A Lot of Ambiguity[*]

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Abstract

We consider a risk averse decision maker who dislikes ambiguity as in the Ellsberg urns. We analyze attitudes to ambiguity when the decision maker is exposed to unrelated sequences of ambiguous situations. We discuss the Choquet expected utility, the smooth, and the maxmin models. Our main results offer conditions under which ambiguity aversion disappears even without learning and conditions under which it does not. An appendix analyzes compound gambles within the expected utility model.

Keywords: Ellsberg urns, repeated ambiguity, compound gambles, Choquet expected utility, maxmin, the smooth model

1   Introduction

A patient suffers from a certain disease. The doctor offers two possible treatments. A standard, well investigated treatment Y, which with probability p leads to a good outcome and with probability $1-p$ leads to a less favorable outcome, which is still better than no treatment.[1] Alternatively, she offers him a new treatment L with somewhat ambiguous probabilities of success.

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[1]For simplicity, suppose that all outcomes are monetary payoffs, for example, the disease only effects people's ability to work.

It is however known that whatever the outcome, it improves over that of the standard treatment. Moreover, although the probabilities are not known for sure, they are believed to be somewhere around $p : 1 - p$. Let X denote the probabilistic lottery yielding the outcomes of L with the probabilities p and $1 - p$, and assume that the expected value of X is zero. Both treatments are preferred to no treatment and the question is which of the two to choose. The patient is ambiguity averse, and as the improvement in the outcomes of the new treatment is not much, he prefers the standard treatment with the known probability of success. In other words, $Y \succ L$.

The doctor does not have any information she did not share with the patient. Moreover, although she knows that she will see many patients like him, she believes that she won't gain any information about the probability of success of the new treatment, as this probability depends entirely on unobservable characteristics of the patients. Her preferences over risk and uncertain prospects are the same as the patient's (alternatively, she adopts the patient's preferences). Does it follow that she too will prefer the standard treatment to the new one?

Although they have exactly the same information and preferences and do not gain any new information by learning, there is one dimension in which the patient and the doctor are different, and this is the number of cases they face. The patient sees only one case, his. Ambiguity aversion can be explained as fear of the unknown. Many people believe that they are unlucky and therefore, if they choose the ambiguous prospect, they'll find out that the winning probabilities took a bad turn and are on the lower side of their expectations. But such people do not necessarily believe that they are always unlucky. Thus the doctor is ambiguity averse, but as she is facing many similar cases, her aversion to each case may diminish. Furthermore, this may lead her to prefer the new treatment over the standard one.

In this paper we formalize this discussion. Suppose that the doctor has to make a decision for $n > 1$ (identical) people. Denote n repetitions of the standard treatment by Y^n and of the new treatment by L^n, where both yield the sum of the outcomes of the respective treatments. Suppose that both are better than no treatment. We show that under some conditions, and for sufficiently large n, $L^n \succ Y^n$. That is, n repetitions of the new treatment are, eventually, preferred over n repetitions of the standard treatment (Theorems 1 and 3).

Next consider an alternative scenario involving, again, a patient and a doctor. This time, avoiding treatment does not lead to a bad outcome but

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may be costly, and only the ambiguous treatment L is available. Suppose that the probabilistic lottery X, yielding the outcomes of L with the probabilities $p : 1 - p$, has a positive expected value and that X and all its repetitions X^n are preferred to no treatment, no matter how small is its cost, while no treatment is preferred to L. We show that under some conditions, n repetitions of the ambiguous treatment are eventually preferred to no treatment (Theorem 2). As long as all outcomes are monetary payoffs, this is quite intuitive. Unless he is extremely ambiguity averse, it seems to make little sense for a decision maker to play a dominated lottery for fear of the unknown probabilities if at the end he receives the sum of everything that is played.

Should society encourage, maybe even enforce, the use of the ambiguous treatment? Patients may be willing to pay the extra price for the unambiguous treatment if it exists, or to bear the cost of no treatment if an alternative treatment does not exist. But if society adopts the point of view of social planners and care takers (even if they do not have any better information), then it may opt out for the ambiguous treatment. Providing general answers to such questions is beyond the scope of the current paper but our aim is to show that, at least under some conditions, such questions are not meaningless.

Theorems 1 and 2 of Section 3 analyze Choquet expected utility preferences (Schmeidler [28]). Under some conditions, similar results hold in the smooth recursive utility model (Klibanoff, Marinacci, and Mukerji [15]), but under some other conditions they do not hold (Theorem 3 in Section 4). On the other hand, in the maxmin expected utility model (Gilboa and Schmeidler [12]) similar results hold only under some extreme conditions (Theorem 4 in Section 5). We discuss some further issues and the literature in Section 6. All claims are proved in the appendixes.

Our analysis requires us to compare ambiguous acts with probabilistic lotteries. In the various models discussed in the paper probabilistic lotteries are evaluated by expected utility functionals, and require the proofs of some results which may be of independent interest. These results are grouped in Appendix B.

2 Setup

One ball is picked at random out of an urn containing balls of γ different colors. Let s_i be the state of nature "color i is picked." Denote $S = \{s_1, \ldots, s_\gamma\}$, and define $\Sigma = 2^S$. The proportion of balls of some colors may be known to be $\frac{1}{\gamma}$. In some cases, this ratio may serve as an anchor for non probabilistic states and events.[2] For example, in the 3-color Ellsberg [4] urn which contains 90 balls, of which 30 are red and each of the other 60 is either black or yellow, the anchoring probabilities are $\frac{1}{3}$ for each of the three colors and $\frac{2}{3}$ for each of complementing events.[3] Denoting the anchor probability measure by P, for $E \subseteq S$, $P(E) = \frac{|E|}{|S|} = \frac{|E|}{\gamma}$.

Assume now the existence of a sequence of such urns. Let $S_i = S$ be the set of states in urn i with the corresponding algebra $\Sigma_i = \Sigma$. The information regarding each of these urns is the same. Moreover, the outcome, or even the mere existence of any urn doesn't change the decision maker's information regarding any other urn. Let $\mathcal{S}^n = S_1 \times \ldots \times S_n$ and $\Omega^n = 2^{\mathcal{S}^n}$ (note that $\Omega^1 = \Sigma$). For $E \in \Omega^n$, define $P^n(E) = \frac{|E|}{|S|^n} = \frac{|E|}{\gamma^n}$.

Consider a non-degenerate act $L = (x_1, E_1; \ldots; x_m, E_m)$ where $x_1, \ldots, x_m \in \Re$, $x_1 < \ldots < x_m$, and $E_1, \ldots, E_m$ is a partition of S. The outcomes $x_1, \ldots, x_m$ denote departures from the current wealth level, which is assumed throughout to be fixed. Define the anchor lottery $X = (x_1, p_1; \ldots; x_m, p_m)$ where $p_i := P^1(E_i) = P(E_i)$ is the anchor probability of E_i and denote its expected value by $\mathrm{E}(X)$. The act L^n is the sequence of act L played once on each of the n urns. We assume that the decision maker is interested in the total sum of outcomes he wins but not in the order or the composition of colors leading to these wins and will therefore view L^n as $(x_1^n, E_1^n; \ldots; x_{k_n}^n, E_{k_n}^n)$, where $x_1^n = nx_1 < \ldots < x_{k_n}^n = nx_m$ and $E_i^n \in \Omega^n$ is the union of events of the form $E_{i_1} \times \ldots \times E_{i_n}$, $E_{i_j} \in \{E_1, \ldots, E_m\}$, such that the sum of their corresponding outcomes is x_i^n. The lottery $X^n = (x_1^n, p_1^n; \ldots; x_{k_n}^n, p_{k_n}^n)$ is a sequence of n independent lotteries of type X where p_i^n is the anchor probability $P^n(E_i^n)$. The lottery X^n serves as a natural anchor for L^n.

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[2]The existence of anchor probabilities in a two-color symmetric situation is in the spirit of Keynes analysis of such urns [15, p. 83]. For more on the anchoring probabilities see Chew and Sagi [3] and Ergin and Gul [7].

[3]More complicated urns are also possible, for example, an urn containing 100 balls. Twenty of which are yellow, and each of the others is either red or green. The anchoring probabilities for (Y,R,G) are $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$, but this situation can easily be described as an urn containing balls of five colors.

Example 1 Consider the following variant of the 3-color Ellsberg urn. Let $S = \{R, B, Y\}$, $E_1 = \{B, Y\}$, $E_2 = \{R\}$, $L = (-1, E_1; 2, E_2)$, and $X = (-1, \frac{2}{3}; 2, \frac{1}{3})$. Then $k_2 = 3$, $E_1^2 = E_1 \times E_1 = \{BB, BY, YB, YY\}$, $E_2^2 = (E_1 \times E_2) \cup (E_2 \times E_1) = \{RB, RY, BR, YR\}$, $E_3^2 = E_2 \times E_2 = \{RR\}$, $L^2 = (-2, E_1^2; 1, E_2^2; 4, E_3^2)$ and $X^2 = (-2, \frac{4}{9}; 1, \frac{4}{9}; 4, \frac{1}{9})$. Observe that $\mathrm{E}(X) = \mathrm{E}(X^2) = 0$. $\square$

Consider a decision maker with preferences $\succeq$ over $\mathcal{L}^n$, the space of all real acts over Ω^n. We assume that the decision maker evaluates lotteries with known probabilities using expected utility theory with the twice differentiable vNM function u. We assume that the decision maker is risk averse (hence his vNM utility u is concave) and ambiguity averse in the sense that he prefers playing X^n to playing L^n. Finally, we assume throughout that $\lim_{x \to -\infty} u'(x)$ and $\lim_{x \to -\infty} -\frac{u''(x)}{u'(x)}$ exist, but not necessarily that they are finite.

3 Choquet Expected Utility

In this section we consider preferences over ambiguous prospects that can be represented by the Choquet expected utility (CEU) model (Schmeidler [28]). According to this theory, there are capacities $\nu^n : \Omega^n \to [0, 1]$ such that $\nu^n(\varnothing) = 0$, $\nu^n(\mathcal{S}^n) = 1$, and the value of L^n, $\mathrm{CEU}^n(L^n)$, is

$$u(x_{k_n}^n)\nu^n(E_{k_n}^n) + \sum_{i=1}^{k_n-1} u(x_i^n) \left[\nu^n \left(\bigcup_{j=i}^{k_n} E_j^n \right) - \nu^n \left(\bigcup_{j=i+1}^{k_n} E_j^n \right) \right] \tag{1}$$

To ensure ambiguity aversion we assume that $\nu^n(E) \leqslant P^n(E)$ for all $E \in \Omega^n$, which is equivalent to $P^n \in \mathrm{Core}(\nu^n)$.[4]

Ambiguity aversion permits the union of two ambiguous events to be non-ambiguous. For example, in the 3-color Ellsberg urn, the union of the two ambiguous colors leads to an event with probability $\frac{2}{3}$. The contribution of an event to the value of a lottery can therefore be larger than its anchor probability. If there is only a finite number of events, then there is of course an upper bound to the ratio between the contribution of the capacities generated by all events and their anchor probabilities. Our main requirement is that

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[4]The core of a capacity ν is the set of all probability distributions q such that for all E, $q(E) \geqslant \nu(E)$.

the following boundedness condition holds uniformly for all n, that is, that the potential over-estimation of the contribution of all events will not go to infinity. Formally:

Boundedness There is K such that for all n and for all disjoint events $E, E' \in \Omega^n$, $\nu^n(E \cup E') - \nu^n(E) \leqslant KP^n(E').$[5]

This condition is satisfied in a trivial way if the capacity is a probability measure. The following is an example of non-probabilistic capacities that satisfy boundedness.

Example 2 Assume urns with 100 balls each of two colors, G and R. When there are n urns, there are 2^n possible outcomes of the samples (that is, $\{G, R\}^n$), with typical elements $t = (t_1, \ldots, t_n)$, where for all i, $t_i \in \{G, R\}$. Recall that the anchor probability P^n of each event E is $|E|/2^n$.

(i) Bounded capacities: Define ν^n by

$$\nu^n(E) = \begin{cases} 0 & P^n(E) \leqslant \frac{1}{2} \\ 2P^n(E) - 1 & \text{otherwise} \end{cases}$$

By definition, $\nu^n(E \cup E') - \nu^n(E) \leqslant \frac{|E'|}{2^{n-1}} = 2P^n(E')$, hence these capacities are bounded with $K = 2$.

Note that this is a product capacity. For all $E = E^1 \times \ldots \times E^n$, $\nu^n(E) = \prod_{i=1}^n \nu^1(E^i) = 0$, unless for all i, $E^i = \{G, R\}$, in which case $\nu^n(E) = \prod_{i=1}^n \nu^1(E^i) = 1$.

(ii) Unbounded capacities: Let $\bar{\nu}^n(E) = \sqrt{P^n(E)}$. These capacities give unproportionally high values to low probability events, hence they do not satisfy the boundedness assumption. To see why, let $E^{n\prime} = \{(G, \ldots, G)\}$ and let $E^n = \varnothing$. We obtain $\bar{\nu}^n(E^n \cup E^{n\prime}) - \bar{\nu}^n(E^n) = \frac{1}{\sqrt{2^n}}$. The ratio between this difference and 2^{-n}, the probability of $E^{n\prime}$, is $\sqrt{2^n}$, which is not bounded by any K. Note that this example does not satisfy ambiguity aversion which requires $\bar{\nu}^n(E) \leqslant P^n(E)$. For unbounded capacities satisfying ambiguity aversion, let $\tilde{\nu}^n(E) = 1 - \sqrt{1 - P^n(E)}$. These capacities give unproportionally high values to low probability events when they are added

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[5]Boundedness requires $K \geqslant 1$, for example, when $E = \varnothing$ and $E' = \mathcal{S}^n$.

to their complements. Let $E^{n'} = \{(G, \ldots, G)\}$ and let $E^n = \neg E^{n'}$. We obtain

$$\tilde{\nu}^n(E^n \cup E^{n'}) - \tilde{\nu}^n(E^n) = 1 - \left(1 - \sqrt{1 - (1 - \frac{1}{2^n})}\right) = \frac{1}{\sqrt{2^n}}$$

As before, the ratio between this difference and the probability of $E^{n'}$ is not bounded by any K. □

Following the discussion in the introduction, consider a given ambiguous act L with the anchor lottery X. Suppose that the expected value of X is zero and let X dominate a lottery Y by first order stochastic dominance (FOSD). Theorem 1 shows that as $n \to \infty$, the decision maker will prefer playing L for n times (that is, L^n) rather than playing Y for n times.

Theorem 1 Suppose that the CEU preferences satisfy ambiguity aversion, risk aversion, and boundedness. Let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) = 0$. Then for every Y dominated by X by strict FOSD there exists n^* such that for all $n \geqslant n^*$, $L^n \succ Y^n$.

Consider now a different case, where $\mathrm{E}(X) > 0$. This of course doesn't mean that the decision maker accepts X, or even that if he accepts it once he would accept it n times. And it may certainly happen that he will accept X, but will decline the corresponding ambiguous act L. For example, the decision maker may accept the lottery $(-100, \frac{1}{2}; 110, \frac{1}{2})$, yet decline the act where in the two-color Ellsberg urn he wins 110 if he correctly guesses the color of the drawn ball, but loses 100 if he does not. If that happens then, by continuity, there are lotteries Y dominated by 0 which are preferred to L. Nevertheless, if for a sufficiently large n, $X^n \succeq 0$, then for any lottery Y dominated by 0, the decision maker prefers L^n to Y^n for all sufficiently large n. Equivalently, assume that avoiding L is costly and the decision maker prefers to bear the cost to L (here Y is the sure cost of avoiding L). Then, no matter how small is its cost, eventually L^n becomes desirable.

Theorem 2 Suppose that the CEU preferences satisfy ambiguity aversion, risk aversion, and boundedness. Let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) > 0$. If there exists n_0 such that for all $n \geqslant n_0$, $X^n \succeq 0$, then for every Y dominated by 0 by strict FOSD, there exists n^* such that for all $n \geqslant n^*$, $L^n \succ Y^n$.

Further results can be obtained with further restrictions on the utility function u and on the lottery X. Assume first that u is bounded from above, which is used to avoid phenomena in the spirit of the St. Petersburg paradox. Proposition 1 shows that under these conditions, from a certain point on the ambiguous acts L^n become strictly desirable.[6]

Proposition 1 Suppose that the CEU preferences satisfy ambiguity aversion, risk aversion, and boundedness and suppose that u is bounded from above. Let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) > 0$. If there exists $\varepsilon > 0$ and n_0 such that for all $n \geqslant n_0$, $X^n \succeq n\varepsilon$, then there exists n^* such that for all $n \geqslant n^*$, $L^n \succ 0$.

Assuming exponential or linear u (thus representing constant absolute risk aversion), the next proposition strengthens Theorems 1 and 2 to general acts L, regardless of the expectation of the anchor lottery X.

Proposition 2 Suppose that the CEU preferences satisfy ambiguity aversion, constant absolute risk aversion, and boundedness. Then for every $Y \prec X$ there exists n^* such that for all $n \geqslant n^*$, $L^n \succ Y^n$.

How restrictive is the boundedness assumption? For example, does boundedness imply that ν^n converge to a capacity ν with a degenerate core, which is equal to the anchor probability measure? If this is the case, then the boundedness assumption makes the analysis trivial, because the limit of the capacities ν^n is just the anchor probability vector and CEU becomes EU. We show however that this is not the case. There are bounded capacities for which the cores do not converge to a singleton.

Example 2 (cont.) Consider ν^n from part (i) that is given by

$$\nu^n(E) = \begin{cases} 0 & P^n(E) \leqslant \frac{1}{2} \\ 2P^n(E) - 1 = 2(\frac{|E|}{2^n} - \frac{1}{2}) & \text{otherwise} \end{cases}$$

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[6]A sufficient condition for boundedness from above is that the Arrow-Pratt measure of absolute risk aversion is bounded away from 0 on an interval $[M, \infty)$. That is, that there exists $\delta > 0$ and M such that for all $z \geqslant M$, $-u''(z)/u'(z) > \delta$. To see it, let $v(z) = -e^{-\delta z}$. Then by Pratt [24], there exists a concave h such that on $[M, \infty)$ $u = h \circ v$. The boundedness of u follows from that of v.

For $s \in \mathcal{S}^n$, define

$$\tilde{P}^n(s) = \begin{cases} 0 & |\{i : s_i = G\}| < \frac{n}{2} \\ 0 & |\{i : s_i = G\}| = \frac{n}{2} \text{ and } s_1 = G \\ \frac{1}{2^{n-1}} & \text{otherwise} \end{cases}$$

For each $E \in \Omega^n$, define $\tilde{P}^n(E) = \sum_{s \in E} \tilde{P}^n(s)$. Then as explained below, for every E,

$$\tilde{P}^n(E) \geqslant \max\left\{2(\tfrac{|E|}{2^n} - \tfrac{1}{2}), 0\right\} = \nu^n(E)$$

To see it, note that for $|E| \leqslant 2^{n-1}$, the rhs is zero and the inequality is obviously satisfied. Otherwise, let $R = \{s : \tilde{P}^n(s) = \frac{1}{2^{n-1}}\}$. The probability $\tilde{P}^n$ of E is the number of sequences in $E \cap R$ times $\frac{1}{2^{n-1}}$. If $|E| \geqslant 2^{n-1}$, then $|E \cap R| \geqslant |E| - 2^{n-1}$, hence $\tilde{P}^n(E) \geqslant (|E| - 2^{n-1})\frac{1}{2^{n-1}} = 2(\frac{|E|}{2^n} - \frac{1}{2})$.

It thus follows that $\tilde{P}^n$ is in the core of ν^n and clearly $\tilde{P}^n$ and P^n do not converge to the same limit. $\square$

As the next example shows, our results do not always hold without the boundedness assumption.

Example 3 Let $u(x) = -e^{-x}$ and let $\tilde{\nu}^n(E) = 1 - \sqrt{1 - P^n(E)}$ (see Example 2 part (ii)).

For Theorem 1, consider the ambiguous act $L = (-0.5, \{R\}; 0.5, \{G\})$ with the anchor lottery $X = (-0.5, \frac{1}{2}; 0.5, \frac{1}{2})$. Let $Y = (-0.55, \frac{1}{2}; 0.45, \frac{1}{2})$. The certainty equivalent of Y^n is $-0.17n$ and that of L^n is $-0.21n$, hence the theorem does not hold.

For the other results, consider the act $L = (-.35, \{R\}; 0.65, \{G\})$ with the anchor lottery $X = (-0.35, \frac{1}{2}; 0.65, \frac{1}{2})$ and let $Y = (-0.02, 1)$. The certainty equivalent of L^n is $-0.06n$ and is smaller than that of Y^n, which is $-0.02n$. $\square$

The requirement that u is bounded from above is needed for Proposition 1. See Example 4 in the appendix. The intuition behind this example is the following. If u is bounded from above, then the EU value of the positive part of X^n is bounded, and since $X^n \succeq n\varepsilon$, the EU value of its negative part is also bounded, and by the boundedness assumption it remains so even after it is evaluated using the CEU model. On the other hand, if u is not bounded, then the EU values of the right and left sides of X^n need not be bounded, and the unbounded value of the left side can become overwhelmingly large compared to the right side even if the capacities satisfy the boundedness assumption.

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4 The Smooth Model

Klibanoff, Marinacci, and Mukerji [17] suggested the following smooth extension of the recursive model of Segal [29]. According to this model, the decision maker has a subjective set of possible probability distributions, and he attaches a subjective probability to each of them. In the smooth model he computes the certainty equivalents of each of the possible distributions using expected utility with the vNM function u and then evaluates the lottery over these values using the vNM function ϕ.[7] Ambiguity aversion in this model is reflected by ϕ being more concave than u. Ambiguity neutrality is obtained when ϕ and u are the same.

Formally, let $L = (x_1, E_1; \ldots; x_m, E_m)$ be an ambiguous act with the anchor lottery $X = (x_1, p_1; \ldots; x_m, p_m)$ where $p_j = P(E_j)$. According to the recursive model, the decision maker believes that with probability μ^i, $i = 1, \ldots, \ell$, the vector of the probabilities of the events of L is given by $p^i = (p_1^i, \ldots, p_m^i)$. Denote $X_{p^i} = (x_1, p_1^i; \ldots; x_m, p_m^i)$. The ambiguous act L is identified with the two-stage lottery $(X_{p^1}, \mu^1; \ldots; X_{p^\ell}, \mu^\ell)$. We assume that these beliefs are consistent with the anchor probabilities, that is, $p = \sum_{i=1}^{\ell} \mu^i p^i$. It follows then that $\sum_{i=1}^{\ell} \mu^i X_{p^i}$ is the anchor lottery X of L. The value of L under the smooth model is given by [8]

$$\text{SM}^{\phi u}(L) = \sum_{i=1}^{\ell} \mu^i \cdot \phi \circ u^{-1}\left(\text{EU}^u(X_{p^i})\right)$$

When there is no ambiguity (that is, the decision maker believes that with probability 1 the probability distribution associated with L is p), then the value of L is $\phi \circ u^{-1}(\text{EU}^u(X))$ which is a monotonic transformation of $\text{EU}^u(X)$. Note that $\text{EU}^u(X)$ is the value attached to L by an ambiguity neutral decision maker for whom $\phi = u$. To see why, observe that

$$\text{SM}^{uu}(L) = \sum_{i=1}^{\ell} \mu^i \cdot \text{EU}^u(X_{p^i}) = \text{EU}^u\left(\sum_{i=1}^{\ell} \mu^i X_{p^i}\right) = \text{EU}^u(X)$$

As before, let X^n and L^n be n-repetitions of X and L. The value of X^n is $\text{EU}^u(X^n)$. Consider L^n. A typical sequence in L^n is a list of n lotteries, each

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[7] The original paper [17] denoted this function v.

[8] Since this model is using two different vNM functions, we add a superscript index (u or ϕ) to indicate the utility function used in the EU operator.

taken from the set $\{X_{p^1}, \ldots, X_{p^\ell}\}$, where X_{p^i} appears j_i times, $i = 1, \ldots, \ell$, and $\sum_i j_i = n$. The probability of such a sequence is the product of the corresponding μ^i probabilities, that is, $\prod_i (\mu^i)^{j_i}$. There are $(\ell)^n$ (ℓ to the power of n) such possible sequences, denote them $\{Y_j^n\}_{j=1}^{(\ell)^n}$ and denote their corresponding probabilities μ_j^n. The act L^n is thus identified with the two-stage lottery $(Y_1^n, \mu_1^n; \ldots; Y_{(\ell)^n}^n, \mu_{(\ell)^n}^n)$. We obtain that

$$\mathrm{SM}^{\phi u}(L^n) = \sum_{j=1}^{(\ell)^n} \mu_j^n \cdot \phi \circ u^{-1}(\mathrm{EU}^u(Y_j^n)) \tag{2}$$

The next theorem shows that the result of Theorem 1 holds if the absolute measures or risk aversion of u and ϕ converge to the same limit as $x \to -\infty$. Although the outcomes of X^n and L^n are spread over the whole range of $[nx_1, nx_m]$, as n increases the probability of falling in any finite segment goes down to zero, and as $\mathrm{E}(X) = 0$ and u and ϕ are concave, most of the EU^u values of the relevant lotteries depend on the behavior of the functions with respect to the (very) negative outcomes. This suggests that the evaluations of L^n and X^n depend on $\lim_{x \to -\infty} -\frac{\phi''(x)}{\phi'(x)}$ and $\lim_{x \to -\infty} -\frac{u''(x)}{u'(x)}$. Observe that although the identity $-\frac{\phi''(x)}{\phi'(x)} \equiv -\frac{u''(x)}{u'(x)}$ implies that ϕ is an affine transformation of u, the restriction $\lim_{x \to -\infty} -\frac{\phi''(x)}{\phi'(x)} = \lim_{x \to -\infty} -\frac{u''(x)}{u'(x)}$ does not imply that in the limit ϕ is an affine transformation of u. For example, let $u(x) = x$ and $\phi(x) = x^3$ on $(-\infty, -1]$.

Theorem 3 Suppose that the SM preferences satisfy ambiguity and risk aversion. Let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) = 0$. If $\lim_{x \to -\infty} -\frac{\phi''(x)}{\phi'(x)} = \lim_{x \to -\infty} -\frac{u''(x)}{u'(x)}$, then for every Y dominated by X by strict FOSD there exists n^* such that for all $n \geqslant n^*$, $L^n \succ Y^n$.

Proposition 1 analyzed conditions under which, within the CEU model, the acts L^n become strictly desirable. The next proposition offers conditions for a similar result under the SM model. For this, we restrict attention to the case where u represents constant absolute risk aversion. Observe that by risk aversion, $X \succ 0$ implies that $\mathrm{E}(X) > 0$.

Proposition 3 Suppose that the SM preferences satisfy ambiguity aversion and constant absolute risk aversion. If $\lim_{x \to -\infty} -\frac{\phi''(x)}{\phi'(x)} = \lim_{x \to -\infty} -\frac{u''(x)}{u'(x)}$, then

for every ambiguous act L with an anchor lottery $X \succ 0$ there exists n^* such that for all $n > n^*$, $L^n \succ 0$.

Theorem 3 and Proposition 3 assume that $\lim_{x \to -\infty} -\frac{\phi''(x)}{\phi'(x)} = \lim_{x \to -\infty} -\frac{u''(x)}{u'(x)}$. The next proposition shows the necessity of this condition. We say that the risk aversion of utility function u is bounded from above [from below] by ζ if for all x, $-u''(x)/u'(x)$ is less than [more than] ζ. We show that if the degree of risk aversion of ϕ is bounded from below by $t^* > 0$, then for u with degree of risk aversion that is bounded from above by a sufficiently small s^*, if Y is sufficiently close to X then $Y^n \succ L^n$, even if Y is FOSD dominated by X.

Proposition 4 Let the SM preferences satisfy ambiguity and risk aversion such that the risk aversion of ϕ is bounded from below by $t^* > 0$. For every ambiguous act L with anchor lottery X such that $\mathrm{E}(X) = 0$ there is $s^* > 0$ and a neighborhood $\mathcal{N}$ of X such that if the risk aversion of u is bounded from above by s^*, then for every $Y \in \mathcal{N}$ there is n^* such that for all $n > n^*$, $Y^n \succ L^n$.

The next proposition shows that if u represents constant absolute risk aversion and the degree of risk aversion of ϕ is bounded from below by slightly more than that of v, then for each ambiguous lottery L, regardless of the expected value of its probabilistic anchor X, for Y sufficiently close to X $Y^n \succ L^n$, even if Y is FOSD dominated by X.

Proposition 5 Let the SM preferences satisfy ambiguity aversion and constant absolute risk aversion with parameter s. If the risk aversion index of ϕ is bounded from below by $t > s$, then for every ambiguous act L with anchor lottery X there is a neighborhood $\mathcal{N}$ of X such that for every $Y \in \mathcal{N}$ there is n^* such that for all $n > n^*$, $Y^n \succ L^n$.

Remark: We presented our results in terms of the relationship between the ambiguous lottery L and its anchor lottery X. Formally, however, the results are about the relationship between L and the lottery $\bar{X} = \sum_{i=1}^{\ell} \mu^i X_{p^i}$, which is the average belief of the decision maker over the true probabilities of L. Although mathematically X and $\bar{X}$ may be different, we believe that similarly to Keynes' [15] arguments, these average beliefs should be equal to the natural anchor probabilities whenever such intuitive probabilities exist.

5   Maxmin Expected Utility

Gilboa and Schmeidler [12] suggested the following maxmin expected utility (MEU) theory. Under ambiguity, the decision maker behaves as if he has a (convex) set of possible probability distributions as well as a utility function u. For each act he computes the expected utility of u with respect to the different possible probability distributions, and evaluates the act as the minimum of these values.

As in the previous sections, let $L = (x_1, E_1; \ldots; x_m, E_m)$ be an ambiguous act and let $X = (x_1, p_1; \ldots; x_m, p_m)$ is the anchor lottery associated with L. Denote the set of possible probability distributions by Q, with typical elements of the form $q = (q_1, \ldots, q_m)$. Denote $X_q = (x_1, q_1; \ldots; x_m, q_m)$. The value of L under the maxmin model is given by

$$\text{MEU}(L) = \min_{q \in Q} \text{EU}(X_q)$$

To facilitate ambiguity aversion, we assume that the anchor probabilities $p = (p_1, \ldots, p_m) \in Q$ and that there is $\hat{q} \in Q$ such that X strictly dominates $X_{\hat{q}}$ by FOSD, hence $\text{MEU}(L) \leqslant \text{EU}(X_{\hat{q}}) < \text{EU}(X)$.

Consider $L^n = \left(x_1^n, E_1^n; \ldots; x_{k_n}, E_{k_n}^n\right)$ and the corresponding anchor lottery $X^n = \left(x_1^n, p_1^n; \ldots; x_{k_n}^n, p_{k_n}^n\right)$ where $p_j^n = P^n(E_j^n)$. As is standard, we assume that the set of possible priors for L^n is $Q^n = Q \times \ldots \times Q$ and define

$$\text{MEU}(L^n) = \min_{q^n \in Q^n} \text{EU}(X_{q^n}^n)$$

As the lottery $(X_{\hat{q}})^n$ is possible under this set of priors, it follows that the priors in Q^n that minimize the MEU value of L^n must yield a value that cannot exceed the EU value of $(X_{\hat{q}})^n$. Since X strictly dominates $X_{\hat{q}}$ by FOSD, the lottery X^n strictly dominates $(X_{\hat{q}})^n$. Hence $\text{MEU}(L^n) < \text{EU}(X^n)$.

Suppose that the decision maker is extremely risk averse, in which case his evaluation of a lottery will be close to his evaluation of its worst outcome. Since L^n cannot be inferior to its worst outcome nx_1, it follows that such a decision maker will be almost indifferent between X^n and L^n. Let $Y = X - \varepsilon$ for some $\varepsilon > 0$. Since the worst outcome of Y^n is $n(x_1 - \varepsilon)$, an extremely risk averse person will eventually prefer L^n to Y^n. Theorem 4 formalizes this argument and shows that this is the only case in which the repeated ambiguous act L^n becomes superior to every such Y. Otherwise, the extreme level of ambiguity aversion generated by the maxmin model will keep L^n less desirable than Y^n for a sufficiently small ε.

Theorem 4 Let the MEU preferences satisfy ambiguity and risk aversion and let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) = 0$.

1. If $\lim_{x \to -\infty} -\frac{u''(x)}{u'(x)} = \infty$, then for every $Y = X - \varepsilon$, $\varepsilon > 0$, there exists n^* such that for all $n \geqslant n^*$, $L^n \succ Y^n$.

2. If $\lim_{x \to -\infty} -\frac{u''(x)}{u'(x)} < \infty$, then there exists $\varepsilon > 0$ such that for all $Y = X - \varepsilon'$, $\varepsilon' < \varepsilon$, there exists n^* such that for all $n \geqslant n^*$, $Y^n \succ L^n$.

Consider now the case $\mathrm{E}(X) > 0$. The following proposition demonstrates that if there exists $\tilde{q} \in Q$ for which the expected value of $X_{\tilde{q}}$ is negative, then the implications of Theorem 2 and Proposition 1 of the CEU model and Proposition 3 of the smooth model do not hold.

Proposition 6 Let the MEU preferences satisfy risk aversion. For every ambiguous act L with an anchor lottery X such that $\mathrm{E}(X) > 0$, if there exists $\tilde{q} \in Q$ such that $\mathrm{E}(X_{\tilde{q}}) < 0$ then for a sufficiently large n, $0 \succ L^n$.

6 Discussion

As early as 1961 did William Fellner [8, pp. 678–9] ask: "there is the question whether, if we observe in him [the decision maker] the trait of nonadditivity, he is or is not likely gradually to lose this trait *as he gets used to the uncertainty with which he is faced*." Fellner pointed out a fundamental problem in answering this question empirically: In an experiment, decision makers may understand that the ambiguity is generated by a randomization mechanism and is therefore not ambiguous, but this is not necessarily the case with processes of nature or social life.

Our analysis shows that a lot depends on the way we choose to model ambiguity. Under some assumptions, some aspects of ambiguity aversion become insignificant when the decision maker is faced with many similar ambiguous situations, at least within the CEU and the smooth models, and sometimes even in the maxmin model. The term "similar" is of course not well defined, but loosely speaking, our analysis shows that even though decision makers don't learn anything new about the world as they face repeated ambiguity, they may still learn not to fear this lack of knowledge.

Throughout the paper we assumed the existence of anchor probabilities. Such probabilities do not always exist. Suppose that the decision maker is

told that a certain urn contains balls of two colors, of which exactly 100 are red and the rest are green. The decision maker may have beliefs about the total number of balls in the urn (and hence about the number of green balls), but there seems to be no natural beliefs to which the ambiguity can be related. On the other hand, in the standard Ellsberg 2- or 3-color urns, whatever can be said about one unknown color can also be said about the other color, thus creating a natural symmetry between them. In our model, the symmetry is translated to the existence of anchor probabilities and to the requirement that these probabilities are the same.

The proofs of Theorems 1, 3, and 4 reveal another property of preferences as n increases to infinity. Denote by c^n and d^n the certainty equivalents of X^n and L^n. These theorems show that $\lim_{n\to\infty} \frac{d^n}{n} = \lim_{n\to\infty} \frac{c^n}{n}$. This interpretation of the theorems emphasizes the certainty equivalents per case. An alternative way to analyze attitudes per case is to divide the acts L^n and the anchoring lotteries X^n by n. Note that by the law of large numbers, the probabilistic lotteries $\frac{X^n}{n}$ converge to the expected value of X. As we now explain, extending our results to the analysis of $\frac{L^n}{n}$ and $\frac{X^n}{n}$ is quite straightforward. In the CEU model, the results hold since the value of $\frac{L^n}{n}$ converges to the value of the average outcome of $\frac{X^n}{n}$ (see Fact 1 in Appendix A). Similarly, the results hold in the smooth model since the reduced forms of $\frac{L^n}{n}$ is $\frac{X^n}{n}$, by the law of large numbers $\lim_{n\to\infty} \frac{X^n}{n} = \mathrm{E}(X)$, and the smooth model is continuous. In the maxmin model, if there is $\hat{q} \in Q$ such that X strictly dominates $X_{\hat{q}}$ by FOSD, then the value of $\frac{L^n}{n}$ is smaller than that of $\frac{(X_{\hat{q}})^n}{n}$ and hence is smaller, even in the limit, than that of $\frac{X^n}{n}$ (their evaluation is the same only if the decision maker is extremely risk averse).

Note however that, as was argued by Samuelson [26], when decision makers are confronted with sequences like X^n they may not evaluate them by looking at the limit of their average distributions. Moreover, the ranking obtained by average distributions is different from the ranking obtained by the distributions of the sum of the outcomes, as the difference in the limits of $\frac{L^n}{n}$ and $\frac{X^n}{n}$ does not determine the decision maker's preferences. For example, consider EU preferences where $X = (-100, \frac{1}{2}; 200, \frac{1}{2})$, $Y = (1, 1)$, and $u(x) = -e^{-ax}$ such that $c^1(X) = 0$. Then, $\lim_{n\to\infty} \frac{X^n}{n} = 50 > 1 \equiv \frac{Y^n}{n}$ while $X^n \sim 0 < n \equiv Y^n$. By continuity, similar examples can be created for all models discussed in this paper.

Maccheroni and Marinacci [19] extended the law of large numbers and proved that as $n \to \infty$, the capacity of the event "the average outcome of the

ambiguous act L is between its CEU value (with the linear utility $u(x) = x$) and minus the CEU value of $-L$" is one. Similarly to this extension of the law of large numbers, the central limit theorem of classical probability theory was also extended to the uncertainty framework. This was done by Marinacci [21], who used a certain set of capacities, and by Epstein, Kaido, and Seo [6], who made use of belief functions. The latter authors also studied confidence regions. Note that all these models assume convex capacities, which we do not.[9]

Very few experiments checked attitudes to repeated ambiguity (although it seems that several more are currently being conducted). Liu and Colman [18] reported that participants chose ambiguous options significantly more frequently in repeated-choice than in single-choice. This suggests that repetition diminishes the effect of ambiguity aversion. Filiz-Ozbay, Gulen, Masatlioglu, and Ozbay [9] asserted that ambiguity aversion diminishes with the size of the urn. The intuition behind their result agrees with our finding, since both are based on the idea that the more options there are (number of balls to draw from or a larger number of urns) the less plausible is the extreme pessimistic view that Nature always acts against the decision-maker. On the other hand, Halevy and Feltkamp [13] and Epstein and Halevy [5] conducted experiments that involve drawing from two urns and report that when no information regarding the dependence between the urns is provided, individuals display higher ambiguity aversion.

Other models imply a connection between CEU and EU. Klibanoff [16] studied the relation between stochastic independence and convexity of the capacity in the CEU model and found that together they imply EU (hence the capacity must be additive). His results are not related to ours since we do not assume stochastic independence and, furthermore, the capacities we analyze are not required to be convex.

Appendix A: Proofs

Denote the variance of X by σ^2 and the certainty equivalents of X^n and L^n by c^n and d^n respectively. In the proofs of this Appendix we show the relationships between $\lim_{n \to \infty} \frac{c^n}{n}$ and $\lim_{n \to \infty} \frac{d^n}{n}$ to obtain the desired results.

---

[9]Convexity of the capacity ν means that $\nu(E) + \nu(E') \leqslant \nu(E \cup E') + \nu(E \cap E')$. For further analysis of these concepts, see Ghirardato and Marinacci [11] and Chateauneuf and Tallon [2]. See also Machina and Siniscalchi [20].

Given the anchor lottery $X^n = (x_1^n, p_1^n; \ldots; x_{k_n}^n; p_{k_n}^n)$, define $g^n : [0,1] \to [0,1]$ such that for $i = 1, \ldots, k_n$,

$$g^n \left(\sum_{j=1}^{i} p_j^n \right) = 1 - \nu^n \left(\bigcup_{j=i+1}^{k_n} E_j^n \right)$$

and let g^n be piecewise linear on the segments $[0, p_1^n]$ and $[\sum_{j=1}^{i} p_j^n, \sum_{j=1}^{i+1} p_j^n]$, $i = 1, \ldots, k_n - 1$. Note that by ambiguity aversion for all E, $\nu^n(E) \leqslant P^n(E)$, hence by the piece-wise linearity of g^n, we have $g^n(p) \geqslant p$. Eq. (1) thus becomes

$$\text{CEU}^n(L^n) = u(x_1^n)g^n(p_1^n) + \sum_{i=2}^{k_n} u(x_i^n) \left[g^n \left(\sum_{j=1}^{i} p_j^n \right) - g^n \left(\sum_{j=1}^{i-1} p_j^n \right) \right]$$

Denote by F_Z the distribution of lottery Z. In the sequel we use the integral versions of the expected utility and the CEU models:

$$\text{EU}(X^n) = \int u(z) \, \mathrm{d}F_{X^n}(z)$$

$$\text{CEU}^n(L^n) = \int u(z) \, \mathrm{d}g^n(F_{X^n}(z)) \tag{3}$$

The value of $\text{CEU}^n(L^n)$ is the same as the rank-dependent value of X^n (Quiggin [25]). Observe that by the boundedness assumption, for each n, g^n is Lipschitz with K. That is, for all $p > p'$, $g^n(p) - g^n(p') \leqslant K(p - p')$. Consider the case where $\lim_{n \to \infty} g^n$ exists, denote it g, and assume that the convergence is uniform. This is the case, for example, when the ν^n functions coincide on common points. For such convergence we obtain:

Fact 1 Let X be the anchor of the ambiguous act L. Then $\lim_{n \to \infty} \text{CEU}^n(\frac{L^n}{n}) = \lim_{n \to \infty} \text{EU}(\frac{X^n}{n}) = u(\text{E}(X))$.

Proof: Let $g = \lim_{n \to \infty} g^n$ and define $\text{CEU}(L^n) = \int u(z) \, \mathrm{d}g(F_{X^n}(z))$. Now

$$
\begin{aligned}
\lim_{n \to \infty} \text{CEU}^n(\tfrac{L^n}{n}) &= \lim_{n \to \infty} \text{CEU}(\tfrac{L^n}{n}) \\
&= \lim_{n \to \infty} \int u(z) \, \mathrm{d}g(F_{X^n/n}(z)) \\
&= \int u(z) \, \mathrm{d}g \left(\lim_{n \to \infty} F_{X^n/n}(z) \right) = u(\text{E}(X))
\end{aligned}
$$

where the first equality follows by the uniform convergence of g^n, the third equality follows since the rank-dependent functional is continuous in the distributions, and the last one follows by the law of large numbers.

Since the expected-utility functional is continuous in the distributions, the equality $\lim_{n\to\infty} \mathrm{EU}(\frac{X^n}{n}) = u(\mathrm{E}(X))$ follows again by the law of large numbers. ∎

The proofs of this appendix use several claims regarding expected utility theory. All these claims are proved as lemmas in Appendix B.

Proof of Theorem 1: We first prove that $\lim_{n\to\infty} \frac{d^n}{n} = \lim_{n\to\infty} \frac{c^n}{n}$. We divide the proof into several cases. Assume throughout, wlg, that $u(0) = 0$ and $u'(0) = 1$. In the proof we use the fact that by ambiguity and risk aversion, for all n, $\mathrm{CEU}^n(L^n) \leqslant \mathrm{EU}(X^n) \leqslant u(\mathrm{E}(X^n)) = u(n\mathrm{E}(X)) = u(0) = 0$, hence $d^n \leqslant c^n \leqslant 0$.

(i) $\lim_{x\to-\infty} u'(x) = H < \infty$: By assumption $h \geqslant 1 = u'(0)$. Since u is concave, $\lim_{x\to-\infty} u''(x) = 0$. By Lemma 6 case 1 (i), $\lim_{n\to\infty} \frac{c^n}{n} = 0$, hence it is enough to prove that $\lim_{n\to\infty} \frac{d^n}{n} \geqslant 0$. Define $w(x) = Hx$ for $x \leqslant 0$ and $w(x) = 0$ for $x > 0$. Since u is concave and $\lim_{x\to-\infty} u'(x) = H$, it follows that $u(x) \geqslant w(x)$ for all x. Let CEU^n_w denote the CEU^n functional with respect to w. Then $\mathrm{CEU}^n(L^n) \geqslant \mathrm{CEU}^n_w(L^n)$. Hence for $\frac{1}{2} < \alpha < 1$ and for sufficiently large n

$$
\begin{aligned}
u(d^n) &= \mathrm{CEU}^n(L^n) \geqslant \mathrm{CEU}^n_w(L^n) \\
&= \int w(z)\, \mathrm{d}g^n(F_{X^n}(z)) = H \int_{z\leqslant 0} z\, \mathrm{d}g^n(F_{X^n}(z)) \\
&\geqslant KH \int_{z\leqslant 0} z\, \mathrm{d}F_{X^n}(z) \geqslant KH \left(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha \right)
\end{aligned}
$$

where the second inequality follows by the assumption that g^n is Lipschitz with K and the last inequality follows by Lemma 2 (all lemmas are in Appendix B). Since u is concave and $u'(0) = 1$, $d^n \geqslant u(d^n) \geqslant KH(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha)$. Therefore, $\lim_{n\to\infty} \frac{d^n}{n} \geqslant KH \lim_{n\to\infty} \left(\frac{x_1\sigma^2}{n^{2\alpha-1}} - \frac{1}{n^{1-\alpha}} \right) = 0$.

(ii) $\lim_{x\to-\infty} u'(x) = \infty$: Recall that $\mathrm{CEU}^n(L^n)$ and $\mathrm{EU}(X^n)$ are negative. We

have

$$\lim_{n\to\infty} \frac{\mathrm{CEU}^n(L^n)}{\mathrm{EU}(X^n)} \leqslant \lim_{n\to\infty} \frac{\int_{x<0} u(x)\,\mathrm{d}g^n(F_{X^n}(x))}{\mathrm{EU}(X^n)}$$

$$\leqslant \lim_{n\to\infty} \frac{K\int_{x<0} u(x)\,\mathrm{d}F_{X^n}(x)}{\mathrm{EU}(X^n)} = K$$

The second inequality follows by the fact that all the g^n functions are Lipschitz with the same value of K and the equality is obtained by Lemma 3. It thus follows that for sufficiently large n,

$$u(d^n) \geqslant (K+1)u(c^n) \tag{4}$$

Let $a = \lim_{x\to-\infty} -\frac{u''(x)}{u'(x)}$. We consider two cases:

(ii-a) $a = 0$: Since u is concave, $u(0) = 0$, and $u'(0) = 1$, $(K+1)u(c^n) \geqslant u((K+1)c^n)$, implying $d^n \geqslant (K+1)c^n$. By Lemma 6 case 1 (ii), $\lim_{n\to\infty} \frac{c^n}{n} = 0$, hence $\lim_{n\to\infty} \frac{d^n}{n} = 0$.

(ii-b) $a > 0$: It follows by the concavity of u and by the fact that $d^n \leqslant c^n$ that

$$\frac{u(c^n) - u(d^n)}{c^n - d^n} \geqslant u'(c^n)$$

hence by inequality (4), for sufficiently large n,

$$c^n - d^n \leqslant \frac{u(c^n) - u(d^n)}{u'(c^n)} \leqslant -\frac{Ku(c^n)}{u'(c^n)}$$

By l'Hopital's rule, since $\lim_{x\to-\infty} u(x) = -\infty$ and $\lim_{x\to-\infty} u'(x) = \infty$,

$$\lim_{x\to-\infty} -\frac{u'(x)}{u(x)} = \lim_{x\to-\infty} -\frac{u''(x)}{u'(x)} = a > 0$$

By Lemma 4, $\lim_{n\to\infty} c^n = -\infty$, hence for a sufficiently large n,

$$0 \leqslant \lim_{n\to\infty} \left[\frac{c^n}{n} - \frac{d^n}{n}\right] \leqslant \lim_{n\to\infty} -\frac{Ku(c^n)}{u'(c^n)} \cdot \frac{1}{n} = -K\lim_{n\to\infty}\frac{u(c^n)}{u'(c^n)} \cdot \lim_{n\to\infty}\frac{1}{n} = 0$$

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It thus follows that in cases (i) and (ii), $\lim\limits_{n\to\infty} \frac{d^n}{n} = \lim\limits_{n\to\infty} \frac{c^n}{n}$.

Denote this common limit $\hat{c}$. By Lemma 6, $\hat{c}$ is the certainty equivalent of X under v, where $v(x) = x$ if $a = 0$, and $v(x) = -e^{-ax}$ if $a > 0$. Consider Y which is dominated by X by strict FOSD, and let $\hat{b} < \hat{c}$ be the certainty equivalent of Y under v. Let b^n be the certainty equivalent of Y^n under u. By Lemma 6, $\lim\limits_{n\to\infty} \frac{b^n}{n} = \hat{b}$, hence $\lim\limits_{n\to\infty} \frac{b^n}{n} < \lim\limits_{n\to\infty} \frac{d^n}{n}$. It thus follows that for sufficiently large n, $d^n > b^n$, hence $L^n \succ Y^n$. ∎

Proof of Theorem 2: We show first that $\lim\limits_{n\to\infty} \frac{d^n}{n} \geqslant 0$. Assume wlg that $u(0) = 0$ and that $u'(0) = 1$. Also, assume wlg that $n_0 = 1$, hence $c^n \geqslant 0$ for all n.

(i) $\lim\limits_{x\to-\infty} u'(x) = \infty$: Define $u^n(x) = u(x) - u(nx_m)$ where as before x_m is the highest possible outcome of X, and note that $u^n(nx_m) = 0$ and $u^n(x) < 0$, for all other outcomes of X^n. As u^n is a positive affine transformation of u, it implies the same values of c^n and d^n as u. As $c^n \geqslant 0$, the above inequalities and the boundedness assumption imply that for $\mathrm{CEU}^n_{u^n}$, the CEU^n functional with respect to u^n,

$$u^n(d^n) = \mathrm{CEU}^n_{u^n}(L^n) = \int u^n(z)\,\mathrm{d}g^n(F_{X^n}(z))$$

$$\geqslant K\int u^n(z)\,\mathrm{d}F_{X^n}(z)$$

$$= Ku^n(c^n) \geqslant Ku^n(0)$$

Going back to u, noting that $1 - K \leqslant 0$ and that, by concavity and the assumption that $u(0) = 0$, $u(nx_m) \leqslant nu(x_m)$,

$$u(d^n) = u^n(d^n) + u(nx_m) \geqslant Ku^n(0) + u(nx_m)$$

$$= -Ku(nx_m) + u(nx_m) = (1 - K)u(nx_m)$$

$$\geqslant n(1 - K)u(x_m)$$

Denote $A = (1 - K)u(x_m)$. By assumption, $A \leqslant 0$. Note that using l'Hopital's rule, $\lim\limits_{x\to-\infty} u'(x) = \infty$ implies

$$\lim\limits_{y\to-\infty} \frac{u^{-1}(y)}{y} = \lim\limits_{x\to-\infty} \frac{x}{u(x)} = \lim\limits_{x\to-\infty} \frac{1}{u'(x)} = 0$$

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Then, $d^n \geqslant u^{-1}(nA)$ implies $\lim\limits_{n \to \infty} \frac{d^n}{n} \geqslant \lim\limits_{n \to \infty} \left(\frac{u^{-1}(nA)}{nA} \right) A = 0$.

(ii) $\lim\limits_{x \to -\infty} u'(x) = H < \infty$: Proceed as in case (i) in the proof of Theorem 1 to show that $\lim\limits_{n \to \infty} \frac{d^n}{n} \geqslant 0$. (Note that since $\mathrm{E}(X) > 0$, Lemma 2 implies $\int_{x \leqslant 0} x \, \mathrm{d}F_{X^n}(x) \geqslant \frac{x_1 \sigma^2}{n^{2(\alpha-1)}} - n^\alpha$ for sufficiently large n.)

We obtain in both case (i) and case (ii) that $\lim\limits_{n \to \infty} \frac{d^n}{n} \geqslant 0$. Similarly to the last paragraph in the proof of Theorem 1, replacing X with 0 implies that for a sufficiently large n, $L^n \succ Y^n$. $\blacksquare$

Proof of Proposition 1: Assume wlg that $u(x) < 0$ for all x and that $\lim\limits_{x \to \infty} u(x) = 0$. Then

$$u(d^n) = \mathrm{CEU}^n(L^n) = \int u(z) \, \mathrm{d}g^n(F_{X^n}(z)) \geqslant K \int u(z) \, \mathrm{d}(F_{X^n}(z)) \geqslant K u(c^n)$$

Since $X^n \succ n\varepsilon$ for a sufficiently large n, we have $c^n > n\varepsilon$. As $n\varepsilon$ goes to infinity, $\lim\limits_{n \to \infty} u(c^n) = 0$ and, by the above argument, $\lim\limits_{n \to \infty} u(d^n) = 0$. This implies the existence of n^* such that for all $n > n^*$, $u(d^n) > u(0)$. For these n, $d^n > 0$ and $L^n \succ 0$. $\blacksquare$

Proof of Proposition 2: As in the proof of Theorem 1, we prove first that $\lim\limits_{n \to \infty} \frac{d^n}{n} = \lim\limits_{n \to \infty} \frac{c^n}{n}$. We consider two cases: (i) u is linear and (ii) u is exponential.

(i) The function u is linear: The case where $\mathrm{E}(X) = 0$ is covered by case (i) in the proof of Theorem 1 (note that, by construction, $c^n = 0$ for all n). Assume that the expected value is not zero. It follows from eq. (1) that since the utility is linear, $\mathrm{CEU}^n(\tilde{L} + \eta) = \mathrm{CEU}^n(\tilde{L}) + \eta$ for all $\tilde{L}$. Denote $\hat{X} = X - \mathrm{E}(X)$ and $\hat{L} = L - \mathrm{E}(X)$, and let $\hat{d}^n \sim \hat{L}^n$. By the above, $\mathrm{E}(\hat{X}) = 0$ implies $\lim\limits_{n \to \infty} \frac{\hat{d}^n}{n} = 0$. Now

$$
\begin{aligned}
d^n &= CEU^n(L^n) = CEU^n((\hat{L} + \mathrm{E}(X))^n) \\
&= CEU^n(\hat{L}^n + n\mathrm{E}(X)) = CEU^n(\hat{L}^n) + n\mathrm{E}(X) = \hat{d}^n + n\mathrm{E}(X)
\end{aligned}
$$

Hence $\lim\limits_{n \to \infty} \frac{d^n}{n} = \mathrm{E}(X)$. Clearly, $\frac{c^n}{n} = \frac{n\mathrm{E}(X)}{n} = \mathrm{E}(X)$ as well.

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(ii) $u(x) = -e^{-ax}$ with $a > 0$. By Lemma 1, $c^n = nc^1$ and hence $\lim_{n\to\infty} \frac{c^n}{n} = c^1$. By the definition of c^1 we have

$$
\begin{aligned}
\mathrm{EU}(X - c^1) &= \int -e^{-az}\,\mathrm{d}F_{X-c^1}(z) = \int -e^{-a(z-c^1)}\,\mathrm{d}F_X(z) \\
&= e^{ac^1} \int -e^{-az}\,\mathrm{d}F_X(z) = e^{ac^1}(-e^{-ac^1}) = -1
\end{aligned}
\tag{5}
$$

and by the definition of d^n we get

$$
\begin{aligned}
\mathrm{CEU}^n\left((L - \tfrac{d^n}{n})^n\right) &= \int -e^{-az}\,\mathrm{d}g^n\left(F_{(X-\frac{d^n}{n})^n}(z)\right) \\
&= \int -e^{-az}\,\mathrm{d}g^n(F_{X^n - d^n}(z)) \\
&= \int -e^{-a(z-d^n)}\,\mathrm{d}g^n\left(F_{X^n}(z)\right) \\
&= e^{ad^n} \int -e^{-az}\,\mathrm{d}g^n\left(F_{X^n}(z)\right) \\
&= e^{ad^n}\left(-e^{-ad^n}\right) = -1
\end{aligned}
\tag{6}
$$

Assume, by way of negation, that the sequence $\left\{\frac{d^n}{n}\right\}_{n=1}^{\infty}$ does not converge to c^1. As this sequence is bounded between the two extreme outcomes of X, there is a converging subsequence d^{n_j} and $\varepsilon > 0$ such that $\lim_{j\to\infty} \frac{d^{n_j}}{n_j} < c^1 - \varepsilon$. Wlg assume that for all j, $\frac{d^{n_j}}{n_j} < c^1 - \varepsilon$. Hence,

$$
\begin{aligned}
\mathrm{CEU}^n\left(\left(L - \tfrac{d^{n_j}}{n_j}\right)^{n_j}\right) &= \int -e^{-az}\,\mathrm{d}g^n\left(F_{(X-d^{n_j}/n_j)^{n_j}}(z)\right) \\
&> \int -e^{-az}\,\mathrm{d}g^n\left(F_{(X-c^1+\varepsilon)^{n_j}}\right)(z) \geqslant -K \int e^{-az}\,\mathrm{d}F_{(X-c^1+\varepsilon)^{n_j}}(z) \\
&= -K\left[\int e^{-az}\,\mathrm{d}F_{X-c^1+\varepsilon}(z)\right]^{n_j} = -K\left[\int e^{-a(z+\varepsilon)}\,\mathrm{d}F_{X-c^1}(z)\right]^{n_j} \\
&= -Ke^{-an_j\varepsilon}\left[\int e^{-az}\,\mathrm{d}F_{X-c^1}(z)\right]^{n_j} = -Ke^{-an_j\varepsilon} \xrightarrow[j\to\infty]{} 0
\end{aligned}
$$

where the second equality follows by Lemma 1 and the last equality follows by eq. (5). Therefore, for sufficiently large j,

$$
\mathrm{CEU}^n\left(\left(L - \tfrac{d^{n_j}}{n_j}\right)^{n_j}\right) > -1
$$

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in contradiction with eq. (6). To conclude, here too $\lim_{n\to\infty} \frac{d^n}{n} = c^1 = \lim_{n\to\infty} \frac{c^n}{n}$.

We now continue as in the last paragraph in the proof of Theorem 1, noting that here $u = v$, hence $Y \prec X$ implies $\hat{b} < \hat{c}$. ∎

Example 4 The boundedness of u from above is required for Proposition 1. Let $X = (-\frac{1}{4}, \frac{1}{2}; \frac{3}{4}, \frac{1}{2})$. Define ν^n as in example 2 by Define ν^n by

$$
\nu^n(E) = \begin{cases} 0 & P^n(E) \leqslant \frac{1}{2} \\ 2P^n(E) - 1 & \text{otherwise} \end{cases}
$$

We get

$$
\mathrm{EU}(X^{4n}) = \sum_{i=-n}^{3n} \binom{4n}{i+n} \frac{1}{2^{4n}} u(i) \tag{7}
$$

$$
\mathrm{CEU}^n(L^{4n}) = 2 \sum_{i=-n}^{n-1} \binom{4n}{i+n} \frac{1}{2^{4n}} u(i) + \binom{4n}{2n} \frac{1}{2^{4n}} u(n) \tag{8}
$$

Let $u(x) = x$ for $x \geqslant 0$. We define $u(-n)$ inductively. Let

$$
v_n = -\sum_{i=-n+1}^{-1} \binom{4n}{i+n} u(i) - \sum_{i=1}^{n-1} \binom{4n}{i+n} i - \binom{4n}{2n} \frac{n}{2} \tag{9}
$$

$$
w_n = 2u(-n+1) - u(-n+2)
$$

and define u for $x < 0$ as follows. For $n = 1, \ldots$, let $u(-n) = \min\{v_n, w_n\}$, and for $x \in (-n, -n+1)$ let $u(x) = u(-n) + (x+n)[u(-n+1) - u(-n)]$. The function u is strictly increasing and weakly concave.

Claim 1 $\lim_{n\to\infty} u(-n)/n = -\infty$.

Proof: Suppose not. Then there exists $A > 0$ such that for all n, $-u(-n)/n \leqslant A$, and since between $-n$ and $-n+1$ the function u is linear, it follows that for all n, $-u(-n)/n \leqslant A$.

By definition, $u(-n) \leqslant v_n$, hence it follows by eqs. (8) and (9) that for all n, $\text{CEU}^n(X^{4n}) \leqslant 0$. On the other hand, by eq. (8),

$$\text{CEU}^n(X^{4n}) = 2\sum_{i=-n}^{-1}\binom{4n}{i+n}\frac{u(i)}{2^{4n}} + 2\sum_{i=1}^{n-1}\binom{4n}{i+n}\frac{i}{2^{4n}} + \binom{4n}{2n}\frac{n}{2^{4n}}$$

$$\geqslant -\frac{(n-1)nA}{2^{4n-1}}\binom{4n}{n-1} + 1 \times \left[\frac{1}{2} - \Pr(X^{4n} \leqslant 0)\right] \quad (10)$$

Let $\beta_n = \frac{(n-1)nA}{2^{4n-1}}\binom{4n}{n-1}$. Clearly

$$\frac{\beta_{n+1}}{\beta_n} = \frac{n(n+1)A2^{4n-1}\binom{4n+4}{n}}{(n-1)nA2^{4n+3}\binom{4n}{n-1}}$$

$$= \frac{(n+1)(4n+4)(4n+3)(4n+2)(4n+1)}{16(n-1)n(3n+4)(3n+3)(3n+2)} \rightarrow \frac{4^4}{16 \times 3^3} = \frac{16}{27}$$

Hence $\lim\limits_{n\to\infty}\beta_n = 0$. Likewise, $\Pr(X^{4n} \leqslant 0) \leqslant \frac{n}{2^{4n}}\binom{4n}{n} \rightarrow 0$, hence the expression of eq. (10) converges to $\frac{1}{2}$; a contradiction. ♦

Define $n_0 = 0$, and let n_i satisfy

1. $u(-n_i) = v_{n_i}$

2. For $n_{i-1} < j < n_i$, $u(-j) < v_j$

It follows by Claim 1 that $\{n_i\}$ is not a finite sequence, as otherwise the function u would become linear from a certain point on to the left and will never intersect the line Ax for sufficiently high A.

By definition, $\text{CEU}^n(X^{4n_i}) = 0$ and $d^{4n_i} = 0$. It thus follows by eq. (7) that

$$u(c^{4n_i}) = \text{EU}(X^{4n_i}) = \left[\binom{4n_i}{2n_i}\frac{n_i}{2} + \sum_{i=n_i+1}^{3n_i}\binom{4n_i}{i+n_i}i\right]\frac{1}{2^{4n_i}}$$

$$> \frac{n_i}{2} \times \Pr\left(X^{4n_i} \geqslant n_i\right) = \frac{n_i}{4}$$

Since it is positive, $u(c^{4n_i}) = c^{4n_i}$, hence for $\varepsilon = \frac{1}{16}$, $X^{4n_i} \succ 4n_i\varepsilon$ while $d^{4n_i} \equiv 0$ implying $L^{4n_i} \sim 0$. □

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Proof of Theorem 3: The certainty equivalents c^n and d^n are defined by $u(c^n) = \mathrm{EU}^u(X^n)$ and $\phi(d^n) = \mathrm{SM}^{\phi u}(L^n)$.[10] By ambiguity aversion, ϕ is more concave than u and $d^n \leqslant c^n$. Let $\bar{d}^n$ be the certainty equivalent of L^n under $\mathrm{SM}^{\phi\phi}$ and since ϕ is more concave than u, $\bar{d}^n \leqslant d^n$. Hence

$$\lim_{n\to\infty} \frac{\bar{d}^n}{n} \leqslant \lim_{n\to\infty} \frac{d^n}{n} \leqslant \lim_{n\to\infty} \frac{c^n}{n}$$

Using $\mathrm{SM}^{\phi\phi}(L^n) = \mathrm{EU}^\phi(X^n)$, Lemma 6 implies $\lim_{n\to\infty} \frac{\bar{d}^n}{n} = \lim_{n\to\infty} \frac{c^n}{n}$. Hence, $\lim_{n\to\infty} \frac{d^n}{n} = \lim_{n\to\infty} \frac{c^n}{n}$. The rest of the proof is similar to the last paragraph in the proof of Theorem 1. ∎

Proof of Proposition 3: As in the proof of Theorem 3, the certainty equivalent of X^n under the expected utility EU^ϕ is below that of the certainty equivalent of L^n under the smooth model.

Consider a linear u. By assumption $\mathrm{E}(X) > 0$ and $\lim_{x\to-\infty} -\frac{\phi''(x)}{\phi'(x)} = 0$. Hence (see Nielsen [23, Prop. 1]) for sufficiently large n, $\mathrm{EU}^\phi(X^n) > \phi(0)$. Together with the observation of the first paragraph, this implies that for all such n, $L^n \succ 0$.

Next suppose that $u(x) = -e^{-sx}$ for some $s > 0$ and define, for $t > 0$, $v_t(x) = -e^{-tx}$. Then $\lim_{x\to-\infty} -\frac{u''(x)}{u'(x)} = \lim_{x\to-\infty} -\frac{\phi''(x)}{\phi'(x)} = s$ and, by assumption, $\mathrm{EU}^u(X) > u(0) = -1$. By continuity, for t close to s, $\mathrm{EU}^{v_t}(X) > v_t(0) = -1$. Choose such $t > s$ and note (Lemmas 7) that there exists y such that $\phi(x) > v_t(x)$ for all $x < y$. Wlg assume $y < 0$ and $\phi(0) = 0$. Then

$$\int_{x<0} \phi(x)\,\mathrm{d}F_{X^n}(x) = \int_{x<y} \phi(x)\,\mathrm{d}F_{X^n}(x) + \int_y^0 \phi(x)\,\mathrm{d}F_{X^n}(x)$$

$$\geqslant \int_{x<y} v_t(x)\,\mathrm{d}F_{X^n}(x) + \phi(y)\Pr(y \leqslant X^n < 0)$$

$$\geqslant \mathrm{EU}^{v_t}(X^n) + \phi(y)\Pr(y \leqslant X^n < 0)$$

$$= -|\mathrm{EU}^{v_t}(X)|^n + \phi(y)\Pr(y \leqslant X^n < 0) \underset{n\to\infty}{\longrightarrow} 0$$

where the limit is 0 because $\mathrm{EU}^{v_t}(X) \in (-1,0)$ and $\lim_{n\to\infty}\Pr(y \leqslant X^n < 0) = 0$. Observe that since v_t is exponential, $\mathrm{EU}^{v_t}(X^n) = -|\mathrm{EU}^{v_t}(X)|^n$ (see

---

[10]The certainty equivalent of the smooth model is computed using ϕ since $\mathrm{SM}^{\phi u}(x, E_1; \ldots; x, E_m) = \phi \circ u^{-1}(u(x)) = \phi(x)$.

Lemma 1). The second inequality follows by the fact that v_t is negative. As $\lim_{n\to\infty} \mathrm{EU}^\phi(X^n)_{x\geqslant 0} = \sup_x \phi(x)$ (see Nielsen [23, Lemma 1]), we conclude that for sufficiently large n, $\mathrm{EU}^\phi(X^n) > \phi(0)$ and as before $L^n \succ 0$. ∎

Proof of Proposition 4: Let $\bar{u}(x) = x$ and $\phi^*(x) = -e^{-t^*x}$. Denote $z_i = \mathrm{E}(X_{p^i})$, $Z = (z_1, \mu^1; ...; z_\ell, \mu^\ell)$ and note that

$$\mathrm{E}(Z) = \sum_{i=1}^\ell \mu^i \mathrm{E}(X_{p^i}) = \mathrm{E}\left(\sum_{i=1}^\ell \mu^i X_{p^i}\right) = \mathrm{E}(X) = 0$$

If the decision maker is using ϕ^* and $\bar{u}$, then

$$
\begin{aligned}
\mathrm{SM}^{\phi^*\bar{u}}(L) &= \sum_{i=1}^\ell \mu^i \cdot \phi^* \circ \bar{u}^{-1}(\mathrm{EU}^{\bar{u}}(X_{p^i})) = \sum_{i=1}^\ell \mu^i \phi^*(\mathrm{E}(X_{p^i})) \\
&= \sum_{i=1}^\ell \mu^i \phi^*(z_i) = \mathrm{EU}^{\phi^*}(Z)
\end{aligned}
$$

Also, it follows from eq. (2) that for all $n > 1$

$$\mathrm{SM}^{\phi^*\bar{u}}(L^n) = \sum_{j=1}^{(\ell)^n} \mu_j^n \cdot \phi^* \circ \bar{u}^{-1}(\mathrm{EU}^{\bar{u}}(Y_j^n)) = \sum_{j=1}^{(\ell)^n} \mu_j^n \phi^*(\mathrm{E}(Y_j^n))$$

The expected value of Y_j^n is the sum of the expected values of the sequence of lotteries it represents. As there are in this sequence j_i lotteries of type X_{p^i}, $i = 1, \ldots, \ell$, the expected value of Y_j^n is $\sum_{i=1}^\ell j_i \mathrm{E}(X_{p^i})$. Hence

$$
\begin{aligned}
\sum_{j=1}^{(\ell)^n} \mu_j^n \phi^*(\mathrm{E}(Y_j^n)) &= \sum_{j=1}^{(\ell)^n} \mu_j^n \phi^*\left(\sum_{i=1}^\ell j_i \mathrm{E}(X_{p^i})\right) \\
&= \sum_{j=1}^{(\ell)^n} \mu_j^n \phi^*\left(\sum_{i=1}^\ell j_i z_i\right) = \mathrm{EU}^{\phi^*}(Z^n)
\end{aligned}
$$

Where the last equation follows by the fact that $\sum_{i=1}^\ell j_i z_i$ is an outcome of the lottery Z^n which is obtained from playing n times lottery Z. Let $\bar{d}^n$ is the certainty equivalent of L^n under the functions $\bar{u}$ and ϕ^* and obtain that for all n

$$\bar{d}^n = (\phi^*)^{-1}(\mathrm{SM}^{\phi^*\bar{u}}(L^n)) = (\phi^*)^{-1}(\mathrm{EU}^{\phi^*}(Z^n))$$

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By the strict concavity of ϕ^*, all $\bar{d}^n$ are negative. As ϕ^* is exponential, Lemma 1 implies that

$$\frac{\bar{d}^n}{n} = \bar{d}^1 \tag{11}$$

Consider the utility function $v^s(x) = -e^{-sx}$. Since this function represents constant absolute risk aversion, it follows that for this function, the average certainty equivalent of X^n, $\frac{\bar{c}^n_s}{n}$, equals the certainty equivalent of X, $\bar{c}^1_s$. As in the proof of Lemma 6 case 1 (ii), as $s \to 0$, $\bar{c}^1_s \to 0$ as well. Let $s^* > 0$ be such that $\bar{d}^1 < \bar{c}^1_{s^*}$ and denote $v^* = v^{s^*}$.

Let u be less risk averse than v^*. Then $\lim\limits_{n\to\infty} \frac{c^n_u}{n}$ computed with respect to u is greater than or equal to $\bar{c}^1_{s^*}$. Let d^n_u be the certainty equivalent of L^n under u and ϕ. As the risk aversion of ϕ is bounded from below by t and u is concave, it follows that for every n, $d^n_u \leqslant \bar{d}^n$. By eq. (11) we get

$$\lim_{n\to\infty} \frac{d^n_u}{n} \leqslant \lim_{n\to\infty} \frac{\bar{d}^n}{n} = \bar{d}^1 < \bar{c}^1_{s^*} = \lim_{n\to\infty} \frac{\bar{c}^n_{s^*}}{n} \leqslant \lim_{n\to\infty} \frac{c^n_u}{n}$$

Let $\mathcal{N} = \{Y : \mathrm{EU}^{v^*}(Y) > v^*(\bar{d}^1)\}$. Clearly, X is in the interior of $\mathcal{N}$. Choose $Y \in \mathcal{N}$. For a utility function w, define b^n_w to be the certainty equivalent of Y^n under w. Since v^* is exponential, $\lim\limits_{n\to\infty} \frac{b^n_{v^*}}{n} = v^{*-1}(\mathrm{EU}^{v^*}(Y)) > \bar{d}^1$.

Consider now u which is less risk averse than v^*. Then for every n, $b^n_u \geqslant b^n_{v^*}$, hence $\lim\limits_{n\to\infty} \frac{b^n_u}{n} \geqslant \lim\limits_{n\to\infty} \frac{b^n_{v^*}}{n} > \bar{d}^1 \geqslant \lim\limits_{n\to\infty} \frac{d^n_u}{n}$. Therefore for every such u we get that for every $Y \in \mathcal{N}$ there is n^* such that for all $n \geqslant n^*$ and for this u, $Y^n \succ L^n$. ∎

Proof of Proposition 5: Consider an ambiguous act L with its anchor lottery X. Our first aim is to show that

$$\lim_{n\to\infty} \frac{d^n}{n} < \lim_{n\to\infty} \frac{c^n}{n} \tag{12}$$

By construction,

$$u(c^1) = \mathrm{EU}^u(X) = \mathrm{EU}^u\left(\sum_{i=1}^{\ell} \mu^i X_{p^i}\right) = \sum_{i=1}^{\ell} \mu^i \mathrm{EU}^u(X_{p^i})$$

and

$$\phi(d^1) = \sum_{i=1}^{\ell} \mu^i (\phi \circ u^{-1})(\mathrm{EU}^u(X_{p^i}))$$

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Rewriting the equations and denoting $h = \phi \circ u^{-1}$ yields

$$u(c^1) = \sum_{i=1}^{\ell} \mu^i \mathrm{EU}^u(X_{p^i}) \ \text{ and } \ h(u(d^1)) = \sum_{i=1}^{\ell} \mu^i h(\mathrm{EU}^u(X_{p^i})) \quad (13)$$

Assume first that ϕ is exponential of the form $\phi(x) = -e^{-tx}$. If u is linear, then similarly to the proof of the first part of Proposition 4, $\frac{d^n}{n} = d^1 < c^1$. Next, consider exponential $u(x) = -e^{-sx}$ where, by assumption, $s > 0$. Since $t > s$, $h(y) = -(-y)^{t/s}$ is strictly concave and increasing. Then equations (13) imply $u(d^1) < u(c^1)$ and $d^1 < c^1$.

By Lemma 1, $\frac{c^n}{n} = c^1$ for all n. Moreover, denoting $c_i = u^{-1}(\mathrm{EU}^u(X_{p^i}))$ and using Lemma 1, for any Y_j^n which is the sum of the sequence of lotteries $(X_{p^1})^{j_1}, \ldots, (X_{p^\ell})^{j_\ell}$ where $\sum_i j_i = n$,

$$\begin{aligned}
\mathrm{EU}^u(Y_j^n) &= -\left|\mathrm{EU}^u(X_{p^1})\right|^{j_1} \times \ldots \times \left|\mathrm{EU}^u(X_{j_\ell})\right|^{j_\ell} \\
&= -\left(e^{-sc_1}\right)^{j_1} \times \ldots \times \left(e^{-sc_\ell}\right)^{j_\ell} \\
&= -e^{-s(j_1 c_1 + \ldots + j_\ell c_\ell)} \\
&= u(j_1 c_1 + \ldots + j_\ell c_\ell)
\end{aligned}$$

Therefore, denoting $C = (c_1, \mu^1; \ldots; c_\ell, \mu^\ell)$, $\mathrm{SM}^{\phi u}(L)$ can be written as $\mathrm{EU}^\phi(C)$:

$$\mathrm{SM}^{\phi u}(L) = \sum_{i=1}^{\ell} \mu^i \phi[u^{-1}(\mathrm{EU}^u(X_{p^i}))] = \sum_{i=1}^{\ell} \mu^i \phi(c_i) = \mathrm{EU}^\phi(C)$$

and for every n, $\mathrm{SM}^{\phi u}(L^n)$ can be written as $\mathrm{EU}^\phi(C^n)$:

$$\begin{aligned}
\mathrm{SM}^{\phi u}(L^n) &= \sum_{j=1}^{(\ell)^n} \mu_j^n \phi[u^{-1}(\mathrm{EU}^u(Y_j^n))] \\
&= \sum_{j=1}^{(\ell)^n} \mu_j^n \phi[j_1 c_1 + \ldots + j_\ell c_\ell] = \mathrm{EU}^\phi(C^n)
\end{aligned}$$

Using $d^1 = \phi^{-1}(\mathrm{EU}^\phi(C))$ and $d^n = \phi^{-1}(\mathrm{EU}^\phi(C^n))$, Lemma 1 implies $\frac{d^n}{n} = d^1$ for all n and hence $\frac{d^n}{n} \equiv d^1 < c^1 \equiv \frac{c^n}{n}$.

Finally, if ϕ is not exponential, then repeat the above analysis for the less concave function $\bar{\phi}(x) = -e^{-tx}$ where $t > s$. Denote by $\bar{d}^n$ the certainty

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equivalent of L^n under $\bar{\phi}$. Then $d^n \leqslant \bar{d}^n$, hence $\frac{d^n}{n} \leqslant \frac{\bar{d}^n}{n} \equiv \bar{d}^1 < c^1 \equiv \frac{c^n}{n}$, which is the desired inequality (12).

The claim of the proposition follows similarly to the last two paragraphs of the proof of Proposition 4. Let $\mathcal{N} = \{Y : \mathrm{EU}^u(Y) > u(\bar{d}^1)\}$. Clearly, X is in the interior of $\mathcal{N}$. Choose $Y \in \mathcal{N}$ and let b^n be the certainty equivalent of Y^n under u. Since u is exponential, $\lim_{n\to\infty} \frac{b^n}{n} = u^{-1}(\mathrm{EU}^u(Y)) > \bar{d}^1 \geqslant \lim_{n\to\infty} \frac{d^n}{n}$. Therefore, there is n^* such that for all $n \geqslant n^*$, $Y^n \succ L^n$. $\blacksquare$

Proof of Theorem 4: Consider first the case $\lim_{x\to-\infty} -\frac{u''(x)}{u'(x)} = \infty$. By Lemma 6 case 3, $\lim_{n\to\infty} \frac{c^n}{n} = x_1$. Similarly, for $Y = X - \varepsilon$, the certainty equivalent b^n of Y^n satisfies $\lim_{n\to\infty} \frac{b^n}{n} = x_1 - \varepsilon$. Now $d^n \geqslant nx_1$ implies $\lim_{n\to\infty} \frac{b^n}{n} = x_1 - \varepsilon < x_1 \leqslant \lim_{n\to\infty} \frac{d^n}{n}$, hence there exists n^* such that for all $n \geqslant n^*$, $L^n \succ Y^n$.

Next, consider the case $\lim_{x\to-\infty} -\frac{u''(x)}{u'(x)} = a \in (0, \infty)$. By Lemma 6 case 2, $\lim_{n\to\infty} \frac{c^n}{n} = \hat{c}$ where $\hat{c}$ is the certainty equivalent of X under the utility $v(x) = -e^{-ax}$. Let $\hat{q} \in Q$ be a probability vector such that X strictly FOSD dominates $X_{\hat{q}}$ and let $\hat{d}$ denote the certainty equivalent of $X_{\hat{q}}$ under v. Clearly, $\hat{d} < \hat{c}$. Define $\hat{d}^n = u^{-1}(\mathrm{EU}(X_{\hat{q}}^n))$ and observe that, by Lemma 6 case 2, $\lim_{n\to\infty} \frac{\hat{d}^n}{n} = \hat{d}$. Since, by construction, $d^n \leqslant \hat{d}^n$, we get $\lim_{n\to\infty} \frac{d^n}{n} \leqslant \hat{d}$. Now let ε such that the certainty equivalent of $X - \varepsilon$ under v is $\hat{d}$. Let $Y = X - \varepsilon'$ where $\varepsilon' < \varepsilon$. Again by Lemma 6 case (iii), b^n, the certainty equivalent of Y^n, satisfies $\lim_{n\to\infty} \frac{b^n}{n} > \hat{d}$. Therefore, for a sufficiently large n, $Y^n \succ L^n$.

The case $\lim_{x\to-\infty} -\frac{u''(x)}{u'(x)} = 0$ is similarly proved (by replacing the exponential function v with a linear function). $\blacksquare$

Proof of Proposition 6: By definition, $X_{\tilde{q}}^n \succeq L^n$. As $\mathrm{E}(X_{\tilde{q}}^n) < 0$, it follows by risk aversion that $0 \succ X_{\tilde{q}}^n \succeq L^n$. $\blacksquare$

Appendix B: Expected Utility

Similarly to Appendix A, we denote the variance of X by σ^2 and by c^n the certainty equivalent of X^n under expected utility theory with the utility u. We assume throughout this appendix that u is twice differentiable and concave, and that $\lim_{x\to-\infty} -\frac{u''(x)}{u'(x)}$ exists.

Lemma 1 Let $u(x) = -e^{-ax}$. Then for all n, $c^n = nc^1$.[11]

Proof: We show first that $\int u(z)\,\mathrm{d}(F_{X^n}(z)) = -\left|\int u(z)\,\mathrm{d}(F_X(z))\right|^n$. Using induction, we get

$$
\int u(z)\,\mathrm{d}(F_{X^n}(z)) = \sum_{i=1}^{m}\left[p_i \int u(z)\,\mathrm{d}(F_{x_i+X^{n-1}}(z))\right]
$$

$$
= \sum_{i=1}^{m}\left[p_i e^{-ax_i}\int u(z)\,\mathrm{d}(F_{X^{n-1}}(z))\right] = \int u(z)\,\mathrm{d}(F_{X^{n-1}}(z))\sum_{i=1}^{m}p_i e^{-ax_i}
$$

$$
= -\left|\int u(z)\,\mathrm{d}(F_X(z))\right|^{n-1}\times\left|\int u(z)\,\mathrm{d}(F_X(z))\right| = -\left|\int u(z)\,\mathrm{d}(F_X(z))\right|^{n}
$$

Then, using $u(c^1) = \int u(z)\,\mathrm{d}(F_X(z))$,

$$
u(c^n) = \int u(z)\,\mathrm{d}(F_{X^n}(z)) \;=\; -\left|u(c^1)\right|^n = -\left(e^{-ac^1}\right)^n
$$

$$
= \; -e^{-anc^1} = u\left(nc^1\right)
$$

Hence $c^n = nc^1$. ∎

Lemma 2 Let $\frac{1}{2} < \alpha < 1$. There exists n_0 such that for all $n > n_0$,

$$
\int_{z\leqslant 0} z\,\mathrm{d}F_{X^n}(z) \geqslant \frac{x_1\sigma^2}{n^{2(\alpha-1)}} + \min\{nE(X),0\} - n^\alpha
$$

Proof: Clearly $E(X^n) = nE(X)$, and as σ^2 is the variance of X, $n\sigma^2$ is the variance of X^n. By Chebyshev's inequality

$$
\Pr\left(|Y - E(Y)| > k\sqrt{\mathrm{Var}(Y)}\right) < \frac{1}{k^2}
$$

Let $Y = X^n$ and $k = n^{\alpha-\frac{1}{2}}/\sigma$ to obtain

$$
\Pr(X^n < nE(X) - n^\alpha) < \frac{\sigma^2}{n^{2\alpha-1}} \tag{14}
$$

---

[11]In general, for lotteries $X_1,\ldots,X_n$, $\mathrm{CE}(\sum_{i=1}^{n}X_i) = \sum_{i=1}^{n}\mathrm{CE}(X_i)$, where $\mathrm{CE}(X)$ is the certainty equivalent of X. This case follows from a property of the moment generating functions (see Bulmer [1]).

Note that when $E(X) > 0$, the inequality implies $\Pr(X^n < -n^\alpha) < \frac{\sigma^2}{n^{2\alpha-1}}$.

Since $\alpha < 1$ and $x_1 < E(X)$, there exists n_0 sufficiently large such that for all $n > n_0$, $nx_1 < \min\{nE(X), 0\} - n^\alpha$. Divide the segment $[nx_1, 0]$ into two parts, with the dividing point being $nE(X) - n^\alpha$ when $E(X) \leqslant 0$ and $-n^\alpha$ when $E(X) > 0$. Then

$$\int_{z \leqslant 0} z \, dF_{X^n}(z) \geqslant \begin{cases} nx_1 \times \frac{\sigma^2}{n^{2\alpha-1}} + [nE(X) - n^\alpha] \times 1 & E(X) \leqslant 0 \\[2mm] nx_1 \times \frac{\sigma^2}{n^{2\alpha-1}} - n^\alpha \times 1 & E(X) > 0 \end{cases}$$

$$= \begin{cases} \frac{x_1 \sigma^2}{n^{2(\alpha-1)}} + nE(X) - n^\alpha & E(X) \leqslant 0 \\[2mm] \frac{x_1 \sigma^2}{n^{2(\alpha-1)}} - n^\alpha & E(X) > 0 \end{cases}$$

The inequality follows by the fact that all outcomes in the segment between nx_1 and $\min\{nE(X), 0\} - n^\alpha$ are replaced with $nx_1 < 0$, which is the lowest possible outcome of X^n, and are then multiplied by higher probability (see eq. (14)), and the value of $\min\{nE(X), 0\} - n^\alpha < 0$ too is multiplied by a higher number. ∎

Lemma 3 Suppose that $E(X) \leqslant 0$. If $\lim\limits_{x \to -\infty} u'(x) = \infty$, then

$$\lim_{n \to \infty} \frac{\displaystyle\int_{x>0} u(x) \, dF_{X^n}(x)}{\displaystyle\int_{x<0} u(x) \, dF_{X^n}(x)} = 0$$

Proof: We assume wlg that $u(0) = 0$ and $u'(0) = 1$. For $\eta > 0$, let $y(\eta) = \max\{y \leqslant 0 : u(y) \leqslant \eta y\}$. Since $\lim\limits_{x \to -\infty} u'(x) = \infty$, it follows that $y(\eta)$ exists. By the Central Limit Theorem, as $n \to \infty$, the probability that X^n will be in any finite segment goes to 0. In particular, $\lim\limits_{n \to \infty} \Pr(X^n \in [y(\eta), 0]) = 0$.

Since $u(0) = 0$ and for positive x, $u'(x) \leqslant 1$, it follows that for such x, $u(x) \leqslant x$. And since for $x < 0$, $u(x) < 0$, we obtain

$$0 \geqslant \frac{\displaystyle\int_{x>0} u(x) \, dF_{X^n}(x)}{\displaystyle\int_{x<0} u(x) \, dF_{X^n}(x)} \geqslant \frac{\displaystyle\int_{x>0} x \, dF_{X^n}(x)}{\displaystyle\int_{x<0} u(x) \, dF_{X^n}(x)}$$

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Since $\mathrm{E}(X^n) \leqslant 0$, it follows that $\int_{x>0} x \, \mathrm{d}F_{X^n}(x) \leqslant - \int_{x<0} x \, \mathrm{d}F_{X^n}(x)$. Therefore

$$\frac{\displaystyle\int_{x>0} x \, \mathrm{d}F_{X^n}(x)}{\displaystyle\int_{x<0} u(x) \, \mathrm{d}F_{X^n}(x)} \geqslant \frac{\displaystyle -\int_{y(\eta)}^{0} x \, \mathrm{d}F_{X^n}(x) - \int_{x<y(\eta)} x \, \mathrm{d}F_{X^n}(x)}{\displaystyle\int_{y(\eta)}^{0} u(x) \, \mathrm{d}F_{X^n}(x) + \int_{x<y(\eta)} u(x) \, \mathrm{d}F_{X^n}(x)}$$

$$> \frac{\displaystyle -\int_{y(\eta)}^{0} x \, \mathrm{d}F_{X^n}(x) - \int_{x<y(\eta)} x \, \mathrm{d}F_{X^n}(x)}{\displaystyle\int_{y(\eta)}^{0} u(x) \, \mathrm{d}F_{X^n}(x) + \eta \times \int_{x<y(\eta)} x \, \mathrm{d}F_{X^n}(x)} \xrightarrow[n\to\infty]{} -\frac{1}{\eta}$$

It thus follows that for all η

$$0 \geqslant \frac{\displaystyle\int_{x>0} u(x) \, \mathrm{d}F_{X^n}(x)}{\displaystyle\int_{x<0} u(x) \, \mathrm{d}F_{X^n}(x)} \geqslant -\frac{1}{\eta} \xrightarrow[\mu\to\infty]{} 0$$

hence the claim. ∎

Lemma 4 Suppose the $\mathrm{E}(X) \leqslant 0$. If $\lim\limits_{x\to-\infty} u'(x) = \infty$, then $\lim\limits_{n\to\infty} c^n = -\infty$.

Proof: By risk aversion, $c^n \leqslant \mathrm{E}(X^n) = n\mathrm{E}(X)$. Therefore, if $\mathrm{E}(X) < 0$, we are through. If $\mathrm{E}(X) = 0$, we show that for every integer $r < 0$, $\lim\limits_{n\to\infty} \mathrm{EU}(X^n) \leqslant u(r-1)$. The value of $\mathrm{EU}(X^n)$ equals

$$\int_{x\leqslant 2(r-1)} u(x) \, \mathrm{d}F_{X^n}(x) \left[1 + \frac{\displaystyle\int_{2(r-1)}^{0} u(x) \, \mathrm{d}F_{X^n}(x)}{\displaystyle\int_{x\leqslant 2(r-1)} u(x) \, \mathrm{d}F_{X^n}(x)} + \frac{\displaystyle\int_{x>0} u(x) \, \mathrm{d}F_{X^n}(x)}{\displaystyle\int_{x\leqslant 2(r-1)} u(x) \, \mathrm{d}F_{X^n}(x)} \right]$$

As in the proof of Lemma 3, it follows by the central limit theorem that $\lim\limits_{n\to\infty} \int_{2(r-1)}^{0} u(x) \, \mathrm{d}F_{X^n}(x) = 0$ and

$$\lim\limits_{n\to\infty} \frac{\displaystyle\int_{x>0} u(x) \, \mathrm{d}F_{X^n}(x)}{\displaystyle\int_{x\leqslant 2(r-1)} u(x) \, \mathrm{d}F_{X^n}(x)} = \lim\limits_{n\to\infty} \frac{\displaystyle\int_{x>0} u(x) \, \mathrm{d}F_{X^n}(x)}{\displaystyle\int_{x\leqslant 0} u(x) \, \mathrm{d}F_{X^n}(x)} = 0$$

where the last equality follows by Lemma 3. By the Central Limit Theorem, the probability of receiving a negative outcome is $\frac{1}{2}$. It thus follows that

$$\begin{aligned}
\lim_{n\to\infty} u(c^n) &= \lim_{n\to\infty} \int u(x)\,\mathrm{d}F_{X^n}(x) \\
&= \lim_{n\to\infty} \int_{x\leqslant 2(r-1)} u(x)\,\mathrm{d}F_{X^n}(x) \leqslant \frac{u(2(r-1))}{2} \leqslant u(r-1)
\end{aligned}$$

Hence $\lim_{n\to\infty} c^n \leqslant r - 1 < r$. ∎

Lemma 5 Suppose that $\mathrm{E}(X) \leqslant 0$, $\lim_{x\to-\infty} u'(x) = \infty$, and that there exists M such that for all $x < M$, $u(x) = v(x)$. Let n_i be such that $\lim_{i\to\infty} \frac{c^{n_i}}{n_i}$ converges to a limit. Then $\lim_{i\to\infty} \frac{\zeta^{n_i}}{n_i}$ converges to the same limit, where ζ^n is the certainty equivalent of X^n under expected utility model with the utility function v.

Proof: Similarly to the proof of Lemma 4, $\lim_{n\to\infty} u(c^n) = \lim_{n\to\infty} \int_{x\leqslant M} u(x)\,\mathrm{d}F_{X^n}(x)$ and $\lim_{n\to\infty} v(\zeta^n) = \lim_{n\to\infty} \int_{x\leqslant M} v(x)\,\mathrm{d}F_{X^n}(x)$, hence, since for $x \leqslant M$, $u(x) = v(x)$,

$$\lim_{n\to\infty} \frac{u(c^n)}{u(\zeta^n)} - 1 = 0 \tag{15}$$

For every n, $c_n \in [nx_1, nx_m]$, hence $\frac{c^n}{n}$, being a sequence in a compact interval, has a converging subsequence $\frac{c^{n_i}}{n_i}$. Take such a converging subsequence. Wlg $\frac{\zeta^{n_i}}{n_i}$ converges, but suppose that $\lim_{i\to\infty} \frac{c^{n_i}}{n_i} \neq \lim_{i\to\infty} \frac{\zeta^{n_i}}{n_i}$, for example, $\lim_{i\to\infty} \frac{c^{n_i}}{n_i} < \lim_{i\to\infty} \frac{\zeta^{n_i}}{n_i}$. Hence for sufficiently large values of i, $c^{n_i} < \zeta^{n_i}$. By the concavity of u

$$\begin{aligned}
\frac{u(c^{n_i})}{u(\zeta^{n_i})} - 1 &= \frac{u(c^{n_i}) - u(\zeta^{n_i})}{u(\zeta^{n_i})} \\
&= \frac{[u(c^{n_i}) - u(\zeta^{n_i})]/[c^{n_i} - \zeta^{n_i}]}{u(\zeta^{n_i})/\zeta^{n_i}} \times \frac{c^{n_i} - \zeta^{n_i}}{\zeta^{n_i}} \\
&> \frac{u'(\zeta^{n_i})}{u(\zeta^{n_i})/\zeta^{n_i}} \times \frac{c^{n_i} - \zeta^{n_i}}{\zeta^{n_i}} \\
&> \frac{c^{n_i} - \zeta^{n_i}}{\zeta^{n_i}} = \frac{c^{n_i}/n_i}{\zeta^{n_i}/n_i} - 1
\end{aligned}$$

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Since $c^{n_i} < \zeta^{n_i} < 0$, the limit of last expression is positive, a contradiction to eq. (15) above. ∎

Lemma 6 Suppose that $E(X) \leqslant 0$ and let $\lim\limits_{x \to -\infty} -\frac{u''(x)}{u'(x)} = a \in [0, \infty]$.

1. $a = 0$: $\lim\limits_{n \to \infty} \frac{c^n}{n} = E(X)$.

2. $0 < a < \infty$: Let $v(x) = -e^{-ax}$ and let $\hat{c}$ be the certainty equivalent of X under v. Then $\lim\limits_{n \to \infty} \frac{c^n}{n} = \hat{c}$.

3. $a = \infty$: $\lim\limits_{n \to \infty} \frac{c^n}{n} = x_1$.

Proof: We assume throughout that $u(0) = 0$ and $u'(0) = 1$.

1. (i) $a = 0$ and $\lim\limits_{x \to -\infty} u'(x) = H < \infty$: Since u is concave, it follows that for all n, $c^n \leqslant E(X^n) = nE(X)$. It is therefore enough to prove that $\lim\limits_{n \to \infty} \frac{c^n}{n} \geqslant E(X)$. Define $w(x) = \min\{Hx, 0\}$. By assumption, $u(x) \geqslant w(x)$ for all x. Let EU^w denote the EU functional with respect to w. Then by Lemma 2 for $\alpha \in (\frac{1}{2}, 1)$ and for sufficiently large n,

$$u(c^n) = EU(X^n) \geqslant EU^w(X^n) = H \int_{z \leqslant 0} z \, dF_{X^n}(z)$$

$$\geqslant H\left(\frac{x_1 \sigma^2}{n^{2(\alpha-1)}} - n^\alpha + nE(X)\right)$$

For $E(X) = 0$, this expression reduces to $u(c^n) \geqslant H(\frac{x_1 \sigma^2}{n^{2(\alpha-1)}} - n^\alpha)$. Since u is concave and $u'(0) = 1$, $c^n \geqslant H(\frac{x_1 \sigma^2}{n^{2(\alpha-1)}} - n^\alpha)$. Hence

$$\lim_{n \to \infty} \frac{c^n}{n} \geqslant H \lim_{n \to \infty} \left(\frac{x_1 \sigma^2}{n^{2\alpha-1}} - \frac{1}{n^{1-\alpha}}\right) = 0$$

If $E(X) < 0$, then $\lim\limits_{n \to \infty} c^n = -\infty$ and, as by l'Hopital's rule $\lim\limits_{x \to -\infty} \frac{x}{u(x)} =$

$\lim_{x\to-\infty} \frac{1}{u'(x)} = \frac{1}{H}$, we obtain that $\lim_{n\to\infty} \frac{c^n}{u(c^n)} = \frac{1}{H}$. Hence,

$$
\begin{aligned}
\lim_{n\to\infty} \frac{c^n}{n} &= \lim_{n\to\infty} \frac{c^n}{u(c^n)} \frac{u(c^n)}{n} = \frac{1}{H} \lim_{n\to\infty} \frac{u(c^n)}{n} \\
&\geqslant \frac{1}{H} \lim_{n\to\infty} \frac{H}{n} \left(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha + nE(X) \right) \\
&= \lim_{n\to\infty} \left(\frac{x_1\sigma^2}{n^{2\alpha-1}} - \frac{1}{n^{1-\alpha}} \right) + E(X) = E(X)
\end{aligned}
$$

1. (ii) $a = 0$ and $\lim_{x\to-\infty} u'(x) = \infty$: Consider the exponential utility $v_\varepsilon(x) = -e^{-\varepsilon x}$ for which $-\frac{v_\varepsilon''}{v_\varepsilon'} \equiv \varepsilon$. Denote by c_ε^n the value of c^n obtained for the function v_ε. By Lemma 1, $\lim_{n\to\infty} \frac{c_\varepsilon^n}{n} = c_\varepsilon^1 < 0$ where c_ε^1, the certainty equivalent of X, satisfies

$$
-e^{-\varepsilon c_\varepsilon^1} = \int -e^{-\varepsilon z} \, dF_X(z) \Longrightarrow c_\varepsilon^1 = -\frac{1}{\varepsilon} \ln \left[\int e^{-\varepsilon z} \, dF_X(z) \right]
$$

Using l'Hopital's rule we obtain

$$
\lim_{\varepsilon\to0} c_\varepsilon^1 = \lim_{\varepsilon\to0} \frac{\int z e^{-\varepsilon z} \, dF_X(z)}{\int e^{-\varepsilon z} \, dF_X(z)} = E(X)
$$

As $\lim_{x\to-\infty} -\frac{u''(x)}{u'(x)} = 0$, it follows that for every $\varepsilon > 0$ there is $x(\varepsilon)$ such that for all $x < x(\varepsilon)$, $-\frac{u''(x)}{u'(x)} < \varepsilon$. Define a function u_ε as follows.

$$
u_\varepsilon = \begin{cases} u(x) & x \leqslant x(\varepsilon) \\ \kappa v_\varepsilon(x) + \theta & x > x(\varepsilon) \end{cases}
$$

where $\kappa = \frac{u'(x(\varepsilon))}{v_\varepsilon'(x(\varepsilon))}$ and $\theta = u(x(\varepsilon)) - \kappa v_\varepsilon(x(\varepsilon))$. Clearly u_ε is less risk averse than v_ε, hence for all n, $c_{u_\varepsilon}^n \geqslant c_\varepsilon^n$. Since $\lim_{n\to\infty} \frac{c_\varepsilon^n}{n} = c_\varepsilon^1$, $\lim_{i\to\infty} \frac{c_{u_\varepsilon}^{n_i}}{n_i} \geqslant c_\varepsilon^1$ for every converging subsequence $\frac{c_{u_\varepsilon}^{n_i}}{n_i}$. By Lemma 5, for all such subsequences, $\frac{c^{n_i}}{n_i} \geqslant c_\varepsilon^1$. The claim now follows by the fact that $\lim_{\varepsilon\to0} c_\varepsilon^1 = E(X)$.

2. $0 < a < \infty$: Note that in this case, $\lim_{x\to-\infty} u'(x) = \infty$. To see why, note that $\lim_{x\to-\infty} u'(x) = H < \infty$ must imply $\lim_{x\to-\infty} u''(x) = 0$ (by concavity, $u'(x)$

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is monotonically increasing towards H when x goes down to $-\infty$) and hence $\lim\limits_{x\to-\infty} -\frac{u''(x)}{u'(x)} = 0$, contradicting $a > 0$.

For any $\varepsilon > 0$ denote $v_{\varepsilon_+}(x) = -e^{-(a+\varepsilon)x}$, $v_{\varepsilon_-}(x) = -e^{-(a-\varepsilon)x}$ and let $\hat{c}_{\varepsilon_+}$ and $\hat{c}_{\varepsilon_-}$, the certainty equivalents of X under $v_{\varepsilon_+}(x)$ and $v_{\varepsilon_-}(x)$, satisfy

$$-e^{-a\hat{c}_{\varepsilon_+}} = \int -e^{-(a+\varepsilon)z}\,\mathrm{d}F_X(z), \qquad -e^{-a\hat{c}_{\varepsilon_-}} = \int -e^{-(a-\varepsilon)z}\,\mathrm{d}F_X(z)$$

Since v_{ε_+} is more concave than v and v is more concave than v_{ε_-}, we have $\hat{c}_{\varepsilon_+} < \hat{c} < \hat{c}_{\varepsilon_-}$. Let $\hat{c}^n_{\varepsilon_+}$ and $\hat{c}^n_{\varepsilon_-}$ denote the certainty equivalents of X^n under v_{ε_+} and v_{ε_-}, respectively. By Lemma 1, $\frac{\hat{c}^n_{\varepsilon_+}}{n} = \hat{c}_{\varepsilon_+}$ and $\frac{\hat{c}^n_{\varepsilon_-}}{n} = \hat{c}_{\varepsilon_-}$.

As $\lim\limits_{x\to-\infty} -\frac{u''(x)}{u'(x)} = a > 0$, for every $\varepsilon \in (0,a)$ there is $x(\varepsilon)$ such that for all $x \leqslant x(\varepsilon)$, $a - \varepsilon < -\frac{u''(x)}{u'(x)} < a + \varepsilon$. Define the functions u_{ε_*}, $* = +,-$, by

$$u_{\varepsilon_*}(x) = \begin{cases} u(x) & x \leqslant x(\varepsilon) \\[2mm] \kappa_* v_{\varepsilon_*}(x) + \theta_* & \text{otherwise} \end{cases}$$

where $\kappa_* = \frac{u'(x(\varepsilon))}{v_{\varepsilon_*}(x(\varepsilon))}$ and $\theta_* = u(x(\varepsilon)) - \kappa_* v_{\varepsilon_*}(x(\varepsilon))$ are defined as to enable continuity and differentiability of these functions.

Clearly, u_{ε_-} is more risk averse than v_{ε_-} and u_{ε_+} is less risk averse than v_{ε_+}. Hence, $c^n_{u_{\varepsilon_+}}$ and $c^n_{u_{\varepsilon_-}}$, the certainty equivalents of X^n under u_{ε_+} and u_{ε_-}, respectively, satisfy $\hat{c}^n_{\varepsilon_-} \geqslant c^n_{u_{\varepsilon_-}}$ and $c^n_{u_{\varepsilon_+}} \geqslant \hat{c}^n_{\varepsilon_+}$. Hence,

$$\hat{c}_{\varepsilon_-} = \frac{\hat{c}^n_{\varepsilon_-}}{n} \geqslant \lim_{n\to\infty} \frac{c^n_{u_{\varepsilon_-}}}{n} = \lim_{n\to\infty} \frac{c^n}{n} = \lim_{n\to\infty} \frac{c^n_{u_{\varepsilon_+}}}{n} \geqslant \frac{\hat{c}^n_{\varepsilon_+}}{n} = \hat{c}_{\varepsilon_+}$$

where the second and third equalities follow from Lemma 5.[12] Finally, note that both $\hat{c}_{\varepsilon_+}$ and $\hat{c}_{\varepsilon_-}$ converge to $\hat{c}$ when $\varepsilon \to 0$, hence $\lim\limits_{n\to\infty} \frac{c^n}{n} = \hat{c}$.

3. $a = \infty$: Let $\hat{c}(s)$ be the certainty equivalent of X under the utility function $-e^{-sx}$, for all $s > 0$. Since there exists M such that on $(-\infty, M)$, u is more concave than $-e^{-sx}$, it follows by Lemma 5 that for all converging

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[12] To be more precise, these equalities apply to all converging subsequences. But as they are all bounded within a compact segment $[\hat{c}_{\varepsilon_+}, \hat{c}_{\varepsilon_-}]$, all of them must converge to the same limit.

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subsequences of $\frac{c^n}{n}$, $\lim\limits_{i\to\infty} \frac{c^{n_i}}{n_i} \leqslant \hat{c}(s)$ for all s. Next we show that $\lim\limits_{s\to\infty} \hat{c}(s) = x_1$ (note that $\hat{c}(s) = -\frac{1}{s}\ln\left(\sum p_i e^{-sx_i}\right)$). Using l'Hopital's rule we get

$$\lim_{s\to\infty} \hat{c}(s) = \lim_{s\to\infty} \frac{\sum p_i x_i e^{-sx_i}}{\sum p_i e^{-sx_i}} = \lim_{s\to\infty} \frac{p_1 x_1 + \sum_{i>1} p_i x_i e^{-s(x_i-x_1)}}{p_1 + \sum_{i>1} p_i e^{-s(x_i-x_1)}} = x_1$$

which, noting that $c^n \geqslant nx_1$ and hence $\frac{c^n}{n} \geqslant x_1$, implies $\lim\limits_{n\to\infty} \frac{c^n}{n} = x_1$. ∎

Conclusion 1 Let $\lim\limits_{x\to-\infty} -\frac{u''(x)}{u'(x)} = a \in [0,\infty]$.

1. $a = 0$ iff for all X, Y such that $E(X), E(Y) \leqslant 0$, $E(X) > E(Y)$ implies that $X^n \succ Y^n$ for all sufficiently large n.

2. $a = a_0 \in (0,\infty)$ iff for all X, Y such that $E(X), E(Y) \leqslant 0$, $EU^v(X) > EU^v(Y)$ where $v(x) = -e^{-a_0 x}$ implies that $X^n \succ Y^n$ for all sufficiently large n.

3. $a = \infty$ iff for all X, Y such that $E(X), E(Y) \leqslant 0$, $x_1 > y_1$ implies that for all sufficiently large n, $X^n \succ Y^n$.

Proof: We prove case 1, the proofs of the other two cases are similar. If $a = 0$, then by Lemma 6 case 1, $\lim\limits_{n\to\infty} \frac{c(X^n)}{n} = E(X)$ and $\lim\limits_{n\to\infty} \frac{c(Y^n)}{n} = E(Y)$. Therefore $E(X) > E(Y)$ implies that for all sufficiently large n, $c(X^n) > c(Y^n)$ and $X^n \succ Y^n$. On the other hand, if $0 < a < \infty$ then there are X, Y such that $E(Y) < E(X) \leqslant 0$, yet $EU^v(Y) > EU^v(X)$ where $v(x) = -e^{-a_0 x}$, hence by Lemma 6 case 2, for all sufficiently large n, $c(Y^n) > c(X^n)$ and $Y^n \succ X^n$. Finally, if $a = \infty$ then there are X, Y such that $E(Y) < E(X) \leqslant 0$ yet $y_1 > x_1$, hence by Lemma 6 case 3, for all sufficiently large n, $c(Y^n) > c(X^n)$ and $Y^n \succ X^n$. ∎

Remark: Hellwig [14] deals with the case $a = 0$, and shows that if $E(X) > 0$ and $E(X) > E(Y)$, then for all sufficiently large n, $X^n \succ Y^n$. Conclusion 1 extends this result to the case $E(X) \leqslant 0$, and shows similar results for all a, provided $E(X), E(Y) \leqslant 0$.

Lemma 7 Suppose that $0 \leqslant \lim\limits_{x\to-\infty} -\frac{u''(x)}{u'(x)} = s^* < t^* \leqslant \lim\limits_{x\to-\infty} -\frac{v''(x)}{v'(x)}$. There is x^* such that for all $x < x^*$, $u(x) > v(x)$.

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Proof: Let s, t such that $s^* < s < t < t^*$ and assume wlg that for all $x < 0$, $-\frac{u''(x)}{u'(x)} < s < t < -\frac{v''(x)}{v'(x)}$. Then, since $-\frac{u''(x)}{u'(x)} = -(\ln u'(x))'$,

$$\ln(u'(0)) - \ln(u'(x)) \geqslant sx \quad \text{and} \quad \ln(v'(0)) - \ln(v'(x)) \leqslant tx \Longrightarrow$$

$$\ln(u'(x)) \leqslant \ln(u'(0)) - sx \quad \text{and} \quad \ln(v'(x)) \geqslant \ln(v'(0)) - tx \Longrightarrow$$

$$u'(x) \leqslant u'(0)e^{-sx} = [-\tfrac{u'(0)}{s}e^{-sx}]' \quad \text{and} \quad v'(x) \geqslant v'(0)e^{-tx} = [-\tfrac{v'(0)}{t}e^{-tx}]' \Longrightarrow$$

$$u(0) - u(x) < \tfrac{u'(0)}{s}(-1 + e^{-sx}) \quad \text{and} \quad v(0) - v(x) > \tfrac{v'(0)}{t}(-1 + e^{-tx}) \Longrightarrow$$

$$u(x) > u(0) + \tfrac{u'(0)}{s}(-e^{-sx} + 1) \quad \text{and} \quad v(x) < v(0) + \tfrac{v'(0)}{t}(-e^{-tx} + 1)$$

Hence

$$u(x) - v(x) > \alpha + \beta(-e^{-sx}) - \gamma(-e^{-tx})$$

where $\alpha = u(0) + \frac{u'(0)}{s} - v(0) - \frac{v'(0)}{t}$, $\beta = \frac{u'(0)}{s} > 0$ and $\gamma = \frac{v'(0)}{t} > 0$. As $x \to -\infty$, the rhs converges to ∞, hence the claim. ∎

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