Invariance of the Equilibrium Set of Games with an Endogenous Sharing Rule*

Guilherme Carmona[†] Konrad Podczeck[‡]

University of Surrey Universität Wien

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Abstract

We consider games with an endogenous sharing rule and provide conditions for the invariance of the equilibrium set, i.e., for the existence of a common equilibrium set for the games defined by each possible sharing rule. Applications of our results include Bertrand competition with convex costs, electoral competition, and contests.

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†Address: University of Surrey, School of Economics, Guildford, GU2 7XH, UK; email: g.carmona@surrey.ac.uk.

‡Address: Institut für Volkswirtschaftslehre, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria; email: konrad.podczeck@univie.ac.at

1 Introduction

Consider the classical problem of a Bertrand duopoly, i.e., two firms set prices to compete for customers. A difficulty in modelling this situation is that when the firms set the same price, customers are indifferent with respect to where to buy, so that it is unclear how to specify the firms' profits (payoffs). Games with an endogenous sharing rule, introduced by Simon and Zame (1990), avoid this difficulty by specifying players' payoffs by a correspondence rather than by a function, thus taking a broad stand on how to specify players' payoffs.

When analyzing a game with an endogenous sharing rule, one may be interested in obtaining the existence of a strategy profile that is an equilibrium for each possible sharing rule, henceforth, an invariant equilibrium. Indeed, whenever such invariant equilibrium exists, the choice of the sharing rule becomes immaterial. Specifically, the predictions provided by such invariant equilibrium are robust to the actual sharing rule that happens to occur, e.g. the actual choice made by consumers regarding which of the two duopolists to buy from.

An appealing scenario occurs when each strategy that is an equilibrium for some sharing rule is an invariant equilibrium, henceforth, when the equilibrium set is invariant. In this case, any equilibrium is robust to the actual sharing rule that happens to occur and, based on this robustness notion, there is no need to select amongst the set of equilibria. Furthermore, as pointed out by Lebrun (1996) and Jackson and Swinkels (2005), the invariance of the equilibrium set is also important from a practical viewpoint. Indeed, it allows us to analyze the equilibrium set of the game defined by a sharing rule we may be interest in by analyzing the equilibrium set of the game defined by any other (simpler, easier to analyze) sharing rule. In particular, in the presence of incomplete information, it is often easier to establish the existence of equilibrium for some type-independent sharing rule by first showing that some type-dependent sharing rule has an equilibrium and then appeal to the invariance of the equilibrium set.

In this paper, we establish results concerning the invariance of the equilibrium set for general games with an endogenous sharing rule. Our first key condition, called "virtual continuity," roughly says that each player can, with a probability close to one, avoid points at which the payoff correspondence is multi-valued while virtually achieving the same payoff given the strategies of the other players, regardless of the particular sharing rule which is in force.[1] We show that, under this condition, the equilibrium set coincides with the set of invariant equilibrium in the class of games defined by efficient sharing rules. More precisely, any strategy that is an equilibrium of the game defined by some sharing rule is also an equilibrium in the game defined by any efficient sharing rule. This means that for equilibrium analysis of virtually continuous games with an endogenous sharing rule, one may focus on efficient sharing rules. In this light, our result has the interesting interpretation that, in equilibrium, indeterminacies are resolved efficiently. Moreover, as we illustrate using simple Bertrand examples, this result is also useful to compute the equilibrium set of games with an endogenous sharing rule.

Our second key condition, called "strong indeterminacy," roughly requires that indeterminacies are not eliminated by focusing on efficient sharing rules. More precisely, if a player has, at some action profile, more than one possible payoff, then there are at least two efficient payoff profiles at that action profile giving different payoffs to that player. Our main result states that each game with an endogenous sharing rule satisfying virtual continuity and strong indeterminacy has an invariant equilibrium set.

The intuition for our results is as follows. First, virtual continuity implies that each player's value function (i.e. the function assigning to each strategy profile of the other players the supremum of the payoffs he can achieve) is the same for all sharing rules. Second, any equilibrium for some sharing rule remains an equilibrium for any efficient sharing rule. This is so because would some player's payoff decrease if the former sharing rule is replaced by the latter, then, by efficiency, some other player's payoff would increase. But this is impossible because, in equilibrium, the payoff of any player must be equal to the payoff as determined by the value function at the strategy profile of the other players, and, by the first point, the value function of any player is the same for all sharing rules. From this we get the invariance of the equilibrium set provided that the set of efficient sharing rules gives the same indeterminacy set for each player as the set of all sharing rules. But this is just the condition of strong

[1] Virtual continuity generalizes analogous conditions that appeared in Dasgupta and Maskin (1986a, Theorem 5), Jackson and Swinkels (2005, Lemma 7) and Bagh (2010, Theorem 4.2).

indeterminacy.

Jackson and Swinkels (2005) have established the invariance of the equilibrium set for a specific setting of private-value auctions. Our contribution is to extend this conclusion to a general framework. The generality of our approach allows us to obtain new equilibrium invariance results for Bertrand competition with convex costs (along the lines of Dasgupta and Maskin (1986b) and Maskin (1986)), for electoral competition (as in Duggan (2007)) and contests (as in Moldovanu and Sela (2001)).

The paper is organized as follows. In Section 2 we consider a simple motivating example to illustrate our results. In Section 3 we present our main results. Section 4 contains applications of our results. In Section 5 we present extensions of our results, in particular to Bayesian contexts. The proof of our results are in Section A.5 of the Appendix.

2 A motivating example

Consider a standard Bertrand duopoly with zero costs and one commodity whose demand is $d(x) = 1 - x$ where x is the lowest price in the market. Each of the two firms sets a price in the unit interval to attract costumers. If prices are different (i.e. $x_1 \neq x_2$), then the firm setting the lowest price, firm i say, receives a profit of $(1-x_i)x_i$ whereas the other firm receives a profit of zero. If $x_1 = x_2$, profits are indeterminate; if θ denotes the fraction of the demand allocated to firm 1, then $\theta(1 - x_1)x_1$ is the profit of firm 1, and $(1 - \theta)(1 - x_1)x_1$ is that of firm 2; θ is allowed to take any value in $[0, 1]$. The situation can thus be described by an endogenous sharing rule with two players (the two firms), each having as its action set the interval $[0, 1]$, and with a payoff correspondence $Q : [0, 1]^2 \to \mathbb{R}^2$ defined by setting, for each $x \in [0, 1]^2$,

$$Q(x) = \begin{cases} (x_1(1 - x_1), 0) & \text{if } x_1 < x_2, \\ (0, x_2(1 - x_2)) & \text{if } x_1 > x_2, \\ \{(x_1\theta(1 - x_1), x_1(1 - \theta)(1 - x_1)) : \theta \in [0, 1]\} & \text{if } x_1 = x_2. \end{cases}$$

It turns out that the discussion of how to specify payoff in this Bertrand duopoly problem is immaterial for equilibrium analysis. Indeed, a particular way of specifying payoffs amounts to choosing a (measurable) selection of the payoff correspondence and, for any such choice, the resulting normal-form game has a unique equilibrium (both in pure and mixed strategies) where both firms set a price of zero. This is well-known for the case of the equal-sharing rule defined by setting $\theta = 1/2$ independently of the price vector x (see, e.g. Kaplan and Wettstein (2000)) and it will follow from our invariance result in the general case.

3 Invariance of the equilibrium set

3.1 Preliminaries

A game $\Gamma = (N, (X_i)_{i\in N}, Q)$ with an endogenous sharing rule is defined by a finite set N of players, a compact metric space X_i of actions for each $i \in N$, and an upper hemicontinuous (uhc in the sequel) payoff correspondence $Q\colon X \to \mathbb{R}^N$ with non-empty compact values, writing $X = \prod_{i\in N} X_i$. The interpretation is that $Q(x)$ is the set of possible payoff vectors for the players when action profile x is played.

We consider mixed strategies. For each player i, we write M_i for the set of mixed strategies available for him. The set M_i is just the set of Borel probability measures on X_i. We write $M = \prod_{i\in N} M_i$ for the set of all mixed strategy profiles. Given $\sigma = (\sigma_1, \ldots, \sigma_n) \in M$, we write τ_σ for the corresponding product measure on X. For each $i \in N$ and $x_i \in X_i$, δ_{x_i} denotes the Dirac measure at x_i, i.e. the measure assigning probability 1 to the singleton $\{x_i\}$.

Let $\Gamma = (N, (X_i)_{i\in N}, Q)$ be an endogenous sharing rule game. A measurable selection of Q is a measurable function $q : X \to \mathbb{R}^N$ such that $q(x) \in Q(x)$ for all $x \in X$. Any such q corresponds to a particular sharing rule of players' payoffs. We write S_Q for the set of all measurable selections of Q. For each $q \in S_Q$, $G_q = (X_i, q_i)_{i\in N}$ is a normal-form game. For each $i \in N$ and $\sigma \in M$, let $\bar{q}_i : M \to \mathbb{R}$ be defined by setting $\bar{q}_i(\sigma) = \int_X q_i \mathrm{d}\tau_\sigma$. The *mixed extension* of G_q is the normal-form game $G_{\bar{q}} = (M_i, \bar{q}_i)_{i\in N}$. A *mixed strategy Nash equilibrium* of $G_q = (X_i, q_i)_{i\in N}$ is a pure strategy Nash equilibrium of $G_{\bar{q}}$. Let $E(G_q)$ denote the set of mixed strategy Nash equilibria of G_q. From now on, we abuse notation and write $q_i(\sigma)$ instead of $\bar{q}_i(\sigma)$ for each $i \in N$ and $\sigma \in M$.

We say that $\sigma \in M$ is an *equilibrium* of the game Γ with an endogenous sharing rule if σ is a mixed strategy Nash equilibrium for some $q \in S_Q$, i.e. $\sigma \in \bigcup_{q\in S_Q} E(G_q)$.

We write $E(\Gamma)$ for the set of all equilibria of Γ.[2] For $\sigma \in E(\Gamma)$, we say that σ is an *invariant equilibrium* of Γ if σ is a mixed strategy Nash equilibrium for each $q \in S_Q$ i.e. $\sigma \in \bigcap_{q \in S_Q} E(G_q)$. We write $I(\Gamma)$ for the set of invariant equilibria of Γ.

Given $x \in X$, we say that a payoff vector $r \in Q(x)$ is efficient if it is not dominated by any other payoff vector $r' \in Q(x)$; formally, $r \in Q(x)$ is *efficient* if $r' \in Q(x)$ and $r' \geq r$ implies $r' = r$. We write Q_{eff} for the correspondence which assigns to each action profile $x \in X$ the set of efficient payoff vectors. Thus, $Q_{\text{eff}} : X \to \mathbb{R}^N$ is given by setting

$$Q_{\text{eff}}(x) = \{r \in Q(x) : r \text{ is efficient}\}$$

for each $x \in X$. We write $S_{Q_{\text{eff}}}$ for the set of all measurable selections of Q_{eff}. Taking $q \in S_Q$ such such that, for each $x \in X$, $q(x)$ solves $\max_{r \in Q(x)} \sum_{i \in N} r_i$ shows that $S_{Q_{\text{eff}}} \neq \emptyset$; see Lemma 5 in Appendix A.1. For $\sigma \in E(\Gamma)$, we say that σ is a Q_{eff}-*invariant equilibrium of* Γ if $\sigma \in \bigcap_{q \in S_{Q_{\text{eff}}}} E(G_q)$. Let $I_{\text{eff}}(\Gamma)$ be the set of Q_{eff}-invariant equilibria.

For each $i \in N$, write π_i for the projection of $\mathbb{R}^n$ onto the i-th copy of $\mathbb{R}$. Let $Q_i = \pi_i \circ Q$ and note that Q_i is uhc with nonempty and compact values. Let D_i be the set of action profiles at which Q_i is multi-valued, i.e.,

$$D_i = \{x \in X : \#(Q_i(x)) > 1\}.$$

We say that $\Gamma = (N, (X_i)_{i \in N}, Q)$ is *strongly indeterminate* if $\#(\pi_i \circ Q_{\text{eff}}(x)) > 1$ for each $i \in N$ and $x \in D_i$. In words, Γ is strongly indeterminate if whenever a player has, at some action profile, more than one possible payoff, then there are at least two efficient payoff vectors at that action profile giving different payoffs to that player. Equivalently, Γ is strongly indeterminate if $D_i = D_i^{\text{eff}}$ for each $i \in N$, where

$$D_i^{\text{eff}} = \{x \in X : \#(\pi_i \circ Q_{\text{eff}}(x)) > 1\}.$$

For each $q \in S_Q$ and $i \in N$, player i's *value function* is the function $v_{q_i} : M \to \mathbb{R}$ defined by setting $v_{q_i}(\sigma) = \sup_{\sigma'_i \in M_i} q_i(\sigma'_i, \sigma_{-i})$ for each $\sigma \in M$.

[2]The set $\{(q, \sigma) \in S_Q \times M : \sigma \in E(G_q)\}$ is the set of solutions of Γ (see Simon and Zame (1990)); the projection of this set in M equals $E(\Gamma)$.

3.2 Virtual continuity

Our definition of virtual continuity is as follows. We say that an endogenous sharing rule game $\Gamma = (N, (X_i)_{i \in N}, Q)$ is *virtually continuous* if for each $q \in S_Q$, $i \in N$, $\varepsilon > 0$ and $\sigma \in M$ there is a $\mu_i \in M_i$ such that

$$\tau_{(\mu_i, \sigma_{-i})}(D_i) < \varepsilon \text{ and } q_i(\mu_i, \sigma_{-i}) > q_i(\sigma) - \varepsilon.$$

Note that, by Lemma 4 in Appendix A.1, D_i is measurable for each $i \in N$, so $\tau_{(\mu_i, \sigma_{-i})}(D_i)$ is defined; recall from Section 3.1 that $\tau_{(\mu_i, \sigma_{-i})}$ denotes the joint probability measure when player i plays according to μ_i and the other players together play according to σ_{-i}. In words, Γ is virtually continuous if each player can, with a probability close to one, avoid points at which his own payoff correspondence is multi-valued while virtually achieving the same payoff given the strategies of the other players, regardless of the particular sharing rule which is in force.

The Bertrand example in Section 2 is easily seen to be virtually continuous as follows. First, for each $i = 1, 2$, player i's payoff correspondence is multi-valued only on a subset of the diagonal $\Delta = \{(t, t) : t \in [0, 1]\}$, i.e. $D_i \subseteq \Delta$. Second, for each $i = 1, 2$, $\varepsilon > 0$ and $\bar{x} \in D_i$, player i can obtain a payoff higher than $\max Q_i(\bar{x}) - \varepsilon$ by deviating to any $x_i < \bar{x}_i$ sufficiently closed to $\bar{x}_i$ (note that $Q(0, 0) = \{(0, 0)\}$ and, hence, $(0, 0) \notin D_i$; thus player i can lower its price at any $\bar{x} \in D_i$). Example 1 below shows that these two properties are indeed sufficient for virtual continuity.

Example 1. Consider an endogenous sharing rule game $\Gamma = (N, (X_i)_{i \in N}, Q)$ where $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, and D_1, $D_2 \subseteq \Delta$, writing $\Delta = \{(t, t) \colon t \in [0, 1]\}$. Consider the following hypothesis. For each $i = 1, 2$, each $\varepsilon > 0$, and each $\bar{x} \in D_i$, there is a set $C_i(\bar{x}) \subseteq [0, 1]$ such that (i) $\bar{x}_i$ is a condensation point of $C_i(\bar{x})$ (i.e. any neighborhood of $\bar{x}_i$ contains uncountably many points of $C_i(\bar{x})$), (ii) $Q_i(x) > \max Q_i(\bar{x}) - \varepsilon$ for each $x \in (C_i(\bar{x}) \times \{\bar{x}_j\}) \backslash \{\bar{x}\}$, where $j \neq i$ (writing (x_2, x_1) instead of (x_1, x_2) if $i = 2$; note also that $\#(Q_i(x)) = 1$ for $x \in X \setminus D_i$, so the inequality in (ii) is defined). Then virtual continuity holds (see Section A.5.1 for a proof).

Verifying virtual continuity in a particular game with an endogenous sharing rule is potentially daunting as one needs to consider all possible selections of the payoff correspondence. The next result may be helpful in this regard. Define $m_i : X \to \mathbb{R}$,

$i \in N$, by setting $m_i(x) = \max_{q \in Q(x)} q_i$ for each $x \in X$, which is possible because Q takes non-empty compact values, and note that because Q is also uhc, m_i is upper semicontinuous, therefore measurable. Lemma 1 states that, to show that Γ is virtually continuous, it is sufficient and necessary to verify the conditions in the definition of virtually continuity for m_i, $i \in N$.

Lemma 1. *Let $\Gamma = (N, (X_i)_{i \in N}, Q)$ be a game with an endogenous sharing rule. Then Γ is virtually continuous if and only if, for each $i \in N$, $\varepsilon > 0$ and $\sigma \in M$ there is a $\mu_i \in M_i$ such that*

$$\tau_{(\mu_i, \sigma_{-i})}(D_i) < \varepsilon \text{ and } m_i(\mu_i, \sigma_{-i}) > m_i(\sigma) - \varepsilon.$$

See Section A.5.2 for a proof of this lemma.

An important implication of virtual continuity is that, in any endogenous sharing rule game $\Gamma = (N, (X_i)_{i \in N}, Q)$ satisfying it and for every player, the value functions defined from the elements of S_Q all agree.

Lemma 2. *If $\Gamma = (N, (X_i)_{i \in N}, Q)$ is virtually continuous, then $v_{q_i} = v_{q'_i}$ for each q, $q' \in S_Q$ and $i \in N$; in particular, if $\sigma \in E(G_q)$, then $\sigma \in E(G_{q'})$ if and only if $q_i(\sigma) = q'_i(\sigma)$ for each $i \in N$.*

This lemma holds because given some strategy profile σ yielding a payoff close to the supremum of what a player i can obtain under a payoff function $q \in S_Q$ given σ_{-i}, there is strategy μ_i for player i with the following properties: First, (μ_i, σ_{-i}) also gives a payoff close to the supremum of what a player i can obtain under a payoff function q and, second, the payoff of (μ_i, σ_{-i}) under q is roughly the same as under any other payoff function $q' \in S_Q$ because, with a probability close to one, (μ_i, σ_{-i}) avoids points at which his own payoff correspondence is multi-valued (see Section A.5.3 for a more detailed proof).

Section A.2 contains additional discussion on virtual continuity.

3.3 Structure of the equilibrium set

The importance of virtual continuity for the invariance of the equilibrium set of a game with an endogenous sharing rule started to appear in the Lemma 2 above as it establishes the invariance of the value functions of each player. While this property is not enough for the invariance of the equilibrium set, it already implies that all efficient selections have the same equilibrium set. Moreover, as this common equilibrium set turns out to be the entire equilibrium set, our first result in this section provides a characterization of equilibria in virtually continuous games.

Theorem 1. *If $\Gamma = (N, (X_i)_{i \in N}, Q)$ is virtually continuous, then $E(\Gamma) = I_{\text{eff}}(\Gamma)$.*

As the proof of this theorem is short and illustrative for our condition of virtual continuity, it is presented here.

Proof of Theorem 1. Clearly $I_{\text{eff}}(\Gamma) \subseteq E(\Gamma)$. For the reverse inclusion, let $q \in S_Q$, $\sigma \in E(G_q)$ and $\hat{q} \in S_{Q_{\text{eff}}}$. Then $\tau_\sigma(\{x \in X : \hat{q}_i(x) > q_i(x)\}) = 0$ for all $i \in N$. Indeed, pick any $i \in N$ and write $F = \{x \in X : \hat{q}_i(x) > q_i(x)\}$. Suppose that $\tau_\sigma(F) > 0$. Set $\tilde{q} = \hat{q}1_F + q1_{X \setminus F}$. Then $\tilde{q} \in S_Q$ and $\tilde{q}_i(\sigma) > q_i(\sigma)$. Hence, by Lemma 2,

$$v_{q_i}(\sigma_{-i}) = v_{\tilde{q}_i}(\sigma_{-i}) \geq \tilde{q}_i(\sigma) > q_i(\sigma),$$

contradicting the assumption that σ is an equilibrium for q. Thus $\tau_\sigma(F) = 0$.

Suppose now that there is an $i \in N$ such that $\tau_\sigma(\{x \in X : \hat{q}_i(x) < q_i(x)\}) > 0$. As $\hat{q} \in S_{Q_{\text{eff}}}$, there must then be a $j \in N$ such that $\tau_\sigma(\{x \in X : \hat{q}_j(x) > q_j(x)\}) > 0$. But this is impossible by what has been shown in the previous paragraph.

It follows that for all $i \in N$, $\tau_\sigma(\{x \in X : \hat{q}_i(x) \neq q_i(x)\}) = 0$. By Lemma 2, we conclude that σ is an equilibrium for $\hat{q}$. □

A simple application of Theorem 1 can be made in the context of the equilibrium existence result for endogenous sharing rule games established by Simon and Zame (1990). Recall that in this latter result, the payoff correspondence is required to take convex values, and that an equilibrium needs to exist just for some selection which cannot be specified. In both of these aspects, Theorem 1 can be used to get an improvement if virtual continuity holds.

Theorem 2. *If $\Gamma = (N, (X_i)_{i \in N}, Q)$ is virtually continuous, then $E(\Gamma) \neq \emptyset$; in fact, $E(G_q) \neq \emptyset$ for every $q \in S_{Q_{\text{eff}}}$.*

The point of Theorem 2 is that virtually continuity dispenses the need for the payoff correspondence to have convex values, as detailed in the following remark (see Section A.5.4 for a proof of Theorem 2).

Remark 1. None of our results requires the payoff correspondence to take convex values. Actually, under our condition of virtual continuity, convexifying payoffs does not alter the equilibrium set, and also not the invariant equilibrium set. Indeed, let $\Gamma = (N, (X_i)_{i\in N}, Q)$ be a virtually continuous game and define $\Gamma' = (N, (X_i)_{i\in N}, Q')$ by letting $Q'(x)$ be the convex hull of $Q(x)$ for each $x \in X$. As shown in the proof of Theorem 2, Γ' is virtually continuous and $S_{Q_{\text{eff}}} \cap S_{Q'_{\text{eff}}} \neq \emptyset$; this, together with Theorem 1, implies that $E(\Gamma) = E(\Gamma')$. Since $S_Q \subseteq S_{Q'}$, it follows that $I(\Gamma') \subseteq I(\Gamma)$. For the converse, let $\sigma \in I(\Gamma)$. Let $q \in S_{Q'}$ and $i \in N$. Let $\underline{q} \in S_Q \cap S_{Q'}$ be such that $\underline{q}_i(x) = \min_{r\in Q(x)} r_i = \min_{r\in Q'(x)} r_i$ for each $x \in X$. Then $q_i(\sigma) \geq \underline{q}_i(\sigma)$. By Lemma 2, applied in Γ', $v_{\underline{q}_i}(\sigma) = v_{q_i}(\sigma)$. As $\sigma \in I(\Gamma)$, $v_{\underline{q}_i}(\sigma) = \underline{q}_i(\sigma)$. It follows that $q_i(\sigma) = v_{q_i}(\sigma)$. As $i \in N$ is arbitrary, $\sigma \in E(G_q)$. As $q \in S_{Q'}$ is arbitrary, $\sigma \in I(\Gamma')$.

Remark 2. Instead of appealing to Simon and Zame (1990) and our invariance result Theorem 1, Theorem 2 can also be proved by using Reny (1999, Corollary 5.2). Indeed, give M the narrow topology. Then virtual continuity implies that, for every $q \in S_Q$, the mixed extension of G_q is payoff secure; see Lemma 9 in the Appendix. Moreover, if $q \in S_Q$ is such that $q(x)$ solves $\max_{r\in Q(x)} \sum_{i\in N} r_i$ for each $x \in X$ (see Lemma 5), then the mixed extension of G_q is reciprocal upper semicontinuous; see Reny (1999, Proposition 5.1). Thus, by Reny (1999, Corollary 5.2), the mixed extension of G_q is better-reply secure and has an equilibrium (see Reny (1999) for the definitions of better-reply security and reciprocal upper semicontinuity).

Heading towards invariance of the equilibrium set, we next provide a characterization of invariant equilibria which we will use in our main result. This characterization has interest in its own right as it shows which equilibria are invariant and which ones are not.

Theorem 3. *If $\Gamma = (N, (X_i)_{i\in N}, Q)$ is virtually continuous, then*

$$I(\Gamma) = \{\sigma \in E(\Gamma) : \tau_\sigma(D_i) = 0 \textit{ for all } i \in N\}\,.$$

Theorem 3 shows that invariant equilibria are precisely those equilibria that assign zero probability to the indeterminacy set of each player (see Section A.5.5 for its proof). Indeed, any such equilibrium σ yields the same payoff for all $q \in S_Q$ and, thus, it is an invariant equilibrium by Lemma 2. Conversely, any invariant equilibrium σ

yields the same payoff for any player and any $q \in S_Q$ (equal to the player's value of the common value function at σ) and this is possible only if σ assigns zero probability to the indeterminacy set of each player.

We now address the question of when $E(\Gamma) = I(\Gamma)$. Having this equality is of interest because it implies that any selection of the payoff correspondence can be used to find an equilibrium for any other selection. A sufficient condition for having $E(\Gamma) = I(\Gamma)$ is that Γ be strongly indeterminate in addition to being virtual continuous.

Theorem 4. *Let $\Gamma = (N, (X_i)_{i \in N}, Q)$ be virtually continuous. Then the following holds:*

1. *If $\sigma \in E(\Gamma)$ then $\tau_\sigma(D_i^{\text{eff}}) = 0$ for all $i \in N$.*

2. *If Γ is strongly indeterminate, then $I(\Gamma) = E(\Gamma)$.*

The first part of Theorem 4 is analogous to Theorem 3: Any equilibrium is, by Theorem 1, a Q_{eff}-invariant equilibrium, and any Q_{eff}-invariant equilibrium σ yields the same payoff for any player and any $q \in S_{Q_{\text{eff}}}$ (equal to the player's value of the common value function at σ); this is possible only if σ assigns zero probability to the indeterminacy set of each player. (The proof of this latter fact is more involved here because, unlike Q, Q_{eff} need not be uhc with compact values; see Section A.5.6 for a proof of Theorem 4.) It is then clear that $I(\Gamma) = E(\Gamma)$ holds whenever $D_i = D_i^{\text{eff}}$ for each player i — but this is precisely the requirement of strong indeterminacy.

A simple class of games with an endogenous sharing rule that satisfy strong indeterminacy is provided in the following example.

Example 2. If $\Gamma = (N, (X_i)_{i \in N}, Q)$ is a constant-sum endogenous sharing rule game (i.e., for some $c \in \mathbb{R}$, $\sum_{i \in N} r_i = c$ for all $r \in Q(x)$ and all $x \in X$) then $Q = Q_{\text{eff}}$ and thus Γ is strongly indeterminate. Consequently, if such a Γ is virtually continuous, then $I(\Gamma) = E(\Gamma)$. Thus, in particular, if Γ is a two-person, constant-sum endogenous sharing rule game which satisfies the hypothesis in Example 1, then $I(\Gamma) = E(\Gamma)$.

Remark 3. In this remark we take a brief look at quasi-concave games. Recall that a game $\Gamma = (N, (X_i)_{i \in N}, Q)$ is quasi-concave if, for each $i \in N$, X_i is a convex subset of an Euclidean space (or, more generally, of a locally convex topological vector space) and there is a $q \in S_Q$ such that, for each $i \in N$ and $x_{-i} \in X_{-i}$, the function $x_i \mapsto$

$q_i(x_i, x_{-i})$ is quasi-concave. Identify, for each $i \in N$, X_i with the set of Dirac measures at the points of this space. Using Bich and Laraki (2017, Theorem 3.4) it follows that if Γ is quasi-concave, then $X \cap E(\Gamma) \neq \emptyset$. Moreover, if Γ is virtually continuous and strongly indeterminate, then $E(\Gamma) = I(\Gamma)$ by Theorem 4 and, therefore, $X \cap E(\Gamma) = X \cap I(\Gamma)$. In words, invariance in mixed strategies implies invariance in pure strategies. However, as it can be easily shown, the latter holds when virtually continuity is required to hold only for pure strategies, in the following sense: For each $q \in S_Q$, $i \in N$, $\varepsilon > 0$ and $x \in X$, there is an $\bar{x}_i \in X_i$ such that $(\bar{x}_i, x_{-i}) \notin D_i$ and $q_i(\bar{x}_i, x_{-i}) > q_i(x) - \varepsilon$.

The following two examples illustrate the scope of Theorem 4. To obtain an invariant equilibrium set, strong indeterminacy cannot be dispensed with (Example 3) and it is not enough to assume that the requirements of virtual continuity hold for only one measurable selection of the payoff correspondence (Example 4).

Example 3. Modify the example in Section 2 by taking $X_2 = [c, 1]$ with $0 < c < 1/2$ (i.e., firm 2 has constant marginal cost c with $0 < c < 1/2$), so that for each $x \in X$,

$$Q(x) = \begin{cases} (x_1(1-x_1), 0) & \text{if } x_1 < x_2, \\ (0, (x_2 - c)(1-x_2)) & \text{if } x_1 > x_2, \\ \{(x_1\theta(1-x_1), (x_1 - c)(1-\theta)(1-x_1)) : \theta \in [0,1]\} & \text{if } x_1 = x_2. \end{cases}$$

Note that $D_1 = \{x \in X : x_1 = x_2\}$.

It follows by Example 1 that Γ is virtually continuous. However, $E(\Gamma) \neq I(\Gamma)$. Indeed, if $q \in S_Q$ is such that $\theta(c, c) = 1$, then $(\delta_c, \delta_c) \in E(G_q) \subseteq E(\Gamma)$. But $(\delta_c, \delta_c) \notin E(G_q)$ if $q \in S_Q$ is such that $\theta(c, c) < 1$. Thus $(\delta_c, \delta_c) \notin I(\Gamma)$.

This shows that virtual continuity is not enough to ensure $E(\Gamma) = I(\Gamma)$, and thus highlights the role of strong indeterminacy as a condition in Theorem 4. Indeed this latter condition fails at (c, c), because $(c, c) \in D_1$ but $\#(\pi_1 \circ Q_{\text{eff}}(x)) = 1$.

Example 4. The following example (whose setup is as in Simon and Zame (1990)) illustrates why our notion of virtual continuity is required to hold for all selections of the payoff correspondence, rather than just for one. Let $N = \{1, 2\}$, $X_1 = [0, 3]$, $X_2 = [3, 4]$, and for each $x \in X$,

$$Q(x) = \begin{cases} \left\{\left(\frac{x_1+x_2}{2}, 4 - \frac{x_1+x_2}{2}\right)\right\} & \text{if } x \neq (3,3), \\ \{(\alpha, 4-\alpha) : \alpha \in [0,4]\} & \text{if } x = (3,3) \end{cases}$$

Note that if $\bar{q} \in S_Q$ is such that $\bar{q}(3,3) = (3,1)$, then $E(G_{\bar{q}}) = \{(\delta_3, \delta_3)\}$.

The function $\bar{q}$ just specified is continuous, so the requirements of virtual continuity hold for $\bar{q}$. Also, strong indeterminacy holds; in fact $Q = Q_{\text{eff}}$. But $I(\Gamma) = \emptyset$. Indeed, suppose $(\sigma_1, \sigma_2) \in I(\Gamma)$. Then, in particular, $(\sigma_1, \sigma_2) \in E(\Gamma_{\bar{q}})$, which implies that $(\sigma_1, \sigma_2) = (\delta_3, \delta_3)$. But $(\delta_3, \delta_3) \notin E(G_{\hat{q}})$ if $\hat{q} \in S_Q$ is such that $\hat{q}(3,3) = (4,0)$. This contradiction shows that $I(\Gamma) = \emptyset$. Finally, note that $\hat{q}_1(x_1, 3) < 3$ whenever $x_1 < 3$, i.e., whenever $(x_1, 3) \in X_1 \backslash D_1$. Thus the requirements of virtual continuity are not satisfied for $\hat{q}$.

4 Applications

4.1 Bertrand-Edgeworth competition with convex costs

In this section we consider a Bertrand-Edgeworth oligopoly with convex costs. The standard formalization of Bertrand-Edgeworth competition, according to which the firm posting the lowest price serves the entire demand, leads to difficulties in this setting. This is so because firms may prefer to tie to reduce the quantity produced. But this desire of a firm to tie is rather artificial and a consequence of the assumption that the firm posting the lowest price must serve the entire demand. In other words, the standard formalization is not appropriate for the case of convex costs; rather, it is more appropriate to allow firms to choose the quantity they want to supply. We allow for this by allowing each firm to choose a price and the maximum production level it is willing to produce. Our formalization is analogous to that of Dasgupta and Maskin (1986b, Section 2.2) where each firm has an exogenously given capacity; here, in contrast, we assume that the capacity of each firm is endogenous, i.e. it is chosen by the firm. Our formalization is also analogous to that of Maskin (1986) where firms choose both prices and quantities, and firms produce to order, i.e., produce only after the entire price profile has been observed.

As in Dasgupta and Maskin (1986b), there is a market for a single commodity with a continuum of consumers represented by the unit interval $[0, 1]$. Consumers are identical, and the representative consumer's demand for the commodity is a continuous and decreasing function $d : \mathbb{R}_+ \to \mathbb{R}_+$ such that there exists $\bar{p} > 0$ satisfying $d(p) > 0$ for all $p < \bar{p}$ and $d(p) = 0$ for all $p \geq \bar{p}$. There is a continuous, increasing and strictly convex cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with $c(0) = 0$.

There are $n \in \mathbb{N}$ firms. Each firm $i \in N = \{1, \ldots, n\}$ chooses a price p_i and a capacity s_i, the latter being the maximum amount the firm is willing to produce. Let $P = [0, \bar{p}]$ and $S = [0, d(0)]$.

To specify how the demand is allocated to firms, it is convenient to consider first the case of two firms. In this case, if one firm offers a price p lower than the price p' offered by the other firm, it serves the entire market up to its capacity s. A fraction $\frac{(d(p)-s)^+}{d(p)} = \frac{\max\{d(p)-s,0\}}{d(p)}$ of consumers is not served and each of these consumers demands $d(p')$ from the firm offering the highest price. When both firms set the same price, then the demand at the common price is split by each firm up to its capacity.

Formally, the quantities produced by firms are described by the correspondence $\Phi : (P \times S)^2 \to \mathbb{R}^2_+$ defined by setting, for each $(p, s) \in (P \times S)^2$,

$$\Phi(p,s) = \begin{cases} (\min\{d(p_1), s_1\}, \min\{\frac{(d(p_1)-s_1)^+}{d(p_1)} d(p_2), s_2\}) & \text{if } p_1 < p_2, \\ \{\phi \in \mathbb{R}^2_+ : \phi_1 + \phi_2 \leq d(p_1), \phi_i \leq s_i \text{ and} & \\ [d(p_1) - \phi_1 - \phi_2][s_i - \phi_i] = 0 \text{ for each } i = 1, 2\} & \text{if } p_1 = p_2, \\ (\min\{\frac{(d(p_2)-s_2)^+}{d(p_2)} d(p_1), s_1\}, \min\{d(p_2), s_2\}) & \text{if } p_1 > p_2. \end{cases}$$

When prices are different, these quantities are the same as in both Dasgupta and Maskin (1986b) and Maskin (1986) (with the proportional rationing rule in the latter). There are, however, differences between our formalization and theirs when prices are equal. First, we allow for indeterminacy, whereas they do not. Second, we rule out the possibility that a firm produces less than its capacity when there is unfulfilled demand (i.e. $\phi_i < s_i$ and $\phi_1 + \phi_2 < d(p_1)$ for some i is not possible); in contrast this is allowed in both Dasgupta and Maskin (1986b) and Maskin (1986).

We now return to the general case of n firms. We start by defining the following correspondence $\Phi \colon (P \times S)^n \to \mathbb{R}^n_+$. Fix any $(p, s) \in (P \times S)^n$. Order the elements of the set $\{p_1, \ldots, p_n\}$ so that $p^{(1)} < \cdots < p^{(L(p))}$. Set $N^{(l)} = \{i \in N \colon p_i = p^{(l)}\}$ for each $l = 1, \ldots, L(p)$. Define numbers $D^{(l)}(p, s)$, $l = 1, \ldots, L(p)$, recursively in the following way. Set $D^{(1)}(p, s) = d(p^{(1)})$; given that $D^{(l')}(p, s)$ has been specified for all l' with $1 \leq l' \leq l - 1 < L(p)$, set

$$D^{(l)}(p,s) = \frac{D^{(l-1)}(p,s) - \min\{D^{(l-1)}(p,s), \sum_{j \in N^{(l-1)}} s_j\}}{d(p^{(l-1)})} d(p^{(l)}).$$

Now set

$$\Phi(p,s)=\Bigg\{\phi\in\mathbb{R}^n_+\colon \phi_i\le s_i,\ i=1,\dots,n,$$
$$\sum_{j\in N^{(l)}}\phi_j=\min\Big\{D^{(l)}(p,s),\sum_{j\in N^{(l)}}s_j\Big\},\ l=1,\dots,L(p)\Bigg\}.$$

The correspondence Φ is closed. To see this, let (p_k,s_k) be a sequence in $(P\times S)^n$ with $(p_k,s_k)\to(p,s)\in(P\times S)^n$, and (ϕ_k) a sequence in $\mathbb{R}^n$ with $\phi_k\in\Phi(p_k,s_k)$ for each k and $\phi_k\to\phi$. We may assume that $L(p_k)$ is constant along the sequence (p_k), say $L(p_k)=K$ for all k. Then for each k, we have $L(p)\le L(p_k)=K$, and we can group the elements of $\{1,\dots,K\}$ into non-empty disjoint sets $A^{(1)},\dots,A^{(L(p))}$ so that for each $1\le l\le L(p)$ and any choice of $p_{k,i}$ with $p_{k,i}=p_k^{(h)}$ for some $h\in A^{(l)}$, $k\in\mathbb{N}$, we have $p_{k,i}\to p^{(l)}$. It is straightforward to check, using induction, together with continuity of d and continuity of taking minima, that for each $l=1,\dots,L(p)$, $\sum_{h\in A^{(l)}}\sum_{i\in N^{(h)}}\phi_{k,i}\to\min\big\{D^{(l)}(p,s),\sum_{j\in N^{(l)}}s_j\big\}$ (just recalculate the limits of the sums $\sum_{h\in A^{(l)}}\sum_{i\in N^{(h)}}\phi_{k,i}$). Consequently $\phi\in\Phi(p,s)$.

Since the values taken by Φ are included in a common compact set, the fact that Φ is closed implies that Φ is uhc and takes compact values. Clearly Φ takes non-empty values.

The above specification of Φ allows firms to choose any capacity. However some choices are easily seen to be redundant. In fact, for each $i\in N$, it suffices to consider capacity choices that are solutions of the problem

$$\max_{0\le s\le d(p_i)} p_i s-c(s).$$

Note that because c is strictly convex, a solution of this problem is unique. Let $s^*\colon P\to\mathbb{R}$ describe this solution as a function on P. Note that s^* is continuous, with $s^*(0)=s^*(\bar{p})=0$, and that, given $p_i\in P$, if $z\ge 0$ is a number with $z\le s^*(p_i)$, then $z'=z$ is the profit maximizing quantity choice of i subject to $z'\in[0,z]$.

This discussion leads to consider the following game Γ with an endogenous sharing rule. For each $i\in N$ let the action set be P and define the payoff correspondence $Q\colon P^n\to\mathbb{R}^n$ by setting, for each $p\in P^n$,

$$Q(p)=\{(p_1\phi_1-c(\phi_1),\dots,p_n\phi_n-c(\phi_n)):\phi\in\Phi(p,\tilde{s}(p))\},$$

writing $\tilde{s}(p) = (s^*(p_1), \ldots, s^*(p_n))$. Then Q takes non-empty values. By the facts that the map c is continuous and the correspondence Φ takes compact values, we see that Q takes compact values, and in addition, using the facts that Φ is uhc, the maps d and s^* are continuous, and that $s^*(\bar{p}) = 0$, we can see that Q is uhc.

To check strong indeterminacy, fix $p \in P^n$ and consider any $i \in N$. Let $p^{(l)}$ be the element of the order $p^{(1)} < \cdots < p^{(L(p))}$ such that $p_i = p^{(l)}$. Suppose $p \in D_i$. Then $p_i > 0$, $s^*(p_i) > 0$, $\#(N^{(l)}) > 1$, and $\sum_{j \in N^{(l)}} s^*(p_j) > D^{(l)}(p, \tilde{s}(p)) > 0$. These facts together imply strong indeterminacy, because (see above) payoffs are strictly increasing on $[0, s^*(p_j)]$.

As for virtual continuity, fix $i \in N$, $p_i \in P$ with $p_i > 0$, and $\varepsilon > 0$. Note that given any $0 \leq p_i' < p_i$ and $p_{-i} \in P^n_{-i}$, and given any $q \in S_Q$, for some numbers $0 \leq \alpha \leq \beta \leq 1$ we have $q_i(p_i, p_{-i}) = p_i \min\{\alpha d(p_i), s^*(p_i)\} - c(\min\{\alpha d(p_i), s^*(p_i)\})$ and $q_i(p_i', p_{-i}) = p_i' \min\{\beta d(p_i'), s^*(p_i')\} - c(\min\{\beta d(p_i'), s^*(p_i')\})$. Continuity of s^*, c, and d, together with compactness of $[0, 1]$, imply that there is a $\delta > 0$ such that whenever $p_i - \delta < p_i' < p_i$ and $\alpha \in [0, 1]$, then

$$\begin{aligned} p_i' \min\{\alpha d(p_i'), s^*(p_i')\} - c(\min\{\alpha d(p_i'), s^*(p_i')\}) \\ > p_i \min\{\alpha d(p_i), s^*(p_i)\} - c(\min\{\alpha d(p_i), s^*(p_i)\}) - \varepsilon. \end{aligned}$$

As payoffs are non-decreasing on $[0, s^*(p_i)]$, it follows that $q_i(p_i', p_{-i}) > q_i(p_i, p_{-i}) - \varepsilon$ for all $p_{-i} \in P^n_{-i}$ whenever $p_i - \delta < p_i' < p_i$. Consequently the hypotheses of Lemma 8 are satisfied. Thus virtual continuity holds.

By Theorems 2 and 4, we conclude that $I(\Gamma) = E(\Gamma) \neq \emptyset$.

4.2 Electoral competition

We consider a location/voting model as in Duggan (2007, Section 6). The setting is as follows. There are 2 players $i = 1, 2$ (e.g. political candidates), choosing locations x_1, x_2, respectively, in a compact and convex subset A of $\mathbb{R}^m$, $m > 0$, with nonempty interior. When these location differ, then, for each $i = 1, 2$, the payoff is given by

$$u_i(x_1, x_2) = \nu(\{\alpha \in A : ||\alpha - x_i|| < ||\alpha - x_j||,\ j \neq i\}),$$

where $|| \cdot ||$ denotes the Euclidean norm and ν is a measure on A which is absolutely continuous with respect to (m-dimensional) Lebesgue measure. The interpretation is that there is a set of individuals whose location in A is distributed according to ν and that each individual is attracted to the player located closest to him. (Note that as long as $x_1 \neq x_2$, absolute continuity of ν with respect to Lebesgue measure implies that points $a \in A$ with $\|a - x_1\| = \|a - x_2\|$ don't matter for the payoffs of the two players.) Now if $x_1 = x_2$, there is no canonical way to determine payoffs; in fact, perturbing such a situation can lead to different payoff sharings in the limit when the perturbations vanish; see the example given in Section 2. It is therefore natural to analyze this situation using a game with an endogenous sharing rule, rather than to make an ad hoc specification of payoffs as in Duggan (2007), where it is assumed that whenever players choose the same locations, payoffs are distributed in equal shares.

Without loss of generality, we assume that $\nu(A) = 1$. Let $S = \{p \in \mathbb{R}^m \colon \|p\| = 1\}$. For each $p \in S$ and each $z \in A$, let $\theta(p, z) = \nu(\{a \in A \colon pa < pz\})$. Note that for x_1, $x_2 \in A$ with $x_1 \neq x_2$, we can write $u_1(x_1, x_2) = \theta\big(\frac{x_2 - x_1}{\|x_2 - x_1\|}, \frac{1}{2}(x_1 + x_2)\big)$ and $u_2(x_1, x_2) = \theta\big(\frac{x_1 - x_2}{\|x_1 - x_2\|}, \frac{1}{2}(x_1 + x_2)\big)$. If x_1, $x_2 \in A$ with $x_1 = x_2 = z$, let

$$Q(x_1, x_2) = \{(r_1, r_2) \colon r_2 = 1 - r_1,\ r_1 = \theta(p, z) \text{ for some } p \in S\}.$$

If x_1, $x_2 \in A$ with $x_1 \neq x_2$, let $Q(x_1, x_2) = \{u_1(x_1, x_2), u_2(x_1, x_2)\}$.

Lemma 3. *(a) The correspondence Q is closed. (b) Virtual continuity is satisfied.*

For a rough intuition for why virtual continuity holds, consider $x \in D_1$ with $x_1 \in \operatorname{int}(A)$. Indeterminacies only arise when candidates choose the same policy, we have $x_1 = x_2$. Letting $p \in S$ be such that $\theta(p, x_1) = \max Q_1(x)$, player 1 satisfies the requirements of virtual continuity provided in Example 1 by deviating to $x_1 - \lambda p$ for all sufficiently small $\lambda > 0$. Indeed, $u_1(x_1 - \lambda p, x_2) = \theta(p, x_1 + \frac{1}{2}\lambda p) \to \theta(p, x_1)$ as $\lambda \to 0$. (See Section A.5.7 for a proof of Lemma 3).

Obviously, the game Γ we have discussed is a constant-sum game and thus satisfies strong indeterminacy. It therefore follows from Lemma 3 and Theorems 2 and 4 that $I(\Gamma) = E(\Gamma) \neq \emptyset$,

Remark 4. (a) Contrary to the case $m = 1$, if $m > 1$ then (because S is connected if $m > 1$) continuity of θ implies that the correspondence Q takes convex values. In particular, because $\max_{p \in S} \theta(p, z) \geq 1/2 \geq \min_{p \in S} \theta(p, z)$ for each $z \in A$, equal sharing is allowed when payoffs are indeterminate and $m > 1$. Equal sharing is also allowed when $m = 1$ by convexifying payoffs as in Remark 1.

(b) The correspondence Q is the smallest closed correspondence which includes the map (u_1, u_2) (in the sense of set inclusion of the graphs). (See Section A.5.8 for a proof).

Remark 5. The analysis of this section does not extend to the case of three or more players. Indeed, suppose $N = \{1, 2, 3\}$, let $X_i = [-1, 1]$ for each $i \in N$, write ν for Lebesgue measure, and let $Q : X \to \mathbb{R}^3$ be the smallest closed correspondence which includes the map $u : X' \to \mathbb{R}^3$, where $X' = \{x \in X : x_i \neq x_j \text{ for each } i \neq j\}$ and

$$u_i(x) = \nu(\{a \in [-1, 1] : |a - x_i| < |a - x_j| \text{ for all } j \neq i\}).$$

Consider $i = 1$. There is a $q \in S_Q$ such that $q_1(0, 0, 1) = q_1(0, 0, -1) = 1$. Let $\sigma = (\delta_0, \delta_0, \frac{\delta_1 + \delta_{-1}}{2})$. Then $q_1(\sigma) = 1$, and a simple calculation shows that for each $x_1 \notin \{-1, 0, 1\}$, $q_1(\delta_{x_1}, \sigma_{-1}) \leq 3/4$. In light of Lemma 7 in Appendix A.2, it follows that Γ is not virtually continuous.

5 Extension: Incomplete Information

In this section we extend our results to the case of incomplete information. Specifically, we consider incomplete information games with indeterminate outcomes, as introduced by Jackson, Simon, Swinkels, and Zame (2002), i.e., games where the assignment of payoffs to type/action profiles factors through a correspondence to some space of possible outcomes. We will present two results. The first one can be interpreted as assuming that the auctioneer knows the realizations of players' types and can use this information when implementing tie breaking rules. In Jackson and Swinkels (2005), this case is called that of an "omniscient auctioneer." In the second one, which is a corollary of the first, we turn to the more realistic case where the auctioneer does not have any information about players' types.

A game with indeterminate outcomes is described as follows. There is a finite set $N = \{1, \ldots, n\}$ of players. For each $i \in N$, there is a compact metric action space A_i and a compact metric type space T_i. Write $A = \prod_{i \in N} A_i$ and $T = \prod_{i \in N} T_i$. Type profiles, i.e., elements of T, are chosen according to a (Borel) probability measure λ on T;[3] write λ_i for the marginal measure on T_i, $i \in N$. As usual in the context of

[3]We use $\mathcal{B}(T)$ and $\mathcal{B}(A)$ to denote the Borel σ-algebra of T and A, respectively.

Bayesian games, it is assumed that λ is absolutely continuous with respect to the product $\lambda_1 \times \cdots \times \lambda_n$. There is an outcome space Ω, assumed to be a compact metric space. Players' actions determine a set of possible outcomes via an uhc correspondence $\Theta : A \to \Omega$ with nonempty and compact values. The payoffs (or utilities) of players are determined by a continuous function $u : T \times \text{graph}(\Theta) \to \mathbb{R}^N$.

The payoff correspondence $Q : T \times A \to \mathbb{R}^N$ is now defined by setting

$$Q(t, a) = \{u(t, a, \omega) : \omega \in \Theta(a)\}$$

for each $(t, a) \in T \times A$. Note that Q is uhc and has nonempty and compact values. Additional notation is as before: given $i \in N$, π_i denotes the projection of $\mathbb{R}^N$ on the ith coordinate, and we write $Q_i = \pi_i \circ Q$, $q_i = \pi_i \circ q$ for $q \in S_Q$, and $D_i = \{(t, a) \in T \times A \colon \#(Q_i(t, a)) > 1\}$.

Following Balder (1988), we describe a mixed strategy σ_i of player i by a Young measure from T_i to A_i, i.e., a map from T_i to the space $M(A_i)$ of probability measures on A_i such that the map $t_i \mapsto \sigma_i(t_i)(B)$ is Borel-measurable for each Borel set B in A_i.[4] As in Section 3, M_i is the set of mixed strategies available for player i, now with the interpretation as a space of Young measures. Again, we write $M = \prod_{i \in N} M_i$ for the set of all profiles of mixed strategies.

As for payoffs, consider any $\sigma = (\sigma_1, \ldots, \sigma_n) \in M$. For every $t = (t_1, \ldots, t_n) \in T$ write $\sigma(t)$ for the Borel measure on A defined by setting $\sigma(t) = \sigma_1(t_1) \times \cdots \times \sigma_n(t_n)$. Then the map $t \mapsto \sigma(t)$ is a Young measure from T to $M(A)$.[5] By Neveu (1965, Proposition III.2.1) it follows that there is a uniquely determined probability measure τ_σ on $T \times A$ such that $\tau_\sigma(E \times B) = \int_E \sigma(t)(B)\, \mathrm{d}\lambda(t)$ for each $E \in \mathcal{B}(T)$ and $B \in \mathcal{B}(A)$. Now, for any $q \in S_Q$ and $i \in N$, the integral $\int_{T \times A} q_i(t, a)\, \mathrm{d}\tau_\sigma(t, a)$ is defined, because q_i is bounded and measurable, and by the generalized version of Fubini's theorem (see again Neveu (1965, Proposition III.2.1)) we have

$$\int_{T \times A} q_i(t, a)\, \mathrm{d}\tau_\sigma(t, a) = \int_T \int_A q_i(t, a)\, \mathrm{d}\sigma(t)(a)\, \mathrm{d}\lambda(t)\,.$$

Because, given any realization $t \in T$ of possible type profiles, the payoff of player i is $\int_A q_i(t, a)\, \mathrm{d}\sigma(t)(a)$ (exactly as in the deterministic framework of the previous sections),

[4] In Milgrom and Weber (1985) such notion of a mixed strategy is called a behavioral strategy and, as they note, is equivalent to the notion of a distributional strategy that they consider.

[5] To see that $t \mapsto \sigma(t)(B)$ is measurable for each $B \in \mathcal{B}(A)$, observe that this is true if B is a product of Borel subsets B_i of T_i, $i = 1, \ldots, n$, and use the monotone class theorem.

we see that in the Bayesian framework considered now, the payoff of this player can be written as $\int_{T\times A} q_i(t,a)\,\mathrm{d}\tau_\sigma(t,a)$. Again, we use the expression $q_i(\sigma)$ as abbreviation.

As in Section 4.1 on Bertrand-Edgeworth competition, one may restrict players' choices of strategies so as that these have certain dominance properties; see also Example 6 below on Bertrand competition where each firm is restricted to set prices above its marginal cost. This can be done by specifying, for each $i \in N$, an uhc correspondence $\Phi_i\colon T_i \to A_i$, with non-empty closed values, and considering only strategy profiles σ such that for each $i \in N$, $\sigma_i(t_i)(\Phi_i(t_i)) = 1$ for λ_i-a.e. $t_i \in T_i$. Write W_i for the set of all $\sigma_i \in M_i$ satisfying this restriction, and let $W = \prod_{i\in N} W_i$.

Given such correspondences Φ_i, it might be of interest to get an invariance result in W. For this the following notion of virtual continuity is appropriate (see Example 6). Writing Φ for the list $(\Phi_1,\ldots,\Phi_n)$, we say that a game is Φ-*virtually continuous* if for any $q \in S_Q$, $i \in N$, $\sigma \in M_i \times W_{-i}$, and $\varepsilon > 0$, there is a $\mu_i \in W_i$ such that $\tau_{(\mu_i,\sigma_{-i})}(D_i) < \varepsilon$ and $q_i(\mu_i,\sigma_{-i}) > q_i(\sigma) - \varepsilon$. Note that when $\Phi_i \equiv A_i$ for each $i \in N$, the requirements of Φ-virtual continuity are the same as those in Section 3; in general, the difference is that now μ_i is required to be in W_i and that $\tau_{(\mu_i,\sigma_{-i})}(D_i) < \varepsilon$ and $q_i(\mu_i,\sigma_{-i}) > q_i(\sigma) - \varepsilon$ need to hold only when $\sigma_{-i} \in W_{-i}$.

Finally, given Φ, some generality can be gained by relaxing strong indeterminacy into the requirement that there be a Borel set $K \subseteq T \times A$ such that both $\tau_\sigma(K) = 0$ for each $\sigma \in W$ and $\#(\pi_i \circ Q_{\text{eff}}(t,a)) > 1$ for each $i \in N$ and each $(t,a) \in D_i \setminus K$ (see Example 6). We will call this notion Φ-*strong indeterminacy.* If T is absent (i.e. if T is a singleton), then this notion is the same as in Section 3 but here we may have $\#(\pi_i \circ Q_{\text{eff}}(t,a)) = 1$ for some $i \in N$ and some $(t,a) \in D_i$, for instance, in a null set of types.

Theorem 5. *Let $\Gamma = (N,(T_i,A_i,\Phi_i,u_i)_{i\in N},\lambda,\Theta)$ be a game with indeterminate outcomes. Suppose that Γ is Φ-virtually continuous and Φ-strongly indeterminate. Then $E(\Gamma) \cap W = I(\Gamma) \cap W \neq \emptyset$.*

The proof of Theorem 5 (in Section A.5.9) is analogous to that of Theorems 1–4, the existence part being now more involved. In particular, the requirement in the definition of Φ-virtual continuity that μ_i belongs to W_i plays a role in guaranteeing that there is an equilibrium in W.

Theorem 5 implies an invariance and existence result for selections of the payoff correspondence which are determined by selections of the outcome correspondence, i.e., for elements q of S_Q which can be written in the form $q(t,a) = u(t,a,\theta(a))$ for some measurable selection θ of Θ. Write S_Q^* for the set of all $q \in S_Q$ which can be written in this form. Let $E^*(\Gamma) = \bigcup_{q \in S_Q^*} E(G_q)$ and $I^*(\Gamma) = \bigcap_{q \in S_Q^*} E(G_q)$. As $I(\Gamma) \cap W \subseteq I^*(\Gamma) \cap W \subseteq E^*(\Gamma) \cap W \subseteq E(\Gamma) \cap W$, just by definition, the following result is an immediate consequence of Theorem 5.

Theorem 6. *Let $\Gamma = (N, (T_i, A_i, \Phi_i, u_i)_{i \in N}, \lambda, \Theta)$ be a game with indeterminate outcomes. Suppose that Γ is Φ-virtually continuous and Φ-strongly indeterminate. Then $E^*(\Gamma) \cap W = I^*(\Gamma) \cap W \neq \emptyset$.*

Such a result does not hold in general as shown by Jackson, Simon, Swinkels, and Zame (2002). Thus the conditions of virtual continuity and strong indeterminacy are important in Theorem 6.

Note, however, that, in contrast with both Jackson, Simon, Swinkels, and Zame (2002) and Jackson and Swinkels (2005), Theorem 6 does not require players' payoff function to be affine in the outcome. Such feature is important as it allows one to cover applications such as Bayesian version of the Bertrand-Edgeworth competition setting of Section 4.1.

We remark that Theorem 6 is important because the type of a player may be his own private information, and because with selections of the payoff correspondence that are obtained via selections of the outcome correspondence no issues concerning type revelation arise.

We illustrate Theorem 5 with two examples. In both of them the next theorem, which provides a way to show that virtual continuity holds in a wide class of games with indeterminate outcomes, is used.

Theorem 7. *Fix $\ell \in \mathbb{N} \setminus \{0\}$ and let $\Gamma = (N, (T_i, A_i, \Phi_i, u_i)_{i \in N}, \lambda, \Theta)$ be a game with indeterminate outcomes such that $A_i \subseteq \mathbb{R}^\ell$ for all $i \in N$. For each $i \in N$, write*

$$\Delta_i = \Big\{ a \in \prod_{j \in N} A_j \colon a_{i,h} = a_{j,h'} \text{ for some } j \in N \setminus \{i\} \text{ and some } 0 \leq h, h' \leq \ell \Big\}.$$

Suppose the following:

(1) A_i is convex and has a non-empty interior for each $i \in N$.

(2) For each $q \in S_Q$, $i \in N$, and $\sigma \in M_i \times W_{-i}$ there is a $\sigma_i' \in W_i$ such that $q_i(\sigma_i', \sigma_{-i}) \geq q_i(\sigma)$.

(3) $D_i \subseteq T \times \Delta_i$ for each $i \in N$.

(4) For each $i \in N$ and each $\varepsilon > 0$, there is a λ_i-null set $C_i \subseteq T_i$, a measurable map $f_i \colon \mathrm{graph}(\Phi_i) \to A_i$ and a correspondence $\Lambda_i \colon \mathrm{graph}(\Phi_i) \to A_i$, with measurable graph, such that for each $(t_i, a_i) \in \mathrm{graph}(\Phi_i) \cap ((T_i \backslash C_i) \times f_i(\mathrm{graph}(\Phi_i))$ the following hold: (i) $f_i(t_i, a_i) \in \Phi_i(t_i)$ and $q_i(t_i, t_{-i}, f_i(t_i, a_i), a_{-i}) \geq q_i(t_i, t_{-i}, a_i, a_{-i})$ for each $q \in S_Q$ and each $(t_{-i}, a_{-i}) \in T_{-i} \times A_{-i}$, (ii) $\Lambda_i(t_i, a_i) \subseteq \Phi_i(t_i)$, (iii) $\Lambda_i(t_i, a_i)$ is open, (iv) a_i is a cluster point of $\Lambda_i(t_i, a_i)$, and (v) for any $a_i' \in \Lambda_i(t_i, a_i)$ and any $(t_{-i}, a_{-i}) \in T_{-i} \times A_{-i}$, if $(t_i, t_{-i}, a_i, a_{-i}) \in D_i$, and $(t_i, t_{-i}, a_i', a_{-i}) \notin D_i$, then $Q_i(t_i, t_{-i}, a_i', a_{-i}) > \max Q_i(t_i, t_{-i}, a_i, a_{-i}) - \varepsilon$ (recall: $\#Q_i(t_i, t_{-i}, a_i', a_{-i}) = 1$ if $(t_i, t_{-i}, a_i', a_{-i}) \notin D_i$).

Then Γ is Φ-virtually continuous.

Theorem 7 is analogous to Example 1. Beside some technical conditions (such as (1)), it requires, in (2) and (3), that each player i has a best-reply in W_i against strategies profiles in W_{-i} (which is, in fact, what is intended with Φ_i) and that his payoff correspondence is multi-valued only when his actions equals, in some coordinate, the action of some other player. Condition (4) requires each player i to have, at each (t_i, a_i) with $a_i \in \Phi_i(t_i)$, a set of actions $\Lambda_i(t_i, a_i)$ which he can use to avoid multiplicities while virtually guaranteeing the best possible payoff. The following remark elaborates on condition 4 of Theorem 7, whose proof is in Section A.5.10.

Remark 6. We now clarify what is intended with condition (4)(i). Without this condition, the restriction that (4) imposes is only the requirement that (ii)–(iv) be satisfied at the same time as (v). Condition (4)(i) helps in this regard as it allows to reduce the set of points at which conditions (ii)-(v) need hold. Note, in particular, that (i) does not impose any restriction in addition to those imposed by (ii)-(v), because one can always set $f_i(t_i, a_i) = a_i$ for each $i \in N$ and $(t_i, a_i) \in \mathrm{graph}(\Phi_i)$.

As a first application of Theorem 5, we consider a general contest with incomplete information.

Example 5. Consider the following game with indeterminate outcomes which models a contest as formalized in Moldovanu and Sela (2001). There are n contestants $i = 1, \ldots, n$ who compete for one of n prizes with values $V_1 \geq V_2 \geq \cdots \geq V_n \geq 0$.[6] The allocation of prizes is determined by the contestants' effort. For example, contestants can be firms investing in *R&D* (their "effort") and prizes be their share of total demand.

Contestants simultaneously choose an effort level. Each contestant suffers a disutility $c(t_i, a_i)$ from his own effort a_i, where $t_i \in \tilde{T}$ denotes his ability, $\tilde{T}$ is a nonempty compact subset of $\mathbb{R}_+$, and $c : \tilde{T} \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies $c(t_i, 0) = 0$ for each $t_i \in \tilde{T}$. Abilities are drawn according to a probability measure λ on $\tilde{T}^n$, which is absolutely continuous with respect to the product of its marginals, and each contestant's ability is his own private information. We assume that there is $\bar{a} > 0$ such that $V_1 < c(t, \bar{a})$ for each $t \in \tilde{T}$.[7]

For each $i \in N$, let $T_i = \tilde{T}$ and $A_i = [0, \bar{a}]$. When all contestants choose different effort levels, then the first prize goes to the player with the highest effort, the second prize goes to the player with the second highest effort and so on. In case of ties in effort levels, randomization is used to determined the allocation of prizes. For example, if players 1, 2 and 4 choose the highest effort level, then the first three prizes are randomly allocated to players 1, 2 and 4. We let H be the set of allocations, i.e. the set of 1-1 functions from N (players) to N (prizes). The outcome space Ω is the set of probability measures on H. Some notation is needed to define the outcome correspondence. Given $a \in A$, order the elements of the set $\{a_1, \ldots, a_n\}$ so that $a^{(1)} > \cdots > a^{(L(a))}$. For each $l = 1, \ldots, L(a)$, set $N_a^{(l)} = \{i \in N : a_i = a^{(l)}\}$ and $n_a^{(l)} = \#(N_a^{(l)})$; furthermore, define $J_a^{(1)} = \{1, \ldots, n_a^{(1)}\}$ and, for each $1 < l \leq L(a)$, $J_a^{(l)} = \{n_a^{(l-1)} + 1, \ldots, n_a^{(l-1)} + n_a^{(l)}\}$. Given $a \in A$, the set of feasible allocations is denoted by $H(a)$ and consists of those $h \in H$ with the property that, for each $i \in N$,

[6] The assumption that the number of prizes equals the number of contestants is without loss of generality. Indeed, the case where there are $p < n$ prizes, which is allowed in Moldovanu and Sela (2001), is identified with $V_j = 0$ for all $j = p + 1, \ldots, n$.

[7] Moldovanu and Sela (2001) assume that $\tilde{T} = [m, 1]$ for some $0 < m < 1$, $c(t_i, a_i) = t_i \gamma(a_i)$ where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and differentiable and satisfies $\gamma(0) = 0$. The existence of $\bar{a}$ then follows when γ is linear or convex. Note also that, unlike Moldovanu and Sela (2001), we do not assume that types are independent with a continuous and strictly positive density.

if $1 \leq l \leq L(a)$ is such that $a_i = a^{(l)}$ then $h_i \in J_a^{(l)}$. We then define $\Theta : A \to \Omega$ in this context by setting, for each $a \in A$,

$$\Theta(a) = \{\omega \in \Omega : \omega_h = 0 \text{ for each } h \notin H(a)\}.$$

The payoff of contestant $i \in N$ equals the expected value of the prize received minus his disutility of effort; thus contestant i's payoff function $u_i \colon T \times A \times \Omega \to \mathbb{R}$ is given by $u_i(t, a, \omega) = \sum_{h \in H(a)} \omega_h V_{h_i} - c(t_i, a_i)$.

Evidently, we have

$$\sum_{i \in N} u_i(t, a, \omega) = \sum_{i=1}^{n} V_i - \sum_{i=1}^{n} c(t_i, a_i)$$

for each $t \in T$, $a \in A$, and $\omega \in \Theta(a)$. This implies easily that the game Γ just defined is strongly indeterminate.

We next show that Γ is virtually continuous by using Theorem 7 with $\Phi_i(t_i) = A_i$ for each $i \in N$ and $t_i \in T_i$ (so that $W_i = M_i$). It is clear that conditions (1)–(3) in Theorem 7 hold. As for condition (4), fix $i \in N$ and $\varepsilon > 0$. Let $C_i = \emptyset$ for each $i \in N$. Let $\eta \in (0, \bar{a})$ be such that $V_1 < c(t, a)$ for each $a > \bar{a} - \eta$, and define f_i by setting, for each $(t_i, a_i) \in T_i \times A_i$,

$$f_i(t_i, a_i) = \begin{cases} a_i & \text{if } a_i \leq \bar{a} - \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Then (i) of condition (4) in Theorem 7 holds. Note that $f_i(T_i \times A_i) \subseteq [0, \bar{a} - \eta]$. Let $\delta \in (0, \eta)$ be such that $|c(t, a) - c(t, a')| < \varepsilon$ whenever $|a - a'| < \delta$, $a, a' \in [0, \bar{a}]$ and $t \in \tilde{T}$, and define Λ_i by setting

$$\Lambda_i(t_i, a_i) = \begin{cases} (a_i, a_i + \delta) & \text{if } a_i \leq \bar{a} - \eta, \\ A_i & \text{otherwise} \end{cases}$$

for each $(t_i, a_i) \in T_i \times A_i$. Clearly Λ_i has a measurable graph and (ii)–(iv) of condition (4) in Theorem 7 are satisfied. As for (v) of that condition, let $t_i \in T_i$, $a_i \in f_i(T_i \times A_i)$, $a_i' \in \Lambda_i(t_i, a_i)$, $(t_{-i}, a_{-i}) \in T_{-i} \times A_{-i}$ and suppose $(t_i, t_{-i}, a_i, a_{-i}) \in D_i$ and $(t_i, t_{-i}, a_i', a_{-i}) \notin D_i$. Let $l \in \{1, \ldots, L(a)\}$ be such that $a_i = a^{(l)}$ and let $l' \in \{1, \ldots, L(a_i', a_{-i})\}$ be such that $a_i' = (a_i', a_{-i})^{(l')}$. Since $a_i' > a_i$, we must have

$\min J^{(l')}_{(a_i',a_{-i})} \geq \max J^{(l)}_a$. Therefore

$$Q_i(t_i,t_{-i},a_i',a_{-i}) - \max Q_i(t_i,t_{-i},a_i,a_{-i}) \\ = V_{\min J^{(l')}_{(a_i',a_{-i})}} - V_{\max J^{(l)}_a} - c(t_i,a_i') + c(t_i,a_i) > -\varepsilon,$$

as desired.

By Theorems 5 we conclude that $E(\Gamma) = I(\Gamma) \neq \emptyset$.

The next theorem provides a way to see that strong indeterminacy holds in a wide class of games with indeterminate outcomes.

Theorem 8. *Let $\Gamma = (N,(T_i,A_i,\Phi_i,u_i)_{i\in N},\lambda,\Theta)$ be a game with indeterminate outcomes. Suppose that, for some $\ell \in \mathbb{N}\backslash\{0\}$, $A_i,T_i \subseteq \mathbb{R}^\ell$ for each $i \in N$. Let Δ_i be as in the statement of Theorem 7 and suppose (2) of Theorem 7 is satisfied. For each $i \in N$ and each $1 \leq h \leq \ell$, let $f_{i,h}\colon A_{i,h} \to \mathbb{R}$ be a measurable function such that $f_{i,h}(a_{i,h}) \geq t_{i,h}$ whenever $a_i \in \Phi_i(t_i)$. For each $i \in N$ and each $1 \leq h \leq \ell$, write*

$$\Delta_{i,h} = \{(t,a) \in T \times A\colon a_{i,h} = a_{j,h'},\ j \neq i,\ 0 \leq h' \leq \ell\}.$$

Suppose the following:

(a) $Q(t,a) = Q_{\text{eff}}(t,a)$ if $(t,a) \in T \times A$ is such that $a_i \in \Phi_i(t_i)$ for each $i \in N$ and such that $(t,a) \in \Delta_{i,h}$ implies $f_{i,h}(a_{i,h}) > t_{i,h}$, $i \in N$, $1 \leq h \leq \ell$.

(b) $\lambda_i(\{t_{i,h}\} \times T_{i,-h}) = 0$ for each $i \in N$, $t_i \in T_i$, and $0 \leq h \leq \ell$.

Then Γ is Φ-strongly indeterminate.

Theorem 8, whose proof is in Section A.5.11, covers situations where Φ is defined via thresholds $f_{i,h}(a_{i,h}) \geq t_{i,h}$ ($i \in N$ and $1 \leq h \leq \ell$), for instance as when one considers, in auctions or Bertrand competition, the closure of the set of undominated actions for each type of each player. Moreover, in such examples, the threshold also means that player i strictly prefer to receive the hth object or to sell more of commodity h whenever $f_{i,h}(a_{i,h}) > t_{i,h}$. Thus, if there is a tie between, say, players 1 and 2 due to $a_{1,h} = a_{2,h'}$ and both $f_{1,h}(a_{1,h}) > t_{1,h}$ and $f_{2,h}(a_{2,h'}) > t_{2,h'}$ hold, then there are multiple possible payoffs for players 1 and 2 but all of them are efficient.

Condition (a) of Theorem 8 covers situations of the above kind. It is not sufficent for Φ-strong indeterminacy as Example 3 with an asymmetric complete-information

Betrand duopoly shows. Condition (b) fills the gap by requiring marginal type distributions to be atomless in each coordinate.

As a second application of Theorem 5, we consider a Bertrand oligopoly with incomplete information.

Example 6. Consider a Bertrand oligopoly with linear cost functions and one commodity whose demand is $d(x)$ where x is the lowest price in the market and $d : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. There is incomplete information regarding firms' marginal costs. For some $\bar{a} > 0$ and each firm $i \in N$, let $A_i = [0, \bar{a}]$ and T_i be a compact subset of $\mathbb{R}_+$. Let λ be a probability measure on T which is absolutely continuous with respect to the product of its marginals and such that λ_i is atomless for each $i \in N$. The outcomes of the game are the fractions of total demand that each firm gets. Thus, we let the outcome space Ω be $\{\omega \in [0,1]^n : \sum_{i \in N} \omega_i = 1\}$, with the interpretation that $\omega_i \in [0,1]$ is the fraction of total demand that firm i satisfies, $i \in N$. The natural choice of the outcome correspondence $\Theta : A \to \Omega$ in the context is given by setting, for each $a \in A$,

$$\Theta(a) = \left\{ \omega \in \Omega : \omega_i > 0 \text{ only if } a_i = \min_{j \in N} a_j \text{ for each } i \in N \right\}.$$

The payoff functions $u_i \colon T \times A \times \Omega \to \mathbb{R}$ are then given by

$$u_i(t, a, \omega) = \omega_i d \left(\min_{j \in N} a_j \right) (a_i - t_i)$$

for each $i \in N$. Finally, let $\Phi_i(t_i) = \{a_i \in A_i : a_i \geq t_i\}$, i.e. $\Phi_i(t_i)$ is the set of prices above marginal cost, which we assume to be nonempty.

To see that the game Γ just defined is virtually continuous, we check that the hypotheses of Theorem 7 are satisfied. Clearly conditions (1) and (3) of that theorem are satisfied. As for condition (2), without loss of generality consider player 1. Fix $q \in S_Q$ and $\sigma \in M$. By Lemma 11 there is a measurable map $g \colon T_1 \to A_1$ such that $q_1(\delta_g, \sigma_{-i}) \geq q_1(\sigma)$. Define $h \colon T_1 \to A_1$ by setting $h(t_1) = \max\{g(t_1), t_1\}$. Then $\delta_h \in W_1$. Moreover, we have $q_1(t_1, t_{-1}, h(t_1), a_{-1}) \geq q_1(t_1, t_{-1}, g(t_1), a_{-1})$ for each

$(t_1, t_{-1}, a_{-1}) \in T_1 \times T_{-1} \times A_{-1}$, so

$$
\begin{aligned}
q_1(\delta_h, \sigma_{-1}) &= \int_T \int_A q_1(t_1, t_{-1}, a_1, a_{-1})\, \mathrm{d}\delta_h(t_1) \times \sigma_{-1}(t_{-1})\, \mathrm{d}\lambda(t) \\
&= \int_T \int_{A_1} q_1(t_1, t_{-1}, h(t_1), a_{-1})\, \mathrm{d}\sigma_{-1}(t_{-1})\, \mathrm{d}\lambda(t) \\
&\geq \int_T \int_{A_1} q_1(t_1, t_{-1}, g(t_1), a_{-1})\, \mathrm{d}\sigma_{-1}(t_{-1})\, \mathrm{d}\lambda(t) \\
&= \int_T \int_A q_1(t_1, t_{-1}, a_1, a_{-1})\, \mathrm{d}\delta_g(t_1) \times \sigma_{-1}(t_{-1})\, \mathrm{d}\lambda(t) \\
&= q_1(\delta_g, \sigma_{-1}) \geq q_1(\sigma_1, \sigma_{-1}).
\end{aligned}
$$

Thus condition (2) of Theorem 7 holds. To see that condition (4) of that theorem holds, fix $\varepsilon > 0$ and $i \in N$. Let $\eta > 0$ be such that $|d(a_i')(a_i' - t_i) - d(a_i)(a_i - t_i)| < \varepsilon$ whenever $t_i \in T_i$ and $a_i, a_i' \in A_i$ are such that $|a_i' - a_i| < \eta$. Let $C_i = \{\bar{a}\} \cap T_i$, and note that $\lambda_i(C_i) = 0$ because λ_i is atomless. Let $f_i(t_i, a_i) = a_i$ for each $i \in N$ and $(t_i, a_i) \in \mathrm{graph}(\Phi_i)$. Define the correspondence $\Lambda_i \colon \mathrm{graph}(\Phi_i) \to A_i$ by setting

$$
\Lambda_i(t_i, a_i) = \begin{cases} (\max\{t_i, a_i - \eta\}, a_i) \text{ if } a_i > t_i \\ (a_i, \bar{a}) \text{ if } a_i = t_i < \bar{a} \\ \{\bar{a}\} \text{ if } a_i = t_i = \bar{a}\,. \end{cases}
$$

Then f_i and Λ_i satisfy the requirements in (4) of Theorem 7 for the given i and ε. As $i \in N$ and $\varepsilon > 0$ are arbitrary, (4) of Theorem 7 is satisfied.

Thus, by Theorem 7, Γ is Φ-virtually continuous. For each $i \in N$, define a map $h_i \colon A_i \to \mathbb{R}$ by setting $h_i(a_i) = a_i$. Using Theorem 8, with $\ell = 1$ and $f_{i,1} = h_i$ for each $i \in N$, we see that Γ is Φ-strongly indeterminate. Now by Theorem 5 we can conclude that $I(\Gamma) \cap W = E(\Gamma) \cap W \neq \emptyset$.

Remark 7. It is straightforward to generalize the above example to the case of more than one commodity by using the full generality of Theorem 7 with $\ell > 1$. It is also interesting to contrast the conclusion of the above Bertrand example with Example 3. In both examples, the two firms have asymmetric costs with probability one; however, invariance of the equilibrium set holds in Example 6 but not in Example 3. The difference is that the possible types of each firm are distributed atomlessly in Example 6, so that Theorem 8 applies to yield strong indeterminacy, which is not the case in the other example.

A Appendix

A.1 Lemmata

This section contains some preliminary lemmata. Let $\Gamma = (N, (X_i)_{i\in N}, Q)$ be a game with an endogenous sharing rule.

Lemma 4. *D_i is measurable for each $i \in N$.*

Proof. Because Q is uhc with non-empty and compact values, there is a sequence $\langle q_k \rangle_{k\in\mathbb{N}}$ of measurable selections of Q such that $\{q_k(x)\colon k \in \mathbb{N}\}$ is dense in $Q(x)$ for each $x \in X$ (see Castaing and Valadier (1977, Corollary III.3, p. 63 and Theorem III.6, p. 65)). Now, for each $i \in N$, $X \setminus D_i = \{x \in X\colon q_{k,i}(x) = q_{0,i}(x) \text{ for all } k \in \mathbb{N}\}$. □

For convenience of later reference, we record the following fact in form of a lemma; the lemma implies in particular that $S_{Q_{\text{eff}}} \neq \emptyset$ (consider $f(r) = \sum_{i\in N} r_i$).

Lemma 5. *If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then there is a $q \in S_Q$ such that, for each $x \in X$, $f(q(x)) = \max_{r\in Q(x)} f(r)$*

Proof. Use Aliprantis and Border (2006, 18.2, 18.19 and 18.20). □

Recall that the σ-algebra of the universally measurable subsets of X equals $\bigcap_\mu \mathcal{B}_\mu$, where the intersection is over all the Borel probability measures μ on X and $\mathcal{B}_\mu$ denotes the μ-completion of the Borel σ-algebra.[8] Each Borel probability measures μ on X has an unique extension to the universally measurable subsets of X, which we also denote by μ.

Lemma 6. *D_i^{eff} is universally measurable for each $i \in N$.*

Proof. **(a)** By Lemma 5, Q_{eff} has nonempty values; we now show that graph(Q_{eff}) is a Borel subset of $X \times \mathbb{R}^N$. To see this, set $A = \mathbb{R}^N$ and $B = \mathbb{R}^N$, and for each $m \in \mathbb{N}\setminus\{0\}$ let

$$G_m = \{(x, a, b) \in X \times A \times B\colon a, b \in Q(x),\ b \geq a,\ b_i \geq a_i + 1/m \text{ for some } i \in N\}.$$

[8] It is easy to see that this definition is equivalent to Castaing and Valadier (1977, Definition 21, p. 73) where the intersection is over all the Borel positive bounded measures μ on X.

Let H_m be the projection of G_m onto $X \times A$. Because Q is uhc with compact values, G_m is compact, and hence so is H_m. Now $\text{graph}(Q_{\text{eff}}) = \text{graph}(Q) \setminus \bigcup_{m \in \mathbb{N} \setminus \{0\}} H_m$.

(b) By Castaing and Valadier (1977, Theorem III.22, p. 74), it follows from (a) that there is a sequence $\langle q_k \rangle_{k \in \mathbb{N}}$ of universally measurable selections of Q such that the set $\{q_k(x) \colon k \in \mathbb{N}\}$ is dense in $Q(x)$ for each $x \in X$. Now, for each $i \in N$, $X \setminus D_i^{\text{eff}} = \{x \in X \colon q_{k,i}(x) = q_{0,i}(x) \text{ for all } k \in \mathbb{N}\}$. □

A.2 Virtual continuous games

Let $\Gamma = (N, (X_i)_{i \in N}, Q)$ be a game with an endogenous sharing rule. One aspect of Example 1 is that Dirac measures (i.e., pure strategies) are taken for the μ_i's of the definition of virtual continuity. In Lemma 7 below it is shown that this is always possible.

Lemma 7. Γ *is virtually continuous if and only if for each* $q \in S_Q$, $i \in N$, $\varepsilon > 0$ *and* $\sigma \in M$, *there exists* $\bar{x}_i \in X_i$ *such that* $\tau_{(\delta_{\bar{x}_i}, \sigma_{-i})}(D_i) < \varepsilon$ *and* $q_i(\bar{x}_i, \sigma_{-i}) > q_i(\sigma) - \varepsilon$.

Proof. The sufficiency part is obvious. For the necessity part, fix $0 < \varepsilon < 1$ and $i \in N$. Choose $a \geq 0$ so that $q_i(x) + a > 0$ for all $x \in X$. Set $k = \max\{1, \sup\{q_i(x) \colon x \in X\}\}$ and $\varepsilon' = \varepsilon^2/(2(k+a))$. Note that $\varepsilon' \leq \varepsilon/2$.

Virtual continuity gives a μ_i so that $\tau_{(\mu_i, \sigma_{-i})}(D_i) < \varepsilon'$ and $q_i(\mu_i, \sigma_{-i}) > q_i(\sigma) - \varepsilon'$. Let $E = \{x_i \in X_i \colon \sigma_{-i}(D_{i,x_i}) \geq \varepsilon\}$ where $D_{i,x_i} \subseteq X_{-i}$ is the section of D_i at x_i. By Fubini's theorem, $\mu_i(E) < \varepsilon'/\varepsilon$. Thus $\mu_i(E) < \varepsilon/(2(k+a))$ by the choice of ε and ε'. Now, again using Fubini's theorem,

$$
\begin{aligned}
&\int_{X_i \setminus E} \int_{X_{-i}} (q_i(x_i, x_{-i}) + a) \, \mathrm{d}\sigma_{-i}(x_{-i}) \, \mathrm{d}\mu_i(x_i) \\
&\qquad > q_i(\sigma) + a - \varepsilon' - \int_E \int_{X_{-i}} (q_i(x_i, x_{-i}) + a) \, \mathrm{d}\sigma_{-i}(x_{-i}) \, \mathrm{d}\mu_i(x_i) \\
&\qquad \geq q_i(\sigma) + a - \varepsilon' - \mu_i(E)(k+a) \\
&\qquad \geq q_i(\sigma) + a - \varepsilon/2 - \varepsilon/2 \\
&\qquad = q_i(\sigma) + a - \varepsilon.
\end{aligned}
$$

There must therefore be an $\bar{x}_i \in X_i \setminus E$ such that

$$
\mu_i(X_i \setminus E) \int_{X_{-i}} (q_i(\bar{x}_i, x_{-i}) + a) \, \mathrm{d}\sigma_{-i}(x_{-i}) > q_i(\sigma) + a - \varepsilon.
$$

Because $\int_{X_{-i}} (q_i(\bar{x}_i, x_{-i}) + a)\,\mathrm{d}\sigma_{-i}(x_{-i}) \geq 0$ by the choice of a, it follows that

$$\int_{X_{-i}} (q_i(\bar{x}_i, x_{-i}) + a)\,\mathrm{d}\sigma_{-i}(x_{-i}) > q_i(\sigma) + a - \varepsilon$$

and hence that $\int_{X_{-i}} q_i(\bar{x}_i, x_{-i})\,\mathrm{d}\sigma_{-i}(x_{-i}) > q_i(\sigma) - \varepsilon$. Finally, we have $\tau_{(\delta_{\bar{x}_i}, \sigma_{-i})}(D_i) < \varepsilon$ because $\bar{x}_i \in X_i \backslash E$. $\square$

Lemma 7 implies that virtual continuity in mixed strategies implies virtual continuity in pure strategies (see Remark 3 for the definition of the latter). However, the converse does not hold, as the following example shows.

Example 7. Suppose $N = \{1, 2\}$ and $X_1 = X_2 = [0, 1]$. Define a correspondence $Q_1 \colon X \to \mathbb{R}$ by setting

$$Q_1(x_1, x_2) = \begin{cases} [0, 1] & \text{if } -1/4 + x_1 \leq x_2 \leq 1/4 + x_1 \\ \{1\} & \text{otherwise.} \end{cases}$$

Define a correspondence $Q_2 \colon X \to \mathbb{R}$ by setting $Q_2(x) = \{0\}$ for all $x \in X$. Let $Q = Q_1 \times Q_2$. Evidently virtual continuity holds for pure strategies. But if σ_2 is the restriction of Lebesgue measure to the Borel sets of $[0, 1]$, then $\tau_\sigma(D_1) \geq 1/4$ for all $\sigma_1 \in M_1$, so virtual continuity fails for mixed strategies.

The next result is used in our treatment of Bertrand-Edgeworth competition.

Lemma 8. *Let $\Gamma = (N, (X_i)_{i \in N}, Q)$ be such that $X_i = A$ for each $i \in N$, where A is a perfect compact subset of $\mathbb{R}^m$, $m \geq 1$. For each $i \in N$, write*

$$\Delta_i = \Big\{x \in X \colon x_i = x_j \text{ for some } j \in N \backslash \{i\}\Big\}.$$

Suppose the following:

(1) $D_i \subseteq \Delta_i$ for each $i \in N$.

(2) Given $\varepsilon > 0$, $i \in N$ and $x_i \in A$, there is an open subset $\Lambda_i(x_i)$ of A such that (i) x_i is a cluster point of $\Lambda_i(x_i)$, (ii) whenever $a_i \in \Lambda_i(x_i)$ and $x_{-i} \in A^{n-1}$ are such that $(x_i, x_{-i}) \in D_i$ and $(a_i, x_{-i}) \notin D_i$, then $Q_i(a_i, x_{-i}) > \max Q_i(x_i, x_{-i}) - \varepsilon$ (recall that $\#(Q_i(a_i, x_{-i})) = 1$ if $(a_i, x_{-i}) \notin D_i$).

Then virtual continuity holds.

Proof. To see this, consider any $q \in S_Q$, $\sigma \in M$, $\varepsilon > 0$, and $i \in N$. Without loss of generality, take $i = 1$ and let $x_1 \in A$ be such that

$$q_1(\delta_{x_1}, \sigma_{-1}) \geq q_1(\sigma).$$

(i) For each $k \in \mathbb{N}\backslash\{0\}$ there is $a_k \in A$ such that

(a) $\tau_{(\delta_{a_k}, \sigma_{-1})}(D_1) = 0$ and

(b) $||a_k - x_1|| < 1/k$.

To see this, first observe that the set $E = \{a_1 \in A \colon \tau_{\sigma_{-1}}(\Delta_{1,a_1}) = 0\}$ is dense, writing Δ_{1,a_1} for the section of Δ_1 at a_1. Indeed, for each $i \in N\backslash\{1\}$, the set of all $r \in \mathbb{R}^m$ such that $\tau_{\sigma_{-1}}(\{x_{-1} \in A^{n-1} \colon x_i = r\}) > 0$ is countable. Now if $a_1 \notin E$ there must be $i \in N\backslash\{1\}$ such that $\tau_{\sigma_{-1}}(\{x_{-1} \in A^{n-1} \colon x_i = a_1\}) > 0$. Thus $A\backslash E$ is countable. Because A is perfect, every point of A is a condensation point for A, and the claim about E follows.

Now for each $k \in \mathbb{N}\backslash\{0\}$, choose $a_k \in \Lambda_1(x_1) \cap E$ such that (b) holds; such a_k exist by condition (2)(i), because E is dense and $\Lambda_1(x_1)$ is open. As for (a), note that we have

$$\tau_{(\delta_{a_k}, \sigma_{-1})}(D_1) \leq \tau_{(\delta_{a_k}, \sigma_{-1})}(\Delta_1) = \tau_{\sigma_{-1}}(\Delta_{1,a_k}) = 0$$

for each k, because $a_k \in E$.

(ii) Using Fubini's theorem and the fact that the countable union of null sets is a null set, we see from (b) that for $\tau_{\sigma_{-1}}$-a.e. $x_{-1} \in A_{-1}$ we have $\delta_{a_k}(D_{1,x_{-1}}) = 0$ for the x_{-1}-section of D_1, i.e., $(a_k, x_{-1}) \notin D_1$, for all $k \in \mathbb{N}\backslash\{0\}$. Hence, from (2)(ii), and because (x_1, x_{-1}) is a continuity point of q_1 whenever $(x_1, x_{-1}) \notin D_1$, we must have

$$\varliminf_{k\to\infty} q_1(a_k, x_{-1}) \geq q_1(x_1, x_{-1}) - \varepsilon$$

for $\tau_{\sigma_{-1}}$-a.e. $x_{-1} \in A^{n-1}$. It follows that

$$\begin{aligned} \varliminf_{k\to\infty} q_1(\delta_{a_k}, \sigma_{-1}) = \varliminf_{k\to\infty} \int_{A^{n-1}} q_1(a_k, x_{-1}) \, \mathrm{d}\tau_{\sigma_{-1}}(x_{-1}) \\ \geq \int_{A^{n-1}} q_1(x_1, x_{-1}) \, \mathrm{d}\tau_{\sigma_{-1}}(x_{-1}) - \varepsilon = q_1(\sigma) - \varepsilon, \end{aligned}$$

by Fatou's lemma. Thus, by (i)(a), and as $\varepsilon > 0$ is arbitrary, the requirements of virtual continuity are satisfied for player 1. $\square$

Returning to the definition of virtual continuity, it would of course be more intuitive, and for typical applications probably also sufficient (this is actually the case for the applications we will consider in this paper), to require $\tau_{(\mu_i,\sigma_{-i})}(D_i) = 0$ in the definition of virtual continuity, rather than just $\tau_{(\mu_i,\sigma_{-i})}(D_i) < \varepsilon$ for $\varepsilon > 0$. However, the "$\tau_{(\mu_i,\sigma_{-i})}(D_i) < \varepsilon$"-clause adds some generality.

Example 8. Let $N = \{1, 2\}$, and $X_1 = X_2 = [0, 1]$. Define a correspondence $Q_1 \colon X \to \mathbb{R}$ by setting

$$Q_1(x_1, x_2) = \begin{cases} [0,1] & \text{if } x_1 \le x_2 \le x_1/2 + 1/2 \ \text{ or } \ x_1 = 1 \\ \{1\} & \text{otherwise.} \end{cases}$$

Define a correspondence $Q_2 \colon X \to \mathbb{R}$ by setting $Q_2(x) = \{0\}$, $x \in X$. Let $Q = Q_1 \times Q_2$. Then Q is uhc with non-empty compact values, and

$$D_1 = \{x \in X \colon x_1 \le x_2 \le x_1/2 + 1/2\} \cup \{x \in X \colon x_1 = 1\}.$$

Pick any $q \in S_Q$. Of course, the requirements of virtual continuity are satisfied for $i = 2$. Consider $i = 1$. Pick any $\sigma_2 \in M_2$. Let $x_{1,k}$, $k \in \mathbb{N}$, be such that $x_{1,k} < 1$ for all k but $x_{1,k} \to 1$. Then, by the choice of Q, $\tau_{(\delta_{x_{1,k}},\sigma_2)}(D_1) \to 0$ and $q_1(\delta_{x_{1,k}}, \sigma_2) \to 1$, from which we can see that the requirements of virtual continuity are satisfied for $i = 1$. However, if σ_2 has full support, then $\tau_{(\sigma_1,\sigma_2)}(D_1) > 0$ for all $\sigma_1 \in M_1$.

Remark 8. Example 8 shows, in particular, that our definition of virtual continuity allows the sets D_i to be quite large. In fact, in that example, D_1 has a non-empty interior.

Lemma 9. *Give M the narrow topology. Then virtual continuity implies that each $q \in S_Q$ is mixed strategy payoff secure, i.e., for all $\sigma \in M$, $i \in N$, and $\varepsilon > 0$ there is a $\mu_i \in M_i$ and a neighborhood V of σ such that $q_i(\mu_i, \sigma'_{-i}) > q_i(\sigma) - \varepsilon$ for all $\sigma' \in V$.*

Proof. Fix $q \in S_Q$, $\sigma \in M$, $i \in N$, and $\varepsilon > 0$. Recall that $\underline{q} \in S_Q$ is such that $\underline{q}_i(x) = \min_{r \in Q(x)} r_i$ for each $x \in X$ and let $\underline{v}_i = v_{\underline{q}_i}$. Let $\mu_i \in M_i$ be such that $\underline{q}_i(\mu_i, \sigma_{-i}) > \underline{v}_i(\sigma) - \varepsilon/2$. Noting that $x \mapsto \underline{q}_i(x)$ is lsc (see Aliprantis and Border (2006, Lemma 17.30)) and, hence, so is $\sigma \mapsto \underline{q}_i(\sigma)$ (see Aliprantis and Border (2006, Theorem

15.5), let V be an open neighborhood of σ such that $\underline{q}_i(\mu_i, \sigma'_{-i}) > \underline{q}_i(\mu_i, \sigma_{-i}) - \varepsilon/2$ for all $\sigma' \in V$. Hence, for each $\sigma' \in O$,

$$q_i(\sigma) \leq v_{q_i}(\sigma) = \underline{v}_i(\sigma) < \underline{q}_i(\mu_i, \sigma_{-i}) + \frac{\varepsilon}{2} < \underline{q}_i(\mu_i, \sigma'_{-i}) + \varepsilon \leq q_i(\mu_i, \sigma'_{-i}) + \varepsilon.$$

$\square$

Lemma 10. *Let M be given the narrow topology. Then virtual continuity implies that v_{q_i} is continuous for each $q \in S_Q$ and each $i \in N$.*

Proof. Fix $q \in S_Q$ and $i \in N$. By Lemma 9 we can see that v_{q_i} is lower semicontinuous. By Lemma 5 there is a $q' \in S_q$ such that q'_i is upper semicontinuous and, thus, $v_{q'_i}$ is also upper semicontinuous. By Lemma 2, v_{q_i} is upper semicontinuous. $\square$

We conclude this section with an example showing that the converse of Lemma 2 does not hold.

Example 9. Let $N = \{1, 2\}$, and $X_1 = X_2 = [0, 1]$. Define a correspondence $Q_1 \colon X \to \mathbb{R}$ by setting

$$Q_1(x) = \begin{cases} [x_1, 1] & \text{if } x_1 < 1, \\ [x_2, 1] & \text{if } x_1 = 1. \end{cases}$$

Define a correspondence $Q_2 \colon X \to \mathbb{R}$ by setting $Q_2(x) = \{0\}$, $x \in X$. Let $Q = Q_1 \times Q_2$. Then Q is uhc with non-empty compact values, and $D_1 = X \setminus \{(1, 1)\}$. If $\sigma_2 = \delta_0$, then $\tau_\sigma(D_1) = 1$ for any σ_1, hence virtual continuity fails. However, there exists a common value function: For each $q \in S_Q$, $v_{q_2} \equiv 0$ and $v_{q_1} \equiv 1$. Indeed, for the latter, note that, as $x_1^k \to 1$ from below, $q_1(x_1^k, \sigma_2) \geq x_1^k \to 1$.

A.3 Young measures

In this section we establish a result on Young measures that is needed for our main results.

Lemma 11. *In the context and notation of Section 5, let $\sigma = (\sigma_1, \ldots, \sigma_n) \in M$ and $q \colon T \times A \to \mathbb{R}$ be a bounded measurable function. Then, for any $i \in N$, there is a measurable map $g \colon T_i \to A_i$ such that $g(t_i) \in \operatorname{supp}(\sigma_i(t_i))$ for λ_i-a.e. in T_i and $q(\delta_g, \sigma_{-i}) \geq q(\sigma_i, \sigma_{-i})$.*

Proof. Without loss of generality consider $i = 1$. Assume first that $q(t, a) \geq 0$ for all $(t, a) \in T \times A$. For each $t_{-1} \in T_{-1}$ write $\sigma_{-1}^{(\times)}(t_{-1})$ for the product measure on A_{-1} defined from the measures $\sigma_2(t_2), \ldots, \sigma_n(t_n)$. Write $\lambda_{-1}^{(\times)}$ for the product measure on T_{-1} defined from the measures $\lambda_2, \ldots, \lambda_n$, and $\tau_{-1}^{(\times)}$ for the uniquely determined probability measure on $T_{-1} \times A_{-1}$ such that $\tau_{-1}^{(\times)}(C \times B) = \int_C \sigma_{-1}^{(\times)}(t_{-1})(B)\, \mathrm{d}\lambda_{-1}^{(\times)}(t_{-1})$ for each $C \in \mathcal{B}(T_{-1})$ and $B \in \mathcal{B}(A_{-1})$. Let $\rho \colon T \to \mathbb{R}_+$ be a Radon-Nikodym derivative of λ with respect to $\lambda_1 \times \ldots \times \lambda_n$. Define $\tilde{q} \colon T \times A \to \mathbb{R}_+$ by setting $\tilde{q}(t, a) = \rho(t) q(t, a)$ for each $(t, a) \in T \times A$. In the sequel of this proof, Tonelli's theorem (in its ordinary and generalized version; see Neveu (1965, Proposition III.2.1)) is used repeatedly and invoked without explicit reference.

Note that whenever $O \subseteq A_1$ is open, then for any $t_1 \in T_1$, $\operatorname{supp}(\sigma_1(t_1)) \cap O \neq \emptyset$ if and only if $\sigma_1(t_1)(O) > 0$. Therefore, by the definition of Young measure, for such sets O, the set $\{t_1 \in T_1 \colon \operatorname{supp}(\sigma_1(t_1)) \cap O \neq \emptyset\}$ is measurable. Using Castaing and Valadier (1977, Proposition III.13, p. 69), this implies that the correspondence $t_1 \mapsto \operatorname{supp}(\sigma_1(t_1)) \colon T_1 \to A_1$ has a measurable graph. Next note that the maps $h \colon T_1 \to \mathbb{R}_+ \cup \{+\infty\}$ and $h_1 \colon T_1 \times A_1 \to \mathbb{R}_+ \cup \{+\infty\}$, defined by setting

$$h(t_1) = \int_{A_1} \int_{T_{-1}} \int_{A_{-1}} \tilde{q}(t_1, t_{-1}, a_1, a_{-1})\, \mathrm{d}\sigma_{-1}^{(\times)}(t_{-1})(a_{-1})\, \mathrm{d}\lambda_{-1}^{(\times)}(t_{-1})\, \mathrm{d}\sigma_1(t_1)(a_1)$$

and

$$h_1(t_1, a_1) = \int_{T_{-1}} \int_{A_{-1}} \tilde{q}(t_1, t_{-1}, a_1, a_{-1})\, \mathrm{d}\sigma_{-1}^{(\times)}(t_{-1})(a_{-1})\, \mathrm{d}\lambda_{-1}^{(\times)}(t_{-1})$$

respectively, are measurable. It follows that the correspondence $F \colon T_1 \to A_1$, defined by setting

$$F(t_1) = \begin{cases} \operatorname{supp}(\sigma_1(t_1)) \cap \{a_1 \in A_1 \colon h_1(t_1, a_1) - h(t_1) \geq 0\} & \text{if } h_1(t_1) < \infty \\ A_1 & \text{otherwise,} \end{cases}$$

has a measurable graph. Clearly $F(t_1) \neq \emptyset$ for all $t_1 \in T_1$. Using Castaing and Valadier (1977, III.22, p. 74), there is a universally measurable map g' such that $g'(t_1) \in F(t_1)$ for all $t_1 \in T_1$. Observe that the set $\left\{t_1 \in T_1 \colon \int_{T_{-1}} \rho(t_1, t_{-1})\, \mathrm{d}\lambda_{-1}^{(\times)}(t_{-1}) = +\infty\right\}$ is a λ_1-null set, so the same must be true of the set $\{t_1 \in T_1 \colon h(t_1) = +\infty\}$ because q is bounded. Modifying g' on a λ_1-null set, if necessary, we can therefore find a measurable map $g \colon T_1 \to A_1$ such that $\int_{T_1} h_1(t_1, g(t_1))\, \mathrm{d}\lambda_1(t_1) \geq \int_{T_1} h(t_1)\, \mathrm{d}\lambda_1(t_1)$ and $g(t_1) \in \operatorname{supp}(\sigma_1(t_1))$ for λ_1-a.e. $t_1 \in T_1$.

Now, for each fixed $t_1 \in T_1$,

$$\begin{aligned}
&\int_{A_1 \times (T_{-1} \times A_{-1})} \tilde{q}(t_1, \cdot) \, \mathrm{d}\sigma_1(t_1) \times \tau_{-1}^{(\times)} \\
&\qquad\qquad = \int_{A_1} \int_{T_{-1} \times A_{-1}} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, \mathrm{d}\tau_{-1}^{(\times)}(t_{-1}, a_{-1}) \, \mathrm{d}\sigma_1(t_1)(a_1) \\
&\qquad\qquad = \int_{T_{-1} \times A_{-1}} \int_{A_1} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, \mathrm{d}\sigma_1(t_1)(a_1) \, \mathrm{d}\tau_{-1}^{(\times)}(t_{-1}, a_{-1}) \, .
\end{aligned}$$

Consequently,

$$\begin{aligned}
&q(\sigma_1, \sigma_{-1}) \\
&\quad = \int_{T_1 \times T_{-1}} \int_A \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, \mathrm{d}\sigma_1(t_1) \times \sigma_{-1}^{(\times)}(t_{-1})(a_1, a_{-1}) \, \mathrm{d}\lambda_1 \times \lambda_{-1}^{(\times)}(t_1, t_{-1}) \\
&\quad = \int_{T_1} \int_{T_{-1}} \int_A \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, \mathrm{d}\sigma_1(t_1) \times \sigma_{-1}^{(\times)}(t_{-1})(a_1, a_{-1}) \, \mathrm{d}\lambda_{-1}^{(\times)}(t_{-1}) \, \mathrm{d}\lambda_1(t_1) \\
&\quad = \int_{T_1} \int_{T_{-1}} \int_{A_{-1}} \int_{A_1} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, \mathrm{d}\sigma_1(t_1)(a_1) \, \mathrm{d}\sigma_{-1}^{(\times)}(t_{-1})(a_{-1}) \, \mathrm{d}\lambda_{-1}^{(\times)}(t_{-1}) \, \mathrm{d}\lambda_1(t_1) \\
&\quad = \int_{T_1} \int_{T_{-1} \times A_{-1}} \int_{A_1} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, \mathrm{d}\sigma_1(t_1)(a_1) \, \mathrm{d}\tau_{-1}^{(\times)}(t_{-1}, a_{-1}) \, \mathrm{d}\lambda_1(t_1) \\
&\quad = \int_{T_1} \int_{A_1} \int_{T_{-1} \times A_{-1}} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, \mathrm{d}\tau_{-1}^{(\times)}(t_{-1}, a_{-1}) \, \mathrm{d}\sigma_1(t_1)(a_1) \, \mathrm{d}\lambda_1(t_1) \\
&\quad = \int_{T_1} \int_{A_1} \int_{T_{-1}} \int_{A_{-1}} \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, \mathrm{d}\sigma_{-1}^{(\times)}(t_{-1})(a_{-1}) \, \mathrm{d}\lambda_{-1}^{(\times)}(t_{-1}) \, \mathrm{d}\sigma_1(t_1)(a_1) \, \mathrm{d}\lambda_1(t_1) \\
&\quad = \int_{T_1} h(t_1) \, \mathrm{d}\lambda_1(t_1) \\
&\quad \leq \int_{T_1} h_1(t_1, g(t_1)) \, \mathrm{d}\lambda_1(t_1) \\
&\quad = \int_{T_1} \int_{T_{-1}} \int_{A_{-1}} \tilde{q}(t_1, t_{-1}, g(t), a_{-1}) \, \mathrm{d}\sigma_{-1}^{(\times)}(t_{-1})(a_{-1}) \, \mathrm{d}\lambda_{-1}^{(\times)}(t_{-1}) \, \mathrm{d}\lambda_1(t_1) \\
&\quad = \int_{T_1} \int_{T_{-1}} \int_A \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, \mathrm{d}\delta_g(t_1) \times \sigma_{-1}^{(\times)}(t_{-1})(a_1, a_{-1}) \, \mathrm{d}\lambda_{-1}^{(\times)}(t_{-1}) \, \mathrm{d}\lambda_1(t_1) \\
&\quad = \int_{T_1 \times T_{-1}} \int_A \tilde{q}(t_1, t_{-1}, a_1, a_{-1}) \, \mathrm{d}\delta_g(t_1) \times \sigma_{-1}^{(\times)}(t_{-1})(a_1, a_{-1}) \, \mathrm{d}\lambda_1 \times \lambda_{-1}^{(\times)}(t_1, t_{-1}) \\
&\quad = q(\delta_g, \sigma_{-1}).
\end{aligned}$$

Thus the lemma is true whenever q is non-negative. But this implies that the lemma is true for any bounded q, by the fact that if $f \colon T \times A \to \mathbb{R}$ is constant-valued, then $f(\sigma_1, \sigma_{-1}) = f(\sigma_1', \sigma_{-1})$ for any $\sigma_1' \in M_1$. $\square$

A.4 Spaces of Young measures

Fix a probability space (T, Σ, ν) and a Polish space X, and let $\mathcal{R}$ denote the set of all Young measures from T to X.

A Carathéodory integrand on $T \times X$, with control measure ν, is a measurable function $q\colon T \times X \to \mathbb{R}$ such that $q(t, \cdot)$ is continuous for each $t \in T$ and such that for some ν-integrable $\theta_q\colon T \to \mathbb{R}_+$, $\sup\{|q(t,x)|\colon x \in X\} \leq \theta_q(t)$ for each $t \in T$. Write $\mathcal{G}_\nu$ for the set of all such functions. Now the narrow topology for Young measures on $\mathcal{R}$, with ν as control measure, is the coarsest topology on $\mathcal{R}$ such that for each $q \in \mathcal{G}_\nu$ the functional

$$\gamma \mapsto \int_T \int_X q(t,x) \mathrm{d}\gamma(t)(x) \mathrm{d}\nu(t)\colon \mathcal{R} \to \mathbb{R}$$

is continuous. With this topology, $\mathcal{R}$ becomes a subset of a locally convex topological vector space (see Balder (2002, Step 2, p. 462)). It should be noted that, in general, the narrow topology for Young measures is not a Hausdorff topology.

If $\kappa\colon T \to X$ is a correspondence, then $\mathcal{R}_\kappa$ denotes the subset of $\mathcal{R}$ defined by setting

$$\mathcal{R}_\kappa = \{\gamma \in \mathcal{R}\colon \operatorname{supp}(\gamma(t)) \subseteq \kappa(t) \text{ for almost all } t \in T\}.$$

The following theorem gathers several properties of the space $\mathcal{R}_\kappa$ (see Carmona and Podczeck (2014, Theorem 10) for a proof).

Theorem 9. *Let $\kappa\colon T \to X$ be a correspondence with measurable graph such that $\kappa(t)$ is non-empty and compact for all $t \in T$. Give $\mathcal{R}$ the narrow topology for Young measures, with ν as control measure. Then the subset $\mathcal{R}_\kappa$ of $\mathcal{R}$ is non-empty, convex, closed, compact, and sequentially compact.*

Now for each $i = 1, \ldots, n$, let $(T_i, \Sigma_i, \lambda_i)$ be a probability measure, X_i a Polish space, and write $\mathcal{R}_i$ for the set of all Young measures from T_i to X_i. Set $X = \prod_{i=1}^n X_i$, $T = \prod_{i=1}^n T_i$, and $\Sigma = \bigotimes_{i=1}^n \Sigma_i$. Let $\mathcal{R}$ be the set of all Young measures from T to X, and $g\colon \prod_{i \in N} \mathcal{R}_i \to \mathcal{R}$ the map defined by setting

$$g(\sigma_1, \ldots \sigma_n)(t) = \sigma_1(t_1) \times \cdots \times \sigma_n(t_n),\ t \in T,$$

for all $(\sigma_1, \ldots \sigma_n) \in \prod_{i \in N} \mathcal{R}_i$. Write $\lambda^{(\times)}$ for the product measure defined from the measures λ_i, $i = 1, \ldots, n$, and let λ be any probability measure on (T, Σ). For each

$i = 1, \ldots, n$, give $\mathcal{R}_i$ the narrow topology for Young measures, with λ_i as control measure. Write $\mathcal{R}_\lambda$ for $\mathcal{R}$ endowed with the narrow topology for Young measures, taking λ as control measure, and $\mathcal{R}_{\lambda^{(\times)}}$ for $\mathcal{R}$ endowed with the narrow topology for Young measures, taking $\lambda^{(\times)}$ as control measure. Then the following three lemmata are true.

Lemma 12. *The map g is continuous.*

Proof. Using induction, this follows from Balder (1988, Theorem 2.5). □

Lemma 13. *If λ is absolutely continuous with respect to $\lambda^{(\times)}$, then the identity from $\mathcal{R}_{\lambda^{(\times)}}$ to $\mathcal{R}_\lambda$ is continuous.*

Proof. Let h be a Radon-Nikodym derivative of λ with respect to $\lambda^{(\times)}$. Note that if q is a Carathéodory integrand on $T \times X$ with control measure λ, then $q \times h$ is a Carathéodory integrand on $T \times X$ with control measure $\lambda^{(\times)}$. □

Lemma 14. *Suppose that T is a Polish space, and that Σ is the Borel σ-algebra. Give $M(T \times X)$ the narrow topology. Then the function $\sigma \mapsto \tau_\sigma$ from $\mathcal{R}_\lambda$ to $M(T \times X)$ is continuous.*

Proof. Let $c : T \times X \to \mathbb{R}$ be bounded and continuous. Then c is a Carathéodory integrand on $T \times X$. Since

$$\int c \mathrm{d}\tau_\sigma = \int_T \left(\int_A c(t, a) \mathrm{d}\sigma(t)(a) \right) \mathrm{d}\lambda(t),$$

the result follows. □

A.5 Proofs

This section contains the proofs of the results in the main text.

A.5.1 Proof of Example 1

Consider any $q \in S_Q$, $\sigma \in M$, $\varepsilon > 0$, and $i \in \{1, 2\}$. Without loss of generality, take $i = 1$. Using Fubini's theorem, we can find an $\bar{x}_1 \in [0, 1]$ such that $\int_{[0,1]} q_1(\bar{x}_1, x_2) \, \mathrm{d}\sigma_2(x_2) \geq q_1(\sigma)$. For any $x_1 \in X_1$, write D_{1,x_1} for the section of D_1 at x_1. If $\sigma_2(D_{1,\bar{x}_1}) = 0$, then $\tau_{(\delta_{\bar{x}_1}, \sigma_2)}(D_1) = 0$, and we are done by taking $\delta_{\bar{x}_1}$ for μ_1. Otherwise, let $\bar{x}$ be the unique point in D_1 determined by $\bar{x}_1$, and choose $C_1(\bar{x})$ corresponding to ε and $\bar{x}$ according to the hypotheses above. By (i) there is a sequence $\langle x_{1,k} \rangle$ in $C_1(\bar{x})$ with $x_{1,k} \to \bar{x}_1$ such that $\sigma_2(D_{1,x_{1,k}}) = 0$ for each k, because $D_1 \subseteq \Delta$. Now $\tau_{(\delta_{x_{1,k}},\sigma_2)}(D_1) = 0$ for all k. Moreover, because each $x \in X \backslash D_1$ is a continuity point of q_1, (ii) and Fatou's lemma ensure that $q_1(\delta_{x_{1,k}}, \sigma_2) > q_1(\sigma) - \varepsilon$ if k is large.

Remark 9. It should be obvious that in Example 1 one may take any compact metric spaces (which need not be the same) for X_1 and X_2, and for Δ any subset of $X_1 \times X_2$ such that the sections $\Delta_{x_1} = \{x_2 \in X_2 : (x_1, x_2) \in \Delta\}$ and $\Delta_{x_2} = \{x_1 \in X_1 : (x_1, x_2) \in \Delta\}$ are empty or singletons for each $x_1 \in X_1$ and $x_2 \in X_2$.

A.5.2 Proof of Lemma 1

The "only if" part is true because for each $i \in N$ there is a $q \in S_Q$ (which may depend on i) such that $q_i(x) = m_i(x)$ for all $x \in X$ (see Lemma 5). For the "if" part, let $i \in N$, $q \in S_Q$, $\varepsilon > 0$ and $\sigma \in M$ be given. Since X is compact and Q is uhc and takes compact values, there is a number B such that $||y|| \leq B$ for all $y \in Q(x)$ and $x \in X$. Let $\eta > 0$ be such that $(1 + 2B)\eta < \varepsilon$. By hypothesis, there is a $\mu_i \in M_i$ such that $m_i(\sigma) < m_i(\mu_i, \sigma_{-i}) + \eta$ and $\tau_{(\mu_i,\sigma_{-i})}(D_i) < \eta$. Now $m_i(\mu_i, \sigma_{-i}) - q_i(\mu_i, \sigma_{-i}) = \int_{D_i}(m_i - q_i)\mathrm{d}\tau_{(\mu_i,\sigma_{-i})} < 2B\eta$, so

$$q_i(\sigma) \leq m_i(\sigma) < m_i(\mu_i, \sigma_{-i}) + \eta < q_i(\mu_i, \sigma_{-i}) + (1 + 2B)\eta < q_i(\mu_i, \sigma_{-i}) + \varepsilon.$$

Also, $\tau_{(\mu_i,\sigma_{-i})}(D_i) < \varepsilon$. Thus the "if" part follows.

A.5.3 Proof of Lemma 2

Fix $i \in N$ and $\sigma \in M$. Let $\varepsilon > 0$ and $H = \{\mu_i \in M_i : \tau_{(\mu_i,\sigma_{-i})}(D_i) < \varepsilon\}$. Consider any q, $q' \in S_Q$. Then, by virtual continuity, since q, q' agree on $X \setminus D_i$,

$$|v_{q_i}(\sigma) - v_{q'_i}(\sigma)| = \left| \sup_{\mu_i \in H} q_i(\mu_i, \sigma_{-i}) - \sup_{\mu_i \in H} q'_i(\mu_i, \sigma_{-i}) \right| \leq \varepsilon 2B$$

where B as in the proof of Lemma 1. As ε is arbitrary, $v_{q_i}(\sigma) = v_{q'_i}(\sigma)$.

A.5.4 Proof of Theorem 2

Define an endogenous sharing rule game $\Gamma' = (N, (X_i)_{i \in N}, Q')$ by letting $Q'(x)$ be the convex hull of $Q(x)$ for each $x \in X$. (By Aliprantis and Border (2006, Theorem 17.35,

p. 573), Q' is uhc and takes non-empty compact values, as required by our definition of a game with an endogenous sharing rule.)

Using Lemma 1, we can see that virtual continuity of Γ implies virtual continuity of Γ', because $\max_{r\in Q(x)} r_i = \max_{r\in Q'(x)} r_i$. Also we have $S_{Q_{\text{eff}}} \cap S_{Q'_{\text{eff}}} \neq \emptyset$ (consider a $q \in S_Q$ such that $q(x)$ solves $\max_{r\in Q(x)} \sum_{i\in N} r_i$ for each $x \in X$).

Now by Simon and Zame (1990), $E(\Gamma') \neq \emptyset$, so by the previous paragraph and Theorem 1, $E(\Gamma) \neq \emptyset$, and in particular, $E(G_q) \neq \emptyset$ for every $q \in S_{Q_{\text{eff}}}$.

A.5.5 Proof of Theorem 3

That $\{\sigma \in E(\Gamma) : \tau_\sigma(D_i) = 0 \text{ for all } i \in N\} \subseteq I(\Gamma)$ is immediate from Lemma 2.

For the reverse inclusion, consider any $i \in N$. There are $\bar{q}$, $\underline{q} \in S_Q$ such that for every $x \in X$, $\bar{q}_i(x) = \max_{r\in Q(x)} r_i$ and $\underline{q}_i(x) = \min_{r\in Q(x)} r_i$ (see Lemma 5). Now if $\sigma \in M$ is such that $\tau_\sigma(D_i) > 0$, then $\bar{q}_i(\sigma) > \underline{q}_i(\sigma)$, so by Lemma 2, $\sigma \notin I(\Gamma)$.

A.5.6 Proof of Theorem 4

(i) Given $\sigma \in M$ and $i \in N$, if $\tau_\sigma(D_i^{\text{eff}}) > 0$, then there are q, $\tilde{q} \in S_{Q_{\text{eff}}}$ such that $q_i(\sigma) \neq \tilde{q}_i(\sigma)$. Indeed, as shown in the proof of Lemma 6, Q_{eff} has a measurable graph and nonempty values, so by Castaing and Valadier (1977, Theorem III.22) that there is a sequence $\langle h_k \rangle_{k\in\mathbb{N}}$ of (Borel) measurable functions $h_k \colon X \to \mathbb{R}^n$ and a Borel set $Y \subseteq X$ with $\tau_\sigma(Y) = 0$ such that $\{h_k(x) \colon k \in \mathbb{N}\}$ is a dense subset of $Q_{\text{eff}}(x)$ for all $x \in X \backslash Y$. Modify each h_k on Y so that it becomes a member of $S_{Q_{\text{eff}}}$ (by making it equal to q on Y for some $q \in S_{Q_{\text{eff}}}$; recall that $S_{Q_{\text{eff}}} \neq \emptyset$). Because D_i^{eff} belongs to the τ_σ-completion of the Borel σ-algebra of X, if $\tau_\sigma(D_i^{\text{eff}}) > 0$ then there a Borel set $H \subseteq D_i^{\text{eff}}$ with $\tau_\sigma(H) > 0$. Set $H' = H \backslash Y$. Then also $\sigma(H') > 0$. Note that $\{h_{k,i}(x) \colon k \in \mathbb{N}\}$ is a dense subset of $Q_{\text{eff},i}(x)$ for each $x \in H'$. Hence, $\#(\{h_{k,i}(x) \colon k \in \mathbb{N}\}) > 1$ for all $x \in H'$. Thus for some $k \in \mathbb{N}\backslash\{0\}$ and some Borel set $B \subseteq H'$ with $\tau_\sigma(B) > 0$ we have $h_{k,i}(x) > h_{0,i}(x)$ for all $x \in B$, or for some $k \in \mathbb{N}\backslash\{0\}$ and some Borel set $B \subseteq H'$ with $\tau_\sigma(B) > 0$ we have $h_{k,i}(x) < h_{0,i}(x)$ for all $x \in B$. In either case, set $q = h_0$ and $\tilde{q} = 1_B h_k + 1_{X\backslash B} h_0$.

(ii) Virtual continuity, Theorem 1, and Lemma 2 combine to say that whenever $\sigma \in E(\Gamma)$ and q, $\tilde{q} \in S_{Q_{\text{eff}}}$, then $q_i(\sigma) = \tilde{q}_i(\sigma)$. Thus (ii) yields part 1.

(iii) If $D_i = D_i^{\text{eff}}$ for each $i \in N$, then part 1 and Theorem 3 imply that $I(\Gamma) = E(\Gamma)$.

A.5.7 Proof of Lemma 3

(i) Given $z \in \text{bd}(A)$, there is a $p \in S$ such that $pz \geq pa$ for all $a \in A$ and such that $z - \lambda p \in A$ for all $\lambda > 0$ sufficiently small. Indeed, let C be the set of all $p \in \mathbb{R}^m$ such that $pz \geq pa$ for all $a \in A$. Then C convex, with $0 \in C$. We must have $(z - C) \cap \text{int}(A) \neq \emptyset$. Otherwise, as $\text{int}(A) \neq \emptyset$, there would be a non-zero $v \in \mathbb{R}^m$ such that $va \leq v(z-p)$ for all $a \in A$ and $p \in C$, by the separation theorem. The fact that $0 \in C$ implies that $v \in C$, and the fact that $z \in A$ implies that $vp \leq 0$ for all $p \in C$. But these implications contradict each other because $v \neq 0$ means $vv > 0$.

(ii) The map θ is continuous. To see this, suppose $p_k \to p$ in S and $z_k \to z$ in A. Set $B_k = \{a \in A\colon p_k a < p_k z_k\}$ and $B = \{a \in A\colon pa < pz\}$. Observe that

$$\bigcap_{m=0}^{\infty} \bigcup_{k \geq m}^{\infty} B \triangle B_k \subseteq \{a \in A\colon pa = pz\}.$$

Because ν is absolutely continuous with respect to Lebesgue measure, it follows that $\nu(B \triangle B_k) \to 0$, and therefore that $\nu(B_k) \to \nu(B)$.

(iii) Using (ii) we see that Q is closed. As for virtual continuity, wlog consider player 1. Suppose that $x_1 = x_2 = z \in \text{int}(A)$. Let $(r_1, r_2) \in Q(x_1, x_2)$, and let $p \in S$ be such that $r_1 = \theta(p, z)$. As $x_1 \in \text{int}(A)$, we have $x_\lambda = x_1 - \lambda p \in A$ for all sufficiently small $\lambda > 0$. Now, for such λ,

$$u_1(x_\lambda, x_2) = \theta\Big(\tfrac{x_2 - x_1 + \lambda p}{\|x_2 - x_1 + \lambda p\|}, \tfrac{1}{2}(x_1 + \lambda p + x_2)\Big) = \theta(p, z + \tfrac{1}{2}\lambda p),$$

and by (ii), $\theta(p, z + \frac{1}{2}\lambda p) \to \theta(p, z)$ as $\lambda \to 0$, so $u_1(x_\lambda, x_2) \to r_1$ as $\lambda \to 0$. This holds, in particular, if $r_1 = \max\{r_1'\colon (r_1', r_2') \in Q(x_1, x_2)\}$.

Suppose next that $x_1 = x_2 = z \in \text{bd}(A)$. Choose $p \in S$ with respect to z according to (i). Then, setting $r_1 = \theta(p, z)$, we have

$$r_1 = \theta(p, z) = 1 = \max\{r_1'\colon (r_1', r_2') \in Q(x_1, x_2)\},$$

and because $z - \lambda p \in A$ for all sufficiently small $\lambda > 0$, we can again choose x_λ for player 1 to get $u_1(x_\lambda, x_2) \to r_1$. In view of Example 1 and Remark 9 (the latter in Section A.5.1) it follows that virtual continuity is satisfied.

A.5.8 Proof of the claim in Remark 4(b)

From (iii) in the proof of Lemma 3 we see that on $(A \times A) \setminus (D \cap (\mathrm{bd}(A) \times \mathrm{bd}(A)))$, Q is the smallest closed correspondence which includes the map (u_1, u_2), writing D for the diagonal in $A \times A$. Now if $x_1 = x_2 = z \in \mathrm{bd}(A)$ and $p \in S$, consider any $z_0 \in \mathrm{int}(A)$. Then by (ii) in the proof of Lemma 3, $\theta(p, \lambda z_0 + (1 - \lambda)z) \to \theta(p, z)$ as $\lambda \to 0$, and it follows that on the entire domain $A \times A$, Q is the smallest closed correspondence which includes the map (u_1, u_2).

A.5.9 Proof of Theorem 5

The proof of the equality $E(\Gamma) \cap W = I(\Gamma) \cap W$ (points (a)-(e) below) amounts, in essence, to a reinterpretation of the proofs of Lemma 2 and Theorems 1, 3, and 4, with $T \times A$ in place of X; note that for the arguments in the proofs of those results it does not matter whether or not the τ_σ's appearing there are product measures.

(a) For each $q \in S_Q$ and $i \in N$, define value functions $v_{q_i} : M \to \mathbb{R}$ in the same way as in Section 3. Then, provided that $\sigma \in W$, Φ-virtual continuity implies that $v_{q_i}(\sigma) = v_{q'_i}(\sigma)$ for any q, $q' \in S_Q$ and any $i \in N$. This follows as in the proof of Lemma 2, just replace M_i by W_i in the definition of the set H there. Consequently, for any q, $q' \in S_Q$, if $\sigma \in W$ then $\sigma \in E(G_q)$ implies $\sigma \in E(G_{q'})$ if and only if $q_i(\sigma) = q'_i(\sigma)$ for each $i \in N$.

(b) From (a) we see that $E(\Gamma) \cap W = I_{\text{eff}}(\Gamma) \cap W$, arguing as in the proof of Theorem 1 (replacing virtual continuity by Φ-virtual continuity).

(c) Next note that $I(\Gamma) \cap W = \{\sigma \in E(\Gamma) \cap W : \tau_\sigma(D_i) = 0 \text{ for all } i \in N\}$; see the proof of Theorem 3.

(d) Putting (a) and (b) together we see that if $\sigma \in E(\Gamma) \cap W$ and $q, \tilde{q} \in S_{Q_{\text{eff}}}$, then $q_i(\sigma) = \tilde{q}_i(\sigma)$. It follows from this by arguments as in (i) of the proof of Theorem 4 that if $\sigma \in E(\Gamma) \cap W$, then $\tau_\sigma(D_i^{\text{eff}}) = 0$ for each $i \in N$.

(e) Φ-strong indeterminacy means that if $\sigma \in W$, then $\tau_\sigma(D_i^{\text{eff}}) = 0$ implies $\tau_\sigma(D_i) = 0$. From (c) and (d) we therefore conclude that $E(\Gamma) \cap W = I(\Gamma) \cap W$.

(f) It remains to see that $E(\Gamma) \cap W \neq \emptyset$. To this end, let $\bar{q} \in S_Q$ be such that $\sum_{i \in N} \bar{q}_i(t, a) = \max_{r \in Q(t,a)} \sum_{i \in N} r_i$ for each $(t, a) \in T \times A$. Note that since Q is uhc with nonempty compact values, $(t, a) \mapsto \sum_{i \in N} \bar{q}_i(t, a)$ is bounded and usc. Taking λ_i as control measure for W_i, give each W_i the narrow topology for Young measures (see Appendix A.4). Then (by Theorem 9 in Appendix A.4) each W_i becomes a non-empty compact convex subset of a locally convex topological vector space. Consider the normal form game $\bar{G} = (W_i, \bar{q}_i)_{i \in N}$, where payoffs are specified as above. Suppose temporarily that $\bar{G}$ has a Nash equilibrium, say $\bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_n)$. Thus, for each $i \in N$, $\bar{\sigma}_i \in W_i$ and $\bar{q}_i(\bar{\sigma}) \geq \bar{q}_i(\sigma_i, \bar{\sigma}_{-i})$ for all $\sigma_i \in W_i$. Pick any $i \in N$ and suppose there is a $\mu_i \in M_i$ with $\bar{q}_i(\mu_i, \bar{\sigma}_{-i}) > \bar{q}_i(\bar{\sigma})$. Then, given $\varepsilon > 0$, Φ-virtual continuity implies that there is a $\sigma_i \in W_i$ such that $\bar{q}_i(\sigma_i, \bar{\sigma}_{-i}) > \bar{q}_i(\mu_i, \bar{\sigma}_{-i}) - \varepsilon$, which implies $q_i(\sigma_i', \bar{\sigma}_{-i}) > q_i(\bar{\sigma})$ if ε is small enough. But this contradicts the fact that $\bar{\sigma}$ is a Nash equilibrium of $\bar{G}$ and we conclude that $\bar{\sigma} \in E(\Gamma) \cap W$. Now by Reny (1999, Theorem 3.1), $\bar{G}$ has a Nash equilibrium if $\bar{G}$ is quasi-concave, payoff secure, and $\sigma \mapsto \sum_{i \in N} \bar{q}_i(\sigma)$ is usc on W. Quasi-concavity is clear. The other facts are established in what follows.

(g) Give $W = \prod_{i \in N} W_i$ the product topology defined from the W_i's. Let $\widetilde{W}$ be the set of all Young measures from T to $M(A)$. Take λ as control measure for $\widetilde{W}$ and give $\widetilde{W}$ the corresponding narrow topology for Young measures. As noted above, given $\sigma \in W$, the map $t \mapsto \sigma_1(t_1) \times \cdots \times \sigma_n(t_n)$ is a Young measure from $T \to M(A)$. We may therefore define a map $f \colon W \to \widetilde{W}$ by setting

$$f(\sigma)(t) = \sigma_1(t_1) \times \cdots \times \sigma_n(t_n), \; t \in T, \; \sigma \in W.$$

It follows from Lemmata 12 and 13 in Appendix A.4 that f is continuous. Now let $\rho \colon T \times A \to \mathbb{R}$ be a bounded and usc. By Balder (1988, Theorem 2.2), the map $\tilde{\sigma} \mapsto \int_T \int_A \rho(t,a) \,\mathrm{d}\tilde{\sigma}(t) \,\mathrm{d}\lambda(t) : \widetilde{W} \to \mathbb{R}$ is usc. Consequently, as f is continuous, the map $\sigma \mapsto \int_T \int_A \rho(t,a) \,\mathrm{d}f(\sigma)(t) \,\mathrm{d}\lambda(t) : W \to \mathbb{R}$ is usc. By the definition of f, it follows that the map $\sigma \mapsto \int_{T \times A} \rho(t,a) \,\mathrm{d}\tau_\sigma(t,a) : W \to \mathbb{R}$ is usc.

(h) As noted above, $(t,a) \mapsto \sum_{i \in N} \bar{q}_i(t,a)$ is bounded and usc. Consequently, in view of (g), the map

$$\sigma \mapsto \sum_{i \in N} \bar{q}_i(\sigma) = \sum_{i \in N} \int_{T \times A} \bar{q}_i(t,a) \,\mathrm{d}\tau_\sigma(t,a) = \int_{T \times A} \sum_{i \in N} \bar{q}_i(t,a) \,\mathrm{d}\tau_\sigma(t,a)$$

is usc on W.

(i) Combining Lemmata 12–14 in Appendix A.4 shows that if $i \in N$, $\mu_i \in W_i$, and $\langle \sigma_k \rangle$ is a sequence in W with $\sigma_k \to \sigma$, then $\tau_{(\mu_i, \sigma_{k,-i})} \to \tau_{(\mu_i, \sigma_{-i})}$. From this and the argument in the proof of Lemma 9 we can see that $\bar{G}$ is payoff secure.

A.5.10 Proof of Theorem 7

Without loss of generality, consider $i = 1$. Fix any $q \in S_Q$, $\sigma \in M_1 \times W_{-1}$, and $\varepsilon > 0$. Use (2) and Lemma 11 to find a $h' \in S_{\Phi_1}$ such that $q_1(\delta_{h'}, \sigma_{-1}) \geq q_1(\sigma)$. Define $h : T_1 \to A_1$ by setting $h(t_1) = f_1(t_1, h'(t_1))$ for $t_1 \in T_1$. By (4)(i), $h \in S_{\Phi_1}$. Also by (4)(i), $q_1(t_1, t_{-1}, h(t_1), a_{-1}) \geq q_1(t_1, t_{-1}, h'(t_1), a_{-1})$ for each $t_1 \in T_1$ and each $(t_{-1}, a_{-1}) \in T_{-1} \times A_{-1}$; thus, $q_1(\delta_h, \sigma_{-1}) \geq q_1(\delta_{h'}, \sigma_{-1}) \geq q_1(\sigma)$.

We claim that for each $k \in \mathbb{N} \setminus \{0\}$ there is a $g_k \in S_{\Phi_1}$ such that

(a) $\|h(t_1) - g_k(t_1)\| < 1/k$ for all $t_1 \in T_1$;

(b) $\tau_{(\delta_g, \sigma_{-1})}(D_1) = 0$;

(c) $g_k(t_1) \in \Lambda_1(t_1, h(t_1))$ for λ_1-a.e. $t_1 \in T_1$.

To see this, let $\lambda_{-1}^{(\times)}$ be the product measure on T_{-1} defined from the measures $\lambda_2, \ldots, \lambda_n$ and let $\tau_{-1}^{(\times)}$ be the uniquely determined probability measure on $T_{-1} \times A_{-1}$ such that $\tau_{-1}^{(\times)}(C \times B) = \int_C \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(B) \, \mathrm{d}\lambda_{-1}^{(\times)}(t)$ for each $C \in \mathcal{B}(T_{-1})$ and $B \in \mathcal{B}(A_{-1})$. We claim that the set $E = \{a_1 \in A_1 \colon \tau_{-1}^{(\times)}(T_{-1} \times \Delta_{1,a_1}) = 0\}$ is a dense G_δ-set, writing Δ_{1,a_1} for the section of Δ_1 at a_1. Indeed, for each $i \in N \backslash \{1\}$ and each $0 \leq h \leq \ell$, the set $R_{i,h} = \{r \in \mathbb{R} \colon \tau_{-1}^{(\times)}(\{(t_{-1}, a_{-1}) \in T_{-1} \times A_{-1} \colon a_{i,h} = r\}) > 0\}$ is countable. Observe that if $a_1 \notin E$, then there must be an $i \in N \backslash \{1\}$ and $0 \leq h, h' \leq \ell$ such that $\tau_{-1}^{(\times)}(\{(t_{-1}, a_{-1}) \in T_{-1} \times A_{-1} \colon a_{i,h} = a_{1,h'}\}) > 0$. We must therefore have $A_1 \setminus E = \bigcup_{h'} \bigcup_{i \neq 1} \bigcup_h \bigcup_{r \in R_{i,h}} \{a_1 \in A_1 \colon a_{1,h'} = r\}$. Because A_1 is convex and has non-empty interior, the set $\{a_1 \in A_1 \colon a_{1,h'} = r\}$ is closed and nowhere dense in A_1 for each $r \in \mathbb{R}$, and the claim about E follows by Baire's category theorem.

Now, for each $k \in \mathbb{N} \setminus \{0\}$, define a correspondence $F_k \colon T_1 \backslash C_1 \to A_1$ by setting

$$F_k(t_1) = \big\{a_1 \in A_1 \colon \|h(t_1) - a_1\| < 1/k\big\} \cap \Lambda_1(t_1, h(t_1)) \cap E$$

for each $t_1 \in T_1 \backslash C_1$. Then F_k has a measurable graph. As $(t_1, h(t_1)) \in \mathrm{graph}(\Phi_1)$ and $h(t_1) \in f_1(\mathrm{graph}(\Phi_1))$ for each $t_1 \in T_1 \backslash C_1$, it follows by (4)(iv) and the properties of E that F_k has non-empty values. Consequently, by Castaing and Valadier (1977, Theorem III.22, p. 74), F_k has a universally measurable selection $g_k' \colon T_1 \backslash C_1 \to A_1$. Choosing a suitable extension to all of T_1, and making modifications on a λ_1-negligible set, if necessary, we obtain a $g_k \in S_{\Phi_1}$ such that (a) and (c) hold.

As for (b), observe that, for each $k \in \mathbb{N} \setminus \{0\}$,

$$\begin{aligned}
&\int_T \delta_{g_k}(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(\Delta_1)\, \mathrm{d}(\lambda_1 \times \ldots \times \lambda_n)(t) \\
&= \int_T \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(\Delta_{1,g_k(t_1)})\, \mathrm{d}(\lambda_1 \times \ldots \times \lambda_n)(t) \\
&= \int_{T_1} \int_{T_{-1}} \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(\Delta_{1,g_k(t_1)})\, \mathrm{d}\lambda_{-1}^{(\times)}(t_{-1})\, \mathrm{d}\lambda_1(t_1) \\
&= \int_{T_1} \tau_{-1}^{(\times)}(T_{-1} \times \Delta_{1,g_k(t_1)})\, \mathrm{d}\lambda_1(t) = 0, \text{ because } g_k(t_1) \in E \text{ for } \lambda_1\text{-a.e. } t_1 \in T_1.
\end{aligned}$$

We must therefore have $\delta_{g_k}(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(\Delta_1) = 0$ for $\lambda_1 \times \ldots \times \lambda_n$-a.e $t \in T$, hence also for λ-a.e. $t \in T$, because λ is absolutely continuous with respect to $\lambda_1 \times \ldots \times \lambda_n$. Consequently $\tau_{(\delta_{g_k}, \sigma_{-1})}(T \times \Delta_1) = 0$, and thus (3) implies (b).

As the countable union of null sets is a null set, there must be a λ-null set $H \subseteq T$ such that $\delta_{g_k}(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(\Delta_1) = 0$ for all $k \in \mathbb{N} \setminus \{0\}$ and all $t \in T \setminus H$. Let $H_1 \subseteq T_1$ be the exceptional set from (c) and let $H' = H \cup (H_1 \times T_{-1}) \cup (C_1 \times T_{-1})$, so that H' is a λ-null set in T. Fix any $t \in T \setminus H'$. Using Fubini's theorem and the fact that the countable union of null sets is a null set, we see that for $\sigma_2(t_2) \times \ldots \times \sigma_n(t_n)$-a.e. $a_{-1} \in A_{-1}$ we have $\delta_{g_k(t_1)}(\Delta_{1,a_{-1}}) = 0$ for all $k \in \mathbb{N} \setminus \{0\}$, i.e., $(g_k(t_1), a_{-1}) \notin \Delta_1$ and thus $(t_1, t_{-1}, g_k(t_1), a_{-1})) \notin D_1$ (as $D_1 \subseteq \Delta_1$). Combining this with (c), (4)((iv)), and the fact that $(t_1, t_{-1}, h(t_1), a_{-1})$ is a continuity point of q_1 if $(t_1, t_{-1}, h(t_1), a_{-1}) \notin D_1$, we see that

$$\varliminf_{k \to \infty} q_1(t_1, t_{-1}, g_k(t_1), a_{-1}) \geq q_1(t_1, t_{-1}, h(t_1), a_{-1}) - \varepsilon$$

for $\sigma_2(t_2) \times \ldots \times \sigma_n(t_n)$-a.e $a_{-1} \in A_{-1}$. Hence, by Fatou's lemma,

$$\begin{aligned}
\varliminf_{k \to \infty} &\int_{A_{-1}} q_1(t_1, t_{-1}, g_k(t_1), a_{-1})\, \mathrm{d}\sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(a_{-1}) \\
&\geq \int_{A_{-1}} q_1(t_1, t_{-1}, h(t_1), a_{-1})\, \mathrm{d}\sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(a_{-1}) - \varepsilon,
\end{aligned}$$

or, in other words,

$$\begin{aligned}
\varliminf_{k \to \infty} &\int_A q_1(t_1, t_{-1}, a_1, a_{-1})\, \mathrm{d}\delta_{g_k}(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(a_{-1}) \\
&\geq \int_A q_1(t_1, t_{-1}, a_1, a_{-1})\, \mathrm{d}\delta_h(t_1) \times \sigma_2(t_2) \times \ldots \times \sigma_n(t_n)(a_{-1}) - \varepsilon.
\end{aligned}$$

Since this is true for λ-a.e. $\in T$, we now see, again using Fatou's lemma, that

$$\begin{aligned}
\varliminf_{k\to\infty} q_1(\delta_{g_k}, \sigma_{-i}) &= \varliminf_{k\to\infty} \int_T \int_A q_1(t,a)\,\mathrm{d}(\delta_{g_k}(t_1)\times\sigma_2(t_2)\times\ldots\times\sigma_n(t_n))\,\mathrm{d}\lambda(t) \\
&\geq \int_T \varliminf_{k\to\infty} \int_A q_1(t,a)\,\mathrm{d}(\delta_{g_k}(t_1)\times\sigma_2(t_2)\times\ldots\times\sigma_n(t_n))\,\mathrm{d}\lambda(t) \\
&\geq \int_T \int_A q_1(t,a)\,\mathrm{d}(\delta_h(t_1)\times\sigma_2(t_2)\times\ldots\times\sigma_n(t_n))\,\mathrm{d}\lambda(t) - \varepsilon \\
&= q_1(\delta_h, \sigma_{-1}) - \varepsilon \geq q_1(\sigma) - \varepsilon, \text{ by the choice of } \delta_h.
\end{aligned}$$

Thus, as $q \in S_Q$, $\sigma \in M_1 \times W_{-1}$, and $\varepsilon > 0$ are arbitrary, the requirements of Φ-virtual continuity are satisfied for player 1. As the consideration of player 1 does not imply any loss of generality, Γ is Φ-virtually continuous.

A.5.11 Proof of Theorem 8

For each $i, j \in N$, $i \neq j$, and each $0 \leq h, h' \leq \ell$, let

$$K_{i,0} = (T \times A) \backslash (\mathrm{graph}(\Phi_i) \times T_{-i} \times A_{-i})$$

and

$$K_{i,j,h,h'} = \{(t,a) \in T \times A\colon f_{i,h}(a_{i,h}) = t_{i,h} \text{ and } a_{i,h} = a_{j,h'}\}.$$

Let

$$K = \bigcup_{i\in N} K_{i,0} \cup \bigcup_{i\in N} \bigcup_{j\in N, j\neq i} \bigcup_{h=1}^{\ell} \bigcup_{h'=1}^{\ell} K_{i,j,h,h'}.$$

By hypothesis, if $(t,a) \in D_i \backslash K$, then $Q(t,a) = Q_{\mathrm{eff}}(t,a)$. We therefore need to show that $\tau_\sigma(K) = 0$ whenever $\sigma \in W$. Thus fix any $\sigma \in W$. By the definition of W,

$\tau_\sigma(K_{i,0}) = 0$ for each $i \in I$. Consider any $K_{i,j,h,h'}$. Then, by the fact that $i \neq j$,

$$\begin{aligned}
\tau_\sigma(K_{i,j,h,h'}) &= \int_T \sigma(t)\big(\{a \in A\colon f_{i,h}(a_{i,h}) = t_{i,h},\ a_{i,h} = a_{j,h'}\}\big)\,\mathrm{d}\lambda(t) \\
&\leq \int_T \sigma(t)\big(\{a \in A\colon f_{i,h}(a_{j,h'}) = t_{i,h}\}\big)\,\mathrm{d}\lambda(t) \\
&= \int_T \left(\int_{A_i} \sigma_{-i}(t_{-i})\big(\{a_{-i} \in A_{-i}\colon f_{i,h}(a_{j,h'}) = t_{i,h}\}\big)\,\mathrm{d}\sigma_i(t_i)(a_i)\right)\mathrm{d}\lambda(t) \\
&= \int_T \sigma_{-i}(t_{-i})\big(\{a_{-i} \in A_{-i}\colon f_{i,h}(a_{j,h'}) = t_{i,h}\}\big)\,\mathrm{d}\lambda(t) \\
&= \int_T \sigma_{-i}(t_{-i})\big(\{a_{-i} \in A_{-i}\colon f_{i,h}(a_{j,h'}) = t_{i,h}\}\big)\rho(t)\,\mathrm{d}\lambda^{(\times)}(t) \\
&= \int_{T_{-i}} \left(\int_{T_i} \sigma_{-i}(t_{-i})(\{a_{-i} \in A_{-i}\colon f_{i,h}(a_{j,h'}) = t_{i,h}\})\rho(t)\mathrm{d}\lambda_i(t_i)\right)\mathrm{d}\lambda_{-i}^{(\times)}(t_{-i}) \\
&= 0,
\end{aligned}$$

because for each $t_{-i} \in T_{-i}$, by Fubini's theorem,

$$\begin{aligned}
&\lambda_i \times \sigma_{-i}(t_{-i})\big(\{(t_i, a_{-i}) \in T_i \times A_{-i}\colon f_{i,h}(a_{j,h'}) = t_{i,h}\}\big) \\
&= \int_{T_i} \sigma_{-i}(t_{-i})\big(\{a_{-i} \in A_{-i}\colon f_{i,h}(a_{j,h'}) = t_{i,h}\}\big)\,\mathrm{d}\lambda_i(t_i) \\
&= \int_{A_{-i}} \lambda_i\big(\{t_i \in T_i\colon f_{i,h}(a_{j,h'}) = t_{i,h}\}\big)\,\mathrm{d}\sigma_{-i}(t_{-i})(a_{-i}) = 0
\end{aligned}$$

by hypothesis (b), so $\sigma_{-i}(t_{-i})(\{a_{-i} \in A_{-i}\colon f_{i,h}(a_{j,h'}) = t_{i,h}\}) = 0$ for λ_i-a.e. $t_i \in T_i$. Since a finite union of null sets is a null set, it follows that $\tau_\sigma(K) = 0$.

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