi_{hh} > 0$, $\phi_{ht} > 0$; $\phi(t, 0) = 1$ ($\forall ta$) and $\phi_h(t, 0) = -\infty$ ($\forall\, t$). Thus, we specify

$$\phi(t, h(t)) = 1 - \sqrt{\frac{h(t)}{z}\frac{t-T}{1-T}} \tag{36}$$

with $z = 3$. The efficiency of health care is decreasing over age, and care becomes entirely ineffective for $t = T$. Furthermore, we specify

$$\nu(S) = \bar{z}(1 - S)$$

with $\bar{z} = 5$. Finally we assume $\alpha = 0.2$ and $r = \rho = 0.03$.

Inefficiencies may arise (a) due to the introduction of an incentive incompatible pension system that induces a lower optimal rate of retirement and (b) the existence of an imperfect annuity market. To disentangle these two effects we arrange our numerical analysis around the five scenarios: The first-best (scenario 0) and the four second-best scenarios depicted in Table 1 all of which relate to a pension scheme with $\alpha = 0.2$.

	$\theta = 0$	$\theta = 1$
$\tau = \tau^*$	scenario 1	scenario 2
τ from (15)	scenario 3	scenario 4

Table 1: Scenarios

Scenarios 1 and 2 isolate the inefficiencies due to health-related moral hazard by considering the solution to the problem in (15) while fixing the retirement age at its first-best level $\tau = \tau^*$. A comparison of scenario 1 with the first-best identifies the distortion in health care due to the pension system. A comparison of scenario 2 with scenario 1 identifies the additional distortion due to an imperfect annuity market. Scenarios 3 and 4 consider distortions due to both health- and retirement-related moral hazard. Thus, a comparison of scenario 3 with scenario 1 allows us to identify the (additional) distortions within a pension system arising from moral hazard with regard to retirement. Finally, a comparison of scenario 4 with scenario 3 identifies the additional distortion arising within an imperfect annuity market. Of course all of the scenarios 1-4 compare directly with the first-best.

We begin by considering scenarios 1 and 2. Figures 1 through 3 plot consumption, health expenditure and aggregate wealth (= savings + retirement account) for the first-best (indexed FB), and the two scenarios with the pension scheme (indexed PS): scenario 1: $\theta = 0$ (indexed 'perf') and scenario 2: $\theta = 1$ (indexed 'imp'). Since we assume $r = \rho$, both the first-best and the outcomes with inefficiencies involve a constant stream of consumption across the life-course (i.e. perfect consumption smoothing) (see Figure 1). For both scenarios 1 and 2, consumption is lower than the first-best and lowest for the case of an imperfect annuity market. The lower consumption level is the result of excessive health expenditures (see Figure 2) and a higher level of overall wealth (see Figure 3). Health investments are greatest for an imperfect annuity market. This is in line with health-related moral hazard tending to increase longevity. From Figure 2 we note that the path of health investments, does indeed take a reversal at the point of retirement from falling to increasing towards a global maximum during the retirement period. As one would expect aggregate wealth peaks at the point of retirement. Finally, we note that the need to accommodate consumption over an inefficiently long life implies that aggregate wealth in the presence of imperfections exceeds the first-best level.

Figures 1 - 3 about here

In a second set of simulations depicted in Figures 4 through 6 we allow for endogenous retirement in the presence of inefficiencies, which as we have seen analytically, will typically deviate from the first-best level. Again, we consider in separate the case of a perfect annuity market (scenario 4: $\theta = 0$) and an imperfect market (scenario 5: $\theta = 1$). As the results indicate (Figures 4 and 5), optimal consumption and health investments now fall short of the first-best level by a sizeable amount. This can be traced back to retirement-related moral hazard leading to retirement at a much too early age. The ensuing loss in life-time income dominates all other effects and implies a much reduced scope to spend both on consumption and health care. The incentive to spend on health care is further lowered relative to the first-best during the phase of 'early-retirement' as for this period there is no value anymore to reducing morbidity (i.e. the disutility of work). Since the earlier age at retirement generates a lower benefit rate, however, overall wealth is raised over and above the first-best level and this despite the shorter length of life (Figure 6).

Figures 4 - 6 about here

A comparison across scenarios 1 and 2 and scenarios 4 and 5, focusing on the allocational impact of an imperfect annuity market (regardless of the retirement scheme) yields a final insight. In case of fixed retirement an imperfect annuity market leads to higher health expenditure but lower consumption. If, however, the individual is able to choose retirement freely both health expenditure and consumption are higher in the presence of an imperfect annuity market. Recall from our previous argument that annuity market imperfections lead to an excess incentive to invest in health. When retirement is fixed - and ignoring the small effects of health investment on survival up to the age of retirement - this implies that in both the perfect and imperfect setting the individual has roughly the same level of life-time wealth available. But then with fixed retirement the scope for consumption is lower in the presence of annuity market imperfections for two reasons. (i) Excessive health investments directly lower the budget available for consumption; (ii) the increase in survival renders it necessary for the individual to accommodate consumption during the additional life-years in retirement. With variable retirement this changes as the reduction in the disutility of work that comes with the excessive health investments induces the individual to retire at a later age. The additional earnings allow the individual to accommodate both higher health investments and a higher level of consumption even over an extended life-time. While one might suspect that the simultaneous increase in consumption, health and longevity must lead to an increase in life-time utility, indeed, this is not the case. This is because the individual incurs the disutility from a longer working-life. Thus, annuity market imperfections lead to the surprising result that they improve the individual's consumption, health and length of life but that nevertheless they are unwarranted as they induce the individual to retire too late.

5 Conclusions

We have examined within a life-cycle model the nexus between health and retirement paying close heed to the fact that both are determined endogenously and simultaneously. In contrast to previous work on health and retirement we take into account that health care contributes both to improved survival and longevity and to a reduction in morbidity, which in our model implies a lower disutility of work. Our analysis shows that within a first-best world with a perfect market for annuities health and retirement are likely to be complementary, the relationship is much more ambiguous in a world, where an imperfect pension system and/or an imperfect annuity market lead to distortions. Imperfections arising both from the pension scheme and the annuity market give rise to two forms of moral hazard: one relating to the retirement age (typically too early) and one relating to the incentive to invest in health and longevity. While previous research (Davies and Kuhn 1992, and Philipson and Becker 1998) has shown that health-related moral hazard within annuity markets / pension schemes typically implies excessive health investments and longevity, our analysis, which takes explicit account of the interaction between the two types of imperfection, suggests a rather more qualified outcome. Indeed, the two imperfections do to some extent offset each other. Even more drastically, if large pension schemes induce individuals to incur a debt, the incentives from health-related moral hazard reverse. Too little is invested in health, and life expectancy is too low. Indeed, poor health may then also trigger too early a time of retirement.

Our numerical analysis suggests that the effects of retirement-related moral hazard tend to dominate the health-related inefficiency. The reduction in life-time income due to early retirement curtails both consumption and the purchase of health care way below their first-best levels. Our numerical analysis also shows that at times the expected distortions can be reversed. This is the case for instance when excessive investments in health induce the individual to retire too late. While we believe these results to be insightful, there are a number of shortcomings and extension to the paper.

First, the numerical results require a robustness analysis. We appreciate that at the moment they are not calibrated and, therefore, may lack realism. Furthermore, some of the parameters are difficult to gauge and we may have set them at levels which bias the results in one direction or other. Indeed, our analytical results suggest the scope for somewhat different patterns of consumption and health spending. Thus, at the moment the numerical analysis should be taken with a grain of caution. It helps to identify and illustrate important channels of transition, and we believe them to be intuitive, but variations in the strengths of the different effects may lead to different outcomes.

Second, our model lacks in as far as we assume morbidity to have only an impact on the disutility of labour but not on earnings. Realistically the latter matters at least as much and opens additional channels for transmission. For instance, while a health-related reduction in the disutility of work typically pushes towards an increase in the retirement age; this may no longer be true if a wealth-effect from greater earnings becomes sufficiently strong. Furthermore, to the extent that earnings do not respond to changes in the individual's health care, this gives rise to an additional inefficiency in health spending: individuals would tend to spend too little when taken earnings as given. Such an effect would arise when earnings predominantly reflect health-related differences in productivity, which is only measured in expected (or average) terms and, therefore, does not respond to individual efforts. In contrast, earnings do directly respond to individual health investments if they increase in the number of healthy hours worked. The latter channel would amend for another shortcoming of our model, namely that so far we only consider the extensive margin of employment but not the intensive margin. A full blown model would take account of endogenous labour supply during the working age; however, as a proxy one could also assume that labour supply within each period increases with health (as typically it would as part of an optimum).

Third, a richer model could distinguish between morbidity and mortality related health care. In reality there is some overlap but certain relevant forms of health care relate either to morbidity (e.g. cataract surgery) or to mortality (e.g. treatment of acute myocardial infection) without necessarily enhancing productivity or perhaps even compromising it. Whether such a model is tractable, however, remains to be seen. Finally, our framework lends itself to the analysis of other retirement schemes, in particular those that allow for improved incentives for the supply of labour.

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A Definition of the pension payment

During the working phase ($t \leq \tau$) tax payments are aggregated in the retirement account. Thus

$$\dot{B}(t) = (r + \mu(h))B(t) + \alpha w(t), \qquad B(t_0) = 0$$

Trivially we have

$$B(t) = \frac{1}{S(t)} \int_{t_0}^{t} e^{r(t-t')} \alpha w(t') S(t') \ dt', \qquad \text{for } t \leq \tau \tag{37}$$

which (for $B(\tau)$) implies one term in the definition of $p(\tau)$ in the paper.

In the retirement phase ($t \geq \tau$) the dynamics change to

$$\dot{B}(t) = (r + \mu(h))B(t) - p, \qquad B(\tau) = \frac{1}{S(\tau)} \int_{t_0}^{\tau} e^{r(\tau-t')} \alpha w(t') S(t') \ dt' \text{ and } B(T) = 0$$

where the retirement payment p is a constant. Solving the above equation implies

$$B(t) = \frac{1}{S(t)} \int_{t}^{T} e^{-r(t'-t)} S(t') p \ dt' \qquad \text{for } t \geq \tau. \tag{38}$$

Evaluating (37) and (38) at τ both must be equal, i.e.

$$\frac{1}{S(\tau)} \int_{t_0}^{\tau} e^{r(\tau-t')} \alpha w(t') S(t') \ dt' = \frac{1}{S(\tau)} \int_{\tau}^{T} e^{-r(t'-\tau)} S(t') p \ dt'$$

Further manipulation yields the expression for p as in the paper, i.e.

$$\begin{aligned} \alpha \int_{t_0}^{\tau} e^{r(\tau-t')} w(t') S(t') \ dt' &= p \int_{\tau}^{T} e^{-r(t'-\tau)} S(t') \ dt' \\ p &= \frac{\alpha \int_{t_0}^{\tau} e^{r(\tau-t')} w(t') S(t') \ dt'}{\int_{\tau}^{T} e^{-r(t'-\tau)} S(t') \ dt'}. \end{aligned}$$

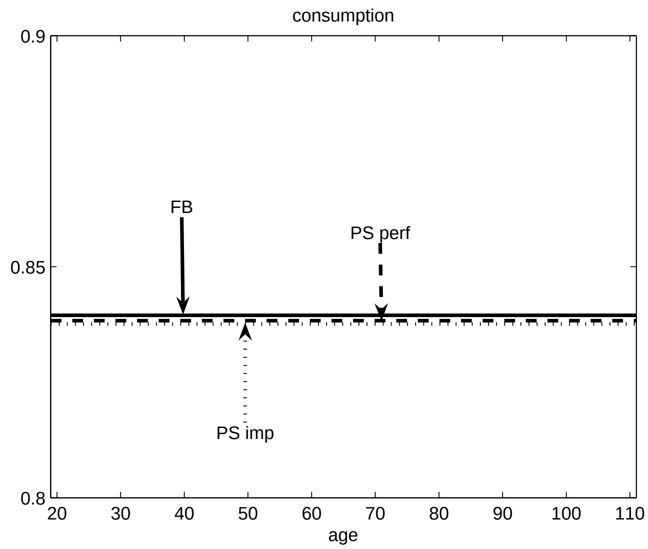

Figure 1: Consumption - scenarios 1 and 2

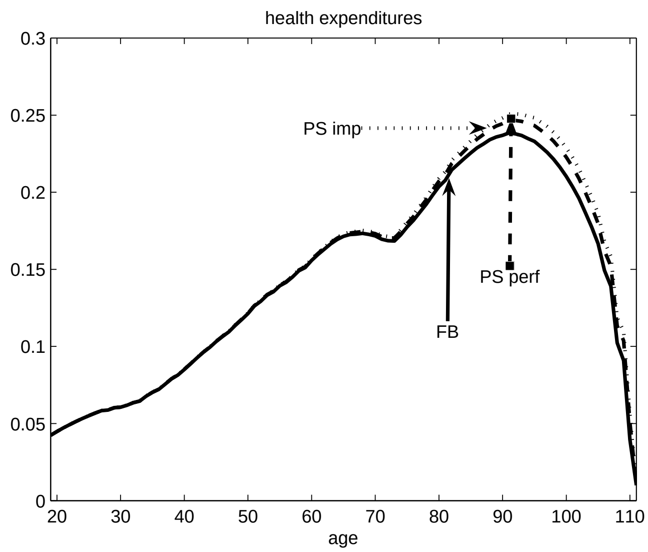

Figure 2: Health expenditure - scenarios 1 and 2

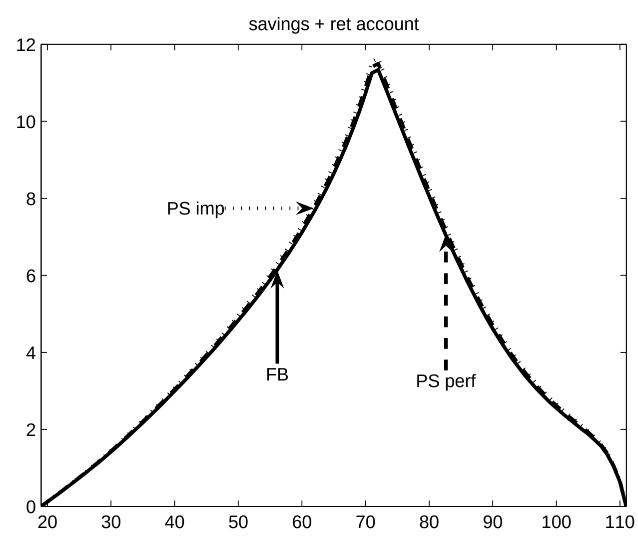

Figure 3: Aggregated wealth - scenarios 1 and 2

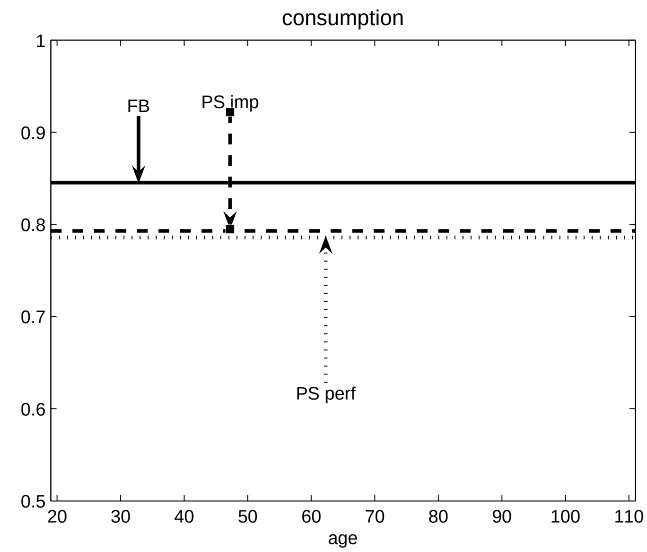

Figure 4: Consumption - scenarios 3 and 4

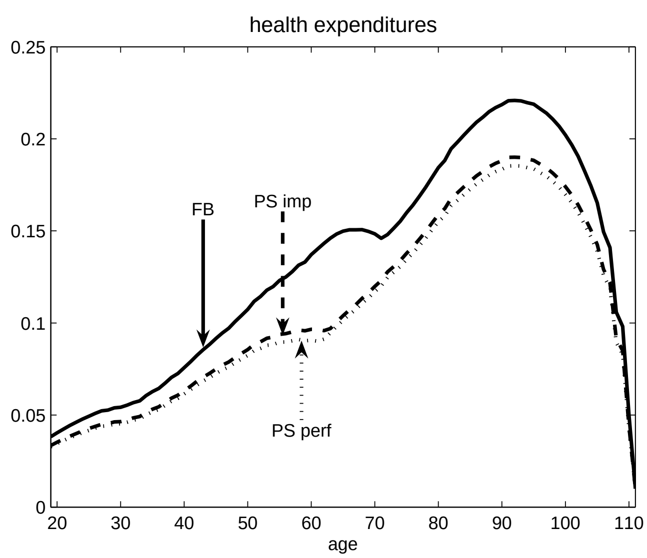

Figure 5: Health expenditure - scenarios 3 and 4

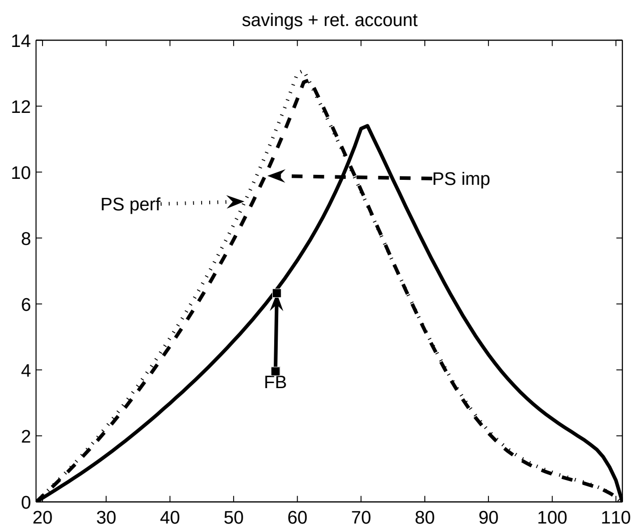

Figure 6: Aggregated wealth - scenarios 3 and 4