# On the least squares estimator in a nearly unstable sequence of stationary spatial AR models 

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#### Abstract

A nearly unstable sequence of stationary spatial autoregressive processes is investigated, when the sum of the absolute values of the autoregressive coefficients tends to one. It is shown that after an appropriate norming the least squares estimator for these coefficients has a normal limit distribution. If none of the parameters equals zero than the typical rate of convergence is $n$.


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## 1. Introduction

Spatial autoregressive models have a great importance in many different fields of science such as geography, geology, biology and agriculture, see e.g. [1 for a detailed discussion, where the authors considered a general unilateral model having the form

$$
\begin{equation*}
X_{k, \ell}=\sum_{i=0}^{p_{1}} \sum_{j=0}^{p_{2}} \alpha_{i, j} X_{k-i, \ell-j}+\varepsilon_{k, \ell}, \quad \alpha_{0,0}=0 \tag{1.1}
\end{equation*}
$$

A particular case of the model (1.1) is the so-called doubly geometric spatial autoregressive model

$$
X_{k, \ell}=\alpha X_{k-1, \ell}+\beta X_{k, \ell-1}-\alpha \beta X_{k-1, \ell-1}+\varepsilon_{k, \ell}
$$

[^0]introduced by Martin [11. In fact, this is the simplest spatial model, since its nice product structure ensures that it can be considered as some kind of combination of two autoregressive processes on the line, and several properties can be derived by the analogy of one-dimensional autoregressive processes. The doubly geometric model was the first one for which the nearly unstability has been studied. Bhattacharyya et al. 7] showed that in the case when a sequence of stable models with $\alpha_{n} \rightarrow 1, \beta_{n} \rightarrow 1$ was considered, in contrast to the $\operatorname{AR}(1)$ model, the sequence of Gauss-Newton estimators ( $\widehat{\alpha}_{n}, \widehat{\beta}_{n}$ ) of ( $\alpha_{n}, \beta_{n}$ ) were asymptotically normal, namely,
$$
n^{3 / 2}\binom{\widehat{\alpha}_{n}-\alpha_{n}}{\widehat{\beta}_{n}-\beta_{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma)
$$
with some covariance matrix $\Sigma$.
The doubly geometric model has several applications. Jain [10] used it in the study of image processing, Martin [12], Cullis and Gleeson [9, Basu and Reinsel [2] in agricultural trials, while Tjøstheim [15] in digital filtering.

In the present paper we study another special case of the model (1.1). We consider the spatial autoregressive process $\left\{X_{k, \ell}: k, \ell \in \mathbb{Z}\right\}$ which is a solution of the spatial stochastic difference equation

$$
\begin{equation*}
X_{k, \ell}=\alpha X_{k-1, \ell}+\beta X_{k, \ell-1}+\varepsilon_{k, \ell} \tag{1.2}
\end{equation*}
$$

with parameters $(\alpha, \beta) \in \mathbb{R}^{2}$. This model is stable (i.e. has a stationary solution) in case $|\alpha|+|\beta|<1$ (see [1]), and unstable if $|\alpha|+|\beta|=1$. In a recent paper Paulauskas [13] determined the exact asymptotic behavior of the variances of a nonstationary solution of (1.2) with $X_{k, \ell}=0$ for $k+\ell \leq 0$, while Baran et al. [5] in the same model clarified the asymptotic properties of the least squares estimator (LSE) of ( $\alpha, \beta$ ) both in stable and unstable cases.

We remark, that in case $|\alpha|+|\beta|<1$, if $\left\{\varepsilon_{k, \ell}: k, \ell \in \mathbb{Z}\right\}$ are independent and identically distributed random variables, a stationary solution can be given by

$$
\begin{equation*}
X_{k, \ell}=\sum_{(i, j) \in U_{k, \ell}}\binom{k+\ell-i-j}{k-i} \alpha^{k-i} \beta^{\ell-j} \varepsilon_{i, j} \tag{1.3}
\end{equation*}
$$

where $U_{k, \ell}:=\left\{(i, j) \in \mathbb{Z}^{2}: i \leq k\right.$ and $\left.j \leq \ell\right\}$ and the convergence of the series is understood in $\mathrm{L}_{2}$-sense.

We are interested in the asymptotic behaviour of the stationary solution of (1.2) in the case when the parameters approach the boundary $|\alpha|+|\beta|=1$. In order to determine the appropriate speed of parameters one may use the idea of Chan and Wei [8] and consider the order of

$$
\mathbb{I}_{n}:=\mathrm{E}\left(\sum_{(k, \ell) \in H_{n}}\left(\begin{array}{cc}
\left(X_{k-1, \ell}\right)^{2} & X_{k-1, \ell} X_{k, \ell-1} \\
X_{k-1, \ell} X_{k, \ell-1} & \left(X_{k, \ell-1}\right)^{2}
\end{array}\right)\right)
$$

that is exactly the observed Fisher information matrix about $(\alpha, \beta)$ when the innovations $\varepsilon_{k, \ell}$ are normally distributed and the process is observed on a set $H_{n} \subset \mathbb{Z}^{2}, n \in \mathbb{N}$. From Theorem 1.1 of 5 we obtain that

$$
\mathbb{I}_{n} \sim \begin{cases}n^{2} \sigma_{\alpha, \beta}^{2} \Gamma_{\alpha, \beta}, & \text { if }|\alpha|+|\beta|<1, \\ n^{5 / 2} \sigma_{\alpha}^{2} \Psi_{\alpha, \beta}, & \text { if }|\alpha|+|\beta|=1,0<|\alpha|<1, \\ n^{3}(4 / 3) \mathcal{I}, & \text { if }|\alpha|+|\beta|=1,|\alpha| \in\{0,1\},\end{cases}
$$

where

$$
\Gamma_{\alpha, \beta}:=2\left(\begin{array}{cc}
1 & -\varrho_{\alpha, \beta} \\
-\varrho_{\alpha, \beta} & 1
\end{array}\right), \quad \Psi_{\alpha, \beta}:=\left(\begin{array}{cc}
1 & \operatorname{sign}(\alpha \beta) \\
\operatorname{sign}(\alpha \beta) & 1
\end{array}\right),
$$

$\mathcal{I}$ denotes the two-by-two unit matrix and

$$
\begin{aligned}
& \sigma_{\alpha, \beta}^{2}:=((1+\alpha+\beta)(1+\alpha-\beta)(1-\alpha+\beta)(1-\alpha-\beta))^{-1 / 2}, \\
& \varrho_{\alpha, \beta}:= \begin{cases}\frac{\left(1-\alpha^{2}-\beta^{2}\right) \sigma_{\alpha, \beta}^{2}-1}{2 \alpha \beta \sigma_{\alpha, \beta}^{2}}, & \text { if } \alpha \beta \neq 0, \\
0 & \text { otherwise, }\end{cases} \\
& \sigma_{\alpha}^{2}:=\frac{2^{9 / 2}}{15 \sqrt{\pi|\alpha|(1-|\alpha|)}} .
\end{aligned}
$$

Now, let $\alpha_{n}:=\alpha-\gamma / a_{n}, \beta_{n}:=\beta-\delta / a_{n},|\alpha|+|\beta|=1,\left|\alpha_{n}\right|+\left|\beta_{n}\right|<1$. As nonstationary behaviour of $X_{k, \ell}$ becomes dominant when $\left(\alpha_{n}, \beta_{n}\right)$ is near the border, a reasonable choice for the sequence $a_{n}$ should retain the order of $\mathbb{I}_{n}$ to be $n^{5 / 2}$ if $0<|\alpha|<1$ and $n^{3}$ if $|\alpha| \in\{0,1\}$. Since we have $\sigma_{\alpha_{n}, \beta_{n}}^{2} \sim a_{n}^{1 / 2}$ for $0<|\alpha|<1$ and $\sigma_{\alpha_{n}, \beta_{n}}^{2} \sim a_{n}$ for $|\alpha| \in\{0,1\}$ while $\varrho_{\alpha_{n}, \beta_{n}} \sim$ const in both cases, the above consideration yields $a_{n}=n$.

In what follows we consider a nearly unstable sequence of stationary processes, i.e. for each $n \in \mathbb{N}$, we take a stationary solution $\left\{X_{k, \ell}^{(n)}: k, \ell \in \mathbb{Z}\right\}$ of equation (1.2) with parameters $\left(\alpha_{n}, \beta_{n}\right)$ defined as

$$
\begin{equation*}
\alpha_{n}:=\alpha-\frac{\gamma_{n}}{n}, \quad \beta_{n}:=\beta-\frac{\delta_{n}}{n}, \quad\left|\alpha_{n}\right|+\left|\beta_{n}\right|<1, \tag{1.4}
\end{equation*}
$$

where $0 \leq|\alpha| \leq 1,|\beta|=1-|\alpha|$ and $\gamma_{n} \rightarrow \gamma, \delta_{n} \rightarrow \delta$ as $n \rightarrow \infty,(\gamma, \delta) \in \mathbb{R}^{2}$. We remark that in an earlier paper [3] the authors considered a similar sequence of stationary processes where the autoregressive parameters were equal and their sum converged to 1 .

For a set $H \subset \mathbb{Z}^{2}$, the LSE $\left(\widehat{\alpha}_{H}^{(n)}, \widehat{\beta}_{H}^{(n)}\right)$ of $\left(\alpha_{n}, \beta_{n}\right)$ based on the observations $\left\{X_{k, \ell}^{(n)}:(k, \ell) \in H\right\}$ has the form

$$
\binom{\widehat{\alpha}_{H}^{(n)}}{\widehat{\beta}_{H}^{(n)}}=\left(\sum_{(k, \ell) \in H}\left(\begin{array}{cc}
\left(X_{k-1, \ell}^{(n)}\right)^{2} & X_{k-1, \ell}^{(n)} X_{k, \ell-1}^{(n)} \\
X_{k-1, \ell}^{(n)} X_{k, \ell-1}^{(n)} & \left(X_{k, \ell-1}^{(n)}\right)^{2}
\end{array}\right)\right)^{-1} \sum_{(k, \ell) \in H}\binom{X_{k-1, \ell}^{(n)} X_{k, \ell}^{(n)}}{X_{k, \ell-1}^{(n)} X_{k, \ell}^{(n)}} .
$$

Consider the triangles $T_{k, \ell}:=\left\{(i, j) \in \mathbb{Z}^{2}: i+j \geq 1, i \leq k\right.$ and $\left.j \leq \ell\right\}$ for $k, \ell \in \mathbb{Z}$. Note that $T_{k, \ell}=\emptyset$ if $k+\ell \leq 0$.

Theorem 1.1 For each $n \in \mathbb{N}$, let $\left\{X_{k, \ell}^{(n)}: k, \ell \in \mathbb{N}\right\}$ be a stationary solution of equation (1.2) with parameters $\left(\alpha_{n}, \beta_{n}\right)$ given by (1.4), and with independent and
identically distributed random variables $\left\{\varepsilon_{k, \ell}^{(n)}: k, \ell \in \mathbb{Z}\right\}$ such that $\mathbb{E} \varepsilon_{0,0}^{(n)}=0$, $\operatorname{Var} \varepsilon_{0,0}^{(n)}=$ 1 and $M:=\sup _{n \in \mathbb{N}} \mathbb{E}\left|\varepsilon_{0,0}^{(n)}\right|^{8}<\infty$. Let $\left(k_{n}\right)$ and $\left(\ell_{n}\right)$ be sequences of integers such that $k_{n}+\ell_{n} \rightarrow \infty$ as $n \rightarrow \infty$.

If $0<|\alpha|<1,|\beta|=1-|\alpha| \quad$ and

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(k_{n}+\ell_{n}\right) n^{-1 / 2}\left(\left|\gamma_{n}\right|+\left|\delta_{n}\right|\right)^{1 / 2}=\infty \tag{1.5}
\end{equation*}
$$

holds then

$$
\left(k_{n}+\ell_{n}\right)\binom{\widehat{\alpha}_{T_{k_{n}, \ell_{n}}}-\alpha_{n}}{\widehat{\beta}_{T_{k_{n}, \ell_{n}}}-\beta_{n}} \xrightarrow{\mathcal{D}} \mathcal{N}_{2}\left(0,|\alpha||\beta| \bar{\Psi}_{\alpha, \beta}\right)
$$

as $n \rightarrow \infty$, where $\bar{\Psi}_{\alpha, \beta}$ denotes the adjoint matrix of $\Psi_{\alpha, \beta}$.
If $|\alpha| \in\{0,1\},|\beta|=1-|\alpha|$ and

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(k_{n}+\ell_{n}\right) n^{-1}\left|\gamma_{n}^{2}-\delta_{n}^{2}\right|^{1 / 2}=\infty \tag{1.6}
\end{equation*}
$$

holds then let

$$
[-\infty, \infty] \ni \omega:=\lim _{n \rightarrow \infty} \omega_{n}, \quad \quad \omega_{n}:=\alpha \frac{\gamma_{n}}{\delta_{n}}+\beta \frac{\delta_{n}}{\gamma_{n}}
$$

If $|\omega|>1$ then

$$
\left(k_{n}+\ell_{n}\right) n^{1 / 2}\left|\gamma_{n}^{2}-\delta_{n}^{2}\right|^{-1 / 4}\binom{\widehat{\alpha}_{T_{k_{n}, \ell_{n}}}-\alpha_{n}}{\widehat{\beta}_{T_{k_{n}, \ell_{n}}}-\beta_{n}} \xrightarrow{\mathcal{D}} \mathcal{N}_{2}\left(0, \Theta_{\alpha, \beta, \omega}^{-1}\right)
$$

as $n \rightarrow \infty$, where

$$
\Theta_{\alpha, \beta, \omega}:=\frac{1}{4}\left(\begin{array}{cc}
1 & \theta(\alpha, \beta, \omega) \\
\theta(\alpha, \beta, \omega) & 1
\end{array}\right)
$$

with

$$
\theta(\alpha, \beta, \omega):= \begin{cases}\frac{-(\alpha+\beta) \operatorname{sign}(\omega)}{|\omega|+\sqrt{\omega^{2}-1}} & \text { if }|\omega|<\infty \\ 0 & \text { if }|\omega|=\infty\end{cases}
$$

Remark 1.2 Obviously, $\left|\omega_{n}\right|>1$, so $|\omega| \geq 1$. Condition $|\omega|>1$ in Theorem 1.1 is needed to ensure the regularity of $\Theta_{\alpha, \beta, \omega}$. However, this condition can be omitted and using similar arguments as in the proof of the second statement of Theorem 1.1] one can easily show that if $|\alpha| \in\{0,1\},|\beta|=1-|\alpha|$ and (1.6) holds then

$$
\left(k_{n}+\ell_{n}\right) n^{1 / 2}\left|\gamma_{n}^{2}-\delta_{n}^{2}\right|^{-1 / 4} \Theta_{\alpha, \beta, \omega_{n}}^{1 / 2}\binom{\widehat{\alpha}_{T_{k_{n}, \ell_{n}}}-\alpha_{n}}{\widehat{\beta}_{T_{k_{n}, \ell_{n}}}-\beta_{n}} \xrightarrow{\mathcal{D}} \mathcal{N}_{2}(0, \mathcal{I}),
$$

where $\Theta_{\alpha, \beta, \omega_{n}}^{1 / 2}$ denotes the symmetric positive semidefinite square root of $\Theta_{\alpha, \beta, \omega_{n}}$.
Remark 1.3 Theorem 1.1 shows that in the typical case $k_{n}=\ell_{n}=n$ and $\gamma_{n}=\gamma \neq 0$, $\delta_{n}=\delta \neq 0$ if $0<|\alpha|<\infty,|\beta|=1-|\alpha|$ then the rate of convergence is $n$.

We may suppose that $\left(k_{n}+\ell_{n}\right)$ is monotone increasing. Observe, that $\left(\widehat{\alpha}_{T_{k_{n}, \ell_{n}}}^{(n)}, \widehat{\beta}_{T_{k_{n}, \ell_{n}}}^{(n)}\right)$ and $\left(\widehat{\alpha}_{T_{\widetilde{k}_{n}, \widetilde{\ell}_{n}}^{(n)}}, \widehat{\beta}_{T_{\tilde{k}_{n}, \widetilde{\ell}_{n}}^{(n)}}^{(n)}\right)$ have the same distribution, where $\widetilde{k}_{n}:=\left[\left(k_{n}+\ell_{n}\right) / 2\right]$ and $\widetilde{\ell}_{n}:=$ $\left[\left(k_{n}+\ell_{n}+1\right) / 2\right]$. As $\widetilde{k}_{n}+\widetilde{\ell}_{n} \underset{\sim}{\sim} k_{n}+\ell_{n}$, in Theorem 1.1 we may substitute $\left(\widetilde{k}_{n}, \widetilde{\ell}_{n}\right)$ for $\left(k_{n}, \ell_{n}\right)$. The sequence $\left(\widetilde{k}_{n}, \widetilde{\ell}_{n}\right)$ can be embedded into the sequence $\left(k_{n}^{\prime}, \ell_{n}^{\prime}\right)$, where $k_{n}^{\prime}:=[n / 2]$ and $\ell_{n}^{\prime}:=[(n+1) / 2]$, namely, $k_{q_{n}}^{\prime}=\widetilde{k}_{n}$ and $\ell_{q_{n}}^{\prime}=\widetilde{\ell}_{n}$ with $q_{n}:=\widetilde{k}_{n}+\widetilde{\ell}_{n}$. Clearly $k_{n}^{\prime}+\ell_{n}^{\prime}=n$. Consider the sequence $\left(r_{n}\right)$ defined by $r_{n}:=k$ for $q_{k} \leq n<q_{k+1}$. Then $r_{q_{n}}=n$, and conditions (1.5) and (1.6) can be replaced by

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n r_{n}^{-1 / 2}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 2}=\infty \tag{1.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n r_{n}^{-1}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{1 / 2}=\infty \tag{1.8}
\end{equation*}
$$

respectively.
Thus, to prove Theorem 1.1 it suffices to show that if $0<|\alpha|<1,|\beta|=1-|\alpha|$ and (1.7) holds then

$$
n\binom{\widehat{\alpha}_{T_{[n / 2],[(n+1) / 2]}}-\alpha_{r_{n}}}{\widehat{\beta}_{T_{[n / 2],[(n+1) / 2]}}-\beta_{r_{n}}} \xrightarrow{\mathcal{D}} \mathcal{N}_{2}\left(0,|\alpha||\beta| \bar{\Psi}_{\alpha, \beta}\right),
$$

while in the case $|\alpha| \in\{0,1\},|\beta|=1-|\alpha|,|\omega|>1$ and (1.8) holds we have

$$
n r_{n}^{1 / 2}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{-1 / 4}\binom{\widehat{\alpha}_{[[n / 2],[(n+1) / 2]}-\alpha_{r_{n}}}{\widehat{\beta}_{T_{[n / 2],[(n+1) / 2]}}-\beta_{r_{n}}} \xrightarrow{\mathcal{D}} \mathcal{N}_{2}\left(0, \Theta_{\alpha, \beta, \omega}^{-1}\right) .
$$

We remark that conditions (1.5) and (1.7) are exactly the same as conditions (4) and (5) of 3], respectively.

To simplify notation we assume $k_{n}=[n / 2], \ell_{n}=[(n+1) / 2]$ and $\left(r_{n}\right)$ is a monotone increasing sequence of positive integers. One can write

$$
\binom{\widehat{\alpha}_{T_{k_{n}, \ell_{n}}}-\alpha_{r_{n}}}{\widehat{\beta}_{T_{k_{n}, \ell_{n}}}-\beta_{r_{n}}}=B_{n}^{-1} A_{n}
$$

with

$$
A_{n}:=\sum_{(k, \ell) \in T_{k_{n}, \ell}}\binom{X_{k-1, \ell}^{\left(r_{n}\right)} \varepsilon_{k, \ell}^{\left(r_{n}\right)}}{X_{k, \ell-1}^{\left(r_{n}\right)} \varepsilon_{k, \ell}^{\left(r_{n}\right)}}, \quad B_{n}:=\sum_{(k, \ell) \in T_{k_{n}, \ell_{n}}}\left(\begin{array}{cc}
\left(X_{k-1, \ell}^{\left(r_{n}\right)}\right)^{2} & X_{k-1, \ell}^{\left(r_{n}\right)} X_{k, \ell-1}^{\left(r_{n}\right)} \\
X_{k-1, \ell}^{\left(r_{n}\right)} X_{k, \ell-1}^{\left(r_{n}\right)} & \left(X_{k, \ell-1}^{\left(r_{n}\right)}\right)^{2}
\end{array}\right) .
$$

Concerning the asymptotic behaviour of the random vector $A_{n}$ and random matrix $B_{n}$ we can formulate the following two propositions.

Proposition 1.4 If $0<|\alpha|<1,|\beta|=1-|\alpha|$ and (1.7) holds then

$$
n^{-2} r_{n}{ }^{-1 / 2}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 2} B_{n} \xrightarrow{\mathrm{~L}_{2}}(32|\alpha||\beta|)^{-1 / 2} \Psi_{\alpha, \beta} \quad \text { as } n \rightarrow \infty
$$

If $|\alpha| \in\{0,1\},|\beta|=1-|\alpha|$ and (1.8) holds then

$$
n^{-2} r_{n}^{-1}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{1 / 2} B_{n} \xrightarrow{\mathrm{~L}_{2}} \Theta_{\alpha, \beta, \omega}
$$

$n \rightarrow \infty$, where

$$
\begin{equation*}
\omega:=\lim _{n \rightarrow \infty} \omega_{r_{n}}, \quad \quad \omega_{r_{n}}:=\alpha \frac{\gamma_{r_{n}}}{\delta_{r_{n}}}+\beta \frac{\delta_{r_{n}}}{\gamma_{r_{n}}} . \tag{1.9}
\end{equation*}
$$

Proposition 1.5 If $0<|\alpha|<1,|\beta|=1-|\alpha|$ and (1.7) holds then

$$
n^{-1} r_{n}^{-1 / 4}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 4} A_{n} \xrightarrow{\mathcal{D}} \mathcal{N}_{2}\left(0,(32|\alpha||\beta|)^{-1 / 2} \Psi_{\alpha, \beta}\right) \quad \text { as } n \rightarrow \infty .
$$

If $|\alpha| \in\{0,1\},|\beta|=1-|\alpha|$ and (1.8) holds then

$$
n^{-1} r_{n}^{-1 / 2}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{1 / 4} A_{n} \xrightarrow{\mathcal{D}} \mathcal{N}_{2}\left(0, \Theta_{\alpha, \beta, \omega}\right) \quad \text { as } n \rightarrow \infty
$$

In case $|\alpha| \in\{0,1\},|\beta|=1-|\alpha|$, and $|\omega| \neq 1, \quad \Theta_{\alpha, \beta, \omega}$ is a regular matrix, so Propositions 1.4 and 1.5 imply the corresponding statement of Theorem 1.1 In the case $0<|\alpha|<1,|\beta|=1-|\alpha|$ we have $B_{n}^{-1}=\bar{B}_{n} / \operatorname{det} B_{n}$, and in this situation the statement of Theorem 1.1 is a consequence of the following propositions.

Proposition 1.6 If $0<|\alpha|<1,|\beta|=1-|\alpha|$ and (1.7) holds then

$$
n^{-4} r_{n}^{-1 / 2}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 2} \operatorname{det} B_{n} \xrightarrow{\mathrm{~L}_{2}} 2(8|\alpha||\beta|)^{-3 / 2} \quad \text { as } n \rightarrow \infty .
$$

Proposition 1.7 If $0<|\alpha|<1,|\beta|=1-|\alpha|$ and (1.7) holds then

$$
n^{-3} r_{n}^{-1 / 2}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 2} \bar{B}_{n} A_{n} \xrightarrow{\mathcal{D}} \mathcal{N}_{2}\left(0,\left(2(8 \alpha \beta)^{2}\right)^{-1} \bar{\Psi}_{\alpha, \beta}\right) \quad \text { as } n \rightarrow \infty .
$$

Obviously, in the case $0 \leq|\alpha| \leq 1,|\beta|=1-|\alpha|$ if $n$ is large enough, the corresponding sequences $\alpha_{r_{n}}$ and $\beta_{r_{n}}$ have the same signs as $\alpha$ and $\beta$, respectively. Hence, similarly to [5], it suffices to prove Propositions 1.6 and 1.7 for $0<\alpha, \beta<1, \alpha+\beta=1$.

## 2. Covariance structure

Let $\left\{X_{k, \ell}: k, \ell \in \mathbb{Z}\right\}$ be a stationary solution of equation (1.2) with parameters $(\alpha, \beta), \quad|\alpha|+|\beta|<1$. Clearly $\operatorname{Cov}\left(X_{i_{1}, j_{1}}, X_{i_{2}, j_{2}}\right)=\operatorname{Cov}\left(X_{i_{1}-i_{2}, j_{1}-j_{2}}, X_{0,0}\right)$ for all $i_{1}, j_{1}, i_{2}, j_{2} \in \mathbb{Z}$. Let $R_{k, \ell}:=\operatorname{Cov}\left(X_{k, \ell}, X_{0,0}\right)$ for $k, \ell \in \mathbb{Z}$. The following lemma is a natural generalization of Lemma 4 of [3] (see also [1).

Lemma 2.1 Let $\alpha \neq 0$ and $\beta \neq 0$. If $k, \ell \in \mathbb{Z}$ with $k \cdot \ell \leq 0$ then

$$
\begin{equation*}
R_{k, \ell}=\sigma_{\alpha, \beta}^{2}\left(\frac{1+\alpha^{2}-\beta^{2}-\sigma_{\alpha, \beta}^{-2}}{2 \alpha}\right)^{|k|}\left(\frac{2 \beta}{1+\beta^{2}-\alpha^{2}+\sigma_{\alpha, \beta}^{-2}}\right)^{|\ell|} . \tag{2.1}
\end{equation*}
$$

If $k, \ell \in \mathbb{Z}$ with $k \cdot \ell \geq 0$ then

$$
\begin{equation*}
R_{k, \ell}=R_{0,|k-\ell|}-\sum_{i=0}^{|k| \wedge|\ell|-1}\binom{|k-\ell|+2 i}{i} \alpha^{i} \beta^{|k-\ell|+i} . \tag{2.2}
\end{equation*}
$$

Remark 2.2 If $\alpha>0$ and $\beta>0$ then $R_{k, \ell} \geq 0$. If $\alpha<0$ or $\beta<0$ we have

$$
0 \leq\left|R_{k, \ell}\right| \leq \widetilde{R}_{k, \ell}:=\operatorname{Cov}\left(\widetilde{X}_{k, \ell}, \widetilde{X}_{0,0}\right), \quad k, \ell \in \mathbb{Z}
$$

where $\left\{\widetilde{X}_{k, \ell}: k, \ell \in \mathbb{Z}\right\}$ is a stationary solution of equation (1.2) with parameters $(|\alpha|,|\beta|)$.

Besides representations (2.1) and (2.2) one can express the covariances as special cases of Appell's hypergeometric series $F_{4}(a, b, c, d ; x, y)$ defined by

$$
F_{4}(a, b, c, d ; x, y):=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_{m}(d)_{n} m!n!} x^{m} y^{n}, \quad \sqrt{|x|}+\sqrt{|y|}<1
$$

where $a, b, c, d \in \mathbb{N}$ and $(a)_{n}:=a(a+1) \ldots(a+n-1)$. 6 .
Lemma 2.3 Let $\alpha \neq 0$ and $\beta \neq 0$. If $k, \ell \in \mathbb{Z}$ with $k \cdot \ell \leq 0$ then

$$
\begin{equation*}
R_{k, \ell}=\alpha^{|k|} \beta^{|\ell|} F_{4}\left(|k|+1,|\ell|+1,|k|+1,|\ell|+1 ; \alpha^{2}, \beta^{2}\right) \tag{2.3}
\end{equation*}
$$

If $k, \ell \in \mathbb{Z}$ with $k \cdot \ell \geq 0$ then

$$
R_{k, \ell}=\alpha^{|k|} \beta^{|\ell|}\binom{|k|+|\ell|}{|k|} F_{4}\left(|k|+|\ell|+1,1,|k|+1,|\ell|+1 ; \alpha^{2}, \beta^{2}\right)
$$

Moreover, in this case we have

$$
\begin{equation*}
R_{k, \ell}=(\operatorname{sign}(\alpha))^{|k|}(\operatorname{sign}(\beta))^{|\ell|} \sum_{i=0}^{\infty}(|\alpha|+|\beta|)^{|k|+|\ell|+2 i} \mathrm{P}\left(S_{i,|k|+|\ell|+i}^{(\nu)}=|\ell|+i\right) \tag{2.4}
\end{equation*}
$$

where $S_{n, m}^{(\nu)}:=S_{n}^{(\nu)}+S_{m}^{(1-\nu)}$, $\nu:=|\alpha| /(|\alpha|+|\beta|)$ and $S_{n}^{(\nu)}$ and $S_{m}^{(1-\nu)}$ are independent binomial random variables with parameters $(n, \nu)$ and $(m, 1-\nu)$, respectively.

Proof. The statements directly follow from representation (1.3) and from the independence of the error terms $\varepsilon_{i, j}$.

We remark, that as

$$
F_{4}\left(a, b, a, b ; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right)=\frac{(1-x)^{b}(1-y)^{a}}{1-x y}
$$

representation (2.1) directly follows from (2.3).

Proposition 2.4 If $\alpha \beta>0,|\alpha|+|\beta|<1$ then there exists a universal positive constant K such that

$$
\left|R_{k-1, \ell+1}-R_{k, \ell}\right| \leq \frac{K}{(\alpha \beta)^{3 / 2}}, \quad \quad k, \ell \in \mathbb{Z}
$$

Proof. Without loss of generality we may assume $\alpha>0$ and $\beta>0$.

Suppose $k>0, \ell \geq 0$, so $(k-1)(\ell+1) \geq 0$ and $k \cdot \ell \geq 0$. Using notations introduced in Lemma 2.3 with the help of (2.4) we obtain

$$
\begin{equation*}
R_{k-1, \ell+1}-R_{k, \ell}=\sum_{i=0}^{\infty}(\alpha+\beta)^{k+\ell+2 i} \Delta_{k, \ell, i}(\nu) \tag{2.5}
\end{equation*}
$$

where

$$
\Delta_{i, k, \ell}(\nu):=\mathrm{P}\left(S_{i, k+\ell+i}^{(\nu)}=\ell+i+1\right)-\mathrm{P}\left(S_{i, k+\ell+i}^{(\nu)}=\ell+i\right)
$$

According to Theorem 2.6 of [5] $\Delta_{i, k, \ell}(\nu)$ can be approximated by

$$
\begin{aligned}
& \widetilde{\Delta}_{i, k, \ell}(\nu):=\frac{1}{(2 \pi \nu(1-\nu)(k+\ell+2 i))^{1 / 2}}\left(\exp \left\{-\frac{(\nu \ell-(1-\nu) k+1)^{2}}{2 \nu(1-\nu)(k+\ell+2 i)}\right\}\right. \\
&\left.-\exp \left\{-\frac{(\nu \ell-(1-\nu) k)^{2}}{2 \nu(1-\nu)(k+\ell+2 i)}\right\}\right)
\end{aligned}
$$

where

$$
\left|\widetilde{\Delta}_{i, k, \ell}(\nu)-\Delta_{i, k, \ell}(\nu)\right| \leq \frac{\widetilde{C}}{(\nu(1-\nu)(k+\ell+2 i))^{3 / 2}}
$$

with some positive constant $\widetilde{C}$. Thus, if in the right hand side of (2.5) we replace $\Delta_{i, k, \ell}(\nu)$ with $\widetilde{\Delta}_{i, k, \ell}(\nu)$, the error of the approximation is

$$
\sum_{i=0}^{\infty}(\alpha+\beta)^{k+\ell+2 i}\left|\widetilde{\Delta}_{i, k, \ell}(\nu)-\Delta_{i, k, \ell}(\nu)\right| \leq \frac{\widetilde{C}}{(\nu(1-\nu))^{3 / 2}} \zeta(3 / 2) \leq \frac{C}{(\alpha \beta)^{3 / 2}}
$$

where $\zeta(x)$ denotes Riemann's zeta function.
To find an upper bound for the approximating sum consider first the case $\nu \ell-(1-\nu) k \geq$ 0 . In this case

$$
\begin{aligned}
\sum_{i=0}^{\infty}(\alpha & +\beta)^{k+\ell+2 i}\left|\widetilde{\Delta}_{i, k, \ell}(\nu)\right| \leq \sum_{i=0}^{\infty} \frac{2(\nu \ell-(1-\nu) k)+1}{\pi^{1 / 2}(2 \nu(1-\nu)(k+\ell+2 i))^{3 / 2}} \exp \left\{-\frac{(\nu \ell-(1-\nu) k)^{2}}{2 \nu(1-\nu)(k+\ell+2 i)}\right\} \\
& \leq \frac{\zeta(3 / 2)+1}{(\nu(1-\nu))^{3 / 2}}+\frac{1}{2 \nu(1-\nu)} \widetilde{\Phi}\left(\frac{\nu \ell-(1-\nu) k}{(2 \nu(1-\nu)(k+\ell))^{1 / 2}}\right) \leq \frac{\zeta(3 / 2)+2}{(\nu(1-\nu))^{3 / 2}} \leq \frac{\zeta(3 / 2)+2}{(\alpha \beta)^{3 / 2}}
\end{aligned}
$$

where $\widetilde{\Phi}(x)$ is the error function defined by

$$
\widetilde{\Phi}(x):=\frac{2}{\pi^{1 / 2}} \int_{0}^{x} \mathrm{e}^{-t^{2} / 2} \mathrm{~d} t, \quad x>0
$$

Case $\nu \ell-(1-\nu) k<0$ follows by symmetry.
In case $k \leq 0, \ell<0$ implying $(k-1)(\ell+1) \geq 0$ and $k \cdot \ell>0$, we have
$R_{k-1, \ell+1}-R_{k, \ell}=\sum_{i=0}^{\infty}(\alpha+\beta)^{-k-\ell+2 i}\left(\mathrm{P}\left(S_{i,-k-\ell+i}^{(\nu)}=-\ell+i-1\right)-\mathrm{P}\left(S_{i,-k-\ell+i}^{(\nu)}=-\ell+i\right)\right)$
and the statement can be proved similarly to the previous case.

Now, suppose $k>0, \ell<0$, so $(k-1)(\ell+1) \leq 0$ and $k \cdot \ell \leq 0$. Using the form (2.1) of the covariances direct calculations show

$$
R_{k-1, \ell+1}-R_{k, \ell}=R_{k, \ell} \frac{1-(\alpha+\beta)^{2}+\sigma_{\alpha, \beta}^{-2}}{2 \alpha \beta}
$$

It is not difficult to see that $1-(\alpha+\beta)^{2} \leq \sigma_{\alpha, \beta}^{-2}$, so we have

$$
\left|R_{k-1, \ell+1}-R_{k, \ell}\right| \leq\left|R_{k, \ell}\right| \frac{\sigma_{\alpha, \beta}^{-2}}{\alpha \beta} \leq \frac{1}{\alpha \beta}
$$

In a similar way one can obtain the result for $k \leq 0, \ell \geq 0$ that completes the proof.
Using the notations of Lemma 2.3 with the help of the exponential approximation one can easily have the analogue of Corollary 2.7 of [5].

Corollary 2.5 If $\alpha \beta>0,|\alpha|+|\beta|<1$ then there exists a constant $C>0$ such that for all $k, \ell>1$ and $0 \leq i \leq k+\ell-1$ we have

$$
\left|\mathrm{P}\left(S_{k, \ell}^{(\nu)}=i+1\right)-\mathrm{P}\left(S_{k, \ell}^{(\nu)}=i\right)\right| \leq \frac{C}{\alpha \beta(k+\ell)}
$$

Remark 2.6 Using Theorem 2.4 of [5] it is not difficult to show that under conditions of Corollary 2.5 there exists a constant $D>0$ such that for all $k, \ell>1$ and $0 \leq i \leq k+\ell$ we have

$$
\left|\mathrm{P}\left(S_{k, \ell}^{(\nu)}=i\right)\right| \leq \frac{D}{\alpha \beta(k+\ell)^{1 / 2}}
$$

Now, let $\left\{X_{k, \ell}^{(n)}: k, \ell \in \mathbb{Z}\right\}, n \in \mathbb{N}$, be a nearly unstable sequence of stationary processes described in Theorem 1.1 For each $n \in \mathbb{N}$ let us introduce the piecewise constant random fields

$$
\begin{array}{ll}
Z_{1,0}^{(n)}(s, t):=r_{n}^{-1 / 4} X_{[n s]+1,[n t]}^{\left(r_{n}\right)}, & Z_{0,1}^{(n)}(s, t):=r_{n}^{-1 / 4} X_{[n s],[n t]+1}^{\left(r_{n}\right)}, \\
Y_{1,0}^{(n)}(s, t):=r_{n}^{-1 / 2} X_{[n s]+1,[n t]}^{\left(r_{n}\right)}, & Y_{0,1}^{(n)}(s, t):=r_{n}^{-1 / 2} X_{[n s],[n t]+1}^{\left(r_{n}\right)}, \quad s, t \in \mathbb{R}
\end{array}
$$

Proposition 2.7 Let $s_{1}, t_{1}, s_{2}, t_{2} \in \mathbb{R}$.
If $0<|\alpha|<1,|\beta|=1-|\alpha|$ and (1.7) holds then for all $\left(i_{1}, j_{1}\right),\left(i_{2}, j_{2}\right) \in$ $\{(1,0),(0,1)\}$ we have

$$
\lim _{n \rightarrow \infty}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 2} \operatorname{Cov}\left(Z_{i_{1}, j_{1}}^{(n)}\left(s_{1}, t_{1}\right), Z_{i_{2}, j_{2}}^{(n)}\left(s_{2}, t_{2}\right)\right)=0 \quad \text { if } s_{1}-s_{2} \neq t_{1}-t_{2}
$$

$$
\limsup _{n \rightarrow \infty}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 2}\left|\operatorname{Cov}\left(Z_{i_{1}, j_{1}}^{(n)}\left(s_{1}, t_{1}\right), Z_{i_{2}, j_{2}}^{(n)}\left(s_{2}, t_{2}\right)\right)\right| \leq \frac{1}{\sqrt{8|\alpha||\beta|}} \quad \text { if } s_{1}-s_{2}=t_{1}-t_{2}
$$

If $|\alpha| \in\{0,1\},|\beta|=1-|\alpha|$ and (1.8) holds then for all $\left(i_{1}, j_{1}\right),\left(i_{2}, j_{2}\right) \in\{(1,0),(0,1)\}$ we have

$$
\begin{aligned}
& \lim _{n \rightarrow \infty}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{1 / 2} \operatorname{Cov}\left(Y_{i_{1}, j_{1}}^{(n)}\left(s_{1}, t_{1}\right), Y_{i_{2}, j_{2}}^{(n)}\left(s_{2}, t_{2}\right)\right)=0 \quad \text { if } s_{1}-s_{2} \neq t_{1}-t_{2} \\
& \limsup _{n \rightarrow \infty}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{1 / 2}\left|\operatorname{Cov}\left(Y_{i_{1}, j_{1}}^{(n)}\left(s_{1}, t_{1}\right), Y_{i_{2}, j_{2}}^{(n)}\left(s_{2}, t_{2}\right)\right)\right| \leq \frac{1}{2} \quad \text { if } s_{1}-s_{2}=t_{1}-t_{2}
\end{aligned}
$$

Moreover, if $s_{1}-s_{2} \neq t_{1}-t_{2}$ then the convergence to 0 in both cases has an exponential rate.

Proof. For simplicity we consider only the case $0 \leq \alpha, \beta \leq 1$. The other cases can be handled in a similar way.

First, let $0<\alpha<1$, so $\beta=1-\alpha$. Without loss of generality we may assume $\alpha_{r_{n}}>0, \beta_{r_{n}}>0$ and $\delta_{r_{n}}>0, \gamma_{r_{n}}>0$. As
$r_{n}^{-1 / 2} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2}=\left(\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)\left(2-\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{r_{n}}\right)\left(2 \alpha-\frac{\gamma_{r_{n}}-\delta_{r_{n}}}{r_{n}}\right)\left(2(1-\alpha)+\frac{\gamma_{r_{n}}-\delta_{r_{n}}}{r_{n}}\right)\right)^{-1 / 2}$
we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} r_{n}^{-1 / 2} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2}=\frac{1}{\sqrt{8 \alpha(1-\alpha)}}=\frac{1}{\sqrt{8 \alpha \beta}} \tag{2.6}
\end{equation*}
$$

Suppose $s_{1}-s_{2} \geq 0 \geq t_{1}-t_{2}$, so $\left[n s_{1}\right]-\left[n s_{2}\right] \geq 0 \geq\left[n t_{1}\right]-\left[n t_{2}\right]$. By (2.1)
$0 \leq \operatorname{Cov}\left(Z_{1,0}^{(n)}\left(s_{1}, t_{1}\right), Z_{1,0}^{(n)}\left(s_{2}, t_{2}\right)\right) \leq r_{n}^{-1 / 2} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2}\left(1-\frac{1}{\varrho_{r_{n}}}\right)^{\frac{n}{2}\left|s_{1}-s_{2}\right|}\left(1+\frac{1}{\tau_{r_{n}}}\right)^{-\frac{n}{2}\left|t_{1}-t_{2}\right|}$ if $n$ is large enough, where

$$
\begin{equation*}
\varrho_{r_{n}}:=\frac{2 \alpha_{r_{n}}}{2 \alpha_{r_{n}}-1-\alpha_{r_{n}}^{2}+\beta_{r_{n}}^{2}+\sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{-2}}, \quad \tau_{r_{n}}:=\frac{2 \beta_{r_{n}}}{1+\beta_{r_{n}}^{2}-\alpha_{r_{n}}^{2}+\sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{-2}-2 \beta_{r_{n}}} \tag{2.7}
\end{equation*}
$$

As

$$
\sigma_{\alpha, \beta}^{2}=\left(\left(1+\alpha^{2}-\beta^{2}\right)^{2}-4 \alpha^{2}\right)^{-1 / 2}
$$

it is easy to see that $\varrho_{r_{n}} \rightarrow \infty$ and $\tau_{r_{n}} \rightarrow \infty$ as $n \rightarrow \infty$. Moreover, condition (1.7) ensures that $n \varrho_{r_{n}}^{-1} \rightarrow \infty$ and $n \tau_{r_{n}}^{-1} \rightarrow \infty$ as $n \rightarrow \infty$. Hence, if $s_{1}=s_{2}$ and $t_{1}=t_{2}$,

$$
\lim _{n \rightarrow \infty}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \operatorname{Cov}\left(Z_{1,0}^{(n)}\left(s_{1}, t_{1}\right), Z_{1,0}^{(n)}\left(s_{2}, t_{2}\right)\right)=\frac{1}{\sqrt{8 \alpha \beta}}
$$

otherwise it converges to 0 in exponential rate.
Further, let $s_{1}-s_{2}>0$ and $t_{1}-t_{2}>0$. In this case $\left[n s_{1}\right]-\left[n s_{2}\right] \geq 0$ and $\left[n t_{1}\right]-\left[n t_{2}\right] \geq 0$, so by (2.2) we have

$$
\begin{equation*}
0 \leq \operatorname{Cov}\left(Z_{1,0}^{(n)}\left(s_{1}, t_{1}\right), Z_{1,0}^{(n)}\left(s_{2}, t_{2}\right)\right) \leq r_{n}^{-1 / 2} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2}\left(1+\frac{1}{\tau_{r_{n}}}\right)^{-\left|\left[n s_{1}\right]-\left[n s_{2}\right]-\left[n t_{1}\right]+\left[n t_{2}\right]\right|} \tag{2.8}
\end{equation*}
$$

If $s_{1}-s_{2} \neq t_{1}-t_{2}$ then similarly to the previous case one can show that the right hand side of (2.8) converges to 0 in exponential rate as $n \rightarrow \infty$.

In case $s_{1}-s_{2}=t_{1}-t_{2}$ we have $\left|\left[n s_{1}\right]-\left[n s_{2}\right]-\left[n t_{1}\right]+\left[n t_{2}\right]\right| \leq 2$, so by (2.8)

$$
\limsup _{n \rightarrow \infty}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \operatorname{Cov}\left(Z_{1,0}^{(n)}\left(s_{1}, t_{1}\right), Z_{1,0}^{(n)}\left(s_{2}, t_{2}\right)\right) \leq \frac{1}{\sqrt{8 \alpha \beta}}
$$

Obviously, the same results hold for the covariances $\operatorname{Cov}\left(Z_{1,0}^{(n)}\left(s_{1}, t_{1}\right), Z_{0,1}^{(n)}\left(s_{2}, t_{2}\right)\right)$, $\operatorname{Cov}\left(Z_{0,1}^{(n)}\left(s_{1}, t_{1}\right), Z_{1,0}^{(n)}\left(s_{2}, t_{2}\right)\right)$ and $\operatorname{Cov}\left(Z_{0,1}^{(n)}\left(s_{1}, t_{1}\right), Z_{0,1}^{(n)}\left(s_{2}, t_{2}\right)\right)$.

Now, consider for example the case $\alpha=1, \beta=0$. Without loss of generality we may assume $\alpha_{r_{n}}>0$. Furthermore, $\left|\alpha_{r_{n}}\right|+\left|\beta_{r_{n}}\right|<1$ implies $\gamma_{r_{n}}>0$ and $\left|\delta_{r_{n}}\right|<\gamma_{r_{n}}$. As

$$
r_{n}^{-1} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2}=\left(\left(\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right)\left(2-\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{r_{n}}\right)\left(2-\frac{\gamma_{r_{n}}-\delta_{r_{n}}}{r_{n}}\right)\right)^{-1 / 2}
$$

we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right)^{1 / 2} r_{n}^{-1} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2}=\frac{1}{2} \tag{2.9}
\end{equation*}
$$

Again, suppose $s_{1}-s_{2} \geq 0 \geq t_{1}-t_{2}$. The form of covariances (2.1) implies that if $n$ is large enough

$$
\begin{equation*}
0 \leq\left|\operatorname{Cov}\left(Y_{1,0}^{(n)}\left(s_{1}, t_{1}\right), Y_{1,0}^{(n)}\left(s_{2}, t_{2}\right)\right)\right| \leq r_{n}^{-1} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2}\left(1-\frac{1}{\varrho_{r_{n}}}\right)^{\frac{n}{2}\left|s_{1}-s_{2}\right|}\left(1+\frac{1}{\left|\tau_{r_{n}}\right|}\right)^{-\frac{n}{2}\left|t_{1}-t_{2}\right|} \tag{2.10}
\end{equation*}
$$

where $\varrho_{r_{n}}$ and $\tau_{r_{n}}$ are defined by (2.7). Obviously, if $s_{1}=s_{2}$ and $t_{1}=t_{2}$ then (2.9) implies

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left(\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right)^{1 / 2}\left|\operatorname{Cov}\left(Y_{1,0}^{(n)}\left(s_{1}, t_{1}\right), Y_{1,0}^{(n)}\left(s_{2}, t_{2}\right)\right)\right| \leq \frac{1}{2} \tag{2.11}
\end{equation*}
$$

Further, we have $\varrho_{r_{n}} \rightarrow \infty$ as $n \rightarrow \infty$ and now (1.8) ensures $n \varrho_{r_{n}}^{-1} \rightarrow \infty$. Thus, as $1+1 /\left|\tau_{r_{n}}\right| \geq 1$, if $s_{1} \neq s_{2}$ then

$$
\begin{equation*}
\left(\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right)^{1 / 2}\left|\operatorname{Cov}\left(Y_{1,0}^{(n)}\left(s_{1}, t_{1}\right), Y_{1,0}^{(n)}\left(s_{2}, t_{2}\right)\right)\right| \rightarrow 0 \tag{2.12}
\end{equation*}
$$

as $n \rightarrow \infty$ in exponential rate. Now, let us assume $s_{1}=s_{2}$ and $t_{1} \neq t_{2}$. Short calculation shows

$$
\begin{equation*}
\left(1+\frac{1}{\left|\tau_{r_{n}}\right|}\right)^{-1}=\frac{2\left|\delta_{r_{n}}\right|}{2 \gamma_{r_{n}}-\frac{\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}}{r_{n}}+\left(\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right)^{1 / 2}\left(\frac{\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}}{r_{n}}-4 \frac{\gamma_{r_{n}}}{r_{n}}+4\right)^{1 / 2}} . \tag{2.13}
\end{equation*}
$$

If $|\delta|<\gamma$ then

$$
\lim _{n \rightarrow \infty}\left(1+\frac{1}{\left|\tau_{r_{n}}\right|}\right)^{-1}=\frac{|\delta|}{\gamma+\left(\gamma^{2}-\delta^{2}\right)^{1 / 2}}<1
$$

so using (2.9) and (2.10) we obtain again (2.12). Further, condition (1.8) implies

$$
\lim _{n \rightarrow \infty} n\left(\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right)^{1 / 2}=\infty
$$

Hence, with the help of (2.13) one can easily see that if $|\delta|=\gamma \neq 0$, or $\delta=\gamma=0$ and $\lim _{n \rightarrow \infty} \gamma_{r_{n}}\left|\delta_{r_{n}}\right|^{-1}=1$, we obtain $\left|\tau_{r_{n}}\right| \rightarrow \infty$ and $n\left|\tau_{r_{n}}\right|^{-1} \rightarrow \infty$ as $n \rightarrow \infty$. Thus, (2.9) and (2.10) imply (2.12) and the rate of convergence is again exponential. In case $\delta=\gamma=0$ and $\lim _{n \rightarrow \infty} \gamma_{r_{n}}\left|\delta_{r_{n}}\right|^{-1}=|\omega|>1$ we have

$$
\lim _{n \rightarrow \infty}\left(1+\frac{1}{\left|\tau_{r_{n}}\right|}\right)^{-1}=\frac{1}{|\omega|+\left(\omega^{2}-1\right)^{1 / 2}}<1
$$

that implies (2.12). Finally, if $\delta=\gamma=0$ and $\lim _{n \rightarrow \infty} \gamma_{r_{n}}\left|\delta_{r_{n}}\right|^{-1}=\infty$ then (2.12) follows from

$$
\lim _{n \rightarrow \infty}\left(1+\frac{1}{\left|\tau_{r_{n}}\right|}\right)^{-1}=0 .
$$

Now, let $s_{1}-s_{2}>0$ and $t_{1}-t_{2}>0$. Lemma 2.1] and Remark 2.2 imply

$$
0 \leq\left|\operatorname{Cov}\left(Y_{1,0}^{(n)}\left(s_{1}, t_{1}\right), Y_{1,0}^{(n)}\left(s_{2}, t_{2}\right)\right)\right| \leq r_{n}^{-1} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2}\left(1+\frac{1}{\left|\tau_{r_{n}}\right|}\right)^{-\left|\left[n s_{1}\right]-\left[n s_{2}\right]-\left[n t_{1}\right]+\left[n t_{2}\right]\right|}
$$

where $\tau_{r_{n}}$ is defined by (2.7). If $s_{1}-s_{2}=t_{1}-t_{2}$ then as $\left|\left[n s_{1}\right]-\left[n s_{2}\right]-\left[n t_{1}\right]+\left[n t_{2}\right]\right| \leq 2$ and $1+1 /\left|\tau_{r_{n}}\right| \geq 1$, using (2.9) we obtain (2.11). Finally, if $s_{1}-s_{2} \neq t_{1}-t_{2}$ then to prove (2.11) one has to do the same considerations as in the case $s_{1}=s_{2}$ and $t_{1} \neq t_{2}$.

In order to estimate the covariances we make use of the following lemma which is a natural generalization of Lemma 2.8 of [5].

Lemma 2.8 Let $\xi_{1}, \xi_{2}, \ldots$ be independent random variables with $\mathrm{E} \xi_{i}=0$, $\mathrm{E} \xi_{i}^{2}=1$ for all $i \in \mathbb{N}$, and $M_{4}:=\sup _{i \in \mathbb{N}} \mathrm{E} \xi_{i}^{4}<\infty$. Let $a_{1}, a_{2}, \ldots, b_{1}, b_{2}, \ldots, c_{1}, c_{2} \ldots, d_{1}, d_{2} \ldots \in$ $\mathbb{R}$, such that $\sum_{i=1}^{\infty} a_{i}^{2}<\infty, \sum_{i=1}^{\infty} b_{i}^{2}<\infty, \sum_{i=1}^{\infty} c_{i}^{2}<\infty$ and $\sum_{i=1}^{\infty} d_{i}^{2}<\infty$. Let

$$
X:=\sum_{i=1}^{\infty} a_{i} \xi_{i}, \quad Y:=\sum_{i=1}^{\infty} b_{i} \xi_{i}, \quad Z:=\sum_{i=1}^{\infty} c_{i} \xi_{i}, \quad W:=\sum_{i=1}^{\infty} d_{i} \xi_{i}
$$

where the convergence of the infinite sums is understood in $\mathrm{L}_{2}$-sense. Then

$$
\begin{equation*}
\operatorname{Cov}(X Y, Z W)=\sum_{i=1}^{\infty}\left(E \xi_{i}^{4}-3\right) a_{i} b_{i} c_{i} d_{i}+\operatorname{Cov}(X, Z) \operatorname{Cov}(Y, W)+\operatorname{Cov}(X, W) \operatorname{Cov}(Y, Z) \tag{2.14}
\end{equation*}
$$

Moreover, if $a_{i}, b_{i}, c_{i}, d_{i} \geq 0$ then

$$
0 \leq \operatorname{Cov}(X Y, Z W) \leq M_{4} \operatorname{Cov}(X, Z) \operatorname{Cov}(Y, W)+M_{4} \operatorname{Cov}(X, W) \operatorname{Cov}(Y, Z)
$$

and

$$
0 \leq \mathrm{E} X Y Z W \leq M_{4}(\mathrm{E} X Z \mathrm{E} Y W+\mathrm{E} X W \mathrm{E} Y Z+\mathrm{E} X Y \mathrm{E} Z W)
$$

Remark 2.9 Using the definitions of Lemma 2.8 from (2.14) one can easily see, that

$$
|\operatorname{Cov}(X Y, Z W)| \leq \operatorname{Cov}(\widetilde{X} \widetilde{Y}, \widetilde{Z} \widetilde{W})
$$

where

$$
\widetilde{X}:=\sum_{i=1}^{\infty}\left|a_{i}\right| \xi_{i}, \quad \widetilde{Y}:=\sum_{i=1}^{\infty}\left|b_{i}\right| \xi_{i}, \quad \widetilde{Z}:=\sum_{i=1}^{\infty}\left|c_{i}\right| \xi_{i}, \quad \widetilde{W}:=\sum_{i=1}^{\infty}\left|d_{i}\right| \xi_{i}
$$

## 3. Proof of Proposition 1.4

Let us assume $\alpha_{r_{n}} \neq 0$ and $\beta_{r_{n}} \neq 0$. Using the stationarity of $\left\{X_{k, \ell}^{\left(r_{n}\right)}: k, \ell \in \mathbb{Z}\right\}$ and Lemma 2.1 we obtain

$$
\begin{aligned}
& \mathrm{E} B_{n}=\sum_{(k, \ell) \in T_{k_{n}, \ell_{n}}\left(\begin{array}{cc}
\operatorname{Var}\left(X_{0,0}^{\left(r_{n}\right)}\right) & \operatorname{Cov}\left(X_{0,0}^{\left(r_{n}\right)}, X_{1,-1}^{\left(r_{n}\right)}\right) \\
\operatorname{Cov}\left(X_{0,0}^{\left(r_{n}\right)}, X_{1,-1}^{\left(r_{n}\right)}\right) & \operatorname{Var}\left(X_{0,0}^{\left(r_{n}\right)}\right)
\end{array}\right)}^{\quad=\frac{\left(k_{n}+\ell_{n}\right)\left(k_{n}+\ell_{n}+1\right)}{2} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2}\left(\begin{array}{cc}
1 & D_{r_{n}} \\
D_{r_{n}} & 1
\end{array}\right)=\frac{n(n+1)}{2} \sigma_{\alpha_{r_{n}, \beta_{r_{n}}}^{2}}\left(\begin{array}{cc}
1 & D_{r_{n}} \\
D_{r_{n}} & 1
\end{array}\right)} \text {, }
\end{aligned}
$$

where

$$
D_{r_{n}}=\left(\frac{1+\alpha_{r_{n}}^{2}-\beta_{r_{n}}^{2}-\sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{-2}}{2 \alpha_{r_{n}}}\right)\left(\frac{2 \beta_{r_{n}}}{1+\beta_{r_{n}}^{2}-\alpha_{r_{n}}^{2}+\sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{-2}}\right)
$$

If $0<|\alpha|<1$ and $|\beta|=1-|\alpha|$ then it is not difficult to see that $\sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{-2} \rightarrow 0$ and in this way $D_{r_{n}} \rightarrow \operatorname{sign}(\alpha \beta)$ as $n \rightarrow \infty$. Hence, using the same arguments as in the proof of (2.6) we obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{-2} r_{n}{ }^{-1 / 2}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 2} \mathrm{E} B_{n}=(32|\alpha||\beta|)^{-1 / 2} \Psi_{\alpha, \beta} \tag{3.1}
\end{equation*}
$$

If $|\alpha| \in\{0,1\}$ and $|\beta|=1-|\alpha|$, again, we have $\sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{-2} \rightarrow 0$ as $n \rightarrow \infty$, and similarly to the proof of (2.9) one can see

$$
\lim _{n \rightarrow \infty} n^{-2} r_{n}{ }^{-1}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{1 / 2} \frac{n(n+1)}{2} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2}=\frac{1}{4}
$$

Concerning the limit of $D_{r_{n}}$ from the four possible cases that can be handled in the same way we consider only the case $\alpha=1, \beta=0$. In this case $\alpha \frac{\gamma_{r_{n}}}{\delta_{r_{n}}}+\beta \frac{\delta_{r_{n}}}{\gamma_{r_{n}}}=\frac{\gamma_{r_{n}}}{\delta_{r_{n}}}$ and we may assume $\alpha_{r_{n}}>0$ and thus $\left|\delta_{r_{n}}\right| \leq \gamma_{r_{n}}$ (hence $\gamma_{r_{n}}>0$ ). Obviously,

$$
\lim _{n \rightarrow \infty} \frac{1+\alpha_{r_{n}}^{2}-\beta_{r_{n}}^{2}-\sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{-2}}{2 \alpha_{r_{n}}}=1
$$

and

$$
\begin{aligned}
\frac{2 \beta_{r_{n}}}{1+\beta_{r_{n}}^{2}-\alpha_{r_{n}}^{2}+\sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{-2}}=\left(\frac{\gamma_{r_{n}}-\delta_{r_{n}}}{2 r_{n}}-\operatorname{sign}(\omega)\left(1-\frac{\gamma_{r_{n}}-\delta_{r_{n}}}{2 r_{n}}\right)^{1 / 2}\right. \\
\left.\quad \times\left(\frac{\gamma_{r_{n}}}{\left|\delta_{r_{n}}\right|}\left(1-\frac{\gamma_{r_{n}}-\delta_{r_{n}}}{2 r_{n}}\right)^{1 / 2}+\left(\frac{\gamma_{r_{n}}^{2}}{\delta_{r_{n}}^{2}}-1\right)^{1 / 2}\left(1-\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{2 r_{n}}\right)^{1 / 2}\right)\right)^{-1} .
\end{aligned}
$$

Hence,

$$
\lim _{n \rightarrow \infty} D_{r_{n}}= \begin{cases}-\operatorname{sign}(\omega)\left(|\omega|+\left(\omega^{2}-1\right)^{1 / 2}\right)^{-1} & \text { if }|\omega|<\infty \\ 0 & \text { if }|\omega|=\infty\end{cases}
$$

where $\omega$ is the limit defined by (1.9) satisfying $|\omega| \geq 1$. Thus, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{-2} r_{n}^{-1}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{1 / 2} \mathrm{E} B_{n}=\Theta_{\alpha, \beta, \omega} . \tag{3.2}
\end{equation*}
$$

Observe, that $\lim _{n \rightarrow \infty} D_{r_{n}}=\lim _{n \rightarrow \infty} \theta\left(\alpha, \beta, \omega_{r_{n}}\right)$.
By Remark 2.9 in the remaining part of the proof we may assume $\alpha_{r_{n}} \geq 0, \beta_{r_{n}} \geq 0$. Hence, using Lemma 2.8 we have

$$
\begin{equation*}
\operatorname{Var}\left(\sum_{(i, j) \in T_{k_{n}, \ell_{n}}}\left(X_{i-1, j}^{\left(r_{n}\right)}\right)^{2}\right) \leq 2 M_{4} \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}} \operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}\right)^{2} \tag{3.3}
\end{equation*}
$$

where $M_{4}:=\sup _{n \in \mathbb{N}} \mathrm{E}\left(\varepsilon_{0,0}^{(n)}\right)^{4}$, and from the stationarity of $\left\{X_{k, \ell}^{\left(r_{n}\right)}: k, \ell \in \mathbb{Z}\right\}$ follows that the triangle $T_{k_{n}, \ell_{n}}$ can be replaced by $T_{n, 0}$.

Now, (3.3) implies that if $0<|\alpha|<1$ and $|\beta|=1-|\alpha|$

$$
\begin{align*}
& n^{-4} r_{n}^{-1}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right) \operatorname{Var}\left(\sum_{(i, j) \in T_{k_{n}, \ell_{n}}}\left(X_{i-1, j}^{\left(r_{n}\right)}\right)^{2}\right)  \tag{3.4}\\
& \quad \leq 2 M_{4} \iint_{T} \iint_{T}\left(\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 2} \operatorname{Cov}\left(Z_{0,1}\left(s_{1}, t_{1}\right), Z_{0,1}\left(s_{2}, t_{2}\right)\right)\right)^{2} \mathrm{~d} s_{1} \mathrm{~d} t_{1} \mathrm{~d} s_{2} \mathrm{~d} t_{2}
\end{align*}
$$

while for $|\alpha| \in\{0,1\},|\beta|=1-|\alpha|$ we have

$$
\begin{align*}
& n^{-4} r_{n}^{-2}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right| \operatorname{Var}\left(\sum_{(i, j) \in T_{k_{n}, \ell_{n}}}\left(X_{i-1, j}^{\left(r_{n}\right)}\right)^{2}\right)  \tag{3.5}\\
& \quad \leq 2 M_{4} \iiint \int_{T}\left(\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{1 / 2} \operatorname{Cov}\left(Y_{0,1}\left(s_{1}, t_{1}\right), Y_{0,1}\left(s_{2}, t_{2}\right)\right)\right)^{2} \mathrm{~d} s_{1} \mathrm{~d} t_{1} \mathrm{~d} s_{2} \mathrm{~d} t_{2},
\end{align*}
$$

where $T:=\left\{(s, t) \in \mathbb{R}^{2}: 0 \leq s \leq 1,-s \leq t \leq 0\right\}$. As the area of the triangle $T$ is finite and the integrands in both cases are uniformly bounded on $T \times T$, Fatou's lemma and Proposition [2.7 imply that the right hand sides of (3.4) and (3.5) converge to 0 as $n \rightarrow \infty$. In a similar way one can show
$n^{-4} \kappa_{n} \operatorname{Var}\left(\sum_{(i, j) \in T_{k_{n}, \ell_{n}}} X_{i-1, j}^{\left(r_{n}\right)} X_{i, j-1}^{\left(r_{n}\right)}\right) \rightarrow 0 \quad$ and $\quad n^{-4} \kappa_{n} \operatorname{Var}\left(\sum_{(i, j) \in T_{k_{n}, \ell_{n}}}\left(X_{i, j-1}^{\left(r_{n}\right)}\right)^{2}\right) \rightarrow 0$,
as $n \rightarrow \infty$, where

$$
\kappa_{n}= \begin{cases}r_{n}^{-1}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right) & \text { if } 0<|\alpha|<1,|\beta|=1-|\alpha|,  \tag{3.6}\\ r_{n}^{-2}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right| & \text { if }|\alpha| \in\{0,1\},|\beta|=1-|\alpha| .\end{cases}
$$

that finishes the proof of Proposition 1.4.

## 4. Proof of Proposition 1.5

To prove Proposition 1.5 we are going to use the same technique as in 315. For a given $n \in \mathbb{N}$ and $1 \leq m \leq n$, let

$$
A_{n, m}=\binom{A_{n, m}^{(1)}}{A_{n, m}^{(2)}}:=\sum_{(k, \ell) \in T_{k_{m}, \ell},}\binom{X_{k-1, \ell_{m}}^{\left(r_{n}\right)} \varepsilon_{k, \ell}^{\left(r_{n}\right)}}{X_{k, \ell-1}^{\left(r_{n}\right)} \varepsilon_{k, \ell}^{\left(r_{n}\right)}},
$$

where $A_{n, 0}:=(0,0)^{\top}$. Let $\mathcal{F}_{m}^{n}$ denote the $\sigma$-algebra generated by the random variables $\left\{\varepsilon_{k, \ell}^{\left(r_{n}\right)}:(k, \ell) \in U_{k_{m}, \ell_{m}}\right\}$. Obviously, $A_{n, n}=A_{n}=\sum_{m=1}^{n}\left(A_{n, m}-A_{n, m-1}\right)$. First we show that $\left(A_{n, m}-A_{n, m-1}, \mathcal{F}_{m}^{n}\right)$ is a square integrable martingale difference. Let $R_{m}:=$ $T_{k_{m}, \ell_{m}} \backslash T_{k_{m-1}, \ell_{m-1}}$, where $\quad R_{1}:=T_{k_{1}, \ell_{1}}$. Short calculation shows

$$
\begin{equation*}
A_{n, m}-A_{n, m-1}=A_{n, m, 1}+\sum_{(k, \ell) \in R_{m}} \varepsilon_{k, \ell}^{\left(r_{n}\right)} A_{n, m, 2, k, \ell}, \tag{4.1}
\end{equation*}
$$

where $A_{n, m, 1}=\left(A_{n, m, 1}^{(1)}, A_{n, m, 1}^{(2)}\right)^{\top}$ and $A_{n, m, 2, k, \ell}=\left(\widetilde{A}_{n, m, 2, k-1, \ell}, \widetilde{A}_{n, m, 2, k, \ell-1}\right)^{\top}$ with

$$
\begin{align*}
A_{n, m, 1}^{(1)} & :=\sum_{(k, \ell) \in R_{m}} \varepsilon_{k, \ell}^{\left(r_{n}\right)} \sum_{(i, j) \in U_{k-1, \ell \backslash U_{k_{m-1}, \ell_{m-1}}}\binom{k+\ell-1-i-j}{k-1-i} \alpha_{r_{n}}^{k-1-i} \beta_{r_{n}}^{\ell-j} \varepsilon_{i, j}^{\left(r_{n}\right)},}^{A_{n, m, 1}^{(2)}}:=\sum_{(k, \ell) \in R_{m}} \varepsilon_{k, \ell}^{\left(r_{n}\right)} \sum_{(i, j) \in U_{k, \ell-1} \backslash U_{k_{m-1}, \ell}, \ell_{m-1}}\binom{k+\ell-1-i-j}{k-i} \alpha_{r_{n}}^{k-i} \beta_{r_{n}}^{\ell-1-j} \varepsilon_{i, j}^{\left(r_{n}\right)}, \\
\widetilde{A}_{n, m, 2, k, \ell} & :=\sum_{(i, j) \in U_{k, \ell} \cap U_{k_{m-1}, \ell_{m-1}}}\binom{k+\ell-i-j}{k-i} \alpha_{r_{n}}^{k-i} \beta_{r_{n}}^{\ell-j} \varepsilon_{i, j}^{\left(r_{n}\right)} .
\end{align*}
$$

We remark that for the odd values of $m$ we have $R_{m}=\bigcup_{i=-\ell_{m}+1}^{k_{m}}\left\{\left(i, \ell_{m}\right)\right\}$, and

$$
\begin{equation*}
A_{n, m, 1}^{(1)}=\sum_{k=-\ell_{m}+2}^{k_{m}} \sum_{i=-\infty}^{k-1} \alpha_{r_{n}}^{k-1-i} \varepsilon_{k, \ell_{m}}^{\left(r_{n}\right)} \varepsilon_{i, \ell_{m}}^{\left(r_{n}\right)}, \quad A_{n, m, 1}^{(2)}=0 \tag{4.3}
\end{equation*}
$$

while for the even values $R_{m}=\bigcup_{j=-k_{m}+1}^{\ell_{m}}\left\{\left(k_{m}, j\right)\right\}$, and

$$
\begin{equation*}
A_{n, m, 1}^{(2)}=\sum_{\ell=-k_{m}+2}^{\ell} \sum_{j=-\infty}^{\ell-1} \beta_{r_{n}}^{\ell-1-j} \varepsilon_{k_{m}, \ell}^{\left(r_{n}\right)} \varepsilon_{k_{m}, j}^{\left(r_{n}\right)}, \quad \quad A_{n, m, 1}^{(1)}=0 . \tag{4.4}
\end{equation*}
$$

The components of $A_{n, m, 1}$ are quadratic forms of the variables $\left\{\varepsilon_{i, j}^{\left(r_{n}\right)}:(i, j) \in\right.$ $\left.R_{m}\right\}$, hence $A_{n, m, 1}$ is independent of $\mathcal{F}_{m-1}^{n}$. Further, the terms $\widetilde{A}_{n, m, 2, k, \ell}$ are linear combinations of the variables $\left\{\varepsilon_{i, j}^{\left(r_{n}\right)}:(i, j) \in U_{k_{m-1}, \ell_{m-1}}\right\}$, thus they are measurable with respect to $\mathcal{F}_{m-1}^{n}$. Hence,

$$
\mathrm{E}\left(A_{n, m}-A_{n, m-1} \mid \mathcal{F}_{m-1}^{n}\right)=\mathrm{E} A_{n, m, 1}+\sum_{(k, \ell) \in R_{m}} A_{n, m, 2, k, \ell} \mathrm{E}\left(\varepsilon_{p, q}^{\left(r_{n}\right)} \mid \mathcal{F}_{m-1}^{n}\right)=0
$$

By the Martingale Central Limit Theorem (see, e.g. [14, Theorem 4, p. 511]), the statement in Proposition 1.5 is a consequence of the following two propositions, where $\mathbb{1}_{H}$ denotes the indicator function of the set $H$.

Proposition 4.1 If $0<|\alpha|<1,|\beta|=1-|\alpha|$ and (1.7) holds then

$$
\begin{aligned}
& n^{-2} r_{n}^{-1 / 2}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 2} \sum_{m=1}^{n} \mathrm{E}\left(\left(A_{n, m}-A_{n, m-1}\right)\left(A_{n, m}-A_{n, m-1}\right)^{\top} \mid \mathcal{F}_{m-1}^{n}\right) \\
& \xrightarrow{\mathrm{L}_{2}}(32|\alpha||\beta|)^{-1 / 2} \Psi_{\alpha, \beta}
\end{aligned}
$$

as $n \rightarrow \infty$.
If $0<|\alpha| \in\{0,1\},|\beta|=1-|\alpha|$ and (1.8) holds then

$$
n^{-2} r_{n}^{-1}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{1 / 2} \sum_{m=1}^{n} \mathrm{E}\left(\left(A_{n, m}-A_{n, m-1}\right)\left(A_{n, m}-A_{n, m-1}\right)^{\top} \mid \mathcal{F}_{m-1}^{n}\right) \xrightarrow{\mathrm{L}_{2}} \Theta_{\alpha, \beta, \omega}
$$

as $n \rightarrow \infty$.

Proposition 4.2 If $0<|\alpha|<1,|\beta|=1-|\alpha|$ and (1.7) holds then for all $\delta>0$

$$
\begin{aligned}
n^{-2} r_{n}^{-1 / 2}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 2} \sum_{m=1}^{n} \mathrm{E} & \left(\left\|A_{n, m}-A_{n, m-1}\right\|^{2}\right. \\
& \left.\times \mathbb{1}_{\left\{\left\|A_{n, m}-A_{n, m-1}\right\| \geq \delta n r_{n}^{1 / 4}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{-1 / 4}\right\}} \mid \mathcal{F}_{m-1}^{n}\right)
\end{aligned}
$$

converges to 0 in probability as $n \rightarrow \infty$.

If $0<|\alpha| \in\{0,1\},|\beta|=1-|\alpha|$ and (1.8) holds then for all $\delta>0$

$$
\begin{aligned}
n^{-2} r_{n}^{-1}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{1 / 2} \sum_{m=1}^{n} \mathrm{E} & \left(\left\|A_{n, m}-A_{n, m-1}\right\|^{2}\right. \\
& \left.\times \mathbb{1}_{\left\{\left\|A_{n, m}-A_{n, m-1}\right\| \geq \delta n r_{n}^{1 / 2}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{-1 / 4}\right\}} \mid \mathcal{F}_{m-1}^{n}\right)
\end{aligned}
$$

converges to 0 in probability as $n \rightarrow \infty$.
Proof of Proposition 4.1. Let $U_{m}^{n}:=\mathrm{E}\left(\left(A_{n, m}-A_{n, m-1}\right)\left(A_{n, m}-A_{n, m-1}\right)^{\top} \mid \mathcal{F}_{m-1}^{n}\right)$. From the definitions of $A_{n, m}$ and $B_{m}$ and from the independence of the error terms $\varepsilon_{k, \ell}^{\left(r_{n}\right)}$ follows that

$$
\mathrm{E} U_{m}^{n}=\mathrm{E}\left(\left(A_{n, m}-A_{n, m-1}\right)\left(A_{n, m}-A_{n, m-1}\right)^{\top}=\mathrm{E} B_{m}-\mathrm{E} B_{m-1}\right.
$$

where $B_{0}$ is the two-by-two matrix of zeros. Thus, if $0<|\alpha|<1$ then (3.1) implies

$$
\lim _{n \rightarrow \infty} n^{-2} r_{n}^{-1 / 2}\left(\left|\gamma_{r_{n}}\right|+\left|\delta_{r_{n}}\right|\right)^{1 / 2} \sum_{m=1}^{n} \mathrm{E} U_{m}^{n} \rightarrow(32|\alpha||\beta|)^{-1 / 2} \Psi_{\alpha, \beta}
$$

while in the case $|\alpha| \in\{0,1\}$ from (3.2) we have

$$
\lim _{n \rightarrow \infty} n^{-2} r_{n}^{-1}\left|\gamma_{r_{n}}^{2}-\delta_{r_{n}}^{2}\right|^{1 / 2} \sum_{m=1}^{n} \mathrm{E} U_{m}^{n} \rightarrow \Theta_{\alpha, \beta, \omega}
$$

where $\omega$ is the limit defined by (1.9).
Further, from the decomposition (4.1) follows

$$
\begin{equation*}
U_{m}^{n}=\mathrm{E} A_{n, m, 1} A_{n, m, 1}^{\top}+\sum_{(k, \ell) \in R_{m}} A_{n, m, 2, k, \ell} A_{n, m, 2, k, \ell}^{\top} \tag{4.5}
\end{equation*}
$$

This means that to complete the proof of the proposition we have to show

$$
\begin{align*}
\lim _{n \rightarrow \infty} n^{-4} \kappa_{n} \operatorname{Var}\left(\sum_{m=1}^{n} \sum_{(k, \ell) \in R_{m}} \widetilde{A}_{n, m, 2, k-1, \ell}^{2}\right) & =0,  \tag{4.6}\\
\lim _{n \rightarrow \infty} n^{-4} \kappa_{n} \operatorname{Var}\left(\sum_{m=1}^{n} \sum_{(k, \ell) \in R_{m}} \widetilde{A}_{n, m, 2, k-1, \ell} \widetilde{A}_{n, m, 2, k, \ell-1}\right) & =0,  \tag{4.7}\\
\lim _{n \rightarrow \infty} n^{-4} \kappa_{n} \operatorname{Var}\left(\sum_{m=1}^{n} \sum_{(k, \ell) \in R_{m}} \widetilde{A}_{n, m, 2, k, \ell-1}^{2}\right) & =0, \tag{4.8}
\end{align*}
$$

where $\kappa_{n}$ is defined by (3.6).
Now, consider

$$
\begin{aligned}
& \operatorname{Var}\left(\sum_{m=1}^{n} \sum_{(k, \ell) \in R_{m}} \widetilde{A}_{n, m, 2, k, \ell}^{2}\right) \\
&=\sum_{m_{1}=1}^{n} \sum_{\left(k_{1}, \ell_{1}\right) \in R_{m_{1}}} \sum_{m_{2}=1}^{n} \sum_{\left(k_{2}, \ell_{2}\right) \in T_{m_{2}}} \operatorname{Cov}\left(\widetilde{A}_{n, m_{1}, 2, k_{1}, \ell_{1}}^{2}, \widetilde{A}_{n, m_{2}, 2, k_{2}, \ell_{2}}^{2}\right) .
\end{aligned}
$$

By Remark 2.9 in the remaining part of the proof we may assume $\alpha_{r_{n}} \geq 0, \beta_{r_{n}} \geq 0$. Hence, as by Lemma 2.8

$$
\operatorname{Cov}\left(\widetilde{A}_{n, m_{1}, 2, k_{1}, \ell_{1}}^{2}, \widetilde{A}_{n, m_{2}, 2, k_{2}, \ell_{2}}^{2}\right) \leq 2 M_{4} \operatorname{Cov}\left(\widetilde{A}_{n, m_{1}, 2, k_{1}, \ell_{1}}, \widetilde{A}_{n, m_{2}, 2, k_{2}, \ell_{2}}\right)^{2}
$$

and representation (1.3) implies

$$
\operatorname{Cov}\left(\widetilde{A}_{n, m_{1}, 2, k_{1}-1, \ell_{1}}, \widetilde{A}_{n, m_{2}, 2, k_{2}-1, \ell_{2}}\right) \leq \operatorname{Cov}\left(X_{k_{1}, \ell_{1}}^{\left(r_{n}\right)}, X_{k_{2}, \ell_{2}}^{\left(r_{n}\right)}\right)
$$

we have

$$
\operatorname{Var}\left(\sum_{m=1}^{n} \sum_{(k, \ell) \in R_{m}} \widetilde{A}_{n, m, 2, k, \ell}^{2}\right) \leq 2 M_{4} \sum_{\left(k_{1}, \ell_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(k_{1}, \ell_{1}\right) \in T_{k_{n}, \ell_{n}}} \operatorname{Cov}\left(X_{k_{1}, \ell_{1}}^{\left(r_{n}\right)}, X_{k_{2}, \ell_{2}}^{\left(r_{n}\right)}\right)^{2}
$$

Thus, using (3.4) and (3.5) for the cases $0<|\alpha|<1$ and $|\alpha| \in\{0,1\}$, respectively, (4.6) follows from Proposition (2.7). In a similar way one can prove (4.7) and (4.8).

Proof of Proposition 4.2. To prove the proposition it suffices to show

$$
\begin{equation*}
n^{-4} \kappa_{n} \sum_{m=1}^{n} \mathrm{E}\left(\left\|A_{n, m}-A_{n, m-1}\right\|^{4} \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathrm{P}} 0 \tag{4.9}
\end{equation*}
$$

as $n \rightarrow \infty$, where $\kappa_{n}$ is defined by (3.6). By the decomposition (4.1)

$$
\left\|A_{n, m}-A_{n, m-1}\right\|^{4} \leq 2^{3}\left\|A_{n, m, 1}\right\|^{4}+2^{3}\left\|\sum_{(k, \ell) \in R_{m}} \varepsilon_{k, \ell}^{\left(r_{n}\right)} A_{n, m, 2, k, \ell}\right\|^{4}
$$

As $A_{n, m, 1}$ is independent from $\mathcal{F}_{m-1}^{n}$ we have $\mathrm{E}\left(\left\|A_{n, m, 1}\right\|^{4} \mid \mathcal{F}_{m-1}^{n}\right)=\mathrm{E}\left\|A_{n, m, 1}\right\|^{4}$, while the measurability of $A_{n, m, 2, k, \ell}$ with respect to $\mathcal{F}_{m-1}^{n}$ implies

$$
\mathrm{E}\left(\left\|\sum_{(k, \ell) \in R_{m}} \varepsilon_{k, \ell}^{\left(r_{n}\right)} A_{n, m, 2, k, \ell}\right\|^{4} \mid \mathcal{F}_{m-1}\right) \leq\left(\left(M_{4}-3\right)^{+}+3\right)\left(\sum_{(k, \ell) \in R_{m}}\left\|A_{n, m, 2, k, \ell}\right\|^{2}\right)^{2}
$$

Hence, in order to prove (4.9), it suffices to show

$$
\begin{align*}
\lim _{n \rightarrow \infty} n^{-4} \kappa_{n} \sum_{m=1}^{n} \mathrm{E}\left\|A_{n, m, 1}\right\|^{4} & =0  \tag{4.10}\\
\lim _{n \rightarrow \infty} n^{-4} \kappa_{n} \sum_{m=1}^{n} \mathrm{E}\left(\sum_{(k, \ell) \in R_{m}}\left\|A_{n, m, 2, k, \ell}\right\|^{2}\right)^{2} & =0 \tag{4.11}
\end{align*}
$$

It is easy to see that using (4.3) and (4.4) we obtain

$$
\left\|A_{n, m, 1}\right\|^{4} \leq 2^{3}\left(\sum_{k=-\ell_{m}+2}^{k_{m}} \sum_{i=-\infty}^{k-1} \alpha_{r_{n}}^{k-1-i} \varepsilon_{k, \ell_{m}}^{\left(r_{n}\right)} \varepsilon_{i, \ell_{m}}^{\left(r_{n}\right)}\right)^{4}+2^{3}\left(\sum_{\ell=-k_{m}+2}^{\ell_{m}} \sum_{j=-\infty}^{\ell-1} \beta_{r_{n}}^{\ell-1-j} \varepsilon_{\left.k_{m}, \ell^{\left(r_{n}\right)} \varepsilon_{k_{m}, j}^{\left(r_{n}\right)}\right)^{4} . . . . ~ . ~}^{4}\right.
$$

Using Lemma 12 of 4 a short calculation shows

$$
\mathrm{E}\left\|A_{n, m, 1}\right\|^{4} \leq\left(\left(1-\alpha_{r_{n}}^{2}\right)^{-1}+\left(1-\beta_{r_{n}}^{2}\right)^{-1}\right) O\left(m^{2}\right), \quad \text { as } n \rightarrow \infty
$$

and as $\kappa_{n}\left(\left(1-\alpha_{r_{n}}^{2}\right)^{-1}+\left(1-\beta_{r_{n}}^{2}\right)^{-1}\right)$ is bounded we obtain (4.10).

Furthermore, we have

$$
\begin{aligned}
& \mathrm{E}\left(\sum_{(k, \ell) \in R_{m}}\left\|A_{n m, 2, k, \ell}\right\|^{2}\right)^{2}=\sum_{\left(i_{1}, j_{1}\right) \in R_{m}} \sum_{\left(i_{2}, j_{2}\right) \in R_{m}} \mathrm{E}\left(\left(\widetilde{A}_{n, m, 2, i_{1}-1, j_{1}}^{2}+\widetilde{A}_{n, m, 2, i_{1}, j_{1}-1}^{2}\right)\right. \\
&\left.\times\left(\widetilde{A}_{n, m, 2, i_{2}-1, j_{2}}^{2}+\widetilde{A}_{n, m, 2, i_{2}, j_{2}-1}^{2}\right)\right)
\end{aligned}
$$

¿From Lemma 2.8 follows

$$
\mathrm{E}\left(\widetilde{A}_{n, m, 2, i_{1}, j_{1}}^{2} \widetilde{A}_{n, m, 2, i_{2}, j_{2}}^{2}\right) \leq 3 M_{4} \mathrm{E} \widetilde{A}_{n, m, 2, i_{1}, j_{1}}^{2} \mathrm{E} \widetilde{A}_{n, m, 2, i_{2}, j_{2}}^{2}
$$

while using (4.2) and representation (1.3) one can see

$$
\mathrm{E} \widetilde{A}_{n, m, 2, k, \ell}^{2} \leq \operatorname{Var} X_{k, \ell}=R_{0,0}
$$

Thus,

$$
\mathrm{E}\left(\sum_{(k, \ell) \in R_{m}}\left\|A_{n, m, 2, k, \ell}\right\|^{2}\right)^{2} \leq 12 M_{4} R_{0,0}^{2} m^{2}=12 M_{4} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{4} m^{2}
$$

that together with (2.6) and (2.9) implies (4.11).

## 5. Proof of Proposition 1.6

In what follows we will assume $0<\alpha<1$ and $\beta=1-\alpha$, so without loss of generality we may suppose $\alpha_{r_{n}}, \beta_{r_{n}}, \gamma_{r_{n}}$ and $\delta_{r_{n}}$ are all positive. Consider the following expression of $\operatorname{det} B_{n}$

$$
\operatorname{det} B_{n}=\sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}} W_{i_{1}, j_{1}, i_{2}, j_{2}}^{(n)}
$$

where

$$
W_{i_{1}, j_{1}, i_{2}, j_{2}}^{(n)}:=\left(X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}\right)^{2}\left(X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}\right)^{2}-X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)} X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)} X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)} X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}
$$

Using representation (1.3) from Lemma 2.8 we obtain.

$$
\begin{equation*}
\mathrm{E} W_{i_{1}, j_{1}, i_{2}, j_{2}}^{(n)}=A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(1, n)}+A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n)}+A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(3, n)}+A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(4, n)} \tag{5.1}
\end{equation*}
$$

where

$$
\begin{aligned}
& A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(1, n)}:=\sum_{(u, v) \in U_{\left(i_{1}-1\right) \wedge i_{2}, j_{1} \wedge\left(j_{2}-1\right)}}\left(\mathrm{E}\left(\varepsilon_{0,0}^{\left(r_{n}\right)}\right)^{4}-3\right)\binom{i_{1}+j_{1}-1-u-v}{i_{1}-1-u}^{2}\binom{i_{2}+j_{2}-1-u-v}{i_{2}-u}^{2} \\
& \times \alpha_{r_{n}}^{2 i_{1}+2 i_{2}-2-4 u} \beta_{r_{n}}^{2 j_{1}+2 j_{2}-2-4 v} \\
& -\sum_{(u, v) \in U_{i_{1} \wedge i_{2}-1, j_{1} \wedge j_{2}-1}}\left(\mathrm{E}\left(\varepsilon_{0,0}^{\left(r_{n}\right)}\right)^{4}-3\right)\binom{i_{1}+j_{1}-1-u-v}{i_{1}-1-u}\binom{i_{1}+j_{1}-1-u-v}{i_{1}-u} \\
& \times\binom{ i_{2}+j_{2}-1-u-v}{i_{2}-1-u}\binom{i_{2}+j_{2}-1-u-v}{i_{2}-u} \alpha_{r_{n}}^{2 i_{1}+2 i_{2}-2-4 u} \beta_{r_{n}}^{2 j_{1}+2 j_{2}-2-4 v}, \\
& A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n)}:=\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)^{2}-\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right) \operatorname{Cov}\left(X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}, X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}\right), \\
& A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(3, n)}:=\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)^{2}-\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}\right) \operatorname{Cov}\left(X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right), \\
& A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(4, n)}:=\operatorname{Var}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}\right) \operatorname{Var}\left(X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)-\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}\right) \operatorname{Cov}\left(X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right) \\
& =\sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2} \frac{1-\alpha_{r_{n}}^{2}-\beta_{r_{n}}^{2}-\sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{-2}}{2 \alpha_{r_{n}}^{2} \beta_{r_{n}}^{2}} .
\end{aligned}
$$

Short calculation shows

$$
\left|A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(1, n)}\right| \leq 2\left(M_{4}+3\right) \operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)
$$

so we have

$$
\begin{aligned}
& n^{-4} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}}\left|A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(1, n)}\right| \\
& \leq 2\left(M_{4}+3\right) \iint_{T} \iint_{T}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \operatorname{Cov}\left(Z_{0,1}\left(s_{1}, t_{1}\right), Z_{0,1}\left(s_{2}, t_{2}\right)\right) \mathrm{d} s_{1} \mathrm{~d} t_{1} \mathrm{~d} s_{2} \mathrm{~d} t_{2} .
\end{aligned}
$$

Hence, using the same arguments as in the proof of Proposition 1.4 Fatou's lemma and Proposition 2.7 imply

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{-4} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}} A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(1, n)}=0 . \tag{5.2}
\end{equation*}
$$

Next consider $A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n)}=A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n, 1)}+A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n, 2)}$, where
$A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n, 1)}:=\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)\left(\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)-\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}\right)\right)$, $A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n, 2)}:=\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)\left(\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}\right)-\operatorname{Cov}\left(X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}, X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}\right)\right)$.
With the help of Proposition 2.4 we can easily show

$$
\begin{aligned}
& n^{-4} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}}\left|A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n, 1)}\right| \\
& \leq \iint_{T} \iint_{T}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2}\left|\operatorname{Cov}\left(Z_{0,1}\left(s_{1}, t_{1}\right), Z_{0,1}\left(s_{2}, t_{2}\right)\right)\right| \\
& \quad \times\left|R_{\left[n s_{1}\right]-\left[n s_{2}\right]-1,\left[n t_{1}\right]-\left[n t_{2}\right]+1}-R_{\left[n s_{1}\right]-\left[n s_{2}\right],\left[n t_{1}\right]-\left[n t_{2}\right]}\right| \mathrm{d} s_{1} \mathrm{~d} t_{1} \mathrm{~d} s_{2} \mathrm{~d} t_{2} \\
& \quad \leq \frac{K}{\left(\alpha_{r_{n}} \beta_{r_{n}}\right)^{3 / 2}} \iiint \int_{T}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2}\left|\operatorname{Cov}\left(Z_{0,1}\left(s_{1}, t_{1}\right), Z_{0,1}\left(s_{2}, t_{2}\right)\right)\right| \mathrm{d} s_{1} \mathrm{~d} t_{1} \mathrm{~d} s_{2} \mathrm{~d} t_{2} \rightarrow 0
\end{aligned}
$$

as $n \rightarrow \infty$. Naturally, the same result can be proved for $A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n, 1)}$, so we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{-4} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}} A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n)}=0 \tag{5.3}
\end{equation*}
$$

Using similar arguments one can also prove

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{-4} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}} A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(3, n)}=0 \tag{5.4}
\end{equation*}
$$

Further, as $k_{n}+\ell_{n}=n$, and $A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(4, n)}$ does not depend on $i_{1}, j_{1}, i_{2}, j_{2}$ using (2.6) we obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{-4} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}} A_{i_{1}, j_{1}, i_{2}, j_{2}}^{(4, n)}=\frac{2}{(8 \alpha \beta)^{3 / 2}} \tag{5.5}
\end{equation*}
$$

Finally, the combination of representation (5.1) and limits (5.2)-(5.5) yields

$$
\lim _{n \rightarrow \infty} n^{-4} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \mathrm{E} \operatorname{det} B_{n}=\frac{2}{(8 \alpha \beta)^{3 / 2}}
$$

Now, let us deal with the variance of $\operatorname{det} B_{n}$. Short calculation shows

$$
\begin{align*}
& n^{-8} r_{n}^{-1}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right) \operatorname{Var}\left(\operatorname{det} B_{n}\right)  \tag{5.6}\\
& =\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{n^{8} r_{n}} \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{3}, j_{3}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{4}, j_{4}\right) \in T_{k_{n}, \ell_{n}}} \operatorname{Cov}\left(W_{i_{1}, j_{1}, i_{2}, j_{2}}^{(n)}, W_{i_{3}, j_{3}, i_{4}, j_{4}}^{(n)}\right) \\
& =\iiint \iiint \int_{T} \iint_{T}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)\left(\Theta_{n}^{(1)}\left(s_{1}, t_{1}, s_{2}, t_{2}, s_{3}, t_{3}, s_{4}, t_{4}\right)+\Theta_{n}^{(2)}\left(s_{1}, t_{1}, s_{2}, t_{2}, s_{3}, t_{3}, s_{4}, t_{4}\right)\right. \\
& \left.+2 \Theta_{n}^{(3)}\left(s_{1}, t_{1}, s_{2}, t_{2}, s_{3}, t_{3}, s_{4}, t_{4}\right)\right) \mathrm{d} s_{1} \mathrm{~d} t_{1} \mathrm{~d} s_{2} \mathrm{~d} t_{2} \mathrm{~d} s_{3} \mathrm{~d} t_{3} \mathrm{~d} s_{4} \mathrm{~d} t_{4},
\end{align*}
$$

where

$$
\begin{aligned}
& \Theta_{n}^{(1)}\left(s_{1}, t_{1}, s_{2}, t_{2}, s_{3}, t_{3}, s_{4}, t_{4}\right) \\
& :=\operatorname{Cov}\left(Z_{1,0}^{\left(r_{n}\right)}\left(s_{1}, t_{1}\right) Z_{0,1}^{\left(r_{n}\right)}\left(s_{2}, t_{2}\right)\left(X_{\left[n s_{1}\right],\left[n t_{1}\right]-1}^{(n)}-X_{\left[n s_{1}\right]-1,\left[n t_{1}\right]}^{\left(r_{n}\right)}\right)\left(X_{\left[n s_{2}\right]-1,\left[n t_{2}\right]}^{(n)}-X_{\left[n s_{2}\right],\left[n t_{2}\right]-1}^{\left(r_{n}\right)}\right),\right. \\
& \quad Z_{1,0}^{(n)}\left(s_{3}, t_{3}\right) Z_{0,1}^{(n)}\left(s_{4}, t_{4}\right)\left(X_{\left[n s_{3}\right],\left[n t_{3}\right]-1}^{(n)}-X_{\left[n s_{3}\right]-1,\left[n t_{3}\right]}^{\left(r_{n}\right)}\right)\left(X_{\left[n s_{4}\right]-1,\left[n t_{4}\right]}^{(n)}-X_{\left[n s_{4}\right],\left[n t_{4}\right]-1}^{\left(r_{n}\right)}\right), \\
& \Theta_{n}^{(2)}\left(s_{1}, t_{1}, s_{2}, t_{2}, s_{3}, t_{3}, s_{4}, t_{4}\right):=\operatorname{Cov}\left(Z_{1,0}^{(n)}\left(s_{1}, t_{1}\right) Z_{0,1}^{(n)}\left(s_{1}, t_{1}\right)\left(X_{\left[n s_{2}\right],\left[n t_{2}\right]-1}^{(n)}-X_{\left[n s_{2}\right]-1,\left[n t_{2}\right]}^{\left(r_{n}\right)}\right),\right. \\
& \left.\quad Z_{1,0}^{(n)}\left(s_{3}, t_{3}\right) Z_{0,1}^{(n)}\left(s_{3}, t_{3}\right)\left(X_{\left[n s_{4}\right],\left[n t_{4}\right]-1}^{(n)}-X_{\left[n s_{4}\right]-1,\left[n t_{4}\right]}^{\left(r_{n}\right)}\right)^{2}\right), \\
& \Theta_{n}^{(3)}\left(s_{1}, t_{1}, s_{2}, t_{2}, s_{3}, t_{3}, s_{4}, t_{4}\right):=\operatorname{Cov}\left(Z_{1,0}^{(n)}\left(s_{1}, t_{1}\right) Z_{0,1}^{(n)}\left(s_{1}, t_{1}\right)\left(X_{\left[n s_{2}\right],\left[n t_{2}\right]-1}^{(n)}-X_{\left[n s_{2}\right]-1,\left[n t_{2}\right]}^{\left(r_{n}\right)}\right)^{2},\right. \\
& \left.\quad Z_{1,0}^{(n)}\left(s_{3}, t_{3}\right) Z_{0,1}^{(n)}\left(s_{4}, t_{4}\right)\left(X_{\left[n s_{3}\right],\left[n t_{3}\right]-1}^{(n)}-X_{\left[n s_{3}\right]-1,\left[n t_{3}\right]}^{\left(r_{n}\right)}\right)\left(X_{\left[n s_{4}\right]-1,\left[n t_{4}\right]}^{(n)}-X_{\left[n s_{4}\right],\left[n t_{4}\right]-1}^{\left(r_{n}\right)}\right)\right) .
\end{aligned}
$$

By representation (1.3) the components $\Theta_{n}^{(q)}, q=1,2,3$, of the integrand in the right hand side of (5.6) are linear combinations of covariances of form

$$
\begin{equation*}
\operatorname{Cov}\left(\varepsilon_{i_{1}, j_{1}} \varepsilon_{i_{2}, j_{2}} \varepsilon_{i_{3}, j_{3}} \varepsilon_{i_{4}, j_{4}}, \varepsilon_{i_{5}, j_{5}} \varepsilon_{i_{6}, j_{6}} \varepsilon_{i_{7}, j_{7}} \varepsilon_{i_{8}, j_{8}}\right) \tag{5.7}
\end{equation*}
$$

where the indices $\left(i_{r}, j_{r}\right) \in \mathbb{Z}^{2}, r=1,2, \ldots, 8$, run either on quarter planes $U_{\left[n s_{q}\right],\left[n t_{q}\right]-1}$ or on $U_{\left[n s_{q}\right]-1,\left[n t_{q}\right]}, q=[(r+1) / 2]$. Using the definitions of Lemma 2.3 we can express
the coefficients of the linear combinations as products of $1 / r_{n}$ and two terms of form $\left(\alpha_{r_{n}}+\beta_{r_{n}}\right)^{\left[n s_{m_{r}}\right]+\left[n t_{m_{r}}\right]-1-i_{r}-j_{r}} \mathrm{P}\left(S_{\left[n s_{m_{r}}\right]+\left[n t_{m_{r}}\right]-1-i_{r}-j_{r}}^{\left(\nu_{r_{n}}\right)}=\left[n s_{m_{r}}\right]-1-i_{r}\right)$, two terms of form $\left(\alpha_{r_{n}}+\beta_{r_{n}}\right)^{\left[n s_{m_{r}}\right]+\left[n t_{m_{r}}\right]-1-i_{r}-j_{r}} \mathrm{P}\left(S_{\left[n s_{m_{r}}\right]+\left[n t_{m_{r}}\right]-1-i_{r}-j_{r}}^{\left(\nu_{r_{n}}\right)}=\left[n s_{m_{r}}\right]-i_{r}\right)$ and four terms of form $\left(\alpha_{r_{n}}+\beta_{r_{n}}\right)^{\left[n s_{m_{r}}\right]+\left[n t_{m_{r}}\right]-1-i_{r}-j_{r}} \widehat{\Delta}_{i_{r}, j_{r}}^{(n)}\left(s_{m_{r}}, t_{m_{r}}\right)$ with

$$
\begin{aligned}
& \widehat{\Delta}_{i_{r}, j_{r}}^{(n)}\left(s_{m_{r}}, t_{m_{r}}\right) \\
& \quad:=\mathrm{P}\left(S_{\left[n s_{m_{r}}\right]+\left[n t_{m_{r}}\right]-1-i_{r}-j_{r}}^{\left(\nu_{r_{n}}\right)}=\left[n s_{m_{r}}\right]-i_{r}\right)-\mathrm{P}\left(S_{\left[n s_{m_{r}}\right]+\left[n t_{m_{r}}\right]-1-i_{r}-j_{r}}^{\left(\nu_{r_{n}}\right)}=\left[n s_{m_{r}}\right]-1-i_{r}\right),
\end{aligned}
$$

where $\nu_{r_{n}}=\alpha_{r_{n}} /\left(\alpha_{r_{n}}+\beta_{r_{n}}\right)$ and $\cup_{r=1}^{8}\left\{m_{r}\right\}=\{1,2,3,4\}$. Corollary 2.5implies that there exists a positive constant $C$ such that

$$
\begin{equation*}
\left|\widehat{\Delta}_{i_{r}, j_{r}}^{(n)}\left(s_{m_{r}}, t_{m_{r}}\right)\right| \leq \frac{C}{\alpha_{r_{n}} \beta_{r_{n}}\left(\left[n s_{m_{r}}\right]+\left[n t_{m_{r}}\right]-1-i_{r}-j_{r}\right)} \tag{5.8}
\end{equation*}
$$

Covariances of form (5.7) are equal to zero if the index sets $\left\{\left(i_{r}, j_{r}\right): r=1,2,3,4\right\}$ and $\left\{\left(i_{r}, j_{r}\right): r=5,6,7,8\right\}$ are disjoint. Besides the nonempty intersection of these sets, to obtain nonzero covariances in (5.7) for each $u \in\{1,2, \ldots, 8\}$ there should exist at least one $v \in\{1,2, \ldots, 8\}$ such that $u \neq v$ and $\left(i_{u}, j_{u}\right)=\left(i_{v}, j_{v}\right)$. Consider first the case, when $\{1,2, \ldots, 8\}$ is divided into two disjoint subsets $\left\{u_{1}, u_{2}, u_{3}, u_{4}\right\}$ and $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\},\left(i_{u_{r}}, j_{u_{r}}\right)=\left(i_{v_{r}}, j_{v_{r}}\right), r=1,2,3,4$, holds and no other index pairs are equal. This configuration yields the highest amount of terms when we express the covariances of $\Theta_{n}^{(q)}, q=1,2,3$. Expression (5.6) shows that the sum of the corresponding terms of $n^{-8} r_{n}^{-1}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right) \operatorname{Var}\left(\operatorname{det} B_{n}\right)$ can be rewritten as the sum of terms of form

$$
\begin{aligned}
& \iint_{T} \iint_{T} \iint_{T} \iint_{T}\left(R_{\left[n s_{m_{1}}\right]-\left[n s_{m_{2}}\right],\left[n t_{m_{1}}\right]-\left[n t_{m_{2}}\right]}-R_{\left[n s_{m_{1}}\right]-\left[n s_{m_{2}}\right] \pm 1,\left[n t_{m_{1}}\right]-\left[n t_{m_{2}}\right] \mp 1}\right) \\
& \quad \times\left(R_{\left[n s_{m_{3}}\right]-\left[n s_{m_{4}}\right],\left[n t_{m_{3}}\right]-\left[n t_{m_{4}}\right]}-R_{\left[n s_{m_{3}}\right]-\left[n s_{m_{4}}\right] \pm 1,\left[n t_{m_{3}}\right]-\left[n t_{m_{4}}\right] \neq 1}\right) \\
& \quad \times\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \operatorname{Cov}\left(Z_{u_{1}, v_{1}}^{(n)}\left(s_{m_{5}}, t_{m_{5}}\right) Z_{u_{2}, v_{2}}^{(n)}\left(s_{m_{6}}, t_{m_{6}}\right)\right) \\
& \quad \times\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \operatorname{Cov}\left(Z_{u_{3}, v_{3}}^{(n)}\left(s_{m_{7}}, t_{m_{3}}\right) Z_{u_{4}, v_{4}}^{(n)}\left(s_{m_{8}}, t_{m_{8}}\right)\right) \mathrm{d} s_{1} \mathrm{~d} t_{1} \mathrm{~d} s_{2} \mathrm{~d} t_{2} \mathrm{~d} s_{3} \mathrm{~d} t_{3} \mathrm{~d} s_{4} \mathrm{~d} t_{4}
\end{aligned}
$$

where $\left\{m_{r}: r=1,2, \ldots, 8\right\}=\{1,2,3,4\},\left(u_{r}, v_{r}\right) \in\{(0,1),(1,0)\}, r=1,2,3,4$. Fatou's lemma, Lemma 2.1 and Propositions 2.4 and 2.7 imply that these terms of the sum $n^{-8} r_{n}^{-1}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right) \operatorname{Var}\left(\operatorname{det} B_{n}\right)$ converge to 0 as $n \rightarrow \infty$.

The next case is when $\{1,2, \ldots, 8\}$ is divided into three disjoint subsets $\left\{u_{1}, u_{2}, u_{3}\right\}$ and $\left\{v_{1}, v_{2}, v_{3}\right\}$ and $\left\{w_{1}, w_{2}\right\}$ and either

$$
\begin{equation*}
\left(i_{u_{r}}, j_{u_{r}}\right)=\left(i_{v_{r}}, j_{v_{r}}\right)=\left(i_{w_{r}}, j_{w_{r}}\right), r=1,2, \quad \text { and } \quad\left(i_{u_{3}}, j_{u_{3}}\right)=\left(i_{v_{3}}, j_{v_{3}}\right) \tag{5.9}
\end{equation*}
$$

or

$$
\begin{equation*}
\left(i_{u_{r}}, j_{u_{r}}\right)=\left(i_{v_{r}}, j_{v_{r}}\right), r=1,2, \quad \text { and } \quad\left(i_{u_{3}}, j_{u_{3}}\right)=\left(i_{v_{3}}, j_{v_{3}}\right)=\left(i_{w_{1}}, j_{w_{1}}\right)=\left(i_{w_{1}}, j_{w_{2}}\right) \tag{5.10}
\end{equation*}
$$

holds and no other index pairs are equal. Inequality (5.8) implies that we have

$$
\begin{aligned}
& \sum \mathrm{P}\left(S_{\left[n s_{1}\right]+\left[n t_{1}\right]-1-i-j}^{\alpha}=\left[n s_{1}\right]-i\right)\left|\widehat{\Delta}_{i, j}^{(n)}\left(s_{2}, t_{2}\right)\right|\left|\widehat{\Delta}_{i, j}^{(n)}\left(s_{3}, t_{3}\right)\right| \\
& (i, j) \in U_{\left[n s_{1}\right] \wedge\left[n s_{2}\right] \wedge\left[n s_{3}\right]-1,\left[n t_{1}\right] \wedge\left[n t_{2}\right] \wedge\left[n t_{3}\right]-1} \\
& \leq \sum_{m=-\infty}^{\left[n s_{1}\right] \wedge\left[n s_{2}\right] \wedge\left[n s_{3}\right]+\left[n t_{1}\right] \wedge\left[n t_{2}\right] \wedge\left[n t_{3}\right]-2} \frac{C^{2}}{\left(\alpha_{r_{n}} \beta_{r_{n}}\right)^{2}\left(\left[n s_{2}\right]+\left[n t_{2}\right]-1-m\right)\left(\left[n s_{3}\right]+\left[n t_{3}\right]-1-m\right)} \\
& \quad \times \sum_{i=m-\left[n t_{1}\right] \wedge\left[n t_{2}\right] \wedge\left[n t_{3}\right]+1}^{\left[n s_{1}\right] \wedge\left[n s_{2}\right] \wedge\left[n s_{3}\right]-1} \mathrm{P}\left(S_{\left[n s_{1}\right]+\left[n t_{1}\right]-1-m}^{\alpha}=\left[n s_{1}\right]-i\right) \leq \frac{C^{2} \zeta(2)}{\left(\alpha_{r_{n}} \beta_{r_{n}}\right)^{2}},
\end{aligned}
$$

so the expressions of the above form are bounded uniformly in $n$ and $\left(s_{r}, t_{r}\right) \in T, r=$ $1,2,3$. Similarly, by Remark 2.6 there exists a constant $D>0$ such that

$$
\begin{equation*}
\sum_{(i, j) \in \bigcap_{r=1}^{4} U_{\left[n s_{r}\right]+u_{r},\left[n t_{r}\right]+v_{r}}}\left(\prod_{r=1}^{4} \mathrm{P}\left(S_{\left[n s_{r}\right]+\left[n t_{r}\right]-1-i-j}^{\alpha}=\left[n s_{r}\right]-i-u_{r}\right)\right) \leq \frac{D^{3} \zeta(3 / 2)}{\left(\alpha_{r_{n}} \beta_{r_{n}}\right)^{3 / 2}}, \tag{5.11}
\end{equation*}
$$

where $\left(u_{r}, v_{r}\right) \in\{(0,1),(1,0)\}, r=1,2,3,4$.
It is not difficult to show that in the case described by (5.9) the corresponding part of the sum $n^{-8} r_{n}^{-1}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right) \operatorname{Var}\left(\operatorname{det} B_{n}\right)$ can always be bounded from above by the sum of components of the form

$$
\begin{aligned}
& \iint_{T} \iint_{T} \iint_{T} \iint_{T} \frac{C^{2} \zeta(2)}{\left(\alpha_{r_{n}} \beta_{r_{n}}\right)^{2}}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2}\left|\operatorname{Cov}\left(Z_{u_{1}, v_{1}}^{(n)}\left(s_{m_{1}}, t_{m_{1}}\right) Z_{u_{2}, v_{2}}^{(n)}\left(s_{m_{2}}, t_{m_{2}}\right)\right)\right| \\
& \quad \times\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2}\left|\operatorname{Cov}\left(Z_{u_{3}, v_{3}}^{(n)}\left(s_{m_{3}}, t_{m_{3}}\right) Z_{u_{4}, v_{4}}^{(n)}\left(s_{m_{4}}, t_{m_{4}}\right)\right)\right| \mathrm{d} s_{1} \mathrm{~d} t_{1} \mathrm{~d} s_{2} \mathrm{~d} t_{2} \mathrm{~d} s_{3} \mathrm{~d} t_{3} \mathrm{~d} s_{4} \mathrm{~d} t_{4}
\end{aligned}
$$

where $\left\{m_{r}: r=1,2,3,4\right\} \subseteq\{1,2,3,4\}$ contains at least 3 different points and $\left(u_{r}, v_{r}\right) \in$ $\{(0,1),(1,0)\}, r=1,2,3,4$. In this way by Fatou's lemma and Proposition 2.7we obtain that the terms of $n^{-8} r_{n}^{-1}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right) \operatorname{Var}\left(\operatorname{det} B_{n}\right)$ corresponding to case (5.9) converge to 0 as $n \rightarrow \infty$. Using similar ideas and (5.11) the same can be proved in the case (5.10).

The remaining terms of $n^{-8} r_{n}^{-1}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right) \operatorname{Var}\left(\operatorname{det} B_{n}\right)$ can be handled in a similar way.

## 6. Proof of Proposition 1.7

Similarly to Section 5 it is enough to consider the case $0<\alpha<1$ and $\beta=1-\alpha$. We have

$$
\begin{aligned}
n^{-3} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \bar{B}_{n} A_{n}=\left(n^{-2} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \bar{B}_{n}\right. & \left.-\frac{1}{\sqrt{32 \alpha \beta}} \overline{1}\right) \frac{1}{n} A_{n} \\
& +\frac{1}{\sqrt{32 \alpha \beta}} \frac{1}{n} \overline{1} A_{n}
\end{aligned}
$$

where 1 denotes the two-by-two matrix of ones. Short straightforward calculations shows

$$
\left(n^{-2} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \bar{B}_{n}-\frac{1}{\sqrt{32 \alpha \beta}} \overline{1}\right) \frac{1}{n} A_{n}=C_{n}+D_{n}
$$

where

$$
\begin{aligned}
C_{n} & :=n^{-1} r_{n}^{-1 / 4}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 4} \operatorname{diag}\left(A_{n}\right) n^{-2} r_{n}^{-1 / 4}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 4} \bar{B}_{n}(1,1)^{\top}, \\
D_{n} & :=\left(n^{-2} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \sum_{(i, j) \in T_{k_{n}, \ell_{n}}} X_{i-1, j}^{\left(r_{n}\right)} X_{i, j-1}^{\left(r_{n}\right)}-\frac{1}{\sqrt{32 \alpha \beta}}\right) \frac{1}{n} Q_{n}(1,-1)^{\top} .
\end{aligned}
$$

Here $\operatorname{diag}\left(A_{n}\right)$ denotes the two-by-two diagonal matrix having $A_{n}$ in its main diagonal and

$$
\begin{equation*}
Q_{n}:=(1,-1) A_{n}=\sum_{(i, j) \in T_{k_{n}}, \ell_{n}}\left(X_{i-1, j}^{\left(r_{n}\right)}-X_{i, j-1}^{\left(r_{n}\right)}\right) \varepsilon_{i, j} . \tag{6.1}
\end{equation*}
$$

By Proposition 1.4

$$
\begin{equation*}
n^{-2} r_{n}^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \sum_{(i, j) \in T_{k_{n}, \ell_{n}}} X_{i-1, j}^{\left(r_{n}\right)} X_{i, j-1}^{\left(r_{n}\right)}-\frac{1}{\sqrt{32 \alpha \beta}} \xrightarrow{\mathrm{~L}_{2}} 0 \quad \text { as } n \rightarrow \infty \tag{6.2}
\end{equation*}
$$

Representation (1.3) and independence of the error terms $\varepsilon_{i, j}^{\left(r_{n}\right)}$ imply $\mathrm{E} Q_{n}=0$ and

$$
\begin{aligned}
\mathrm{E} Q_{n}^{2} & =\sum_{(i, j) \in T_{k_{n}, \ell_{n}}} \mathrm{E}\left(X_{i-1, j}^{\left(r_{n}\right)}-X_{i, j-1}^{\left(r_{n}\right)}\right)^{2}=\left(k_{n}+\ell_{n}\right)\left(k_{n}+\ell_{n}+1\right)\left(R_{0,0}-R_{-1,1}\right) \\
& =\frac{n(n+1)}{4 \alpha_{r_{n}} \beta_{r_{n}}}\left(1+\left(\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{r_{n}}\right)^{1 / 2} \sigma_{\alpha_{r_{n}}, \beta_{r_{n}}}^{2}\left(\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{r_{n}}\right)^{1 / 2}\left(\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{r_{n}}-2\right)\right) .
\end{aligned}
$$

Taking into account (2.6) we obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n^{2}} \mathrm{E} Q_{n}^{2}=\frac{1}{4 \alpha \beta} \tag{6.3}
\end{equation*}
$$

that together with (6.2) implies $D_{n} \xrightarrow{\mathrm{P}}(0,0)^{\top}$ as $n \rightarrow \infty$.

$$
\begin{aligned}
& \frac{1}{n^{2}}\left(\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{r_{n}}\right)^{1 / 4} \mathrm{E}\left(\bar{B}_{n}(1,1)^{\top}\right)=\frac{1}{n^{2}}\left(\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{r_{n}}\right)^{1 / 4} \sum_{(k, \ell) \in T_{k_{n}, \ell}} \mathrm{E}\binom{\left(X_{k, \ell-1}^{\left(r_{n}\right)}\right)^{2}-X_{k-1, \ell}^{\left(r_{n}\right)} X_{k, \ell-1}^{\left(r_{n}\right)}}{\left(X_{k, \ell-1}^{\left(r_{n}\right)}\right)^{2}-X_{k-1, \ell}^{\left(r_{n}\right)} X_{k, \ell-1}^{\left(r_{n}\right)}} \\
& \quad=\left(\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{r_{n}}\right)^{1 / 4} \frac{n+1}{2 n}\left(R_{0,0}-R_{-1,1}\right)\binom{1}{1}=\left(\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{r_{n}}\right)^{1 / 4} \frac{1}{2 n^{2}} \mathrm{E} Q_{n}^{2}\binom{1}{1} \rightarrow\binom{0}{0}
\end{aligned}
$$

as $n \rightarrow \infty$. Furthermore, with the help of Lemma 2.8 we obtain

$$
\begin{aligned}
& \operatorname{Var}\left(\sum_{(k, \ell) \in T_{k_{n}, \ell_{n}}}\left(\left(X_{k, \ell-1}^{\left(r_{n}\right)}\right)^{2}-X_{k-1, \ell}^{\left(r_{n}\right)} X_{k, \ell-1}^{\left(r_{n}\right)}\right)\right) \\
& \quad=\sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}} B_{i_{1}, j_{1}, i_{2}, j_{2}}^{(1, n)}+2 B_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n)}+B_{i_{1}, j_{1}, i_{2}, j_{2}}^{(3, n)}+B_{i_{1}, j_{1}, i_{2}, j_{2}}^{(4, n)},
\end{aligned}
$$

where

$$
\begin{aligned}
& B_{i_{1}, j_{1}, i_{2}, j_{2}}^{(1, n)}:=\sum_{(u, v) \in U_{i_{1} \wedge i_{2}, j_{1} \wedge j_{2}-1}}\left(\mathrm{E}\left(\varepsilon_{0,0}^{\left(r_{n}\right)}\right)^{4}-3\right)\binom{i_{1}+j_{1}-1-u-v}{i_{1}-u}^{2}\binom{i_{2}+j_{2}-1-u-v}{i_{2}-u}^{2} \\
& \times \alpha_{r_{n}}^{2 i_{1}+2 i_{2}-4 u} \beta_{r_{n}}^{2 j_{1}+2 j_{2}-4-4 v} \\
& -\sum_{(u, v) \in U_{i_{1} \wedge\left(i_{2}-1\right), j_{1} \wedge j_{2}-1}} 2\left(\mathrm{E}\left(\varepsilon_{0,0}^{\left(r_{n}\right)}\right)^{4}-3\right)\binom{i_{1}+j_{1}-1-u-v}{i_{1}-u}^{2}\binom{i_{2}+j_{2}-1-u-v}{i_{2}-1-u} \\
& \times\binom{ i_{2}+j_{2}-1-u-v}{i_{2}-u} \alpha_{r_{n}}^{2 i_{1}+2 i_{2}-1-4 u} \beta_{r_{n}}^{2 j_{1}+2 j_{2}-3-4 v} \\
& +\sum_{(u, v) \in U_{i_{1} \wedge i_{2}-1, j_{1} \wedge j_{2}-1}}\left(\mathrm{E}\left(\varepsilon_{0,0}^{\left(r_{n}\right)}\right)^{4}-3\right)\binom{i_{1}+j_{1}-1-u-v}{i_{1}-1-u}\binom{i_{1}+j_{1}-1-u-v}{i_{1}-u} \\
& \times\binom{ i_{2}+j_{2}-1-u-v}{i_{2}-1-u}\binom{i_{2}+j_{2}-1-u-v}{i_{2}-u} \alpha_{r_{n}}^{2 i_{1}+2 i_{2}-2-4 u} \beta_{r_{n}}^{2 j_{1}+2 j_{2}-2-4 v}, \\
& B_{i_{1}, j_{1}, i_{2}, j_{2}}^{(2, n)}:=\operatorname{Cov}\left(X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)\left(\operatorname{Cov}\left(X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)-\operatorname{Cov}\left(X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}, X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}\right)\right), \\
& B_{i_{1}, j_{1}, i_{2}, j_{2}}^{(3, n)}:=\operatorname{Cov}\left(X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)\left(\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}\right)-\operatorname{Cov}\left(X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}, X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}\right)\right), \\
& B_{i_{1}, j_{1}, i_{2}, j_{2}}^{(4, n)}:=\operatorname{Cov}\left(X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}, X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}\right)\left(\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)-\operatorname{Cov}\left(X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}, X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)\right) .
\end{aligned}
$$

Hence, using the same arguments as in the proof of Proposition 1.6 (see (5.2) and (5.3)) one can verify

$$
\lim _{n \rightarrow \infty} \frac{1}{n^{4}}\left(\frac{\gamma_{r_{n}}+\delta_{r_{n}}}{r_{n}}\right)^{1 / 2} \operatorname{Var}\left(\sum_{(k, \ell) \in T_{k_{n}, \ell_{n}}}\left(\left(X_{k, \ell-1}^{\left(r_{n}\right)}\right)^{2}-X_{k-1, \ell}^{\left(r_{n}\right)} X_{k, \ell-1}^{\left(r_{n}\right)}\right)\right)=0
$$

Naturally, the same holds for the second component of $n^{-2} r_{n}^{-1 / 4}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 4} \bar{B}_{n}(1,1)^{\top}$, that means

$$
\begin{equation*}
n^{-2} r_{n}^{-1 / 4}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 4} \bar{B}_{n}(1,1)^{\top} \xrightarrow{\mathrm{L}_{2}}(0,0)^{\top} \quad \text { as } n \rightarrow \infty \tag{6.4}
\end{equation*}
$$

Proposition 1.5 and (6.4) imply $C_{n} \xrightarrow{\mathrm{P}}(0,0)^{\top}$ as $n \rightarrow \infty$, so to prove the asymptotic normality of $n^{-3} n^{-1 / 2}\left(\gamma_{r_{n}}+\delta_{r_{n}}\right)^{1 / 2} \bar{B}_{n} A_{n}$ it suffices to show the asymptotic normality of $n^{-1} \overline{\mathbf{1}} A_{n}=n^{-1} Q_{n}(1,-1)^{\top}$.

For a given $n \in \mathbb{N}$ and $1 \leq m \leq n$ let $Q_{n, m}:=(1,-1) A_{n, m}$. Obviously $Q_{n, n}=Q_{n}$ and from (4.1) we have

$$
\begin{equation*}
Q_{n, m}-Q_{n, m-1}=A_{n, m, 1}^{(1)}-A_{n, m, 1}^{(2)}+\sum_{(k, \ell) \in R_{m}} \varepsilon_{k, \ell}^{\left(r_{n}\right)}\left(\widetilde{A}_{n, m, 2, k-1, \ell}-\widetilde{A}_{n, m, 2, k, \ell-1}\right) \tag{6.5}
\end{equation*}
$$

As $\left(Q_{n, m}-Q_{n, m-1}, \mathcal{F}_{m}^{n}\right)$ is a square integrable martingale difference, similarly to the proof of Proposition 1.5 the statement of Proposition 1.7 follows from the propositions below.

Proposition 6.1 If $0<\alpha<1, \beta=1-\alpha$ and (1.7) holds then

$$
\frac{1}{n^{2}} \sum_{m=1}^{n} \mathrm{E}\left(\left(Q_{n, m}-Q_{n, m-1}\right)^{2} \mid \mathcal{F}_{m-1}^{n}\right) \xrightarrow{\mathrm{P}} \frac{1}{4 \alpha \beta} \quad \text { as } n \rightarrow \infty
$$

Proposition 6.2 If $0<\alpha<1, \beta=1-\alpha$ and (1.7) holds then for all $\delta>0$

$$
\frac{1}{n^{2}} \sum_{m=1}^{n} \mathrm{E}\left(\left(Q_{n, m}-Q_{n, m-1}\right)^{2} \mathbb{1}_{\left\{\left|Q_{n, m}-Q_{n, m-1}\right| \geq \delta n\right\}} \mid \mathcal{F}_{m-1}^{n}\right) \xrightarrow{\mathrm{P}} 0 \quad \text { as } n \rightarrow \infty
$$

Proof of Proposition 6.1, The proof is very similar to that of Proposition 4.1. Let $V_{m}^{n}:=\mathrm{E}\left(\left(Q_{n, m}-Q_{n, m-1}\right)^{2} \mid \mathcal{F}_{m-1}^{n}\right)$. The statement of Proposition 6.1 will follow from

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n^{2}} \sum_{m=1}^{n} \mathrm{E} V_{m}^{n}=\frac{1}{4 \alpha \beta} \quad \text { and } \quad \lim _{n \rightarrow \infty} \frac{1}{n^{4}} \operatorname{Var}\left(\sum_{m=1}^{n} V_{m}^{n}\right)=0 \tag{6.6}
\end{equation*}
$$

By the martingale property of $Q_{n, m}$ we have

$$
\sum_{m=1}^{n} \mathrm{E} V_{m}^{n}=\sum_{m=1}^{n}\left(\mathrm{E} Q_{n, m}^{2}-\mathrm{E} Q_{n, m-1}^{2}\right)=\mathrm{E} Q_{n}^{2}
$$

that together with (6.3) implies the convergence of the means in (6.6). Furthermore, representations (6.1) of $Q_{n}^{m}$ and (4.5) of $U_{n}^{m}$ imply

$$
V_{m}^{n}=(1,-1) U_{n} U_{m}^{n}(1,-1)^{\top}=\mathrm{E}\left(A_{n, m, 1}^{(1)}-A_{n, m, 1}^{(2)}\right)^{2}+\sum_{(k, \ell) \in R_{m}}\left(\widetilde{A}_{n, m, 2, k-1, \ell}-\widetilde{A}_{n, m, 2, k, \ell-1}\right)^{2}
$$

Using representation (1.3), definition (4.2) and Lemma 2.8 one can verify

$$
\begin{aligned}
\operatorname{Var}\left(\sum_{m=1}^{n} V_{m}^{n}\right) & =\operatorname{Var}\left(\sum_{m=1}^{n} \sum_{(k, \ell) \in R_{m}}\left(\widetilde{A}_{n, m, 2, k-1, \ell}-\widetilde{A}_{n, m, 2, k, \ell-1}\right)^{2}\right) \\
& \leq \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}} G_{n, i_{1}, j_{1}, i_{2}, j_{2}}+H_{n}
\end{aligned}
$$

where

$$
G_{n, i_{1}, j_{1}, i_{2}, j_{2}}:=\operatorname{Cov}\left(\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}-X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}\right)^{2},\left(X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}-X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)^{2}\right)
$$

and $n^{-4} H_{n} \rightarrow 0$ as $n \rightarrow \infty$. As $X_{k-1, \ell}^{\left(r_{n}\right)}-X_{k, \ell-1}^{\left(r_{n}\right)}$ is also a linear combination of the variables $\left\{\varepsilon_{i, j}^{\left(r_{n}\right)}:(i, j) \in U_{k, \ell}\right\}$, by Lemma 2.8 we have

$$
\begin{aligned}
& \quad \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}} G_{n, i_{1}, j_{1}, i_{2}, j_{2}} \\
& \leq \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}}\left(2 M_{4} L_{n, i_{1}, j_{1}, i_{2}, j_{2}}^{(1)}+\left(M_{4}-3\right)^{+} L_{n, i_{1}, j_{1}, i_{2}, j_{2}}^{(2)}\right) \\
& \quad+\left(M_{4}-3\right)^{+}\left(\sum_{i=-\ell_{n}+1}^{k_{n}} \sum_{j_{1}=-i+1}^{\ell_{n}} \sum_{j_{2}=-i+1}^{\ell_{n}} L_{j_{1}, j_{2}}^{(3)}\left(\alpha_{r_{n}}\right)+\sum_{j=-k_{n}+1}^{\ell_{n}} \sum_{i_{1}=-j+1}^{k_{n}} \sum_{i_{2}=-j+1}^{k_{n}} L_{i_{1}, i_{2}}^{(3)}\left(\beta_{r_{n}}\right)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
& L_{n, i_{1}, j_{1}, i_{2}, j_{2}}^{(1)}:=\operatorname{Cov}\left(X_{i_{1}-1, j_{1}}^{\left(r_{n}\right)}-X_{i_{1}, j_{1}-1}^{\left(r_{n}\right)}, X_{i_{2}-1, j_{2}}^{\left(r_{n}\right)}-X_{i_{2}, j_{2}-1}^{\left(r_{n}\right)}\right)^{2}, \\
& L_{n, i_{1}, j_{1}, i_{2}, j_{2}}^{(2)}:=\sum_{(u, v) \in U_{i_{1} \wedge i_{2}-1, j_{1} \wedge j_{2}-1}}\left(\alpha_{r_{n}}+\beta_{r_{n}}\right)^{2\left(i_{1}+j_{2}+i_{2}+j_{2}-2-2 u-2 v\right)} \\
& \times\left(\mathrm{P}\left(S_{i_{1}+j_{1}-1-u-v}^{\left(\nu_{r_{n}}\right)}=i_{1}-u\right)-\mathrm{P}\left(S_{i_{1}+j_{1}-1-u-v}^{\left(\nu_{r_{n}}\right)}=i_{1}-1-u\right)\right)^{2} \\
& \times\left(\mathrm{P}\left(S_{i_{2}+j_{2}-1-u-v}^{\left(\nu_{r_{n}}\right)}=i_{2}-u\right)-\mathrm{P}\left(S_{i_{2}+j_{2}-1-u-v}^{\left(\nu_{r_{n}}\right)}=i_{2}-1-u\right)\right)^{2}, \\
& L_{i_{1}, i_{2}}^{(3)}(\nu):=\sum_{u=-\infty}^{i_{2} \wedge i_{2}-1} \nu^{2\left(i_{1}+i_{2}-2-2 u\right)} \leq \frac{1}{1-\nu^{2}}, \quad 0<|\nu|<1 .
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
& \frac{1}{n^{4}} \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} \sum_{\left(i_{2}, j_{2}\right) \in T_{k_{n}, \ell_{n}}} L_{n, i_{1}, j_{1}, i_{2}, j_{2}}^{(1)} \\
& =\iint_{T} \iint_{T}\left(r_{n}^{1 / 2} \operatorname{Cov}\left(Z_{0,1}^{(n)}\left(s_{1}, t_{1}\right)-Z_{1,0}^{(n)}\left(s_{1}, t_{1}\right), Z_{0,1}^{(n)}\left(s_{2}, t_{2}\right)-Z_{1,0}^{(n)}\left(s_{2}, t_{2}\right)\right)\right)^{2} \mathrm{~d} s_{1} \mathrm{~d} t_{1} \mathrm{~d} s_{2} \mathrm{~d} t_{2},
\end{aligned}
$$

where due to (1.7), Propositions 2.4, 2.7 and Fatou's lemma the right hand side converges to 0 as $n \rightarrow \infty$. Furthermore, using Remark [2.6 one can find an upper bound for $L_{n, i_{1}, j_{1}, i_{2}, j_{2}}^{(2)}$, namely

$$
L_{n, i_{1}, j_{1}, i_{2}, j_{2}}^{(2)} \leq \frac{D \zeta(5 / 4)}{\left(\alpha_{r_{n}} \beta_{r_{n}}\right)^{3}\left(i_{1} \vee i_{2}+j_{1} \vee j_{2}\right)^{1 / 4}}
$$

with some positive constant $D$. Hence,

$$
\frac{1}{n^{4}} \sum_{\left(i_{1}, j_{1}\right) \in T_{k_{n}, \ell_{n}}} L_{n, i_{1}, j_{1}, i_{2}, j_{2}}^{(2)} \leq \frac{20 D \zeta(5 / 4)}{\left(\alpha_{r_{n}} \beta_{r_{n}}\right)^{3} n^{1 / 4}} \rightarrow 0
$$

as $n \rightarrow \infty$. Finally, if $\nu_{r_{n}}$ denotes one of the sequences $\alpha_{r_{n}}$ or $\beta_{r_{n}}$ we have

$$
\lim _{n \rightarrow \infty} \frac{1}{n^{4}} \sum_{i=-\ell_{n}+1}^{k_{n}} \sum_{j_{1}=-i+1}^{\ell_{n}} \sum_{j_{2}=-i+1}^{\ell_{n}} L_{j_{1}, j_{2}}^{(3)}\left(\nu_{r_{n}}\right) \leq \lim _{n \rightarrow \infty} \frac{1}{n\left(1-\nu_{r_{n}}^{2}\right)}=0
$$

that completes the proof.
Proof of Proposition 6.2. Using the same techniques as in the proof of Proposition 4.2 with the help of representation (6.5) one can show that

$$
\frac{1}{n^{4}} \sum_{m=1}^{n} \mathrm{E}\left(\left(Q_{n, m}-Q_{n, m-1}\right)^{4} \mid \mathcal{F}_{m-1}^{n}\right) \xrightarrow{\mathrm{P}} 0 \quad \text { as } n \rightarrow \infty
$$

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