WEAK CONVERGENCE OF MULTIVARIATE PARTIAL MAXIMA PROCESSES

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Abstract. For a strictly stationary sequence of $\mathbb{R}_+^d$–valued random vectors we derive functional convergence of partial maxima stochastic processes under joint regular variation and weak dependence conditions. The limit process is an extremal process and the convergence takes place in the space of $\mathbb{R}_+^d$–valued càdlàg functions on $[0,1]$, with the Skorohod weak M_1 topology. We also show that this topology in general can not be replaced by the stronger (standard) M_1 topology. The theory is illustrated on three examples, including the multivariate squared GARCH process with constant conditional correlations.

1. Introduction

A classical question in extreme value theory is under what assumptions the scaled maximum

$$\bigvee_{i=1}^{n} \frac{X_i - b_n}{a_n}$$

of i.i.d. random variables $(X_i)_{i \in \mathbb{N}}$ converges weakly, for some $a_n > 0$ and $b_n \in \mathbb{R}$. Also what are the possible limit distributions? Answers to these questions were given by Fisher and Tippet [12], Gnedenko [13] and de Haan [14]. Introducing a time variable, Lamperti [18] studied the asymptotical distributional behavior of partial maxima stochastic processes

$$\bigvee_{i=1}^{\lfloor nt \rfloor} \frac{X_i - b_n}{a_n}, \quad t \geq 0.$$

Extension of the theory to dependent random variables, and then to multivariate and spatial settings were particularly stimulating and useful in applications, we refer here only to Adler [1], Leadbetter [19], [20], Beirlant et al. [7], de Haan and Ferreira [15] and Resnick [22].

In this paper we focus on the multivariate case in the weakly dependent setting. Let $\mathbb{R}_+^d = [0, \infty)^d$. We consider a stationary sequence of $\mathbb{R}_+^d$–valued random vectors (X_n). In the i.i.d. case it is well known that weak convergence of the scaled maximum is equivalent to the regular variation of the distribution of X_1, i.e.

$$M_n = \bigvee_{i=1}^{n} \frac{X_i}{a_n} \xrightarrow{d} Y_0$$

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if and only if
$$nP\Big(\frac{X_1}{a_n} \in \cdot\,\Big) \xrightarrow{v} \mu(\,\cdot\,), \tag{1.1}$$
where Y_0 is a random vector with distribution function $F_0(x) = e^{-\mu([[0,x]]^c)}$, $x \in \mathbb{R}_+^d$, μ is a Radon measure and (a_n) a sequence of positive real numbers such that
$$nP(\|X_1\| > a_n) \to 1 \qquad \text{as } n \to \infty,$$
see Proposition 7.1 in Resnick [22]. The arrow " $\xrightarrow{v}$ " above denotes vague convergence of measures, and $[[a,b]]$ the product segment, i.e.
$$[[a,b]] = [a^1, b^1] \times [a^2, b^2] \times \cdots \times [a^d, b^d]$$
for $a = (a^1, \ldots, a^d), b = (b^1, \ldots, b^d) \in \mathbb{R}_+^d$.

In the i.i.d. case relation (1.1) is also equivalent to the functional convergence of stochastic processes of partial maxima of (X_n), i.e.
$$M_n(\,\cdot\,) = \bigvee_{i=1}^{\lfloor n\cdot \rfloor} \frac{X_i}{a_n} \xrightarrow{d} Y_0(\,\cdot\,) \tag{1.2}$$
in $D([0,1], \mathbb{R}_+^d)$, the space of $\mathbb{R}_+^d$–valued càdlàg functions on $[0,1]$, with the Skorohod J_1 topology, with the limit $Y_0(\,\cdot\,)$ being an extremal process, see Proposition 7.2 in Resnick [22].

In this paper we are interested in the investigation of the asymptotic distributional behavior of the processes $M_n(\,\cdot\,)$ for a sequence of weakly dependent $\mathbb{R}_+^d$–valued random vectors that are jointly regularly varying. Since we study extremes of random processes, nonnegativity of the components of random vectors X_n in reality is not a restrictive assumption.

First, we introduce the essential ingredients about regular variation, weak dependence and Skorohod topologies in Section 2. In Section 3 we prove the so called timeless result on weak convergence of scaled extremes M_n, based on a point process convergence obtained by Davis and Mikosch [10]. Using this result and a multivariate version of the limit theorem derived by Basrak et al. [5] for a certain time-space point processes, in Section 4 we prove a functional limit theorem for processes of partial maxima $M_n(\,\cdot\,)$ in the space $D([0,1], \mathbb{R}_+^d)$ endowed with the Skorohod weak M_1 topology. This topology is weaker than the standard M_1 topology (when $d > 1$). The used methods are partly based on the work of Basrak and Krizmanić [3] for partial sums. Finally, in Section 5 the theory is applied to m–dependent processes, stochastic recurrence equations and multivariate squared GARCH (p,q) with constant conditional correlations. We also illustrate by an example that the weak M_1 convergence in our main theorem, in general, can not be replaced by the standard M_1 convergence.

2. Preliminaries

In this section we introduce some basic notions and results on regular variation and point processes that will be used in the following sections.

2.1. **Regular variation.** Regular variation on $\mathbb{R}_+^d$ for random vectors is typically formulated in terms of vague convergence on $\mathbb{E}^d = [0,\infty]^d \setminus \{\mathbf{0}\}$. The topology on $\mathbb{E}^d$ is chosen so that a set $B \subseteq \mathbb{E}^d$ has compact closure if and only if it is bounded away from zero, that is, if there exists $u > 0$ such that $B \subseteq \mathbb{E}_u^d = \{x \in \mathbb{E}^d : \|x\| > u\}$.

Here $\|\cdot\|$ denotes the max-norm on $\mathbb{R}_+^d$, i.e. $\|x\| = \max\{x^i : i = 1, \ldots, d\}$ where $x = (x^1, \ldots, x^d) \in \mathbb{R}_+^d$. Denote by $C_K^+(\mathbb{E}^d)$ the class of all $\mathbb{R}_+$–valued continuous functions on $\mathbb{E}^d$ with compact support.

The vector ξ with values in $\mathbb{R}_+^d$ is (multivariate) regularly varying with index $\alpha > 0$ if there exists a random vector Θ on the unit sphere $\mathbb{S}_+^{d-1} = \{x \in \mathbb{R}_+^d : \|x\| = 1\}$ in $\mathbb{R}_+^d$, such that for every $u \in (0, \infty)$

$$\frac{\mathrm{P}(\|\xi\| > ux,\, \xi/\|\xi\| \in \cdot\,)}{\mathrm{P}(\|\xi\| > x)} \xrightarrow{w} u^{-\alpha}\mathrm{P}(\Theta \in \cdot\,) \tag{2.1}$$

as $x \to \infty$, where the arrow "$\xrightarrow{w}$" denotes weak convergence of finite measures. Regular variation can be expressed in terms of vague convergence of measures on $\mathcal{B}(\mathbb{E}^d)$:

$$n\mathrm{P}(a_n^{-1}\xi \in \cdot\,) \xrightarrow{v} \mu(\,\cdot\,),$$

where (a_n) is a sequence of positive real numbers tending to infinity and μ is a non-null Radon measure on $\mathcal{B}(\mathbb{E}^d)$.

We say that a strictly stationary $\mathbb{R}_+^d$–valued process $(\xi_n)_{n\in\mathbb{Z}}$ is *jointly regularly varying* with index $\alpha > 0$ if for any nonnegative integer k the kd-dimensional random vector $\xi = (\xi_1, \ldots, \xi_k)$ is multivariate regularly varying with index α.

Theorem 2.1 in Basrak and Segers [6] provides a convenient characterization of joint regular variation: it is necessary and sufficient that there exists a process $(Y_n)_{n\in\mathbb{Z}}$ with $\mathrm{P}(\|Y_0\| > y) = y^{-\alpha}$ for $y \geq 1$ such that as $x \to \infty$,

$$\big((x^{-1}\ \xi_n)_{n\in\mathbb{Z}} \,\big|\, \|\xi_0\| > x\big) \xrightarrow{\text{fidi}} (Y_n)_{n\in\mathbb{Z}}, \tag{2.2}$$

where "$\xrightarrow{\text{fidi}}$" denotes convergence of finite-dimensional distributions. The process $(Y_n)_{n\in\mathbb{Z}}$ is called the *tail process* of $(\xi_n)_{n\in\mathbb{Z}}$.

2.2. **Point processes and dependence conditions.** Let (X_n) be a strictly stationary sequence of $\mathbb{R}_+^d$–valued random vectors and assume it is jointly regularly varying with index $\alpha > 0$. Let (Y_n) be the tail process of (X_n). In order to obtain weak convergence of the scaled extremes M_n and the partial maxima processes $M_n(\,\cdot\,)$ we will use limit results for the corresponding point processes of jumps and then by the continuous mapping theorem transfer this convergence results to extremes and maxima processes. In order to establish these point process convergence we introduce the following processes

$$N_n = \sum_{i=1}^{n} \delta_{X_i/a_n}, \qquad N_n^* = \sum_{i=1}^{n} \delta_{(i/n,\, X_i/a_n)} \qquad \text{for all } n \in \mathbb{N},$$

where (a_n) is a sequence of positive real numbers such that

$$n\mathrm{P}(\|X_1\| > a_n) \to 1, \tag{2.3}$$

as $n \to \infty$. The point process convergence for the sequence (N_n) was obtained by Davis and Mikosch [10], while the convergence for the sequence (N_n^*) in the univariate case was established by Basrak et al. [5], but with straightforward adjustments it carries over to the multivariate case, see Theorem 2.3 below. The appropriate weak dependence conditions for this convergence results are given below. With them we will be able to control the dependence in the sequence (X_n).

Condition 2.1. There exists a sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$ and such that for every $f \in C_K^+([0,1] \times \mathbb{E}^d)$, denoting $k_n = \lfloor n/r_n \rfloor$, as $n \to \infty$,

$$\mathrm{E}\biggl[\exp\biggl\{-\sum_{i=1}^{n} f\biggl(\frac{i}{n}, \frac{X_i}{a_n}\biggr)\biggr\}\biggr] - \prod_{k=1}^{k_n} \mathrm{E}\biggl[\exp\biggl\{-\sum_{i=1}^{r_n} f\biggl(\frac{kr_n}{n}, \frac{X_i}{a_n}\biggr)\biggr\}\biggr] \to 0. \tag{2.4}$$

It can be shown that Condition 2.1 is implied by the strong mixing property (cf. Krizmanić [17]). Condition 2.1 is slightly stronger than the condition $\mathcal{A}(a_n)$ introduced by Davis and Mikosch [10].

Condition 2.2. There exists a sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$ and such that for every $u > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathrm{P}\biggl(\max_{m \le |i| \le r_n} \|X_i\| > ua_n \,\bigg|\, \|X_0\| > ua_n \biggr) = 0. \tag{2.5}$$

By Proposition 4.2 in Basrak and Segers [6], under Condition 2.2 the following holds

$$\theta = \mathrm{P}(\sup_{i\ge1}\|Y_i\| \le 1) = \mathrm{P}(\sup_{i\le-1}\|Y_i\| \le 1) > 0, \tag{2.6}$$

and θ is the extremal index of the univariate sequence $(\|X_n\|)$. Recall that a strictly stationary sequence of nonnegative random variables (ξ_n) has extremal index θ if for every $\tau > 0$ there exists a sequence of real numbers (u_n) such that

$$\lim_{n\to\infty} n\mathrm{P}(\xi_1 > u_n) \to \tau \qquad \text{and} \qquad \lim_{n\to\infty} \mathrm{P}\biggl(\max_{1\le i\le n} \xi_i \le u_n\biggr) \to e^{-\theta\tau}. \tag{2.7}$$

It holds that $\theta \in [0,1]$. In particular, if the ξ_n are i.i.d. then (2.7) can hold only for $\theta = 1$. For a detailed discussion on joint regular variation and dependence Conditions 2.1 and 2.2 we refer to Basrak et al. [5], Section 3.4.

Under joint regular variation and Conditions 2.1 and 2.2, by Theorem 2.8 in Davis and Mikosch [10] we obtain the convergence in distribution of point processes N_n to some N, which by Theorem 2.2 and Corollary 2.4 in [10] has the following cluster representation

$$N \stackrel{d}{=} \sum_i \sum_j \delta_{P_i Q_{ij}}, \tag{2.8}$$

where $\sum_{i=1}^{\infty} \delta_{P_i}$ is a Poisson process on $\mathbb{R}_+$ with intensity measure κ given by $\kappa(dy) = \theta\alpha y^{-\alpha-1} 1_{(0,\infty)}(y)\,dy$, and $\sum_{j=1}^{\infty} \delta_{Q_{ij}}$, $i \ge 1$, are i.i.d. point processes whose points satisfy $\sup_j \|Q_{ij}\| = 1$, and all point processes are mutually independent. For a more precise description of the distribution of point process $\sum_{j=1}^{\infty} \delta_{Q_{ij}}$ see [10].

Then by the same arguments as in the proof of Theorem 2.3 in [5] one obtains the following result (cf. also Basrak and Krizmanić [3]).

Theorem 2.3. *Assume that Conditions 2.1 and 2.2 hold for the same sequence (r_n). Then for every $u \in (0,\infty)$ and as $n \to \infty$,*

$$N_n^* \bigg|_{[0,1]\times\mathbb{E}_u^d} \xrightarrow{d} N^{(u)} = \sum_i \sum_j \delta_{(T_i^{(u)}, uZ_{ij})} \bigg|_{[0,1]\times\mathbb{E}_u^d}, \tag{2.9}$$

in $[0,1] \times \mathbb{E}_u^d$ and

(1) $\sum_i \delta_{T_i^{(u)}}$ *is a homogeneous Poisson process on $[0,1]$ with intensity $\theta u^{-\alpha}$,*

(2) $(\sum_j \delta_{Z_{ij}})_i$ *is an i.i.d. sequence of point processes in* $\mathbb{E}^d$*, independent of* $\sum_i \delta_{T_i^{(u)}}$*, and with distribution equal to* $(\sum_{j\in\mathbb{Z}} \delta_{Y_j} \mid \sup_{i\le -1} \|Y_i\| \le 1)$.

2.3. **The weak M_1 topology.** The stochastic processes that we consider have discontinuities, and therefore it is natural for the function space of sample paths of these stochastic processes to take the space $D([0,1],\mathbb{R}_+^d)$ of all right-continuous $\mathbb{R}_+^d$–valued functions on $[0,1]$ with left limits.

In the one dimensional case (cf. Krizmanić [16]) the partial maxima processes $M_n(\,\cdot\,)$ converge to an extremal process in the space $D([0,1],\mathbb{R}_+)$ equipped with the Skorohod M_1 topology. In this paper we extend this result to the multivariate setting, but with the weak M_1 topology, since as we show later the direct generalization of the one-dimensional result to random vectors fails in the standard M_1 topology on $D([0,1],\mathbb{R}_+^d)$ for $d \ge 2$. In the sequel we give the definition of the weak M_1 topology.

For $x \in D([0,1],\mathbb{R}_+^d)$ the completed graph of x is the set

$$G_x = \{(t,z) \in [0,1] \times \mathbb{R}_+^d : z \in [[x(t-),x(t)]]\},$$

where $x(t-)$ is the left limit of x at t. We define an order on the graph G_x by saying that $(t_1,z_1) \le (t_2,z_2)$ if either (i) $t_1 < t_2$ or (ii) $t_1 = t_2$ and $|x^j(t_1-) - z_1^j| \le |x^j(t_2-) - z_2^j|$ for all $j = 1,\ldots,d$. Note that the relation $\le$ induces only a partial order on the graph G_x. A weak parametric representation of the graph G_x is a continuous nondecreasing function (r,u) mapping $[0,1]$ into G_x, with $r \in C([0,1],[0,1])$ being the time component and $u = (u^1,\ldots,u^d) \in C([0,1],\mathbb{R}_+^d)$ being the spatial component, such that $r(0)=0, r(1)=1$ and $u(1)=x(1)$. Let $\Pi_w(x)$ denote the set of weak parametric representations of the graph G_x. For $x_1,x_2 \in D([0,1],\mathbb{R}_+^d)$ define

$$d_w(x_1,x_2) = \inf\{\|r_1-r_2\|_{[0,1]} \vee \|u_1-u_2\|_{[0,1]} : (r_i,u_i) \in \Pi_w(x_i), i=1,2\},$$

where $\|x\|_{[0,1]} = \sup\{\|x(t)\| : t \in [0,1]\}$. Now we say that $x_n \to x$ in $D([0,1],\mathbb{R}_+^d)$ for a sequence (x_n) in the weak Skorohod M_1 (or shortly WM_1) topology if $d_w(x_n,x) \to 0$ as $n \to \infty$. The WM_1 topology is weaker than the standard (or strong) M_1 topology on $D([0,1],\mathbb{R}_+^d)$. For $d=1$ the two topologies coincide. The WM_1 topology coincides with the topology induced by the metric

$$d_p(x_1,x_2) = \max\{d_{M_1}(x_1^j,x_2^j) : j=1,\ldots,d\} \tag{2.10}$$

for $x_i = (x_i^1,\ldots,x_i^d) \in D([0,1],\mathbb{R}_+^d)$ and $i=1,2$ (here d_{M_1} denotes the standard Skorohod M_1 metric on $D([0,1],\mathbb{R}_+)$). The metric d_p induces the product topology on $D([0,1],\mathbb{R}_+^d)$. For detailed discussion of the strong and weak M_1 topologies we refer to Whitt [24], sections 12.3–12.5. Recall here the definition of the metric d_{M_1}. For $x \in D([0,1],\mathbb{R}_+^d)$ we define the set

$$\Gamma_x = \{(t,z) \in [0,1] \times \mathbb{R}_+^d : z \in [x(t-),x(t)]\},$$

where $[a,b] = \{\lambda a + (1-\lambda)b : 0 \le \lambda \le 1\}$ for $a,b \in \mathbb{R}_+^d$. We say (r,u) is a parametric representation of Γ_x if it is a continuous nondecreasing function mapping $[0,1]$ onto Γ_x. Denote by $\Pi(x)$ the set of all parametric representations of the graph Γ_x. Then for $x_1,x_2 \in D([0,1],\mathbb{R}_+^d)$

$$d_{M_1}(x_1,x_2) = \inf\{\|r_1-r_2\|_{[0,1]} \vee \|u_1-u_2\|_{[0,1]} : (r_i,u_i) \in \Pi(x_i), i=1,2\}.$$

3. Weak convergence of partial maxima M_n

In this section we establish weak convergence of the multivariate partial maxima M_n by generalizing the corresponding one dimensional result given in Krizmanić [16]. Let (X_n) be a strictly stationary sequence of $\mathbb{R}_+^d$–valued random vectors, jointly regularly varying with index $\alpha \in (0,\infty)$ and assume Conditions 2.1 and 2.2 hold. Then by (2.8) it holds that, as $n \to \infty$,

$$N_n = \sum_{i=1}^{n} \delta_{X_i/a_n} \xrightarrow{d} N = \sum_i \sum_j \delta_{P_i Q_{ij}},$$

where (a_n) is chosen as in (2.3). Denote by $\mathbf{M}_p(\mathbb{E}^d)$ the space of Radon point measures on $\mathbb{E}^d$ equipped with the vague topology. Recall $M_n = a_n^{-1} \bigvee_{i=1}^{n} X_i = \big(a_n^{-1} \bigvee_{i=1}^{n} X_i^k \big)_{k=1,\ldots,d}$.

Theorem 3.1. *Let (X_n) be a strictly stationary sequence of $\mathbb{R}_+^d$–valued random vectors, jointly regularly varying with index $\alpha \in (0,\infty)$. Suppose that Conditions 2.1 and 2.2 hold. Then, as $n \to \infty$,*

$$M_n \xrightarrow{d} M = \bigvee_i \bigvee_j P_i Q_{ij}.$$

Proof. Let $\epsilon > 0$ be arbitrary. The mapping $T_\epsilon \colon \mathbf{M}_p(\mathbb{E}^d) \to \mathbb{R}_+^d$ defined by

$$T_\epsilon\Big(\sum_{i=1}^{\infty} \delta_{x_i} \Big) = \bigg(\bigvee_{i=1}^{\infty} x_i^k 1_{\{x_i^k \in [\epsilon,\infty)\}} \bigg)_{k=1,\ldots,d}$$

is continuous on the set

$$\Lambda_\epsilon = \{ \eta \in \mathbf{M}_p(\mathbb{E}^d) : \eta(\{(y_1,\ldots,y_d) : y_i = \epsilon \text{ for some } i\}) = 0 \}.$$

One can see this by showing the continuity of the components

$$T_\epsilon^k\Big(\sum_{i=1}^{\infty} \delta_{x_i^k} \Big) = \bigvee_{i=1}^{\infty} x_i^k 1_{\{x_i^k \in [\epsilon,\infty)\}}$$

(cf. the one dimensional case in Krizmanić [16]).

N has no fixed atoms (see Lemma 2.1 in Davis and Mikosch [10]), i.e. $\mathrm{P}(N \in \Lambda_\epsilon) = 1$, and therefore by the continuous mapping theorem we get

$$M_n[\epsilon,\infty) = T_\epsilon(N_n) \xrightarrow{d} T_\epsilon(N) = M[\epsilon,\infty) \qquad \text{as } n \to \infty, \tag{3.1}$$

with the notation

$$M_n B = (M_n^k B)_{k=1,\ldots,d} = \bigg(a_n^{-1} \bigvee_{i=1}^{n} X_i^k 1_{\{a_n^{-1} X_i^k \in B\}} \bigg)_{k=1,\ldots,d},$$

and

$$M B = (M^k B)_{k=1,\ldots,d} = \bigg(\bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} P_i Q_{ij}^k 1_{\{P_i Q_{ij}^k \in B\}} \bigg)_{k=1,\ldots,d}$$

for any Borel set B in $\mathbb{R}_+$. Obviously

$$M[\epsilon,\infty) \to M(0,\infty) = M \tag{3.2}$$

almost surely as $\epsilon \to 0$.

In order to obtain $M_n \xrightarrow{d} M$, i.e. $M_n(0,\infty) \xrightarrow{d} M(0,\infty)$ as $n \to \infty$, by Theorem 3.5 in Resnick [22] it suffices to prove that

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \mathrm{P}(\|M_n[\epsilon,\infty) - M_n(0,\infty)\| > \delta) = 0 \tag{3.3}$$

for any $\delta > 0$. Since for arbitrary real numbers $x_1,\ldots,x_n,y_1,\ldots,y_n$ the following inequality

$$\Big| \bigvee_{i=1}^{n} x_i - \bigvee_{i=1}^{n} y_i \Big| \le \bigvee_{i=1}^{n} |x_i - y_i| \tag{3.4}$$

holds, note that

$$|M_n^k[\epsilon,\infty) - M_n^k(0,\infty)| \le M_n^k(0,\epsilon)$$

for all $k=1,\ldots,d$, and this yields

$$\|M_n[\epsilon,\infty) - M_n(0,\infty)\| = \bigvee_{k=1}^{d} |M_n^k[\epsilon,\infty) - M_n^k(0,\infty)| \le \|M_n(0,\epsilon)\|. \tag{3.5}$$

Take now an arbitrary $s > \alpha$. Then using stationarity and Markov's inequality we get the bound

$$\begin{aligned}
\mathrm{P}(\|M_n(0,\epsilon)\| > \delta) \quad &\le \quad n\mathrm{P}\bigg(\bigvee_{k=1,\ldots,d} \frac{X_1^k}{a_n} 1_{\{X_1^k < \epsilon a_n\}} > \delta \bigg) \\
&\le \quad n \sum_{k=1}^{d} \mathrm{P}\bigg(\frac{X_1^k}{a_n} 1_{\{X_1^k < \epsilon a_n\}} > \delta \bigg) \le n \sum_{k=1}^{d} \frac{1}{\delta^s a_n^s} \mathrm{E}((X_1^k)^s 1_{\{X_1^k < \epsilon a_n\}}) \\
&= \quad \frac{n}{\delta^s a_n^s} \sum_{k=1}^{d} \Big[\mathrm{E}((X_1^k)^s 1_{\{X_1^k < \epsilon a_n, \|X_1\| > \epsilon a_n\}}) + \mathrm{E}((X_1^k)^s 1_{\{X_1^k < \epsilon a_n, \|X_1\| \le \epsilon a_n\}}) \Big] \\
&\le \quad \frac{n}{\delta^s} \sum_{k=1}^{d} \Big[\epsilon^s \mathrm{P}(\|X_1\| > \epsilon a_n) + \mathrm{E}\Big(\frac{\|X_1\|^s}{a_n^s} 1_{\{\|X_1\| \le \epsilon a_n\}} \Big) \Big] \\
&= \quad \frac{\epsilon^s d}{\delta^s} \cdot n\mathrm{P}(\|X_1\| > \epsilon a_n) \bigg[1 + \frac{\mathrm{E}(\|X_1\|^s 1_{\{\|X_1\| < \epsilon a_n\}})}{\epsilon^s a_n^s \mathrm{P}(\|X_1\| > \epsilon a_n)} \bigg].
\end{aligned} \tag{3.6}$$

Since the distribution of $\|X_1\|$ is regularly varying with index α, using (2.3) it follows immediately that

$$n\mathrm{P}(\|X_1\| > \epsilon a_n) \to \epsilon^{-\alpha}$$

as $n \to \infty$. By Karamata's theorem

$$\lim_{n \to \infty} \frac{\mathrm{E}(\|X_1\|^s \, 1_{\{\|X_1\| < \epsilon a_n\}})}{\epsilon^s a_n^s \mathrm{P}(\|X_1\| > \epsilon a_n)} = \frac{\alpha}{s - \alpha}.$$

Thus from (3.6) we get

$$\limsup_{n \to \infty} \mathrm{P}(\|M_n(0,\epsilon)\| > \delta) \le \frac{\epsilon^{s-\alpha} d}{\delta^s} \Big[1 + \frac{\alpha}{s - \alpha} \Big].$$

Letting $\epsilon \to 0$ we finally obtain

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \mathrm{P}(\|M_n(0,\epsilon)\| > \delta) = 0,$$

and taking into account (3.5), relation (3.3) follows. Hence $M_n \xrightarrow{d} M$ as $n \to \infty$. $\square$

Remark 3.2. From the representation in (2.8) and the fact that $\sup_j \|Q_{ij}\| = 1$ it follows that $\|M\|$ is a Fréchet random variable, since

$$\begin{aligned}
\mathrm{P}(\|M\| \leq x) &= \mathrm{P}\Big(\max_{k=1,\ldots,d} \bigvee_i \bigvee_j P_i Q_{ij}^k \leq x \Big) = \mathrm{P}\Big(\bigvee_i P_i \leq x \Big) \\
&= \mathrm{P}\Big(\sum_i \delta_{P_i}(x,\infty) = 0 \Big) = e^{-\kappa(x,\infty)} = e^{-\theta x^{-\alpha}}
\end{aligned}$$

for $x > 0$.

4. Functional convergence of partial maxima processes $M_n(\,\cdot\,)$

In this section we show the convergence of the partial maxima process

$$M_n(t) = \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{X_i}{a_n} = \bigg(\bigvee_{i=1}^{\lfloor nt \rfloor} \frac{X_i^k}{a_n} \bigg)_{k=1,\ldots,d}, \quad t \in [0,1],$$

to an extremal process in the space $D([0,1], \mathbb{R}_+^d)$ equipped with Skorohod weak M_1 topology. Similar to the one dimensional case treated in Krizmanić [16] we first represent $M_n(\,\cdot\,)$ as the image of the time-space point process N_n^* under a certain maximum functional. Then, using certain continuity properties of this functional, the continuous mapping theorem and the standard "finite dimensional convergence plus tightness" procedure we transfer the weak convergence of N_n^* in (2.9) to weak convergence of $M_n(\,\cdot\,)$.

Extremal processes can be derived from Poisson processes in the following way. Let $\xi = \sum_k \delta_{(t_k, j_k)}$ be a Poisson process on $[0,\infty) \times \mathbb{E}^d$ with mean measure $\lambda \times \nu$, where λ is the Lebesgue measure and ν is a measure on $\mathbb{E}^d$ satisfying

$$\nu(\{x \in \mathbb{E}^d : \|x\| > \delta\}) < \infty$$

for any $\delta > 0$. The extremal process $\widetilde{M}(\,\cdot\,)$ generated by ξ is defined by

$$\widetilde{M}(t) = \bigvee_{t_k \leq t} j_k, \qquad t > 0.$$

Then for $x \in \mathbb{E}^d$ and $t > 0$ it holds that

$$\mathrm{P}(\widetilde{M}(t) \leq x) = e^{-t\nu([[0,x]]^c)},$$

with the notation that for two vectors $y = (y^1, \ldots, y^d)$ and $z = (z^1, \ldots, z^d)$, $y \leq z$ means $y^k \leq z^k$ for all $k = 1, \ldots, d$ (cf. Resnick [22], section 5.6). The measure ν is called the exponent measure.

Now fix $0 < v < u < \infty$ and define the maximum functional

$$\phi^{(u)} \colon \mathbf{M}_p([0,1] \times \mathbb{E}_v^d) \to D([0,1], \mathbb{R}_+^d)$$

by

$$\phi^{(u)}\Big(\sum_i \delta_{(t_i, (x_i^1, \ldots, x_i^d))} \Big)(t) = \Big(\bigvee_{t_i \leq t} x_i^k \, 1_{\{u < x_i^k < \infty\}} \Big)_{k=1,\ldots,d}, \qquad t \in [0,1],$$

where the supremum of an empty set may be taken, for convenience, to be 0. $\phi^{(u)}$ is well defined because $[0,1] \times \mathbb{E}_u^d$ is a relatively compact subset of $[0,1] \times \mathbb{E}_v^d$. The space $\mathbf{M}_p([0,1]\times\mathbb{E}_v^d)$ of Radon point measures on $[0,1]\times\mathbb{E}_v^d$ is equipped with the vague topology and $D([0,1],\mathbb{R}_+^d)$ is equipped with the weak M_1 topology. Let

$$\Lambda = \{\eta \in \mathbf{M}_p([0,1]\times\mathbb{E}_v^d) : \eta(\{0,1\}\times\mathbb{E}_u^d)=0 \text{ and}$$
$$\eta([0,1]\times\{x=(x^1,\ldots,x^d) : x^i\in\{u,\infty\} \text{ for some } i\})=0\}.$$

Then the point process $N^{(v)}$ defined in (2.9) almost surely belongs to the set Λ, see Lemma 3.1 in Basrak and Krizmanić [3]. Now we will show that $\phi^{(u)}$ is continuous on the set Λ.

Lemma 4.1. *The maximum functional $\phi^{(u)}\colon \mathbf{M}_p([0,1]\times\mathbb{E}_v^d)\to D([0,1],\mathbb{R}_+^d)$ is continuous on the set Λ, when $D([0,1],\mathbb{R}_+^d)$ is endowed with the weak M_1 topology.*

Proof. Take an arbitrary $\eta\in\Lambda$ and suppose that $\eta_n \xrightarrow{v} \eta$ in $\mathbf{M}_p([0,1]\times\mathbb{E}_v^d)$. We need to show that $\phi^{(u)}(\eta_n)\to\phi^{(u)}(\eta)$ in $D([0,1],\mathbb{R}_+^d)$ according to the WM_1 topology. By Theorem 12.5.2 in Whitt [24], it suffices to prove that, as $n\to\infty$,

$$d_p(\phi^{(u)}(\eta_n),\phi^{(u)}(\eta)) = \max_{k=1,\ldots,d} d_{M_1}(\phi^{(u)\,k}(\eta_n),\phi^{(u)\,k}(\eta)) \to 0,$$

where $\phi^{(u)}(\xi)=(\phi^{(u)\,k}(\xi))_{k=1,\ldots,d}$ for $\xi\in\mathbf{M}_p([0,1]\times\mathbb{E}_v^d)$.

Now one can follow, with small modifications, the lines in the proof of Lemma 4.1 in Krizmanić [16] to obtain $d_{M_1}(\phi^{(u)\,k}(\eta_n),\phi^{(u)\,k}(\eta))\to 0$ as $n\to\infty$. Therefore $d_p(\phi^{(u)}(\eta_n),\phi^{(u)}(\eta))\to 0$ as $n\to\infty$, and we conclude that $\phi^{(u)}$ is continuous at η. ☐

Lemma 4.2. *Assume $\xi_n=\sum_i \delta_{(t_i^{(n)},j_i^{(n)})}$, $n\ge 0$, are Poisson processes on $[0,\infty)\times\mathbb{E}^d$ with mean measures $\lambda\times\beta_n$, and let H_n be the corresponding extremal processes generated by the ξ_n's. If*

$$\beta_n \xrightarrow{v} \beta_0 \qquad \text{as } n\to\infty, \tag{4.1}$$

then the finite dimensional distributions of $H_n(\,\cdot\,)$ converge to the finite dimensional distributions of $H_0(\,\cdot\,)$ as $n\to\infty$.

Proof. By Lemma 6.1 in Resnick [22], from (4.1) we obtain that, as $n\to\infty$,

$$\beta_n([[0,x]]^c)\to\beta_0([[0,x]]^c) \tag{4.2}$$

for all continuity points x of $\beta_0([[0,\,\cdot\,]]^c)$.

Similar to the univariate case, the finite dimensional distributions of $H_n(\,\cdot\,)=\bigvee_{t_i^{(n)}\le\,\cdot} j_i^{(n)}$ are of the form

$$\begin{aligned}&\mathrm{P}(H_n(t_1)\le x_1,\ldots,H_n(t_m)\le x_m)\\ &\quad = e^{-t_1\beta_n([[0,\bigwedge_{i=1}^m x_i]]^c)}\cdot e^{-(t_2-t_1)\beta_n([[0,\bigwedge_{i=2}^m x_i]]^c)}\cdot\ldots\cdot e^{-(t_m-t_{m-1})\beta_n([[0,x_m]]^c)},\end{aligned}$$

for $0\le t_1<t_2<\ldots<t_m\le 1$ and $x_1,\ldots,x_m\in\mathbb{E}^d$. Letting $n\to\infty$ and using (4.2) we immediately obtain that the right hand side in the last equation above converges (in the continuity points $x_1,\ldots,x_m$ of $\beta_0([[0,\cdot]]^c)$) to

$$e^{-t_1\beta_0([[0,\bigwedge_{i=1}^m x_i]]^c)}\cdot e^{-(t_2-t_1)\beta_0([[0,\bigwedge_{i=2}^m x_i]]^c)}\cdot\ldots\cdot e^{-(t_m-t_{m-1})\beta_0([[0,x_m]]^c)}.$$

But since this limit is in fact $\mathrm{P}(H_0(t_1)\le x_1,\ldots,H_0(t_m)\le x_m)$, we conclude that the finite dimensional distributions of $H_n(\,\cdot\,)$ converge to the finite dimensional distributions of $H_0(\,\cdot\,)$ as $n\to\infty$. ☐

Theorem 4.3. *Let (X_n) be a strictly stationary sequence of $\mathbb{R}_+^d$–valued random vectors, jointly regularly varying with index $\alpha > 0$. Suppose that Conditions 2.1 and 2.2 hold. Then the partial maxima stochastic process*

$$M_n(t) = \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{X_i}{a_n}, \qquad t \in [0,1],$$

satisfies

$$M_n(\,\cdot\,) \xrightarrow{d} \widetilde{M}(\,\cdot\,) \qquad \text{as } n \to \infty,$$

in $D([0,1], \mathbb{R}_+^d)$ endowed with the weak M_1 topology, where $\widetilde{M}(\,\cdot\,)$ is an extremal process.

Remark 4.4. The exponent measure ν of the limiting process $\widetilde{M}(\,\cdot\,)$ in the theorem is the vague limit of the sequence of measures $(\nu^{(u)})$ $(u > 0)$ as $u \downarrow 0$, with $\nu^{(u)}$ being defined by

$$\nu^{(u)}(((x,y]]) \quad = u^{-\alpha}\, \mathrm{P}\bigg(u \bigvee_{i \geq 0} \big(Y_i^j \, 1_{\{Y_i^j > 1\}} \big)_{j=1,\ldots,d} \in ((x,y]],\ \sup_{i \leq -1} \|Y_i\| \leq 1 \bigg),$$

for $x = (x^1, \ldots, x^d)$, $y = (y^1, \ldots, y^d) \in \mathbb{E}^d$ such that $((x,y]] = (x^1, y^1] \times \cdots \times (x^d, y^d]$ is bounded away from zero. Here (Y_n) is the tail process of the sequence (X_n).

Proof. (*Theorem 4.3*) Using the techniques from the proof of Theorem 3.4 in Basrak and Krizmanić [3] we obtain that the point process

$$\widehat{N}^{(u)} = \sum_i \delta_{(T_i^{(u)},\, u \bigvee_j (Z_{ij}^k 1_{\{Z_{ij}^k > 1\}})_{k=1,\ldots,d})}$$

is a Poisson process with mean measure $\lambda \times \nu^{(u)}$.

Consider now $0 < v < u$ and

$$\phi^{(u)}(N_n^* \,|_{\,[0,1] \times \mathbb{E}_u^d})(\,\cdot\,) = \phi^{(u)}(N_n^* \,|_{\,[0,1] \times \mathbb{E}_v^d})(\,\cdot\,) = \bigvee_{i/n \leq \,\cdot} \Big(\frac{X_i^k}{a_n} 1_{\big\{ \frac{X_i^k}{a_n} > u \big\}} \Big)_{k=1,\ldots,d},$$

which by Theorem 2.3, Lemma 4.1 and the continuous mapping theorem converges in distribution in $D([0,1], \mathbb{R}_+^d)$ under the WM_1 topology to

$$\phi^{(u)}(N^{(v)})(\,\cdot\,) \stackrel{d}{=} \phi^{(u)}(N^{(v)} \,|_{\,[0,1] \times \mathbb{E}_u^d})(\,\cdot\,) \stackrel{d}{=} \bigvee_{T_i^{(u)} \leq \,\cdot} \bigvee_j u (Z_{ij}^k 1_{\{Z_{ij}^k > 1\}})_{k=1,\ldots,d}.$$

This can be rewritten as

$$M_n^{(u)}(\,\cdot\,) := \bigvee_{i=1}^{\lfloor n\,\cdot\, \rfloor} \Big(\frac{X_i^k}{a_n} 1_{\big\{ \frac{X_i^k}{a_n} > u \big\}} \Big)_{k=1,\ldots,d} \xrightarrow{d} M^{(u)}(\,\cdot\,) := \bigvee_{T_i \leq \,\cdot} K_i^{(u)} \quad \text{as } n \to \infty, \tag{4.3}$$

in $D([0,1], \mathbb{R}_+^d)$ under the WM_1 metric, since $\phi^{(u)}(N^{(u)}) = \phi^{(u)}(\widehat{N}^{(u)}) \stackrel{d}{=} \phi^{(u)}(\widetilde{N}^{(u)})$, where

$$\widetilde{N}^{(u)} = \sum_i \delta_{(T_i,\, K_i^{(u)})}$$

is a Poisson process with mean measure $\lambda \times \nu^{(u)}$.

Note that the limiting process $M^{(u)}(\,\cdot\,)$ is an extremal process with exponent measure $\nu^{(u)}$, and therefore

$$\mathrm{P}(M^{(u)}(t) \leq x) = \mathrm{P}(\widetilde{N}^{(u)}((0,t] \times [[0,x]]^c) = 0) = e^{-t\nu^{(u)}([[0,x]]^c)} \tag{4.4}$$

for $t \in [0,1]$ and $x \in \mathbb{E}^d$. Since the function $\pi \colon D([0,1],\mathbb{R}_+^d) \to \mathbb{R}_+^d$ defined by $\pi(y) = y(1)$ is continuous (see Theorem 12.5.2 (iii) in Whitt [24]), an application of the continuous mapping theorem to relation (4.3) yields

$$M_n^{(u)}(1) \xrightarrow{d} M^{(u)}(1) \qquad \text{as } n \to \infty. \tag{4.5}$$

Apply now the notation from the proof of Theorem 3.1 to see that $M_n^{(u)}(1) = M_n(u,\infty)$. Hence comparing (3.1) and (4.5) we conclude that $M^{(u)}(1) \stackrel{d}{=} M(u,\infty)$. Further, from (3.2) it follows that

$$M^{(u)}(1) \xrightarrow{d} M \qquad \text{as } u \to 0, \tag{4.6}$$

which means that

$$F_u(x) := \mathrm{P}(M^{(u)}(1) \le x) \to F(x) := \mathrm{P}(M \le x) \qquad \text{as } u \to 0, \tag{4.7}$$

for all $x \in \mathbb{E}^d$ that are continuity points of F. From (4.4) we obtain

$$F_u^t(x) = \mathrm{P}(M^{(u)}(t) \le x)$$

for $t \in [0,1]$ and $x \in \mathbb{E}^d$, which implies that the multivariate distribution function F_u is max-infinitely divisible (cf. Resnick [22], Section 5.6). Since the class of max-infinitely divisible distributions is closed in $\mathbb{R}^d$ with respect to weak convergence (cf. Proposition 5.1 in Resnick [21]), relation (4.7) implies that F is max-infinitely divisible, and hence by Proposition 5.8 in Resnick [21] there exists an exponent measure μ on $\mathbb{E}^d$ such that

$$F(x) = e^{-\nu([[0,x]]^c)}, \qquad x \in \mathbb{E}^d.$$

Therefore, from (4.7) we obtain, as $u \to 0$,

$$\nu^{(u)}([[0,x]]^c) \to \nu([[0,x]]^c)$$

for all continuity points x of $\nu([[0,\,\cdot\,]]^c)$. Now an application of Lemma 6.1 in Resnick [22] yields that $\nu^{(u)} \xrightarrow{v} \nu$ as $u \to 0$. Therefore, by Lemma 4.2 it follows that the finite dimensional distributions of $M^{(u)}(\,\cdot\,)$ converge to the finite dimensional distributions of $\widetilde{M}(\,\cdot\,)$ as $u \to 0$, where $\widetilde{M}(\,\cdot\,)$ is the extremal process generated by the Poisson process $T = \sum_i \delta_{(T_i,K_i)}$ with mean measure $\lambda \times \nu$, i.e. $\widetilde{M}(t) = \bigvee_{T_i \le t} K_i$, $t \in [0,1]$.

This implies that the finite dimensional distributions of each coordinate $M^{(u)k}(\,\cdot\,)$ $(k = 1,\ldots,d)$ converge to the finite dimensional distributions of $\widetilde{M}^k(\,\cdot\,)$ as $u \to 0$. According to the arguments used in the univariate case (see the proof of Theorem 4.3 in Krizmanić [16]) this suffices to conclude that $M^{(u)k}(\,\cdot\,) \xrightarrow{d} \widetilde{M}^k(\,\cdot\,)$ in $D([0,1],\mathbb{R}_+)$ with the M_1 topology. Hence $\{M^{(u)k} : u > 0\}$ is tight, and thus by Lemma 3.2 in Whitt [25] it follows that $\{M^{(u)} : u > 0\}$ is also tight (in the space $D([0,1],\mathbb{R}_+^d)$ with the product topology generated by the metric d_p).

From the convergence of finite dimensional distributions and tightness for processes $M^{(u)}(\,\cdot\,)$ we obtain the convergence in distribution, i.e. as $u \to 0$,

$$M^{(u)}(\,\cdot\,) \xrightarrow{d} \widetilde{M}(\,\cdot\,) \tag{4.8}$$

in $D([0,1],\mathbb{R}_+^d)$ with the WM_1 topology.

If we show that

$$\lim_{u \to 0} \limsup_{n \to \infty} \mathrm{P}(d_p(M_n(\,\cdot\,), M_n^{(u)}(\,\cdot\,)) > \epsilon) = 0$$

for any $\epsilon > 0$, from (4.3) and (4.8) by a variant of Slutsky's theorem (see Theorem 3.5 in Resnick [22]) it will follow that $M_n(\,\cdot\,) \xrightarrow{d} \widetilde{M}(\,\cdot\,)$ as $n \to \infty$, in $D([0,1], \mathbb{R}_+^d)$ with the WM_1 topology.

Since the metric d_p on $D([0,1], \mathbb{R}_+^d)$ is bounded above by the uniform metric on $D([0,1], \mathbb{R}_+^d)$ (see Theorem 12.10.3 in Whitt [24]), it suffices to show that

$$\lim_{u \downarrow 0} \limsup_{n \to \infty} \mathrm{P}\bigg(\sup_{0 \le t \le 1} \|M_n^{(u)}(t) - M_n(t)\| > \epsilon \bigg) = 0.$$

Recalling the definitions and using the inequality (3.4) , we have

$$\begin{aligned}
&\mathrm{P}\bigg(\sup_{0 \le t \le 1} \|M_n^{(u)}(t) - M_n(t)\| > \epsilon \bigg) \\
&\qquad = \mathrm{P}\bigg(\sup_{0 \le t \le 1} \max_{k=1,\ldots,d} \bigg| \bigvee_{i=1}^{\lfloor nt \rfloor} \bigg(\frac{X_i^k}{a_n} 1_{\big\{ \frac{X_i^k}{a_n} > u \big\}} - \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{X_i^k}{a_n} \bigg) \bigg| > \epsilon \bigg) \\
&\qquad \le \mathrm{P}\bigg(\sup_{0 \le t \le 1} \max_{k=1,\ldots,d} \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{X_i^k}{a_n} 1_{\big\{ \frac{X_i^k}{a_n} \le u \big\}} > \epsilon \bigg) \\
&\qquad = \mathrm{P}\bigg(\bigg\| \bigvee_{i=1}^{n} \bigg(\frac{X_i^k}{a_n} 1_{\big\{ \frac{X_i^k}{a_n} \le u \big\}} \bigg)_{k=1,\ldots,d} \bigg\| > \epsilon \bigg) \\
&\qquad \le \sum_{k=1}^{d} \mathrm{P}\bigg(\bigvee_{i=1}^{n} \frac{X_i^k}{a_n} 1_{\big\{ \frac{X_i^k}{a_n} \le u \big\}} > \epsilon \bigg).
\end{aligned}$$

Since the last term above is equal to zero for $u \in (0, \epsilon)$, it holds that

$$\lim_{u \to 0} \limsup_{n \to \infty} \mathrm{P}(d_p(M_n(\,\cdot\,), M_n^{(u)}(\,\cdot\,)) > \epsilon) = 0,$$

and this concludes the proof. $\square$

Remark 4.5. The WM_1 convergence in Theorem 4.3 in general can not be replaced by the standard M_1 convergence. This is shown in Example 5.1.

The problem in our proof if we consider the standard M_1 topology is Lemma 4.1, which in this case does not hold. To see this, fix $u > 0$ and define

$$\eta_n = \delta_{(\frac{1}{2} - \frac{1}{n}, (2u, 0))} + \delta_{(\frac{1}{2} - \frac{1}{2n}, (0, 2u))} \qquad \text{for } n \ge 3.$$

Then $\eta_n \xrightarrow{v} \eta$, where

$$\eta = \delta_{(\frac{1}{2}, (2u, 0))} + \delta_{(\frac{1}{2}, (0, 2u))} \in \Lambda.$$

It is easy to compute

$$\phi^{(u)\,1}(\eta_n)(t) = 2u\, 1_{[\frac{1}{2} - \frac{1}{n}, 1]}(t) \quad \text{and} \quad \phi^{(u)\,2}(\eta_n)(t) = 2u\, 1_{[\frac{1}{2} - \frac{1}{2n}, 1]}(t).$$

Then

$$y_n(t) := \phi^{(u)\,1}(\eta_n)(t) - \phi^{(u)\,2}(\eta_n)(t) = 2u\, 1_{[\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} - \frac{1}{n})}(t), \quad t \in [0,1],$$

and similarly

$$y(t) := \phi^{(u)\,1}(\eta)(t) - \phi^{(u)\,2}(\eta)(t) = 0, \quad t \in [0,1].$$

For all parametric representations $(r_n, u_n) \in \Pi(y_n)$ and $(r, u) \in \Pi(y)$ we have

$$\|u_n - u\|_{[0,1]} = 2u.$$

Hence $d_{M_1}(y_n, y) \geq 2u$ for all $n \geq 3$, which means that $d_{M_1}(y_n, y)$ does not converge to zero as $n \to \infty$. Since

$$d_{M_1}(y_n, y) \leq d_{M_1}(\phi^{(u)}(\eta_n), \phi^{(u)}(\eta))$$

(see Theorem 12.7.1 in Whitt [24]), we conclude that $d_{M_1}(\phi^{(u)}(\eta_n), \phi^{(u)}(\eta))$ does not converge to zero. Therefore the maximum functional $\phi^{(u)}$ is not continuous at η with respect to the standard M_1 topology. Since $\eta \in \Lambda$ we conclude that $\phi^{(u)}$ is not continuous on the set Λ.

5. Examples

Example 5.1. (A m-dependent process). Let $(Z_n)_{n \in \mathbb{Z}}$ be a sequence of i.i.d. unit Fréchet random variables, i.e. $\mathrm{P}(Z_n \leq x) = e^{-1/x}$ for $x > 0$. Hence Z_n is regularly varying with index $\alpha = 1$. Take a sequence of positive real numbers (a_n) such that $n\mathrm{P}(Z_1 > a_n) \to 1$ as $n \to \infty$. Now let

$$X_n = (Z_n, Z_{n-1}, \ldots, Z_{n-m}), \quad n \in \mathbb{Z}.$$

Then every X_n is also regularly varying with index $\alpha = 1$. By an application of Proposition 5.1 in Basrak et al. [4] it can be seen that the random process (X_n) is jointly regularly varying. Since the sequence (X_n) is m–dependent, it follows immediately that Conditions 2.1 and 2.2 hold (cf. Basrak and Krizmanić [3]).

Therefore (X_n) satisfies all the conditions of Theorem 4.3, and the corresponding partial maxima process $M_n(\,\cdot\,)$ converge in distribution in $D([0,1], \mathbb{R}_+^{m+1})$ to an extremal process $\widetilde{M}(\,\cdot\,)$ under the weak M_1 topology.

Next we show that $M_n(\,\cdot\,)$ does not converge in distribution under the standard M_1 topology on $D([0,1], \mathbb{R}_+^{m+1})$. This shows that the weak M_1 topology in Theorem 4.3 in general can not be replaced by the standard M_1 topology. In showing this we use, with appropriate modifications, a combination of arguments used by Basrak and Krizmanić [3] in their Example 4.1 and Avram and Taqqu [2] in their Theorem 1 (cf. also Example 5.1 in Krizmanić [16]).

For simplicity take $m = 1$. We have $M_n(t) = (M_n^1(t), M_n^2(t))$, where

$$M_n^1(t) = \bigvee_{j=1}^{\lfloor nt \rfloor} \frac{Z_j}{a_n} \quad \text{and} \quad M_n^2(t) = \bigvee_{j=1}^{\lfloor nt \rfloor} \frac{Z_{j-1}}{a_n}.$$

Let

$$V_n(t) := M_n^1(t) - M_n^2(t), \quad t \in [0,1].$$

The first step is to show that $V_n(\,\cdot\,)$ does not converge in distribution in $D([0,1], \mathbb{R}_+)$ endowed with the (standard) M_1 topology. For this, according to Skorohod [23] (cf. also Proposition 2 in Avram and Taqqu [2]), it suffices to show that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathrm{P}(\omega_\delta(V_n(\,\cdot\,)) > \epsilon) > 0 \tag{5.1}$$

for some $\epsilon > 0$, where

$$\omega_\delta(x) = \sup_{\substack{t_1 \leq t \leq t_2 \\ 0 \leq t_2 - t_1 \leq \delta}} M(x(t_1), x(t), x(t_2))$$

$(x \in D([0,1], \mathbb{R}_+), \delta > 0)$ and

$$M(x_1, x_2, x_3) = \begin{cases} 0, & \text{if } x_2 \in [x_1, x_3], \\ \min\{|x_2 - x_1|, |x_3 - x_2|\}, & \text{otherwise,} \end{cases}$$

Note that $M(x_1, x_2, x_3)$ is the distance form x_2 to $[x_1, x_3]$, and $\omega_\delta(x)$ is the M_1 oscillation of x.

Let $i' = i'(n)$ be the index at which $\max_{1 \le i \le n-1} Z_i$ is obtained. Fix $\epsilon > 0$ and introduce the events

$$A_{n,\epsilon} = \{Z_{i'} > \epsilon a_n\} = \Big\{ \max_{1 \le i \le n-1} Z_i > \epsilon a_n \Big\}$$

and

$$B_{n,\epsilon} = \{Z_{i'} > \epsilon a_n \text{ and } \exists\, l \neq 0, -i' \le l \le 1, \text{ such that } Z_{i'+l} > \epsilon a_n/4\}.$$

Using the facts that (Z_i) is an i.i.d. sequence and $n\mathrm{P}(Z_1 > c a_n) \to 1/c$ as $n \to \infty$ for $c > 0$ (which follows from the regular variation property of Z_1) we get

$$\lim_{n \to \infty} \mathrm{P}(A_{n,\epsilon}) = 1 - e^{-1/\epsilon} \tag{5.2}$$

and

$$\limsup_{n \to \infty} \mathrm{P}(B_{n,\epsilon}) \le \frac{4}{\epsilon^2} \tag{5.3}$$

(see Example 5.1 in Krizmanić [16]).

On the event $A_{n,\epsilon} \setminus B_{n,\epsilon}$ one has $Z_{i'} > \epsilon a_n$ and $Z_{i'+l} \le \epsilon a_n/4$ for every $l \neq 0$, $-i' \le l \le 1$, so that

$$\bigvee_{j=1}^{i'} \frac{Z_j}{a_n} = \frac{Z_{i'}}{a_n} > \epsilon$$

and

$$\max\bigg\{ \bigvee_{j=1}^{i'-1} \frac{Z_j}{a_n}, \bigvee_{j=1}^{i'} \frac{Z_{j-1}}{a_n}, \bigvee_{j=1}^{i'-1} \frac{Z_{j-1}}{a_n} \bigg\} \le \frac{\epsilon}{4}.$$

Therefore

$$V_n\Big(\frac{i'}{n}\Big) = \bigvee_{j=1}^{i'} \frac{Z_j}{a_n} - \bigvee_{j=1}^{i'} \frac{Z_{j-1}}{a_n} > \epsilon - \frac{\epsilon}{4} = \frac{3\epsilon}{4},$$

$$V_n\Big(\frac{i'-1}{n}\Big) = \bigvee_{j=1}^{i'-1} \frac{Z_j}{a_n} - \bigvee_{j=1}^{i'-1} \frac{Z_{j-1}}{a_n} \in \Big[-\frac{\epsilon}{4}, \frac{\epsilon}{4} \Big],$$

$$V_n\Big(\frac{i'+1}{n}\Big) = \bigvee_{j=1}^{i'+1} \frac{Z_j}{a_n} - \bigvee_{j=1}^{i'+1} \frac{Z_{j-1}}{a_n} = \frac{Z_{i'}}{a_n} - \frac{Z_{i'}}{a_n} = 0,$$

and these imply

$$\Big| V_n\Big(\frac{i'}{n}\Big) - V_n\Big(\frac{i'-1}{n}\Big) \Big| > \frac{3\epsilon}{4} - \frac{\epsilon}{4} = \frac{\epsilon}{2} \tag{5.4}$$

and

$$\Big| V_n\Big(\frac{i'+1}{n}\Big) - V_n\Big(\frac{i'}{n}\Big) \Big| > \frac{3\epsilon}{4}. \tag{5.5}$$

Note that on the set $A_{n,\epsilon} \setminus B_{n,\epsilon}$ it also holds that

$$V_n\Big(\frac{i'}{n}\Big) \notin \Big[V_n\Big(\frac{i'-1}{n}\Big), V_n\Big(\frac{i'+1}{n}\Big) \Big],$$

which implies that

$$
\begin{aligned}
M\Big(V_n\Big(\frac{i'-1}{n}\Big), V_n\Big(\frac{i'}{n}\Big), V_n\Big(\frac{i'+1}{n}\Big)\Big) \\
= \min\bigg\{\Big|V_n\Big(\frac{i'}{n}\Big) - V_n\Big(\frac{i'-1}{n}\Big)\Big|, \Big|V_n\Big(\frac{i'+1}{n}\Big) - V_n\Big(\frac{i'}{n}\Big)\Big|\bigg\}.
\end{aligned}
$$

Taking into account (5.4) and (5.5) we obtain

$$
\begin{aligned}
\omega_{2/n}(V_n(\,\cdot\,)) &= \sup_{\substack{t_1 \leq t \leq t_2 \\ 0 \leq t_2 - t_1 \leq 2/n}} M(V_n(t_1), V_n(t), V_n(t_2)) \\
&\geq M\Big(V_n\Big(\frac{i'-1}{n}\Big), V_n\Big(\frac{i'}{n}\Big), V_n\Big(\frac{i'+1}{n}\Big)\Big) > \frac{\epsilon}{2}
\end{aligned}
$$

on the event $A_{n,\epsilon} \setminus B_{n,\epsilon}$. Therefore, since $\omega_\delta(\,\cdot\,)$ is nondecreasing in δ, it holds that

$$
\begin{aligned}
\liminf_{n\to\infty} \mathrm{P}(A_{n,\epsilon} \setminus B_{n,\epsilon}) &\leq \liminf_{n\to\infty} \mathrm{P}(\omega_{2/n}(V_n(\,\cdot\,)) > \epsilon/2) \\
&\leq \lim_{\delta\to 0} \limsup_{n\to\infty} \mathrm{P}(\omega_\delta(V_n(\,\cdot\,)) > \epsilon/2). \qquad (5.6)
\end{aligned}
$$

Note that $x^2(1-e^{-1/x})$ tends to infinity as $x \to \infty$, and therefore we can find $\epsilon > 0$ such that $\epsilon^2(1-e^{-1/\epsilon}) > 4$, i.e.

$$
1 - e^{-1/\epsilon} > \frac{4}{\epsilon^2}.
$$

For this ϵ, by relations (5.2) and (5.3), it holds that

$$
\lim_{n\to\infty} \mathrm{P}(A_{n,\epsilon}) > \limsup_{n\to\infty} \mathrm{P}(B_{n,\epsilon}),
$$

i.e.

$$
\liminf_{n\to\infty} \mathrm{P}(A_{n,\epsilon} \setminus B_{n,\epsilon}) \geq \lim_{n\to\infty} \mathrm{P}(A_{n,\epsilon}) - \limsup_{n\to\infty} \mathrm{P}(B_{n,\epsilon}) > 0.
$$

Thus by (5.6) we obtain

$$
\lim_{\delta\to 0} \limsup_{n\to\infty} \mathrm{P}(\omega_\delta(V_n(\,\cdot\,)) > \epsilon/2) > 0
$$

and (5.1) holds, i.e. $V_n(\,\cdot\,)$ does not converge in distribution in $D([0,1],\mathbb{R}_+)$ endowed with the (standard) M_1 topology.

If $M_n(\,\cdot\,)$ would converge in distribution to some $\widetilde{M}(\,\cdot\,)$ in the standard M_1 topology on $D([0,1],\mathbb{R}_+^2)$, then using the fact that linear combinations of the coordinates are continuous in the same topology (cf. Theorem 12.7.1 and Theorem 12.7.2 in Whitt [24]) and the continuous mapping theorem, we would obtain that $V_n(\,\cdot\,) = M_n^1(\,\cdot\,) - M_n^2(\,\cdot\,)$ converges to $\widetilde{M}^1(\,\cdot\,) - \widetilde{M}^2(\,\cdot\,)$ in $D([0,1],\mathbb{R}_+)$ endowed with the standard M_1 topology, which is impossible, as is shown above.

Example 5.2. (Stochastic recurrence equation). Suppose the d–dimensional random process (X_n) satisfies a stochastic recurrence equation

$$
X_n = A_n X_{n-1} + B_n, \quad n \in \mathbb{Z},
$$

for some i.i.d. sequence $((A_n, B_n))$ of random $d \times d$ matrices A_n and d–dimensional vectors B_n, all with nonnegative components. Then it can be shown that under relatively general conditions the process (X_n) satisfies all conditions of Theorem 4.3

(see Example 4.2 in Basrak and Krizmanić [3]), and hence the corresponding partial maxima process $M_n(\,\cdot\,)$ converges in $D([0,1],\mathbb{R}_+^d)$ with the weak M_1 topology.

Example 5.3. (Multivariate squared GARCH process). We consider the multivariate GARCH (p, q) model with constant conditional correlations, which is defined as follows; see Fernández and Muriel [11]. Let $(\eta_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. random vectors with mean vector 0 and covariance matrix R such that $R(i,i)=1$ for all $i=1,\ldots,d$. The stochastic process $(X_n)_{n\in\mathbb{Z}}$ is a CCC-GARCH (p,q) process if it satisfies the following equations

$$\begin{aligned}
\delta(H_n) &= C + \sum_{i=1}^{p} A_i \delta(X_{n-i}X_{n-i}^T) + \sum_{j=1}^{q} B_j \delta(H_{n-j}), \\
D_n &= \operatorname{diag}(H_n(1,1)^{1/2}, H_n(2,2)^{1/2}, \ldots, H_n(d,d)^{1/2}), \\
H_n &= D_n R D_n, \\
X_n &= D_n \eta_n,
\end{aligned}$$

where for a square $d\times d$ matrix M, $\delta(M)$ denotes the vector whose entries are $\delta(M)(i)=M(i,i)$ for $i=1,\ldots,d$ (i.e. the main diagonal of M), and $\operatorname{diag}(M)$ denotes the diagonal matrix with the same diagonal as M. The vector C is assumed to be positive and the matrices A_i, B_j are assumed to be nonnegative for $i=1,\ldots,p$ and $j=1,\ldots,q$.

Assume now the matrices A_i, B_j have no zero rows, η_1 has a strictly positive density on $\mathbb{R}^d$ and for any $\gamma\geq 1$ there exists $h>1$ such that $\gamma^h \leq \mathrm{E}[(\eta_1^j)^{2h}] \leq \infty$ for all $j=1,\ldots,d$. Put

$$Y_n = (\delta(H_{n+1})^T, \ldots, \delta(H_{n-q+2})^T, \delta(X_n X_n^T)^T, \ldots, \delta(X_{n-p+2}X_{n-p+2}^T)^T)^T.$$

Then by Theorem 5 in [11] there exists $\alpha>0$ such that for every $x\in\mathbb{R}^{d(p+q-1)}\setminus\{0\}$, $\sum_{i=1}^{d(p+q-1)} x^i Y_1^i$ is regularly varying with index α. If α is not an even integer and η_1 has symmetric marginal distributions, from Corollary 6 in [11] we know that the process (X_n) is jointly regularly varying with index 2α. Further (X_n) is β–mixing (see Remark 4 in [11], cf. also Boussama [8]), and since β–mixing implies strong mixing (cf. Bradley [9]), Condition 2.1 holds. As in the one-dimensional case in Basrak et al. [4] it can be proved that (X_n) satisfies Condition 2.2.

The joint regular variation property and Conditions 2.1 and 2.2 transfer immediately to the squared CCC-GARCH (p,q) process

$$X_n^2 = ((X_n^1)^2, \ldots, (X_n^d)^2)$$

(with the remark that this process is jointly regularly varying with index α), and from Theorem 4.3, we conclude that the corresponding partial maxima process of (X_n^2) converges in distribution in $D([0,1],\mathbb{R}_+^d)$ to an extremal process under the weak M_1 topology.

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