# An Independence Test Based on Recurrence Rates 

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#### Abstract

A new test of independence between random elements is presented in this article. The test is based on a functional of the Cramér-von Mises type, which is applied to a $U$-process that is defined from the recurrence rates. Theorems of asymptotic distribution under $H_{0}$, and consistency under a wide class of alternatives are obtained. The results under contiguous alternatives are also shown. The test has a very good behaviour under several alternatives, which shows that in many cases there is clearly larger power when compared to other tests that are widely used in literature. In addition, the new test could be used for discrete or continuous time series.


Keywords: independence tests, recurrence rates, U-process. $62 \mathrm{H} 15,62 \mathrm{H} 20$

## 1 Introduction

Let $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right)$ i.i.d. sample of $(X, Y), X \in S_{X}$ and $Y \in S_{Y}$, where $S_{X}$ and $S_{Y}$ are metric spaces. When we have the following hypothesis test: $H_{0}: X$ and $Y$ are independent random elements, we are under the so called independent tests. The independence tests have been developed in the first instance for the $S_{X}=S_{Y}=\mathbb{R}$ case, based on the pioneering work of Galton [10] and Pearson [23] (this is the famous correlation test, which is widely used today). The limitations of this hypothesis test are well known and they have motivated several different proposals in this topic, such as the classical rank test (e.g. Spearman, [24], Kendall, [19] or Blomqvist, [6]). Another classic and intuitive result can be found in Hoeffding [15], where the test statistic is defined by $\iint\left(F_{X, Y}(x, y)-F_{X}(x) F_{Y}(y)\right)^{2} d F_{X, Y}(x, y)$, although it is not widely used. Independence between random vectors is addressed for the first time in Wilks [27]. Genest and Rémillard [16] propose a test based on copulas for continuous random variables. Kojadinovic and Holmes [18], generalize this result for random vectors using a Cramér-von Mises type statistic. Bilodeau and Lafaye de Micheaux [5], propose a test of independence between random vectors, each of which has a normal marginal distribution. Continuing in some sense this work, Beran et al. [4] propose a universally consistent test for random vectors, from empirical multidimensional distributions. Gretton et al. [12] propose a universally consistent test based on Hilbert-Schmidt norms. Another consistent test is proposed by Székely et al. [25, 26], which defines the concept of distance covariance. This test has its origin in [3] and it has since become very popular. It has been used and has had a considerable impact from the moment that it was proposed. More recently, Heller et al. [13] propose a test that in many cases has much more powerfull than the distance covariance test. In his monograph, Boglioni [7] compares several alternatives of these tests by means of intense work of power calculations. Because the tests proposed in Beran et al. [4] and Heller et al. [13] have very good performance under several alternatives, in Section 4 we will compare them with the test that we propose in our work.

Starting from another point of view, Eckman et al. [9] introduce the recurrence plot (RP). This is a very important graphical tool to understand the dynamics of a time series in high dimension. Eckman et al.'s [9] generated an appreciable amount of work and is currently applied in many different areas in which mathematical models are used, whether probabilistic or deterministic. The RP is a graphical tool that shows the recurrence in a time series $(X)$ and it is constructed using the recurrence matrix $R M(X)$ as defined by $R M_{i j}(X)=\mathbf{1}_{\left\{\left\|X_{i}-X_{j}\right\|<r\right\}}$, where $r$ is an appropriate parameter. The objective of this tool is to determine the patterns in a time series. The choice of $r$ is a key point to detect patterns and several suggestions have been made on how to appropriately find it. Marwan 21 gives a historical review of recurrence plots techniques, together with everything developed from them. However, the potential of these techniques has not yet been studied in depth from the point of view of mathematical statistics.

The main objective of this article is to propose a hypothesis test to detect dependence between two random elements, $X$ and $Y$, based on recurrence rates by using the
information of $\mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r\right\}}$ and $\mathbf{1}_{\left\{d\left(Y_{i}, Y_{j}\right)<s\right\}}$ for any values of $r$ and $s$. One advantage of our test is that instead of choosing appropriate values of $r$ and $s$, we use the information generated by both samples for all of the possible values of $r$ and $s$. In our test, $X$ and $Y$ can take values in any metric space. Therefore, our test can be used to test if $X$ and $Y$ are independent in the case where $X$ and $Y$ are random variables, random vectors or time series. We can then replace the norms by distances.

The rest of this paper is organized as follows. In Section 2, we give the definitions of recurrence rates for $X$, for $Y$ and for joint $(X, Y)$ and we propose the statistical procedure to make the decision between $H_{0}$ vs $H_{1}$. The statistics are based on a functional of the Cramér von-Mises type applied to a $U$-process defined from the recurrence rates of $X, Y$ and $(X, Y)$. We also give the theoretical results, which are the asymptotic distribution and consistency of the test statistic (Subsection 2.1), and the behavior under contiguous alternatives (Subsection 2.2). In Section 3, we describe how the test can be implemented, including a formula to obtain the statistic for the test. In Section 4, we use simulations to show the performance of the test against others by power comparison in the cases where $X$ and $Y$ are random variables or random vectors. We also compute power in the case where $X$ and $Y$ are discrete and continuous time series. Like Heller et al.'s $[13$ test, our test is based on distances between the elements of the sample. Likewise, our test had very good performance under several alternatives. Our concluding remarks are given in Section 5. Appendix gives the proofs of the results that are established in Section 2.

## 2 Test approach and theoretical results

Given $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right)$ i.i.d. sample of $(X, Y)$ where $X \in S_{X}, Y \in S_{Y}$ where $S_{X}$ and $S_{Y}$ are metric spaces, and given $r, s>0$. To simplify the notation and without risk of confusion, we will use the same letter $d$ for the distance function in both metric spaces $S_{X}$ and $S_{Y}$.

We define the recurrence rate for the sample of $X$ and $Y$ as

$$
\begin{aligned}
R R_{n}^{X}(r) & :=\frac{1}{n^{2}-n} \sum_{i \neq j} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r\right\}} \\
R R_{n}^{Y}(s) & :=\frac{1}{n^{2}-n} \sum_{i \neq j} \mathbf{1}_{\left\{d\left(Y_{i}, Y_{j}\right)<s\right\}}
\end{aligned}
$$

respectively, and the joint recurrence rate for $(X, Y)$ as

$$
R R_{n}^{X, Y}(r, s):=\frac{1}{n^{2}-n} \sum_{i \neq j} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right\}}
$$

We define $p_{X}(r):=P\left(d\left(X_{1}, X_{2}\right)<r\right)$ the probability that the distance between any two elements of the sample $X$ is less than $r$. Similarly, we define the probability between
three points as $p_{X}^{(3)}(r):=P\left(d\left(X_{1}, X_{2}\right)<r, d\left(X_{1}, X_{3}\right)<r\right)$ and analogously $p_{Y}$ and $p_{Y}^{(3)}$.

We also need to define $p_{X, Y}(r, s):=P\left(d\left(X_{1}, X_{2}\right)<r, d\left(Y_{1}, Y_{2}\right)<s\right)$.
The strong law of large numbers for $U$-statistics ([14]) allows us to affirm that for any $r, s>0$,

$$
\begin{equation*}
R R_{n}^{X}(r) \xrightarrow{\text { a.s. }} p_{X}(r), \quad R R_{n}^{Y}(s) \xrightarrow{\text { a.s. }} p_{Y}(s) \text { and } R R_{n}^{X, Y}(r, s) \xrightarrow{\text { a.s. }} p_{X, Y}(r, s) . \tag{1}
\end{equation*}
$$

We want to test $H_{0}: X$ and $Y$ are independent, against $H_{1}: H_{0}$ does not hold.
If $H_{0}$ is true, then $p_{X, Y}(r, s)=p_{X}(r) p_{Y}(s)$ for all $r, s>0$, and we expect that if $n$ is large, $R R_{n}^{X, Y}(r, s) \cong R R_{n}^{X}(r) R R_{n}^{Y}(s)$ for any $r, s>0$. Then, we propose to build the test statistic, to work with the process $\left\{E_{n}(r, s)\right\}_{r, s>0}$ where

$$
\begin{equation*}
E_{n}(r, s):=\sqrt{n}\left(R R_{n}^{X, Y}(r, s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)\right) . \tag{2}
\end{equation*}
$$

Therefore, it is natural to reject $H_{0}$ when $T_{n}>c$ where

$$
\begin{equation*}
T_{n}:=n \int_{0}^{+\infty} \int_{0}^{+\infty}\left(R R_{n}^{X, Y}(r, s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)\right)^{2} d G(r, s) \tag{3}
\end{equation*}
$$

where $c$ is a constant and $G$ is a distribution function.
Throughout this work, we use the notation $\phi$ and $\varphi$ for distribution and density function of $N(0,1)$ random variable respectively, and for each $m$, the set

$$
I_{m}^{n}:=\left\{\left(i_{1}, \ldots, i_{m}\right): i_{j} \neq i_{k} \text { for all } j \neq k, \text { and } i_{j} \in\{1, \ldots, n\} \text { for all } j=1, \ldots, m\right\}
$$

Now we will formulate the asymptotic results of our test statistic. First, we will show a result that guarantees the asymptotic distribution of $T_{n}$ under $H_{0}$. We will also present a result that establishes a consistency of our test under a wide class of alternatives. Second, we will analyze the asymptotic bias when we consider contiguous alternatives.

### 2.1 Asymptotic results under $H_{0}$ and consistency

We start with the next lemma, in which we obtain the formula for the asymptotic autocovariance function of the process $\left\{E_{n}(r, s)\right\}_{r, s>0}$ under $H_{0}$.

Lemma 1. Given $r, r^{\prime}, s, s^{\prime}>0$, and $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right)$ i.i.d. in $S_{X} \times S_{Y}$ where $X$ and $Y$ are independent, then

$$
\begin{gather*}
\lim _{n \rightarrow+\infty} \mathbb{C O V}\left(E_{n}(r, s), E_{n}\left(r^{\prime}, s^{\prime}\right)\right)= \\
4\left(p_{X}^{(3)}\left(r \wedge r^{\prime}\right)-p_{X}(r) p_{X}\left(r^{\prime}\right)\right)\left(p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)-p_{Y}(s) p_{Y}\left(s^{\prime}\right)\right) . \tag{4}
\end{gather*}
$$

The following lemma will be useful to reduce asymptotic convergence of the process $\left\{E_{n}(r, s)\right\}_{r, s>0}$ to the convergence of an approximate $U$ - process that we will call $\left\{E_{n}^{\prime}(r, s)\right\}_{r, s>0}$ and is defined as follows

$$
\begin{gather*}
E_{n}^{\prime}(r, s):=\frac{\sqrt{n}}{n(n-1)(n-2)(n-3)} \times \\
\sum_{(i, j, k, h) \in I_{4}^{n}}\left(\mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right\}}-\mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{h}, Y_{k}\right)<s\right\}}\right) \tag{5}
\end{gather*}
$$

Lemma 2. Given $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right)$ i.i.d. in $S_{X} \times S_{Y}$, then

$$
E_{n}(r, s)=\sqrt{n}\left(R R_{n}^{X, Y}(r, s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)\right)=E_{n}^{\prime}(r, s)-H_{n}(r, s)
$$

where

$$
0 \leq H_{n}(r, s) \leq \frac{4}{\sqrt{n}} \text { for all } r, s>0
$$

To obtain the weak convergence of the process $\left\{E_{n}(r, s)-\mathbb{E}\left(E_{n}(r, s)\right)\right\}_{r, s>0}$ to a centered Gaussian process (therefore the asymptotic distribution of the statistics $T_{n}$ defined in (3) is determined), we will use Theorem 4.10 obtained by Arcones \& Giné [1]:

Let $(S, \mathcal{S}, P)$ be a probability space, and for all $i \in \mathbb{N}, X_{i}: S \rightarrow S$ are i.i.d. sequence with $\mathcal{L}\left(X_{i}\right)=P$. Given $m$, let $\mathcal{F}$ be a class of measurable functions on $S^{m}$, the $U$-process based on $P$ and indexed by $\mathcal{F}$ is

$$
U_{m}^{n}(f)=\frac{(n-m)!}{m!} \sum_{\left(i_{1}, \ldots, i_{m}\right) \in I_{m}^{n}} f\left(X_{i_{1}}, X_{i_{2}}, \ldots, X_{i_{m}}\right)
$$

where $f \in \mathcal{F}$.
Given $\varepsilon>0$, assume that exists $\mathcal{L}=\left\{l_{1}, l_{2}, \ldots, l_{v}\right\}, \mathcal{U}=\left\{u_{1}, u_{2}, \ldots, u_{v}\right\}$ such that $\mathcal{L}, \mathcal{U} \subset L^{2}$ and for all
$f \in \mathcal{F}$, exists $l_{f} \in \mathcal{L}$ and $u_{f} \in \mathcal{U}$ where $l_{f} \leq f \leq u_{f}$ a.s. and $\mathbb{E}\left(u_{f}-l_{f}\right)^{2}<\varepsilon^{2}$.

$$
\begin{equation*}
N_{[]}^{(2)}\left(\varepsilon, \mathcal{F}, P^{m}\right)=\min \{v:(6) \text { holds }\} \tag{7}
\end{equation*}
$$

Theorem (Arcones \& Giné 1993)

If

$$
\begin{equation*}
\int_{0}^{+\infty}\left(\log N_{[]}^{(2)}\left(\varepsilon, \mathcal{F}, P^{m}\right)\right)^{1 / 2} d \varepsilon<+\infty \tag{8}
\end{equation*}
$$

then

$$
\begin{equation*}
\mathcal{L}\left(\sqrt{n}\left(U_{m}^{n}-P^{m}\right) f\right) \xrightarrow{w} \mathcal{L}\left(m G_{p} \circ P^{m-1} f\right) \quad \text { in } l^{\infty}(\mathcal{F}) \tag{9}
\end{equation*}
$$

where $G_{P}$ is the Brownian bridge associated with $P$.
Convergence in the space $l^{\infty}(\mathcal{F})$, is in the sense of Hoffmann-Jørgensen, see ([11]).

Theorem 3. Given $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right)$ i.i.d. in $S_{X} \times S_{Y}$. If the distribution functions of $d\left(X_{1}, X_{2}\right)$ and $d\left(Y_{1}, Y_{2}\right)$ are continuous, then

$$
\begin{equation*}
\left\{E_{n}(r, s)-\mathbb{E}\left(E_{n}(r, s)\right)\right\}_{r, s>0} \xrightarrow{w}\{E(r, s)\}_{r, s>0} \tag{10}
\end{equation*}
$$

where $\{E(r, s)\}_{r, s>0}$ is a centered Gaussian process.
Remark 1. Observe that our process $\left\{E_{n}(r, s)\right\}_{r, s>0}$ lies in $L^{2}(d G)$ (because $G$ is a probability measure). Therefore, our test statistic $T_{n}$ is $\left\|\{E(r, s)\}_{r, s>0}\right\|$, thus, the functional is continuous.

Remark 2. Given $r, s>0$ and $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right) \in \mathbb{R}^{2}$ i.i.d. sample of $(X, Y)$ where the marginals $X, Y$ are $N(0,1)$ independent. Then

$$
\sqrt{n}\left(R R_{n}^{X, Y)}(r, s)-R R_{n}(r) R R_{n}^{Y}(s)\right) \xrightarrow{w} N\left(0, \sigma_{X, Y}^{2}(r, s)\right)
$$

where

$$
\begin{gather*}
\sigma_{X, Y}^{2}(r, s)=4\left(\int_{-\infty}^{+\infty}(\phi(x+r)-\phi(x-r))^{2} \varphi(x) d x-(2 \phi(r / \sqrt{2})-1)^{2}\right) \times \\
\left(\int_{-\infty}^{+\infty}(\phi(x+s)-\phi(x-s))^{2} \varphi(x) d x-(2 \phi(s / \sqrt{2})-1)^{2}\right) \tag{11}
\end{gather*}
$$

If $d\left(X_{1}, X_{2}\right)$ and $d\left(Y_{1}, Y_{2}\right)$ are not independent, then our test is consistent.
Theorem 4. Given $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right)$ i.i.d. in $S_{X} \times S_{Y}$. If $d G(r, s)=$ $g(r, s) d r d s, g(r, s)>0$ for all $r, s>0$, and $d\left(X_{1}, X_{2}\right), d\left(Y_{1}, Y_{2}\right)$ are continuous and not independent random variables, then $T_{n} \xrightarrow{P}+\infty$ as $n \rightarrow+\infty$.

The next corollary follows from Theorem 4 .
Corollary 1. If $(X, Y) \sim N(0, \Sigma)$, where $X$ and $Y$ are not independent, and $d G(r, s)=$ $g(r, s) d r d s, g(r, s)>0$ for all $r, s>0$, then $T_{n} \xrightarrow{P}+\infty$ as $n \rightarrow+\infty$.

Remark 3. Consider $\left(X_{1}, Y_{1}\right)$, $\left(X_{2}, Y_{2}\right)$ in $\mathbb{R}^{2}$ i.i.d. with joint density $f_{X, Y}$ and joint distribution $F$ such that $\left|X_{1}-X_{2}\right|$ and $\left|Y_{1}-Y_{2}\right|$ are independent.

Then

$$
\begin{gathered}
\alpha(r, s):=P\left(\left|X_{1}-X_{2}\right| \leq r,\left|Y_{1}-Y_{2}\right| \leq s\right)= \\
\iint_{\mathbb{R}^{2}} f_{X, Y}\left(x_{1}, y_{1}\right) d x_{1} d y_{1} \int_{x_{1}-r}^{x_{1}+r} d x_{2} \int_{y_{1}-s}^{y_{1}+s} f_{X, Y}\left(x_{2}, y_{2}\right) d y_{2}= \\
\iint_{\mathbb{R}^{2}} P\left(x_{1}-r \leq X_{1} \leq x_{1}+r, y_{1}-s \leq Y_{2} \leq y_{1}+s\right) f_{X, Y}\left(x_{1}, y_{1}\right) d x_{1} d y_{1}= \\
\mathbb{E}(F(X+r, Y+s)-F(X+r, Y-s)-F(X-r, Y+s)+F(X-r, Y-s))
\end{gathered}
$$

Similarly,

$$
\beta(r, s):=P\left(\left|X_{1}-X_{2}\right| \leq r\right) P\left(\left|Y_{1}-Y_{2}\right| \leq s\right)=
$$

$$
\mathbb{E}\left(F_{X}(X+r)-F_{X}(X-r)\right) \mathbb{E}\left(F_{Y}(Y+r)-F_{Y}(Y-r)\right)
$$

Then, $\alpha(r, s)=\beta(r, s)$ for all $r, s>0$.
Of course, it could happen that condition $\alpha(r, s)=\beta(r, s)$ for all $r, s>0$ is fulfilled, and nevertheless $X$ and $Y$ are not independent. This is the restricted type of distributions that do not satisfy the conditions of our consistency theorem.

### 2.2 Contiguous alternatives

In this subsection we will analyze the behavior of this test under contiguous alternatives.
More explicitly, given $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right)$ i.i.d. in $\mathbb{R}^{p} \times \mathbb{R}^{q}$, consider

$$
H_{0}: f_{X, Y}(x, y)=f_{X}(x) f_{Y}(y) \quad \text { for all }(x, y)
$$

(i.e. $X$ and $Y$ are independent), vs

$$
H_{n}: f_{X, Y}(x, y)=f_{X, Y}^{(n)}(x, y) \quad \text { for all }(x, y)
$$

where $f_{X, Y}^{(n)}(x, y)=c_{n}(\delta) f_{X}(x) f_{Y}(y)\left(1+\frac{\delta}{2 \sqrt{n}} k_{n}(x, y)\right)^{2}, \delta>0, c_{n}(\delta)$ is a constant such that $f_{X, Y}^{(n)}(x, y)$ be a density, and the functions $k_{n}$ verify the conditions (i) and (ii) that are given below:

Define $L_{0}^{2}=L^{2}\left(d F_{0}\right)$ for $d F_{0}(x, y)=f_{X}(x) f_{Y}(y) d x d y$, the distribution function of ( $X, Y$ ) under $H_{0}$, analogously define $L_{0}^{1}$.
(i) Exists a function $K \in L_{0}^{1}$ such that $k_{n} \leq K$ for all $n$
(ii) Exists $k \in L_{0}^{2}$ such that $k_{n} \xrightarrow{L_{0}^{2}} k,\|k\|=1$.

It can be proven that conditions (i) and (ii) imply contiguity (Cabaña 8]).
The $\delta$ coefficient is introduced so that $\|k\|=1$. The function $\delta k$ is called asymptotic drift.

We will show in the following lines that under $H_{n}$, the process $\left\{E_{n}(r, s)\right\}_{r, s>0}$ has the same asymptotic limit as under $H_{0}$ plus a deterministic drift.

We use the notation $\mathbb{E}^{(n)}(T)$ and $P^{(n)}((X, Y) \in A)$ for the expectation value of $T$, and the probability of the set $\{(X, Y) \in A\}$ under $H_{n}$ respectively. Analogously we use $\mathbb{E}^{(0)}(T)$ and $P^{(0)}((X, Y) \in A)$ under $H_{0}$.

## Proposition 1.

Under $H_{n}$

$$
\mathbb{E}^{(n)}\left(E_{n}(r, s)\right) \rightarrow \delta \mu(r, s) \quad \text { as } n \rightarrow+\infty \quad \text { for all } r, s>0
$$

where $\mu(r, s)=$

$$
\begin{equation*}
\iiint \int_{A_{r, s}}\left(k\left(x_{1}, y_{1}\right)+k\left(x_{2}, y_{2}\right)\right) f_{X}\left(x_{1}\right) f_{Y}\left(y_{1}\right) f_{X}\left(x_{2}\right) f_{Y}\left(y_{2}\right) d x_{1} d x_{2} d y_{1} d y_{2} \tag{12}
\end{equation*}
$$

and $A_{r, s}:=\left\{\left(x_{1}, y_{1}, x_{2}, y_{2}\right) \in \mathbb{R}^{2 p+2 q}: d\left(x_{1}, x_{2}\right)<r, d\left(y_{1}, y_{2}\right)<s\right\}$.

With a little more work, using the Le Cam third lemma (Le Cam \& Yang, [20] and Oosterhoff \& Van Zwet, [22]) it is possible to prove that under $H_{n}$,

$$
\left\{E_{n}(r, s)\right\}_{r, s>0} \xrightarrow{w}\{E(r, s)+\delta \mu(r, s)\}_{r, s>0}
$$

where $\{E(r, s)\}_{r, s>0}$ is the limit process under $H_{0}$ and $\mu(r, s)=$

$$
\iiint \int_{A_{r, s}}\left(k\left(x_{1}, y_{1}\right)+k\left(x_{2}, y_{2}\right)\right) f_{X}\left(x_{1}\right) f_{Y}\left(y_{1}\right) f_{X}\left(x_{2}\right) f_{Y}\left(y_{2}\right) d x_{1} d x_{2} d y_{1} d y_{2}
$$

Therefore, under $H_{n}$

$$
T_{n} \xrightarrow{w} \int_{0}^{+\infty} \int_{0}^{+\infty}(E(r, s)+\delta \mu(r, s))^{2} d G(r, s) .
$$

## 3 Implementation of the test

## 3.1 $X$ and $Y$ are random variables

In the case where $X$ and $Y$ are continuous random variables, we observe that $X$ and $Y$ are independent; it is equivalent to say that $X^{\prime}=\phi^{-1}\left(F_{X}(X)\right)$ and $Y^{\prime}=\phi^{-1}\left(F_{Y}(Y)\right)$ are independent, where $F_{X}$ and $F_{Y}$ are the distribution functions of $X$ and $Y$, respectively. If we apply the test procedure to $X^{\prime}$ and $Y^{\prime}$, then we have the advantage that now the variables are on the same scale and each has a normal centered distribution that approximates to the hypotheses of Remark 2. In addition, in this case the formula (11) for $\sigma_{X^{\prime}, Y^{\prime}}^{2}(r, s)$ is completely determined. Another additional advantage is that under $H_{0}$ ( $X^{\prime}$ and $Y^{\prime}$ are independent and $N(0,1)$ ), for small values of $n$, we can calculate the critical values at $5 \%$ or another level because we will know the distribution of $T_{n}$ under $H_{0}$. Where $X$ and $Y$ are random vectors, the same transformation can be applied in each coordinate. To give an idea of the variability of the process $\left\{E_{n}(r, s)\right\}_{r, s>0}$, in Figure 1 we show the values of $\sigma_{X^{\prime}, Y^{\prime}}^{2}(r, r)$ for different values of $r$. The maximum is 0.06409 and is reached in $r=1.3488$.

### 3.2 General case

As happens in many statistical applications, we are able to have a moderately small sample size. However, an erroneous decision can be made if the researcher uses the p -value (or the critical value) obtained through the asymptotic distribution to make the decision in the hypothesis test. Therefore, when we have a sample of size $n$, it is preferable to estimate the p-value (or the critical value) by estimating the distribution of the $T_{n}$ for this value of $n$. Moreover, in our test, the asymptotic distribution is difficult to obtain because we need to conduct several simulations of a centered continuous Gaussian processes indexed in $D=(0,+\infty) \times(0,+\infty)$. We then need to calculate the integral in D.

To calculate the p-value or the critical value of the test for fixed $n$ we can proceed as explained in the following lines. Fixed $n$, if $H_{0}$ is true, we do not know the distribution of


Figure 1: $\sigma_{X, Y}^{2}(r, r)$ in function of $r$, in the case $X$ and $Y$ are independent and $N(0,1)$.
$T_{n}$, but given the observed value from our sample that we call $t_{\text {obs }}$, we could generate, by a permutation procedure, a large sample of $T_{n}$ with which we can estimate $P\left(T_{n} \geq t_{o b s}\right)$. Given $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right)$ i.i.d. sample of $(X, Y)$. Observe that the distribution of $T_{n}$ depends of the joint distribution of $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right)$. If $H_{0}$ is true, and if we consider any $\sigma:\{1,2,3, \ldots, n\} \rightarrow\{1,2,3, \ldots, n\}$ permutation of the index set, then the joint distribution of $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right)$ and the joint distribution of $\left(X_{\sigma(1)}, Y_{1}\right),\left(X_{\sigma(2)}, Y_{2}\right), \ldots,\left(X_{\sigma(n)}, Y_{n}\right)$ are the same. Consider $S(n)=$ $\left\{\sigma_{1}, \sigma_{2}, \ldots, \sigma_{n!}\right\}$ the set of all the permutation $\sigma:\{1,2, \ldots, n\} \rightarrow\{1,2, \ldots, n\}$. Suppose that the sample $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right)$ is fixed and consider $Z$ defined by $Z=$ $T_{n}\left(\left(X_{\sigma_{i}(1)}, Y_{1}\right), \ldots,\left(X_{\sigma_{i}(n)}, Y_{n}\right)\right)$ with probability $1 / n$ ! for each $i=1,2, \ldots, n!$. If we take $Z_{1}, Z_{2}, \ldots, Z_{m}$ i.i.d. sample of $Z$, we can estimate the value of $p_{n}=P\left(T_{n} \geq t_{\text {obs }}\right)$ simply by using $\widehat{p}_{n}^{(m)}=\frac{1}{m} \sum_{i=1}^{m} \mathbf{1}_{\left\{Z_{i} \geq t_{\text {obs }}\right\}}$ for $m$ large enough. Define the random variables $B_{i}=\sum_{j=1}^{m} \mathbf{1}_{\left\{Z_{j}=T_{n}\left(\left(X_{\sigma_{i}(1)}, Y_{1}\right), \ldots,\left(X_{\sigma_{i}(n)}, Y_{n}\right)\right)\right\}}$. for $i=1,2,, \ldots, n!$. Observe that $B_{i}$ is distributed as $\operatorname{Bin}(m, 1 / n!)$ for each $i=1,2, \ldots, n!$. Then

$$
\widehat{p}_{n}^{(m)}=\frac{1}{m} \sum_{j=1}^{m} \mathbf{1}_{\left\{Z_{j} \geq t_{o b s}\right\}}=\frac{1}{m} \sum_{i=1}^{n!} B_{i} \mathbf{1}_{\left\{T _ { n } \left(\left(X_{\left.\left.\left.\sigma_{i}(1), Y_{1}\right), \ldots,\left(X_{\sigma_{i}(n)}, Y_{n}\right)\right) \geq t_{o b s}\right\}}, ~\right.\right.\right.}
$$

converges as $m \rightarrow+\infty$ to $\frac{1}{n!} \sum_{i=1}^{n!} \mathbf{1}_{\left\{T_{n}\left(\left(X_{\sigma_{i}(1)}, Y_{1}\right), \ldots,\left(X_{\left.\left.\left.\sigma_{i}(n), Y_{n}\right)\right) \geq t_{o b s}\right\}} \text { a.s. If we now con- }\right.\right.\right.}$ sider that $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{n}, Y_{n}\right)$ are random elements that can take an expected value, and we obtain (using dominated convergence) $\mathbb{E}\left(\widehat{p}_{n}^{(m)}\right) \underset{m \rightarrow+\infty}{\rightarrow} p_{n}$, then $\widehat{p}_{n}^{(m)}$ is an asymptotically unbiased estimator of $p_{n}$.

### 3.3 A simple method to choose the weight function

The performance of our test depends on the choice of the weight function. The weight function can be chosen by the researcher in each particular case. According to Theorem 4. we can use any function $G$ such that $d G(r, s)=g(r, s) d r d s$ where $g(r, s)>0$ for any $r, s>0$. It would be interesting to study some kind of optimality in the choice of the $G$ function, under certain kind of alternatives. Consequently, we propose a simple method to chose the $G$ function. As will be seen in the next section, this simple choice of $G$, has very good performance under the alternatives studied in this work.

Define $d G(r, s)=g_{1}(r) g_{2}(s) d r d s$, where $g_{1}$ and $g_{2}$ are Gaussian densities. In the case of $g_{1}$ we can use $\mu_{1}=\mathbb{E}\left(d\left(X_{1}, X_{2}\right)\right)$ and $\sigma_{1}^{2}=\mathbb{V}\left(d\left(X_{1}, X_{2}\right)\right)$. The values of $\mu_{1}$ and $\sigma_{1}$ can easily be estimated by the sample $d\left(X_{i}, X_{j}\right)$ with $(i, j) \in I_{2}^{n}$. We can proceed similarly with the election of $\mu_{2}$ and $\sigma_{2}$ for the density $g_{2}$. In this way, we give more weight in the neighbourhoods of the average distance between two independent observations $X_{1}$ and $X_{2}$ for $g_{1}$, and analogously for $g_{2}$. Meanwhile, observe that we can avoid the problem of choosing $G$, if we use $T_{n}^{\prime}=\sqrt{n} \sup _{r, s>0}\left|R R_{n}^{X, Y}(r, s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)\right|$ to test independence because all of the theoretical results obtained in this work for $T_{n}$ are still valid for $T_{n}^{\prime}$.

### 3.4 Computing the statistic

In this subsection we will see how to calculate the statistic $T_{n}$. We will consider the case in which $d G(r, s)=g_{1}(r) g_{2}(s) d r d s$ where $g_{1}$ and $g_{2}$ are density functions with $G_{1}$ and $G_{2}$ their respective distribution functions.

$$
\begin{gather*}
\int_{0}^{+\infty} \int_{0}^{+\infty}\left(R R_{n}^{X, Y}(r, s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)\right)^{2} g_{1}(r) g_{2}(s) d r d s= \\
\int_{0}^{+\infty} \int_{0}^{+\infty}\left[R R_{n}^{X, Y}(r, s)\right]^{2} g_{1}(r) g_{2}(s) d r d s+ \\
\int_{0}^{+\infty}\left[R R_{n}^{X}(r)\right]^{2} g_{1}(r) d r \int_{0}^{+\infty}\left[R R_{n}^{Y}(s)\right]^{2} g_{2}(s) d s \\
-2 \int_{0}^{+\infty} \int_{0}^{+\infty} R R_{n}^{X, Y}(r, s) R R_{n}^{X}(r) R R_{n}^{Y}(s) g_{1}(r) g_{2}(s) d r d s:=A_{n}+B_{n}-2 C_{n} \tag{13}
\end{gather*}
$$

To simplify the notation and for the rest of this section, we will call $N=n(n-1)$. We will also index $d\left(X_{i}, X_{j}\right)$ with $(i, j) \in I_{2}^{n}$ in the form $Z_{1}, Z_{2}, \ldots, Z_{N}$. Analogously, we use the same indexes as $Z^{\prime} s, T_{1}, T_{2}, \ldots, T_{N}$ to the values $d\left(Y_{i}, Y_{j}\right)$. We will also call $Z_{1}^{*}, Z_{2}^{*}, \ldots, Z_{N}^{*}$ to the order statistics of $Z^{\prime} s$, and analogously $T_{1}^{*}, T_{2}^{*}, \ldots, T_{N}^{*}$.

$$
\begin{aligned}
& \int_{0}^{+\infty}\left[R R_{n}^{X}(r)\right]^{2} g_{1}(r) d r=\frac{1}{N^{2}} \sum_{i \neq j} \sum_{k \neq h} \int_{0}^{+\infty} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(X_{h}, X_{k}\right)<r\right\}} g_{1}(r) d r= \\
& \frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{0}^{+\infty} \mathbf{1}_{\left\{Z_{i}<r, Z_{j}<r\right\}} g_{1}(r) d r=\frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N}\left(1-G_{1}\left(\max \left\{Z_{i}, Z_{j}\right\}\right)\right)= \\
& \frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N}\left(1-G_{1}\left(\max \left\{Z_{i}^{*}, Z_{j}^{*}\right\}\right)\right)=1-\frac{1}{N^{2}} \sum_{i=1}^{N}\left(2 \sum_{j=1}^{i-1} G_{1}\left(Z_{i}^{*}\right)+G_{1}\left(Z_{i}^{*}\right)\right)= \\
& 1-\frac{1}{N^{2}} \sum_{i=1}^{N}\left(2(i-1) G_{1}\left(Z_{i}^{*}\right)+G_{1}\left(Z_{i}^{*}\right)\right)=1-\frac{1}{N^{2}} \sum_{i=1}^{N}(2 i-1) G_{1}\left(Z_{i}^{*}\right) .
\end{aligned}
$$

Analogously

$$
\int_{0}^{+\infty}\left[R R_{n}^{Y}(s)\right]^{2} g_{2}(s) d s=1-\frac{1}{N^{2}} \sum_{i=1}^{N}(2 i-1) G_{2}\left(T_{i}^{*}\right) .
$$

Then

$$
\begin{gather*}
B_{n}=\left(1-\frac{1}{N^{2}} \sum_{i=1}^{N}(2 i-1) G_{1}\left(Z_{i}^{*}\right) .\right)\left(1-\frac{1}{N^{2}} \sum_{i=1}^{N}(2 i-1) G_{2}\left(T_{i}^{*}\right) .\right)  \tag{14}\\
A_{n}=\int_{0}^{+\infty} \int_{0}^{+\infty}\left[R R_{n}^{X, Y}(r, s)\right]^{2} g_{1}(r) g_{2}(s) d r d s= \\
\frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{0}^{+\infty} \mathbf{1}_{\left\{Z_{i}<r, Z_{j}<r\right\}} g_{1}(r) d r \int_{0}^{+\infty} \mathbf{1}_{\left\{T_{i}<s, T_{j}<s\right\}} g_{2}(s) d s= \\
\frac{1}{N^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N}\left(1-G_{1}\left(\max \left\{Z_{i}, Z_{j}\right\}\right)\right)\left(1-G_{2}\left(\max \left\{T_{i}, T_{j}\right\}\right)\right) .  \tag{15}\\
C_{n}=\int_{0}^{+\infty} \int_{0}^{+\infty} R R_{n}^{X, Y}(r, s) R R_{n}^{X}(r) R R_{n}^{Y}(s) g_{1}(r) g_{2}(s) d r d s= \\
\frac{1}{N^{3}} \sum_{i \neq j} \sum_{k \neq h} \sum_{l \neq m} \int_{0}^{+\infty} \int_{0}^{+\infty} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s, d\left(X_{h}, X_{k}\right)<r, d\left(Y_{l}, Y_{m}\right)<s\right\}} g_{1}(r) g_{2}(s) d r d s= \\
\frac{1}{N^{3}} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N}\left(1-G_{1}\left(\max \left\{Z_{i}, Z_{j}\right\}\right)\right)\left(1-G_{2}\left(\max \left\{T_{i}, T_{k}\right\}\right)\right) . \tag{16}
\end{gather*}
$$

Then

$$
\begin{equation*}
T_{n}=n\left(A_{n}+B_{n}-2 C_{n}\right) \tag{17}
\end{equation*}
$$

where $A_{n}, B_{n}$ and $C_{n}$ are given in the formulas (15), (14) and (16) respectively.

## 4 A simulation study

In this section we will compare the performance of our test with respect to other recently proposed tests that have good performance. Tables 1 to 6 show the power of our test for different functions $G$ and also for other tests, for $n=30, n=50$ and $n=80$ sample sizes. All power calculations that we have considered have been calculated at the significance level of $5 \%$. The calculations were made using 17) and taking as a function of weights $d G(r, s)=g_{1}(r) g_{2}(s) d r d s$ where $g_{1}=g_{2}=g$ is the density function of a $N\left(\mu, \sigma^{2}\right)$ random variable for some values of $\mu$ and $\sigma^{2}$, except for the last column, where we take the functions $g_{1}$ and $g_{2}$ suggested in Subsection 3.3. We will compare the power of our test with respect to the test proposed in Heller et al. [13] (which we will call HHG), the test of covariance distance proposed in Székely et al. [25] (which we will call DCOV) and the test proposed in Gretton et al. [12] (which we will call HSIC). In Subsection 4.1 we will consider the case in which $X$ and $Y$ are random variables; that is, $(X, Y) \in \mathbb{R}^{2}$. Meanwhile, in Subsection 4.2 we consider examples in dimensions greater than two. Lastly, in Subsection 4.3 we simulate discrete and continuous time series for certain alternatives and representspower as a function of sample size. In this case, we take the functions $g_{1}$ and $g_{2}$ suggested in Subsection 3.3.

## 4.1 $\quad X$ and $Y$ are random variables

Table 3 considers Heller et al.'s [13] tests, which are called "Parabola", "Two parabolas", "Circle", "Diamond", "W-shape" and "Four independent clouds" and which are defined as follows:
Parabola: $X \sim U(-1,1), Y=\left(X^{2}+U(0,1)\right) / 2$.
Two parabolas: $X \sim U(-1,1), Y=\left(X^{2}+U(0,1) / 2\right)$ with probability $1 / 2$ and $Y=-\left(X^{2}+U(0,1) / 2\right)$ with probability $1 / 2$.
Circle: $U \sim U(-1,1), X=\sin (\pi U)+N(0,1) / 8, Y=\cos (\pi U)+N(0,1) / 8$.
Diamond: $U_{1}, U_{2} \sim U(-1,1)$ independent, $X=\sin (\theta) U_{1}+\cos (\theta) U_{2}, Y=-\sin (\theta) U_{1}+$ $\cos (\theta) U_{2}$ for $\theta=\pi / 4$.
W-shape: $U \sim U(-1,1), U_{1}, U_{2} \sim U(0,1)$ independent. $X=U+U_{1} / 3$ and $Y=$ $4\left(U^{2}-1 / 2\right)^{2}+U_{2} / n$.
Four independent clouds: $X=1+Z_{1} / 3$ with probability $1 / 2, X=-1+Z_{2} / 3$ with probability $1 / 2$ and $Y=1+Z_{3} / 3$ with probability $1 / 2, Y=-1+Z_{4} / 3$ with probability $1 / 2$, where $Z_{1}, Z_{2}, Z_{3}, Z_{4} \sim N(0,1)$ are independent.
Observe that in "Four independent clouds", $H_{0}$ is true, and the power in all the cases should be around 0.05 . In all cases, the critical values of our test were calculated through 50000 replications and the power of all of the tests considered from 10000 replications. The first three columns of Table 1 give the power of the HHG, DCV and HSIC tests. Column 4 gives the maximum power among the classic correlation test: Pearson, Spearman and Kendall, which we call PSK. Columns 5, 6 and 7 give the power of our test for different $g=g_{1}=g_{2}$ function considered in the weight function $G$. In column 8, we use the function $g_{1}$ and $g_{2}$ proposed in Subsection 3.3, analogously in Table 2 and Table 3. Figure 2 give us $n=1000$ simulations of the alternatives considered in this subsection.

Table 1: Power comparison for the different test for sample size of $n=30$.

| Test | HHG | DCOV | HSIC | PSK | $\mathrm{N}(1,1)$ | $\mathrm{N}(0,1)$ | $\mathrm{N}(1,4)$ | $g_{1}, g_{2}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Parabola | 0.791 | 0.522 | 0.733 | 0.103 | 0.824 | $\mathbf{0 . 8 3 1}$ | 0.814 | 0.817 |
| 2 parabolas | 0.962 | 0.204 | 0.849 | 0.194 | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ |
| Circle | 0.646 | 0.051 | 0.488 | 0.096 | 0.923 | 0.716 | $\mathbf{0 . 9 4 7}$ | 0.823 |
| Diamond | 0.283 | 0.030 | 0.262 | 0.016 | 0.422 | 0.139 | $\mathbf{0 . 4 7 7}$ | 0.395 |
| W-shape | $\mathbf{0 . 9 0 8}$ | 0.569 | 0.856 | 0.179 | 0.788 | 0.887 | 0.782 | 0.874 |
| 4 clouds | 0.052 | 0.053 | 0.053 | 0.046 | 0.052 | 0.052 | 0.051 | 0.051 |

Table 2: Power comparison for the different test for sample size of $n=50$.

| Test | HHG | DCOV | HSIC | PSK | N $(1,1)$ | $\mathrm{N}(0,1)$ | $\mathrm{N}(1,4)$ | $g_{1}, g_{2}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Parabola | 0.983 | 0.854 | 0.957 | 0.114 | 0.979 | 0.983 | $\mathbf{1 . 0 0 0}$ | 0.975 |
| 2 parabolas | $\mathbf{1 . 0 0 0}$ | 0.354 | 0.997 | 0.198 | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ |
| Circle | 0.985 | 0.075 | 0.914 | 0.008 | 0.999 | 0.997 | $\mathbf{1 . 0 0 0}$ | 0.995 |
| Diamond | 0.664 | 0.048 | 0.545 | 0.013 | 0.836 | 0.630 | $\mathbf{0 . 8 8 4}$ | 0.761 |
| W-shape | $\mathbf{0 . 9 9 9}$ | 0.935 | 0.988 | 0.077 | 0.989 | 0.998 | 0.987 | 0.979 |
| 4 clouds | 0.050 | 0.047 | 0.048 | 0.046 | 0.512 | 0.055 | 0.054 | 0.051 |

## 4.2 $\quad X$ and $Y$ are random vectors

In our test, the distance considered for the calculations of recurrences measures is given for the Euclidean norm. Because the Euclidean distance increases with the dimension, the densities of $N(0,4)$ and $N(2,4)$ were aggregated in the columns 6 and 7 . In this subsection, we consider the last two alternatives in Table 3, and in Table 4 of Heller et al. [13], which we will call "Logarithmic", "Epsilon" and "Quadratic" tests and which are defined as follows:
Logarithmic: $X, Y \in \mathbb{R}^{5}$ where $X_{i} \sim N(0,1)$ are independent, $Y_{i}=\log \left(X_{i}^{2}\right)$ for $i=1,2,3,4,5$.
Epsilon: $X, Y, \varepsilon \in \mathbb{R}^{5}$ where $X_{i}, \varepsilon_{i} \sim N(0,1)$ are independent, $Y_{i}=\varepsilon_{i} X_{i}$ for $i=$ $1,2,3,4,5$.
Quadratic: $X, Y, \varepsilon \in \mathbb{R}^{5}$ where $X_{i}, \varepsilon_{i}$ are independent, $X_{i} \sim N(0,1), \varepsilon_{i} \sim N(0,3)$, $Y_{i}=X_{i}+4 X_{i}^{2}+\varepsilon_{i} i=1,2, Y_{i}=\varepsilon_{i}$ for all $i=3,4,5$.
We also add the alternatives considered in Boglioni, which are called "2D-pairwise independent" and are defined as follows:
2D-pairwise independent: $X, Z_{0}, Y_{1} \sim N(0,1)$ independent, $Y=\left(Y_{1}, Y_{2}\right)$ where $Y_{2}=$ $\left|Z_{0}\right| \operatorname{sign}\left(X Y_{1}\right)$.
In all cases, the critical values of our test were calculated through 50000 replications and the power of all of the tests were considered from 10000 replications.

To have an idea of the size of the test for random vectors, we have simulated $X, Y \in$ $\mathbb{R}^{5}$ using $g_{1}$ and $g_{2}$ proposed in Subsection 3.3. The power of the test were $0.051,0.048$ and 0.052 for sample sizes of 30,50 and 80 , respectively.

Table 3: Power comparison for the different test for sample size of $n=80$.

| Test | HHG | DCOV | HSIC | PSK | $\mathrm{N}(1,1)$ | $\mathrm{N}(0,1)$ | $\mathrm{N}(1,4)$ | $g_{1}, g_{2}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Parabola | $\mathbf{1 . 0 0 0}$ | 0.994 | $\mathbf{1 . 0 0 0}$ | 0.105 | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ |
| 2 parabolas | $\mathbf{1 . 0 0 0}$ | 0.700 | $\mathbf{1 . 0 0 0}$ | 0.201 | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ |
| Circle | $\mathbf{1 . 0 0 0}$ | 0.196 | 0.999 | 0.004 | 0.999 | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ |
| Diamond | 0.948 | 0.096 | 0.853 | 0.003 | 0.836 | 0.953 | $\mathbf{1 . 0 0 0}$ | 0.999 |
| W-shape | $\mathbf{1 . 0 0 0}$ | 0.999 | $\mathbf{1 . 0 0 0}$ | 0.085 | 0.988 | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ |
| 4 clouds | 0.047 | 0.047 | 0.047 | 0.049 | 0.051 | 0.049 | 0.055 | 0.057 |

Table 4: Power comparison for the different test for sample size of $n=30$.

| Test | HHG | DCOV | HSIC | $\mathrm{N}(1,1)$ | $\mathrm{N}(1,4)$ | $\mathrm{N}(0,4)$ | $\mathrm{N}(2,4)$ | $g_{1}, g_{2}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Log | 0.594 | 0.154 | 0.610 | 0.710 | 0.759 | 0.321 | $\mathbf{0 . 8 8 5}$ | 0.813 |
| Epsilon | 0.784 | 0.226 | 0.484 | 0.470 | 0.576 | 0.194 | 0.749 | $\mathbf{0 . 8 5 8}$ |
| Quadratic | $\mathbf{0 . 6 8 7}$ | 0.302 | 0.530 | 0.197 | 0.155 | 0.170 | 0.147 | 0.144 |
| 2D-indep | 0.161 | 0.175 | $\mathbf{0 . 4 0 3}$ | 0.177 | 0.264 | 0.106 | 0.263 | 0.112 |

## 4.3 $\quad X$ and $Y$ are time series

In this subsection, we consider the case in which $X$ and $Y$ are time series. In all cases $X$ and $Y$ are time series of length 100 and the power (due to the computational cost) were calculated by a permutation method for $m=1.000$ replications (Table 7 and Table 8 ) and $m=100$ replications (Table 9). All the power were calculated using $g_{1}$ and $g_{2}$ proposed in Subsection 4.3. The power for different alternatives and sample sizes in the discrete case are given in Table 7. The $\operatorname{AR}(0.1)$ and $\operatorname{AR}(0.9)$ means that the time series $X$ is an $\operatorname{AR}(1)$ with parameter 0.1 and 0.9 , respectively. The case called $\operatorname{ARMA}(2,1)$, is an $\operatorname{ARMA}(2,1)$ model with parameters $\phi=(0.2,0.5)$ and $\theta=0.2$. In column 4 of Table $7, Z$ represents a white noise where $\sigma$ is the standard deviation of $\sqrt{|X|}$. In Table 7 and Table 8, $\varepsilon$ and $\varepsilon^{\prime}$ are independent white noises with $\sigma=1$. In Table 8 are given the power for different alternatives and sample sizes in the continuous case. In this table, $B m$ represents that $X$ is a Brownian motion with $\sigma=1$ observed in $[0,1]$ (at times $0,1 / 100,2 / 100, \ldots, 99 / 100$ ) and $f B m$ is a fractional Brownian motion with Hurst parameter $H=0.7$. Finally, Table 9 shows the power for cases in which the dependency between $X$ and $Y$ is more difficult to detect. In these cases, $Y$ is a fractional Ornstein-Uhlenbeck process driven by a $f B m(X)$ for $H=0.5$ and $H=0.7$, which we call $O U$ and $F O U$, respectively. A particular linear combination of $F O U$, which we call $F O U(2)$, and whose definition and theoretical developed is found in [17], is a particular case of the models proposed in [2]. Table 9 considers the parameters $\sigma=1, \lambda=0.3$ (column 3) and $\sigma=1, \lambda_{1}=0.3, \lambda_{2}=0.8$ (column 4). More explicitly, $Y_{t}=\sigma \int_{-\infty}^{t} e^{-\lambda(t-s)} d X_{s}$ in column 3 (where $X=\left\{X_{t}\right\}$ is a fBm), and $Y_{t}=\frac{\lambda_{1}}{\lambda_{1}-\lambda_{2}} \sigma \int_{-\infty}^{t} e^{-\lambda_{1}(t-s)} d X_{s}+\frac{\lambda_{2}}{\lambda_{2}-\lambda_{1}} \sigma \int_{-\infty}^{t} e^{-\lambda_{2}(t-s)} d X_{s}$ in column 4 (where $X=\left\{X_{t}\right\}$ is a fBm ). To give an idea of the size of the test, in column $5 Y$ is a $B m$ independent of $X$.

Table 5: Power comparison for the different test for sample size of $n=50$.

| Test | HHG | DCOV | HSIC | $\mathrm{N}(1,1)$ | $\mathrm{N}(1,4)$ | $\mathrm{N}(0,4)$ | $\mathrm{N}(2,4)$ | $g_{1}, g_{2}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Log | 0.936 | 0.386 | 0.958 | 0.998 | 0.999 | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | 0.995 |
| Epsilon | 0.969 | 0.298 | 0.689 | 0.895 | 0.967 | 0.968 | $\mathbf{0 . 9 9 9}$ | 0.984 |
| Quadratic | $\mathbf{0 . 9 3 4}$ | 0.485 | 0.904 | 0.362 | 0.293 | 0.315 | 0.733 | 0.236 |
| 2D-indep | 0.27 | 0.359 | $\mathbf{0 . 7 9 8}$ | 0.281 | 0.219 | 0.261 | 0.198 | 0.172 |

Table 6: Power comparison for the different test for sample size of $n=80$.

| Test | HHG | DCOV | HSIC | $\mathrm{N}(1,1)$ | $\mathrm{N}(1,4)$ | $\mathrm{N}(0,4)$ | $\mathrm{N}(2,4)$ | $g_{1}, g_{2}$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Log | $\mathbf{1 . 0 0 0}$ | 0.793 | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ |
| Epsilon | 0.999 | 0.382 | 0.896 | 0.998 | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ | $\mathbf{1 . 0 0 0}$ |
| Quadratic | $\mathbf{0 . 9 9 6}$ | 0.725 | 0.971 | 0.595 | 0.545 | 0.535 | 0.480 | 0.416 |
| 2D-indep | 0.544 | 0.751 | $\mathbf{0 . 9 9 3}$ | 0.489 | 0.348 | 0.466 | 0.263 | 0.284 |

## 5 Conclusions

In this work we have presented a new test of independence between two random elements lying in metric spaces. Our test is based on percentages of recurrences for which we need, for each sample, only the information obtained by the distance between points. We have obtained the asymptotic distribution of our statistic and we have shown that the limit distribution under contiguous alternatives has a bias. We have also proven the consistency of the test for a wide class of alternatives, which include the particular case in which $(X, Y)$ follows a multivariate normal distribution. The performance of the test measured through the calculation of power through several alternatives has shown very good results, clearly improving on others in many cases for different dimensions of the spaces. In future work, we think that the result can be generalized to the case in which there is some kind of dependence between the observation of the sample. In addition, the work of the simulations should be expanded and deepened.

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## 6 Proofs

Proof of Lemma 1.

Table 7: Power for the case of discrete time series and different sample sizes.

| $n$ | $X$ | $Y=X^{2}+3 \varepsilon$ | $Y=\sqrt{\|X\|}+Z$ | $Y=\varepsilon X$ | $Y=\varepsilon$ |
| ---: | :---: | ---: | ---: | ---: | ---: |
| 30 | $\operatorname{AR}(0,1)$ | 0.350 | 0.214 | 0.772 | 0.051 |
| 50 | $\operatorname{AR}(0,1)$ | 0.592 | 0.402 | 0.962 | 0.050 |
| 100 | $\operatorname{AR}(0,1)$ | 0.999 | 0.698 | 1.000 | 0.046 |
| 30 | $\operatorname{AR}(0,9)$ | 1.000 | 0.903 | 1.000 | 0.035 |
| 50 | $\operatorname{AR}(0,9)$ | 1.000 | 0.998 | 1.000 | 0.053 |
| 100 | $\operatorname{AR}(0,9)$ | 1.000 | 1.000 | 1.000 | 0.039 |
| 30 | $\operatorname{ARMA}(2,1)$ | 0.817 | 0.323 | 0.925 | 0.057 |
| 50 | $\operatorname{ARMA}(2,1)$ | 0.986 | 0.566 | 0.996 | 0.047 |
| 100 | $\operatorname{ARMA}(2,1)$ | 1.000 | 0.921 | 1.000 | 0.051 |

Table 8: Power for the case of continuous time series and different sample sizes.

| $n$ | $X$ | $Y=X^{2}+3 \varepsilon$ | $Y=\sqrt{\|X\|}+\varepsilon$ | $Y=\varepsilon X+3 \varepsilon^{\prime}$ | $Y=\varepsilon$ |
| ---: | ---: | ---: | ---: | ---: | ---: |
| 30 | $B m$ | 0.770 | 0.519 | 0.402 | 0.060 |
| 50 | $B m$ | 0.924 | 0.752 | 0.656 | 0.052 |
| 80 | $B m$ | 0.994 | 0.923 | 0.839 | 0.040 |
| 30 | $f B m$ | 0.732 | 0.550 | 0.366 | 0.039 |
| 50 | $f B m$ | 0.883 | 0.805 | 0.586 | 0.040 |
| 80 | $f B m$ | 0.987 | 0.930 | 0.804 | 0.051 |

Observe that as $\left(X_{1}, Y_{1}\right),\left(X_{2}, Y_{2}\right), \ldots,\left(X_{n}, Y_{n}\right)$ i.i.d, then

$$
P\left(d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right)=p_{X, Y}(r, s)
$$

for all $i, j$ such that $i \neq j$. Therefore

$$
\begin{gathered}
\mathbb{E}\left(R R_{n}^{X, Y}(r, s)\right)=\mathbb{E}\left(\frac{1}{n^{2}-n} \sum_{i \neq j} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right\}}\right)= \\
\frac{1}{n^{2}-n} \sum_{i \neq j} P\left(d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right)=p_{X, Y}(r, s)
\end{gathered}
$$

Analogously, $\mathbb{E}\left(R R_{n}^{X}(r)\right)=p_{X}(r)$ and $\mathbb{E}\left(R R_{n}^{Y}(s)\right)=p_{Y}(s)$. Given that $X$ and $Y$ are independent, then

$$
\mathbb{E}\left(R R_{n}^{X, Y}(r, s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)\right)=0
$$

Thus,

$$
\begin{gathered}
\mathbb{C O V}\left(E_{n}(r, s), E_{n}\left(r^{\prime}, s^{\prime}\right)\right)= \\
\mathbb{E}\left[\left(R R_{n}^{X, Y}(r, s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)\right)\left(R R_{n}^{X, Y}\left(r^{\prime}, s^{\prime}\right)-R R_{n}^{X}\left(r^{\prime}\right) R R_{n}^{Y}\left(s^{\prime}\right)\right)\right]= \\
\mathbb{E}\left[\left(R R_{n}^{X, Y}(r, s)\right)\left(R R_{n}^{X, Y}\left(r^{\prime}, s^{\prime}\right)\right)\right]-\mathbb{E}\left(R R_{n}^{X, Y}(r, s) R R_{n}^{X}\left(r^{\prime}\right) R R_{n}^{Y}\left(s^{\prime}\right)\right)
\end{gathered}
$$

Table 9: Power where the dependence is between a fractional Brownian motion and its associated FOU and FOU(2), for the cases $H=0.5(B m)$ and $H=0.7(f B m)$.

| $n$ | $X$ | $Y=\mathrm{FOU}$ | $Y=\mathrm{FOU}(2)$ | $Y=B m$ |
| ---: | :---: | ---: | ---: | ---: |
| 30 | $B m$ | 0.775 | 0.183 | 0.053 |
| 50 | $B m$ | 0.906 | 0.541 | 0.046 |
| 80 | $B m$ | 0.986 | 0.880 | 0.056 |
| 30 | $f B m$ | 0.380 | 0.106 | 0.045 |
| 50 | $f B m$ | 0.516 | 0.282 | 0.039 |
| 80 | $f B m$ | 0.707 | 0.542 | 0.042 |

$$
\begin{equation*}
-\mathbb{E}\left(R R_{n}^{X, Y}\left(r^{\prime}, s^{\prime}\right) R R_{n}^{X}(r) R R_{n}^{Y}(s)\right)+\mathbb{E}\left[R R_{n}^{X}(r) R R_{n}^{X}\left(r^{\prime}\right)\right] \mathbb{E}\left[R R_{n}^{Y}(s) R R_{n}^{Y}\left(s^{\prime}\right)\right] \tag{18}
\end{equation*}
$$

$$
\begin{gather*}
\mathbb{E}\left(R R_{n}^{X}(r) R R_{n}^{X}\left(r^{\prime}\right)\right)=\mathbb{E}\left(\frac{1}{n^{2}(n-1)^{2}} \sum_{i \neq j} \sum_{h \neq k} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(X_{h}, X_{k}\right)<r^{\prime}\right\}}\right)= \\
\frac{1}{n^{2}(n-1)^{2}} \sum_{i \neq j} \sum_{h \neq k} P\left(d\left(X_{i}, X_{j}\right)<r, d\left(X_{h}, X_{k}\right)<r^{\prime}\right) . \tag{19}
\end{gather*}
$$

Decomposing (19) in the terms in which $i, j, k, h$ are pairwise different, $\{i, j\}=\{h, k\}$ and $\{i, j, h, k\}$ has three elements, and using that the $X$-random vectors are i.i.d, we obtain that 19 is equal to

$$
\begin{gather*}
\frac{n(n-1)(n-2)(n-3) p_{X}(r) p_{X}\left(r^{\prime}\right)+2 n(n-1) p_{X}(r)+4 n(n-1)(n-2) p_{X}^{(3)}\left(r \wedge r^{\prime}\right)}{n^{2}(n-1)^{2}}= \\
\frac{n-2}{n(n-1)}\left[(n-3) p_{X}(r) p_{X}\left(r^{\prime}\right)+4 p_{X}^{(3)}\left(r \wedge r^{\prime}\right)\right]+o\left(\frac{1}{n}\right) \tag{20}
\end{gather*}
$$

Analogously

$$
\begin{equation*}
\mathbb{E}\left[\left(R R_{n}^{Y}(s)\right) R R_{n}^{Y}\left(s^{\prime}\right)\right]=\frac{n-2}{n(n-1)}\left((n-3) p_{Y}(s) p_{Y}\left(s^{\prime}\right)+4 p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)\right)+o\left(\frac{1}{n}\right) \tag{21}
\end{equation*}
$$

Similarly, using that the $(X, Y)$-random vectors are i.i.d. and also that $X$ and $Y$ are independent,

$$
\mathbb{E}\left[R R_{n}^{X, Y}(r, s) R R_{n}^{X, Y}\left(r^{\prime}, s^{\prime}\right)\right]=
$$

$\frac{(n-2)(n-3) p_{X}(r) p_{X}\left(r^{\prime}\right) p_{Y}(s) p_{Y}\left(s^{\prime}\right)+2 p_{X}(r) p_{Y}(s)+4(n-2) p_{X}^{(3)}\left(r \wedge r^{\prime}\right) p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)}{n(n-1)}=$

$$
\begin{equation*}
\frac{n-2}{n(n-1)}\left[(n-3) p_{X}(r) p_{X}\left(r^{\prime}\right) p_{Y}(s) p_{Y}\left(s^{\prime}\right)+4 p_{X}^{(3)}\left(r \wedge r^{\prime}\right) p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)\right]+o\left(\frac{1}{n}\right) \tag{22}
\end{equation*}
$$

With the same technique as in 20 and 21 , we obtain $\mathbb{E}\left[R R_{n}^{X, Y}(r, s) R R_{n}^{X}\left(r^{\prime}\right) R R_{n}^{Y}\left(s^{\prime}\right)\right]=$

$$
\begin{aligned}
& \mathbb{E}\left(\frac{1}{n^{3}(n-1)^{3}} \sum_{i \neq j} \sum_{h \neq k} \sum_{l \neq m} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s, d\left(X_{h}, X_{k}\right)<r^{\prime}, d\left(Y_{l}, Y_{m}\right)<s^{\prime}\right\}}\right)= \\
& \frac{1}{n^{3}(n-1)^{3}} \sum_{i \neq j} \sum_{h \neq k} \sum_{l \neq m} P\left(d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s, d\left(X_{h}, X_{k}\right)<r^{\prime}, d\left(Y_{l}, Y_{m}\right)<s^{\prime}\right)= \\
& \frac{1}{n^{3}(n-1)^{3}} \sum_{i \neq j} \sum_{h \neq k} \sum_{l \neq m} P\left(d\left(X_{i}, X_{j}\right)<r, d\left(X_{h}, X_{k}\right)<r^{\prime}\right) P\left(d\left(Y_{i}, Y_{j}\right)<s, d\left(Y_{l}, Y_{m}\right)<s^{\prime}\right)= \\
& \frac{1}{n^{3}(n-1)^{3}}\left[n(n-1)(n-2)^{2}(n-3)^{2} p_{X}(r) p_{X}\left(r^{\prime}\right) p_{Y}(s) p_{Y}\left(s^{\prime}\right)\right]+ \\
& \frac{1}{n^{3}(n-1)^{3}}\left[4 n(n-1)(n-2)^{2}(n-3) p_{X}(r) p_{X}\left(r^{\prime}\right) p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)\right]+ \\
& \frac{1}{n^{3}(n-1)^{3}}\left[4 n(n-1)(n-2)^{2}(n-3) p_{X}^{(3)}\left(r \wedge r^{\prime}\right) p_{Y}(s) p_{Y}\left(s^{\prime}\right)+8 n(n-1)(n-2) p_{X}(r) p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)\right]+ \\
& \frac{1}{n^{3}(n-1)^{3}} {\left[8 n(n-1)(n-2) p_{X}^{(3)}\left(r \wedge r^{\prime}\right) p_{Y}(s)+2 n(n-1)(n-2)(n-3) p_{X}(r) p_{X}\left(r^{\prime}\right) p_{Y}(s)\right]+} \\
& \frac{1}{n^{3}(n-1)^{3}} {\left[2 n(n-1)(n-2)(n-3) p_{X}(r) p_{Y}(s) p_{Y}\left(s^{\prime}\right)+16 n(n-1)(n-2)^{2} p_{X}^{(3)}\left(r \wedge r^{\prime}\right) p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)\right]+} \\
& \frac{1}{n^{3}(n-1)^{3}} 4 n(n-1) p_{X}(r) p_{Y}(s) .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \mathbb{E}\left[R R_{n}^{X, Y}(r, s) R R_{n}^{X}\left(r^{\prime}\right) R R_{n}^{Y}\left(s^{\prime}\right)\right]= \\
& \frac{1}{n^{2}(n-1)^{2}}\left[(n-2)^{2}(n-3)^{2} p_{X}(r) p_{X}\left(r^{\prime}\right) p_{Y}(s) p_{Y}\left(s^{\prime}\right)+4(n-2)^{2}(n-3) p_{X}(r) p_{X}\left(r^{\prime}\right) p_{Y}^{(3)}(s)\right]+ \\
& \frac{1}{n^{2}(n-1)^{2}}\left[4(n-2)^{2}(n-3) p_{X}^{(3)}\left(r \wedge r^{\prime}\right) p_{Y}(s) p_{Y}\left(s^{\prime}\right)+8(n-2) p_{X}(r) p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)+8(n-2) p_{X}^{(3)}(r) p_{Y}(s)\right]+ \\
& \frac{1}{n^{2}(n-1)^{2}}\left[2(n-2)(n-3) p_{X}(r) p_{X}\left(r^{\prime}\right) p_{Y}(s)+2(n-2)(n-3) p_{X}(r) p_{Y}(s) p_{Y}\left(s^{\prime}\right)\right]+ \\
& \frac{1}{n^{2}(n-1)^{2}}\left[16(n-2)^{2} p_{X}^{(3)}\left(r \wedge r^{\prime}\right) p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)+4 p_{X}(r) p_{Y}(s)\right]= \\
& \frac{(n-2)^{2}(n-3)}{n^{2}(n-1)^{2}}\left[(n-3) p_{X}(r) p_{X}\left(r^{\prime}\right) p_{Y}(s) p_{Y}\left(s^{\prime}\right)+4\left(p_{X}^{(3)}\left(r \wedge r^{\prime}\right) p_{Y}(s) p_{Y}\left(s^{\prime}\right)+p_{X}\left(r \wedge r^{\prime}\right) p_{Y}^{(3)}(s)\right)\right] \\
& +o\left(\frac{1}{n}\right) .
\end{aligned}
$$

Putting (20), (21) and (22) in (18), we obtain that 18 is equal to

$$
\begin{gathered}
\frac{1}{n^{2}(n-1)^{2}}\left[(n-2)(n-3)(4 n-6) p_{X}(r) p_{X}\left(r^{\prime}\right) p_{Y}(s) p_{Y}\left(s^{\prime}\right)\right]+ \\
\frac{1}{n^{2}(n-1)^{2}}\left[4(n-2)\left(n^{2}+3 n-8\right) p_{X}^{(3)}\left(r \wedge r^{\prime}\right) p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)\right]+ \\
\frac{-4(n-2)^{2}(n-3)}{n^{2}(n-1)^{2}}\left(p_{X}(r) p_{X}\left(r^{\prime}\right) p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)+p_{X}^{(3)}\left(r \wedge r^{\prime}\right) p_{Y}^{2}(s)\right)+o\left(\frac{1}{n}\right) .
\end{gathered}
$$

Then

$$
\begin{gathered}
\lim _{n \rightarrow+\infty} \mathbb{C O V}\left(E_{n}(r, s), E_{n}\left(r^{\prime}, s^{\prime}\right)\right)= \\
4\left(p_{X}^{(3)}\left(r \wedge r^{\prime}\right)-p_{X}(r) p_{X}\left(r^{\prime}\right)\right)\left(p_{Y}^{(3)}\left(s \wedge s^{\prime}\right)-p_{Y}(s) p_{Y}\left(s^{\prime}\right)\right)
\end{gathered}
$$

Proof of Lemma 2.

$$
\begin{gathered}
\sqrt{n}\left(R R_{n}^{X, Y}(r, s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)\right)= \\
\frac{\sqrt{n}}{n(n-1)} \sum_{(i, j) \in I_{2}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right\}}-\sqrt{n} R R_{n}^{X}(r) R R_{n}^{Y}(s)= \\
\frac{\sqrt{n}}{n(n-1)(n-2)(n-3)} \sum_{(i, j, h, k) \in I_{4}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right\}}-\sqrt{n} R R_{n}^{X}(r) R R_{n}^{Y}(s)= \\
E_{n}^{\prime}(r, s)-H_{n}(r, s)
\end{gathered}
$$

where $H_{n}(r, s)=$

$$
\sqrt{n}\left(R R_{n}^{X}(r) R R_{n}^{Y}(s)-\frac{1}{n(n-1)(n-2)(n-3)} \sum_{(i, j, k, h) \in I_{4}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{h}, Y_{k}\right)<s\right\}}\right) .
$$

Then, $H_{n}(r, s)$ is equal to

$$
\begin{gathered}
\frac{\sqrt{n}}{n^{2}(n-1)^{2}} \sum_{(i, j) \in I_{2}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r\right\}} \sum_{(h, k) \in I_{2}^{n}} \mathbf{1}_{\left\{d\left(Y_{h}, Y_{k}\right)<s\right\}} \\
-\frac{\sqrt{n}}{n(n-1)(n-2)(n-3)} \sum_{(i, j, k, h) \in I_{4}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{h}, Y_{k}\right)<s\right\}}= \\
\frac{\sqrt{n}}{n^{2}(n-1)^{2}} \frac{1}{n(n-1)} \sum_{(i, j) \in I_{2}^{n}} \sum_{(h, k) \in I_{2}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{h}, Y_{k}\right)<s\right\}}
\end{gathered}
$$

$$
\begin{equation*}
-\frac{\sqrt{n}}{n(n-1)(n-2)(n-3)} \sum_{(i, j, k, h) \in I_{4}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{h}, Y_{k}\right)<s\right\}} \tag{23}
\end{equation*}
$$

Now, we decompose

$$
\begin{gathered}
\sum_{(i, j) \in I_{2}^{n}} \sum_{(h, k) \in I_{2}^{n}} 1_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{h}, Y_{k}\right)<s\right\}}= \\
\sum_{(i, j, k, h) \in I_{4}^{n}} 1_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{h}, Y_{k}\right)<s\right\}}+4 \sum_{(i, j, k) \in I_{3}^{n}} 1_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{k}\right)<s\right\}}+ \\
2 \sum_{(i, j) \in I_{2}^{n}} 1_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right\}}
\end{gathered}
$$

and substituting in (23) we obtain that (23) is equal to

$$
\begin{align*}
& \frac{\sqrt{n}}{n(n-1)}\left(\left(\frac{1}{n(n-1)}-\frac{1}{(n-2)(n-3)}\right) \sum_{(i, j, k, h) \in I_{4}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{h}, Y_{k}\right)<s\right\}}\right)+ \\
& \frac{\sqrt{n}}{n^{2}(n-1)^{2}}\left(4 \sum_{(i, j, k) \in I_{3}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{k}\right)<s\right\}}+2 \sum_{(i, j) \in I_{2}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right\}}\right)= \\
& \frac{\sqrt{n}}{n^{2}(n-1)^{2}(n-2)(n-3)} \sum_{(i, j, k, h) \in I_{4}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{h}, Y_{k}\right)<s\right\}}+ \\
& \frac{\sqrt{n}}{n^{2}(n-1)^{2}}\left(4 \sum_{(i, j, k) \in I_{3}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{k}\right)<s\right\}}+2 \sum_{(i, j) \in I_{2}^{n}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right\}}\right) \tag{24}
\end{align*}
$$

Observe that (24) it is bounded between 0 and

$$
\frac{\sqrt{n}}{n^{2}(n-1)^{2}}(4 n(n-1)(n-2)+2 n(n-1))=\frac{1}{\sqrt{n}} \frac{4 n-6}{n-1}<\frac{4}{\sqrt{n}}
$$

## Proof of Theorem 3 .

Every continuous function $h: \mathbb{R} \rightarrow \mathbb{R}$ with finit limits as $x \rightarrow \pm \infty$ is uniformly continuous. Therefore given $\varepsilon>0$, exist $\delta>0$ such that $|F(x)-F(y)| \leq \varepsilon^{2} / 8$ and $|G(x)-G(y)| \leq \varepsilon^{2} / 8$ for all $(x, y)$ such that $|x-y|<\delta$, where $F$ and $G$ are the distribution functions of $d\left(X_{1}, X_{2}\right)$ and $d\left(Y_{1}, Y_{2}\right)$ respectively. If $H_{0}$ is true, consider for each $r, s>0$ the functions $f_{r, s}:\left(S_{X} \times S_{Y}\right)^{4} \rightarrow \mathbb{R}$ defined by

$$
f_{r, s}\left(x, y, x^{\prime}, y^{\prime}, x^{\prime \prime}, y^{\prime \prime}, x^{\prime \prime \prime}, y^{\prime \prime \prime}\right)=\mathbf{1}_{\left\{d\left(x, x^{\prime}\right)<r, d\left(y, y^{\prime}\right)<s\right\}}-\mathbf{1}_{\left\{d\left(x, x^{\prime}\right)<r, d\left(y^{\prime \prime}, y^{\prime \prime \prime}\right)<s\right\}}
$$

where $x, x^{\prime}, x^{\prime \prime}, x^{\prime \prime \prime} \in S_{X}$ and $y, y^{\prime}, y^{\prime \prime}, y^{\prime \prime \prime} \in S_{Y}$. and consider the family $\mathcal{F}=\left\{f_{r, s}\right\}_{r, s>0}$. To simplify the notation, we call $z=\left(x, y, x^{\prime}, y^{\prime}, x^{\prime \prime}, y^{\prime \prime}, x^{\prime \prime \prime}, y^{\prime \prime \prime}\right)$ throughout the demonstration.

Observe that

$$
E_{n}^{\prime}(r, s)=\frac{\sqrt{n}}{n(n-1)(n-2)(n-3)} \sum_{(i, j, j, k, h) \in I_{4}^{n}} f_{r, s}\left(X_{i}, Y_{i}, X_{j}, Y_{j}, X_{h}, Y_{h}, X_{k}, Y_{k}\right)
$$

then the process $\left\{E_{n}^{\prime}(r, s)\right\}_{r, s>0}$ is an $U$-process of order 4.
To obtain the convergence, according to Arcones \& Giné's Theorem 4.10, it is enough to prove that

$$
\int_{0}^{+\infty}\left(\log N_{[]}^{(2)}\left(\varepsilon, \mathcal{F}, P^{4}\right)\right)^{1 / 2} d \varepsilon<+\infty
$$

If $\varepsilon \geq 2$, then $-1 \leq f_{r, s}(z) \leq 1$ for all $z \in\left(S_{X} \times S_{Y}\right)^{4}$ and $r, s>0$. Then $\mathcal{L}=\{-1\}$, $\mathcal{U}=\{1\}$ satisfied 6) Thus, $N_{[]}^{(2)}\left(\varepsilon, \mathcal{F}, P^{4}\right)=1$, therefore $\int_{0}^{+\infty}\left(\log N_{[]}^{(2)}\left(\varepsilon, \mathcal{F}, P^{4}\right)\right)^{1 / 2} d \varepsilon=$ $\int_{0}^{2}\left(\log N_{[]}^{(2)}\left(\varepsilon, \mathcal{F}, P^{4}\right)\right)^{1 / 2} d \varepsilon$.

If $\varepsilon<2$, we take $T>0$ such that $\max \{1-F(T), 1-G(T)\}<\varepsilon^{2} / 8$, then we partition $[0,+\infty)$ into $m+1$ subintervals of the form $\left[\frac{i T}{m}, \frac{(i+1) T}{m}\right)$ such that $\frac{T}{m}<\delta$, where $\frac{(m+1) T}{m}$ is interpreted as $+\infty$. Define the following functions

$$
g_{i, j}(z)=\left\{\begin{array}{ccc}
\left.\mathbf{1}_{\left\{d\left(x, x^{\prime}\right)<\frac{i T}{m}, d\left(y, y^{\prime}\right)<\frac{j T}{m}\right\}}\right\} & \text { for } \quad i, j=1,2, \ldots, m \\
0 & \text { for } \quad i=0 \text { or } j=0
\end{array}\right.
$$

and

$$
h_{i, j}(z)=\left\{\begin{array}{ccc}
\mathbf{1}_{\left\{d\left(x, x^{\prime}\right)<\frac{i T}{m}, d\left(y^{\prime \prime}, y^{\prime \prime \prime}\right)<\frac{j T}{m}\right\}} & \text { for } & i, j=1,2, \ldots, m \\
\mathbf{1}_{\left\{d\left(x, x^{\prime}\right)<\frac{i T}{m}\right\}} & \text { for } & i=1,2, \ldots, m, j=m+1 \\
\mathbf{1}_{\left\{d\left(y^{\prime \prime}, y^{\prime \prime \prime}\right)<\frac{j T}{m}\right\}} & \text { for } & j=1,2, \ldots, m, i=m+1 \\
1 & \text { for } & i=j=m+1
\end{array} .\right.
$$

Observe that for each $r, s>0$ there exists $i, j \in\{0,1,2, \ldots, m\}$ such that $\frac{i T}{m} \leq r<\frac{(i+1) T}{m}$ and $\frac{j T}{m} \leq s<\frac{(j+1) T}{m}$.

Then

$$
g_{i, j}(z)-h_{i+1, j+1}(z) \leq f_{r, s}(z) \leq g_{i+1, j+1}(z)-h_{i, j}(z) \text { for all } z \in\left(S_{X} \times S_{Y}\right)^{4},
$$

Thus $\mathcal{L}=\left\{l_{i, j}\right\}$ and $\mathcal{U}=\left\{u_{i, j}\right\}$ where $l_{i, j}(z)=g_{i, j}(z)-h_{i+1, j+1}(z)$ and $u_{i, j}(z)=$ $g_{i+1, j+1}(z)-h_{i, j}(z)$ for $i, j=0,1,2, \ldots, m$. Also

$$
\begin{equation*}
\mathbb{E}\left(u_{i, j}(Z)-l_{i, j}(Z)\right)^{2} \leq 2\left(\mathbb{E}\left(g_{i+1, j+1}(Z)-g_{i, j}(Z)\right)^{2}+\mathbb{E}\left(h_{i+1, j+1}(Z)-h_{i, j}(Z)\right)^{2}\right) . \tag{25}
\end{equation*}
$$

Define the sets $A_{i, j}:=\left[0, \frac{(i+1) T}{m}\right) \times\left[0, \frac{(j+1) T}{m}\right)-\left[0, \frac{i T}{m}\right) \times\left[0, \frac{j T}{m}\right)$, then

$$
\begin{gather*}
\mathbb{E}\left(g_{i+1, j+1}(Z)-g_{i, j}(Z)\right)^{2}=\mathbb{E}\left(\mathbf{1}_{A_{i, j}}(Z)\right) \\
\leq P\left(\frac{i T}{m} \leq d\left(X_{1}, X_{2}\right)<\frac{(i+1) T}{m}\right)+P\left(\frac{j T}{m} \leq d\left(Y_{1}, Y_{2}\right)<\frac{(j+1) T}{m}\right) \leq \\
F\left(\frac{(i+1) T}{m}\right)-F\left(\frac{i T}{m}\right)+G\left(\frac{(j+1) T}{m}\right)-G\left(\frac{j T}{m}\right) \leq \varepsilon^{2} / 4 . \tag{26}
\end{gather*}
$$

Analogously,

$$
\begin{equation*}
\mathbb{E}\left(h_{i+1, j+1}(Z)-h_{i, j}(Z)\right)^{2} \leq \varepsilon^{2} / 4 . \tag{27}
\end{equation*}
$$

putting (27) and 26) in (25) we obtain that $\mathbb{E}\left(u_{i, j}(Z)-l_{i, j}(Z)\right)^{2} \leq \varepsilon^{2}$. Lastly, observe that the cardinal of $\mathcal{L}$ and $\mathcal{U}$ is $(m+1)^{2}$, then

$$
N_{[]}^{(2)}\left(\varepsilon, \mathcal{F}, P^{4}\right) \leq \frac{c t e}{\varepsilon^{4}} \text {, thus } \int_{0}^{2}\left(\log N_{[]}^{(2)}\left(\varepsilon, \mathcal{F}, P^{4}\right)\right)^{1 / 2} d \varepsilon<+\infty .
$$

## Proof of Theorem 4.

Define $\mu(r, s)=P\left(d\left(X_{1}, X_{2}\right)<r, d\left(Y_{1}, Y_{2}\right)<s\right)-P\left(d\left(X_{1}, X_{2}\right)<r\right) P\left(d\left(Y_{1}, Y_{2}\right)<s\right)$. Then, $r_{0}, s_{0}>0$ exist, such that $\mu^{2}\left(r_{0}, s_{0}\right)>0$, thus $\varepsilon>0$ exist and $A \subset[0,+\infty)^{2}$ such that $\left(r_{0}, s_{0}\right) \in A$ and $\mu^{2}(r, s)>\varepsilon$ for all $(r, s) \in A$. Then, as $n \rightarrow+\infty$,

$$
n \int_{0}^{+\infty} \int_{0}^{+\infty} \mu^{2}(r, s) g(r, s) d r d s \geq n \varepsilon \iint_{A} g(r, s) d r d s \rightarrow+\infty
$$

Now, using that $(a+b)^{2} \leq 2\left(a^{2}+b^{2}\right)$ we obtain that $n \int_{0}^{+\infty} \int_{0}^{+\infty} \mu^{2}(r, s) g(r, s) d r d s \leq$

$$
\begin{gathered}
2 n \int_{0}^{+\infty} \int_{0}^{+\infty}\left(R R_{n}^{X, Y}(r, s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)-\mu(r, s)\right)^{2} g(r, s) d r d s+ \\
2 n \int_{0}^{+\infty} \int_{0}^{+\infty}\left(R R_{n}^{X, Y}(r, s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)\right)^{2} g(r, s) d r d s
\end{gathered}
$$

Thus

$$
T_{n}=n \int_{0}^{+\infty} \int_{0}^{+\infty}\left(R R_{n}^{X, Y}(r, s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)\right)^{2} g(r, s) d r d s \xrightarrow{P}+\infty \text { as } n \rightarrow+\infty .
$$

## Proof of Corollary 1.

Because all of the norms in $\mathbb{R}^{p}$ and $\mathbb{R}^{q}$ are equivalent, it is enough to give the proof for the Euclidean norm case. We use that if $(Z, T)$ has centered normal bivariate distribution, then $\mathbb{C O V}\left(Z^{2}, T^{2}\right)=2(\mathbb{C O V}(Z, T))^{2}$.

Let us call $X=\left(X_{(1)}, X_{(2)}, \ldots, X_{(p)}\right)$ and $Y=\left(Y_{(1)}, Y_{(2)}, \ldots, Y_{(q)}\right)$. Then

$$
\mathbb{C O V}\left(\|X\|^{2},\|Y\|^{2}\right)=\mathbb{C O V}\left(\sum_{i=1}^{p} X_{(i)}^{2}, \sum_{j=1}^{q} Y_{(j)}^{2}\right)=2 \sum_{i=1}^{p} \sum_{j=1}^{q}\left(\mathbb{C O V}\left(X_{(i)}, Y_{(j)}\right)\right)^{2}
$$

If $X$ and $Y$ are not independent, then $i$ and $j$ exist such that $\mathbb{C O V}\left(X_{(i)}, Y_{(j)}\right) \neq 0$, then $\mathbb{C O V}\left(\|X\|^{2},\|Y\|^{2}\right)>0$, then $\|X\|^{2}$ and $\|Y\|^{2}$ are not independent, therefore $\|X\|$ and $\|Y\|$ are not independent, and then exist $r$ and $s$ positive numbers such that $P(\|X\|<r, \quad\|Y\|<s) \neq P(\|X\|<r) P(\|Y\|<s)$. If we apply this argument for $X_{1}-$ $X_{2}$ and $Y_{1}-Y_{2}$ instead $X$ and $Y$, then we obtain that

$$
P\left(\left\|X_{1}-X_{2}\right\|<r, d\left\|Y_{1}-Y_{2}\right\|<s\right) \neq P\left(\left\|X_{1}-X_{2}\right\|<r\right) P\left(\left\|Y_{1}-Y_{2}\right\|<s\right)
$$

Lastly, the result follows from Theorem 2.
Proof of Proposition 1.

$$
\begin{gather*}
\mathbb{E}^{(n)}\left(R R_{n}^{X, Y}(r, s)\right)=\mathbb{E}^{(n)}\left(\frac{1}{N} \sum_{(i, j) \in I_{2}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right\}}\right)= \\
P^{(n)}\left(d\left(X_{i}, X_{j}\right)<r, d\left(Y_{i}, Y_{j}\right)<s\right) \tag{28}
\end{gather*}
$$

Define $A_{r, s}:=\left\{\left(x_{1}, y_{1}, x_{2}, y_{2}\right) \in \mathbb{R}^{2 p+2 q}: d\left(x_{1}, x_{2}\right)<r, d\left(y_{1}, y_{2}\right)<s\right\}$, then 28) is equal to

$$
\begin{gathered}
c_{n}^{2}(\delta) \iiint \int_{A_{r, s}} f_{X, Y}^{(n)}\left(x_{1}, y_{1}\right) f_{X, Y}^{(n)}\left(x_{2}, y_{2}\right) d x_{1} d x_{2} d y_{1} d y_{2}= \\
c_{n}^{2}(\delta) \iiint \int_{A_{r, s}} f_{X}\left(x_{1}\right) f_{Y}\left(y_{1}\right) f_{X}\left(x_{2}\right) f_{Y}\left(y_{2}\right) \times \\
\left(1+\frac{\delta}{2 \sqrt{n}} k_{n}\left(x_{1}, y_{1}\right)\right)^{2}\left(1+\frac{\delta}{2 \sqrt{n}} k_{n}\left(x_{2}, y_{2}\right)\right)^{2} d x_{1} d x_{2} d y_{1} d y_{2}= \\
c_{n}^{2}(\delta) p_{X}^{(0)}(r) p_{Y}^{(0)}(s)+c_{n}^{2} \frac{\delta}{\sqrt{n}} \times \\
\iiint \int_{A_{r, s}}\left(k_{n}\left(x_{1}, y_{1}\right)+k_{n}\left(x_{2}, y_{2}\right)\right) f_{X}\left(x_{1}\right) f_{Y}\left(y_{1}\right) f_{X}\left(x_{2}\right) f_{Y}\left(y_{2}\right) d x_{1} d x_{2} d y_{1} d y_{2} \\
+\varepsilon_{n}(r, s)
\end{gathered}
$$

where $\left|\varepsilon_{n}(r, s)\right| \leq \frac{c}{\sqrt{n}}$ for all $r, s>0$ and $c$ is a constant.

$$
\begin{gathered}
\mathbb{E}^{(n)}\left(R R_{n}^{X}(r) R R_{n}^{Y}(s)\right)=\frac{1}{N^{2}} \mathbb{E}^{(n)}\left(\sum_{(i, j) \in I_{2},(h, k) \in I_{2}} \mathbf{1}_{\left\{d\left(X_{i}, X_{j}\right)<r, d\left(Y_{h}, Y_{k}\right)<s\right\}}\right)= \\
c_{n}^{2}(\delta) \frac{(n-2)(n-3)}{N} p_{X}^{(0)}(r) p_{Y}^{(0)}(s)+\frac{2}{N} P^{(n)}\left(A_{r, s}\right)+ \\
\frac{4(n-2)}{N} P^{(n)}\left(d\left(X_{1}, X_{2}\right)<r, d\left(Y_{1}, Y_{3}\right)<s\right) .
\end{gathered}
$$

Therefore

$$
\begin{gathered}
\mathbb{E}^{(n)}\left(E_{n}(r, s)\right)=\sqrt{n} \mathbb{E}^{(n)}\left(R R_{n}^{X}(r) R R_{n}^{Y}(s)-R R_{n}^{X}(r) R R_{n}^{Y}(s)\right)= \\
\sqrt{n} c_{n}^{2}(\delta) \frac{4 n-6}{N} p_{X}^{(0)}(r) p_{Y}^{(0)}(s)+\delta c_{n}^{2}(\delta) \times \\
\iiint \int_{A_{r, s}}\left(k_{n}\left(x_{1}, y_{1}\right)+k_{n}\left(x_{2}, y_{2}\right)\right) f_{X}\left(x_{1}\right) f_{Y}\left(y_{1}\right) f_{X}\left(x_{2}\right) f_{Y}\left(y_{2}\right) d x_{1} d x_{2} d y_{1} d y_{2}+\varepsilon_{n}(r, s) .
\end{gathered}
$$

$$
\text { Then } \mathbb{E}^{(n)}\left(E_{n}(r, s)\right) \rightarrow
$$

$$
\delta \iiint \int_{A_{r, s}}\left(k\left(x_{1}, y_{1}\right)+k\left(x_{2}, y_{2}\right)\right) f_{X}\left(x_{1}\right) f_{Y}\left(y_{1}\right) f_{X}\left(x_{2}\right) f_{Y}\left(y_{2}\right) d x_{1} d x_{2} d y_{1} d y_{2}
$$

as $n \rightarrow+\infty$.

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Figure 2: Parabola, Two parabolas, circle, diamond, wshape and four independent clouds.

