

Parallel Global Edge Switching for the Uniform Sampling of Simple Graphs with Prescribed Degrees*

Daniel Allendorf ✉ 

Goethe University Frankfurt, Germany

Ulrich Meyer ✉

Goethe University Frankfurt, Germany

Manuel Penschuck ✉ 

Goethe University Frankfurt, Germany

Hung Tran ✉

Goethe University Frankfurt, Germany

Abstract

The uniform sampling of simple graphs matching a prescribed degree sequence is an important tool in network science, e.g. to construct graph generators or null-models. Here, the Edge Switching Markov Chain (ES-MC) is a common choice. Given an arbitrary simple graph with the required degree sequence, ES-MC carries out a large number of small changes, called edge switches, to eventually obtain a uniform sample. In practice, reasonably short runs efficiently yield approximate uniform samples.

In this work, we study the problem of executing edge switches in parallel. We discuss parallelizations of ES-MC, but find that this approach suffers from complex dependencies between edge switches. For this reason, we propose the Global Edge Switching Markov Chain (G-ES-MC), an ES-MC variant with simpler dependencies. We show that G-ES-MC converges to the uniform distribution and design shared-memory parallel algorithms for ES-MC and G-ES-MC. In an empirical evaluation, we provide evidence that G-ES-MC requires not more switches than ES-MC (and often fewer), and demonstrate the efficiency and scalability of our parallel G-ES-MC implementation.

2012 ACM Subject Classification Mathematics of computing → Random graphs

Keywords and phrases Random Graph, Uniform Sampling, Markov Chain, Edge Switching, Parallelism

Supplementary Material Source code: <https://github.com/manpen/global-edge-switching>.

1 Introduction

In network science there are various measures, so-called centralities, to quantify the importance of nodes [1]. The degree centrality, for instance, suggests that a node's importance is proportional to its degree, i.e., the number of neighbors it has (see also [2]). This leads to the natural question whether graphs with matching degrees share structural properties. While this is not the case in general, a reoccurring task in practice is to quantify the statistical significance of some property observed in a network. Given an observed graph with degrees \mathbf{d} , a popular null-model is the uniform distribution over all simple graphs $\mathcal{G}(\mathbf{d})$ with matching degrees [3–5].

In this context, the *Edge Switching Markov Chain* (ES-MC) is a common choice to obtain an approximate uniform sample from $\mathcal{G}(\mathbf{d})$. In each so-called *edge switch*, two edges are selected uniformly at random and modified by exchanging their endpoints. This process

* Accepted manuscript. Link to final version: <https://doi.org/10.1016/j.jpdc.2022.12.010>

preserves the degrees of all nodes involved. We further keep the graph simple by rejecting all edge switches that introduce self-loops or multi-edges.

There exist different variants of ES-MC catering to various graph classes (e.g. Carstens [6] considers directed/undirected graphs, with/without loops, with/without multi-edges). Here, we focus on simple and undirected graphs. It is, however, straight-forward to adopt our findings to the other cases (some of which even lead to easier algorithms).

1.1 Related Work

Various methods to obtain a graph from a prescribed degree sequence have been studied [7].

Havel [8] and Hakimi [9] independently lay the foundation for a deterministic linear time generator. The algorithm, however, does not yield random graphs; while randomizations (e.g. [10, 11]) are available, they produce non-uniform samples.

The Chung-Lu Model [12] constructs graphs that match the prescribed degrees only in expectation. Under reasonable assumptions [7] it can be sampled in linear time [13].

The Configuration Model [14] outputs a random, but possibly non-simple, graph in linear time; adding rejection-sampling yields simple graphs in polynomial time if the maximum degree is $\mathcal{O}(\sqrt{\log n})$ (cf. [15–18]). Efficient and exact uniform generators can be obtained by adding a repair step between the Configuration Model and the rejection-sampling to boost the acceptance probability. Such algorithms are available for several degree sequence classes including bounded regular graphs or power-law sequences with sufficiently large exponents [19–21].

Further a plethora of Markov Chain Monte-Carlo (MCMC) algorithms have been proposed and analyzed (e.g. [22–31]). In comparison to the aforementioned exactly uniform generators, these algorithms allow for larger families of degree sequences, topological restrictions (e.g. connected graphs [24, 31]), or more general characterizations (e.g. joint degrees [27, 28]). Switch Markov Chains such as ES-MC have been shown to be rapidly mixing for classes of undirected graphs such as bounded-degree or power-law graphs (c.f. [23, 32–37]). While these bounds remain impractical for everyday use due to the high degrees of the polynomials involved, empirical studies suggest that a number of steps linear in the number of edges yields samples which are sufficiently uncorrelated to the input graph [24, 38, 39].

The only prior parallelization of ES-MC we are aware of was given by [40]. While the proposed algorithm has the advantage that it can be used in a distributed setting, it only avoids conflicts between edge switches that arise due to concurrent accesses to the same edge. This however is not enough to ensure that the algorithm faithfully implements ES-MC, e.g. the graphs generated by the parallel process differ from the ones generated by a sequential process. To address this issue the authors conduct an empirical error analysis. However, the issue remains that the graphs generated by such a process may not converge to the uniform distribution due to asymmetrical transition probabilities [41]. In addition, [42] proposed an external memory parallelization of the Curveball Markov Chain for sampling undirected graphs. Note however, that this algorithm requires sorting as a subroutine, which may be too expensive for usage in internal memory. In contrast to switch Markov Chains, Curveball has also not been shown to be rapidly mixing for undirected graphs, and despite preliminary empirical results [42], it remains an open question to relate the mixing times of Curveball and ES-MC for undirected graphs.

1.2 Our Contribution

We propose the Global Edge Switching Markov Chain (G-ES-MC), a new switch Markov Chain for sampling simple undirected graphs with prescribed degrees and show that it converges to the uniform distribution. This Markov Chain is designed with parallel algorithms in mind and exhibits less complex dependencies between edge switches. Consequently, we describe and analyze an exact shared-memory parallel algorithm for G-ES-MC. In addition, we describe an exact parallel algorithm for ES-MC.

In an empirical study, we provide evidence that G-ES-MC mixes faster than or equally well as standard ES-MC, and demonstrate the efficiency and scalability of our parallel G-ES-MC implementation. To the best of our knowledge, our implementation outperforms all openly available solutions by up to two orders of magnitude using 32 threads.

1.3 Outline

The article is structured as follows. Section 2 covers the notation used and the necessary background for our results, such as a definition of the Edge Switching Markov Chain (ES-MC). In Section 3, we introduce the Global Edge Switching Markov Chain (G-ES-MC) and prove its convergence to the uniform distribution. Section 4 contains a description of our parallel algorithms for ES-MC and G-ES-MC, and a formal analysis of the G-ES-MC algorithm. We discuss implementation details of the parallel G-ES-MC algorithm, as well as sequential and parallel baseline/reference implementations, in Section 5. In Section 6, we relate the mixing time of ES-MC and G-ES-MC empirically and investigate the efficiency and scalability of our implementations. Section 7 concludes the article with a summary of the results and an outlook on potential future work.

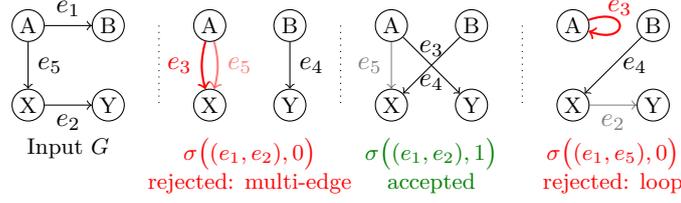
2 Preliminaries

2.1 Notation and Definitions

Define the short-hands $[k..n] := \{k, \dots, n\}$ and $[n] := [1..n]$. A graph $G = (V, E)$ has n nodes $V = \{v_1, \dots, v_n\}$ and m undirected edges E . We assume that edges are indexed (e.g. by their position in an edge list) and denote the i -th edge of a graph as $E[i]$. Given an undirected edge $e = \{v_i, v_j\}$ we denote a directed representation as \vec{e} . We treat both as the same object and defining one implies the other; we default to the canonical orientation $\vec{e} = (v_{\min(i,j)}, v_{\max(i,j)})$ whenever the direction is ambiguous. An edge (v, v) is called a *loop* at v ; an edge that appears more than once is called a *multi-edge*. A graph is *simple* if it contains neither multi-edges nor loops.

Given a graph $G = (V, E)$ and a node $v \in V$, define the *degree* $\deg(v) = |\{u : \{u, v\} \in E\}|$ as the number of edges incident to node v . Let $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$ be a *degree sequence* and denote $\mathcal{G}(\mathbf{d})$ as the set of simple graphs on n nodes with $\deg(v_i) = d_i$ for all $v_i \in V$. The degree sequence \mathbf{d} is *graphical* if $\mathcal{G}(\mathbf{d})$ is non-empty.

A commonly considered class of graphs are power-law graphs where the degrees follow a power-law distribution. To this end, let $\text{PLD}([a..b], \gamma)$ refer to an integer Power-Law Distribution with exponent $-\gamma \in \mathbb{R}$ for $\gamma \geq 1$ and values from the interval $[a..b]$; let X be an integer random variable drawn from $\text{PLD}([a..b], \gamma)$, then $\mathbb{P}[X = k] \propto k^{-\gamma}$ (proportional to) if $a \leq k \leq b$ and $\mathbb{P}[X = k] = 0$ otherwise.



■ **Figure 1** Edge Switch on an undirected graph $G = (V, E)$. To avoid ambiguity, we indicate for each $e \in E$ the orientation \vec{e} used in Definition 1.

2.2 The Edge Switching Markov Chain (ES-MC)

Definition 1 (Edge Switch, ES-MC). Let $G = (V, E) \in \mathcal{G}(\mathbf{d})$ be a simple undirected graph. We represent an edge switch $\sigma = (i, j, g)$ by two indices $i, j \in [m]$ and a direction bit g . Then, we compute $G' \in \mathcal{G}(\mathbf{d})$ based on σ as follows:

1. Let $e_1 = E[i]$ and $e_2 = E[j]$.
2. Compute new edges $(\vec{e}_3, \vec{e}_4) = \tau(\vec{e}_1, \vec{e}_2, g)$ where

$$\tau((u, v), (x, y), g) = \begin{cases} ((u, x), (v, y)) & \text{if } g = 0 \\ ((u, y), (v, x)) & \text{if } g = 1 \end{cases}.$$

3. *Reject* if either of e_3 or e_4 is a loop or already exists in E ; otherwise *accept* and set $E[i] \leftarrow e_3$ and $E[j] \leftarrow e_4$.

The *Edge Switching Markov Chain* (ES-MC) transitions from state G to state $G' \in \mathcal{G}(\mathbf{d})$ by sampling i, j , and g uniformly at random. \triangle

Observation 1. Any edge switch $\sigma = (i, j, g)$ that translates G into G' can be reversed in a single step, i.e., there exists an inverse edge switch $\tilde{\sigma}$ that translates G' back into G . If σ is rejected, it does not alter the graph ($G = G'$). Thus the claim trivially holds with $\tilde{\sigma} = \sigma$. For an accepted edge switch σ , it is easy to verify that $\tilde{\sigma} = (i, j, 0)$ reverses the effects of σ . Further observe that the probability of choosing σ in state G equals the probability of choosing $\tilde{\sigma}$ in state G' [36]. \triangle

3 Global Edge Switching Markov Chain (G-ES-MC)

Hamann et al. [44] consider the out-of-order execution of a batch consisting of ℓ edge switches. To this end, they classify the dependencies within the batch that arise if the switches were executed in-order. We adopt this characterization distinguishing between source- and target dependencies:

Definition 2 (Source/target dependencies). Two switches $\sigma_1 = (i_1, j_1, \cdot)$ and $\sigma_2 = (i_2, j_2, \cdot)$ are *source dependent* if they share at least one source index, i.e., $\{i_1, j_1\} \cap \{i_2, j_2\} \neq \emptyset$. Two switches $(e_1, f_1) \leftarrow \sigma(\cdot, \cdot, \cdot)$ and $(e_2, f_2) \leftarrow \sigma(\cdot, \cdot, \cdot)$ have a *target dependency* if they try to produce at least one common edge, i.e., $\{e_1, f_1\} \cap \{e_2, f_2\} \neq \emptyset$; this dependency is counted even if one or both edge switches are rejected. \triangle

Source dependencies can be modelled by a balls-into-bins process where edges correspond to bins and each edge switch throws a linked pair of balls into two bins chosen uniformly at random. A source dependency arises whenever a ball falls into a non-empty bin. Czumaj and Lingas [45] analyse this process in a different context. Interpreted for ES-MC, they show that

for $\ell = m$ the longest source dependency chain has an expected length of $\Theta(\log m / \log \log m)$. The distribution of target dependencies, on the other hand, depends on the graph's degree sequence as the probability that a random edge switch produces the edge $\{u, v\}$ is proportional to $\deg(u) \cdot \deg(v)$.

From an algorithmic point of view, source dependencies are more difficult to deal with if we want to process edge switches out-of-order (e.g. for parallel execution). Since each previous edge switch may or may not change the edge associated with the colliding edge index, the number of possible assignments may grow exponentially in the length of the dependency chain (if multiple chains cross). Consequently, we need to either serialize such edge switches or accommodate all possible assignments. In contrast, target dependencies only imply a binary predicate, namely whether a previous edge switch already introduced the target edge.

The above discussion suggests that it is most reasonable to parallelize ES-MC by parallelizing batches of edge switches without source dependencies. Naturally, the scalability of such a parallelization depends on the size of these batches. For ES-MC, each switch selects its source edges uniformly at random, and the probability of a source dependency between two edge switches is $\Theta(1/m)$. Thus, the expected batch size for ES-MC is $\Theta(\sqrt{m})$.

As the expected batch size for ES-MC is rather small, it is natural to ask if there exist other switch Markov Chains where the size is larger, and which thereby exhibit more parallelism. To answer this question, we define a *global switch*¹ — a batch of up to $\lfloor m/2 \rfloor$ edge switches where each edge participates in an edge switch exactly once; conceptually, we place all edges into an urn and iteratively draw without replacement pairs of edges until the urn is empty; each pair implies an edge switch. It is folklore to encode such a process in a permutation that captures the order of edges drawn [7].

Similarly to techniques of [42, 46], our proof of Theorem 1 requires a small positive probability that any global switch collapses into a single switch. We implement this by independently rejecting each switch with probability P_L .

Definition 3 (Global Switch, G-ES-MC). Let $G = (V, E)$ be a simple undirected graph. A *global switch* is represented by $\Gamma = (\pi, \ell)$ where π is a permutation of $[m]$ and ℓ an integer with $0 \leq \ell \leq \lfloor m/2 \rfloor$. The global switch Γ consists of ℓ edge switches $\sigma_1, \dots, \sigma_\ell$ that are executed in sequence, where $\sigma_k = (\pi(2k-1), \pi(2k), g_k)$ and $g_k = \mathbf{1}_{\pi(2k-1) < \pi(2k)}$, and where $\mathbf{1}$ denotes the indicator function.

The *Global Edge Switching Markov Chain* (G-ES-MC) transitions from graph G using a random global switch $\Gamma = (\pi, \ell)$. To this end, π is drawn uniformly from all permutations on $[m]$ and ℓ is drawn from a binomial distribution of $\lfloor m/2 \rfloor$ trials with success probability $0 < P_L < 1$. \triangle

By selecting ℓ from a binomial distribution and executing the first ℓ edge switches of a random permutation, we simulate $\lfloor m/2 \rfloor$ edge switches that are each executed only with probability $1 - P_L$. Also note, that the direction bits $g_k = \mathbf{1}_{\pi(2k-1) < \pi(2k)}$ are independent and unbiased random bits because the permutation π is drawn uniformly at random.

Theorem 1. Let $G \in \mathcal{G}(\mathbf{d})$ be a simple undirected graph with degree sequence \mathbf{d} . The Global Edge Switching Markov Chain started at G converges to the uniform distribution on $\mathcal{G}(\mathbf{d})$.

¹ We adapt the term *global* from [46] where it is used to describe a variant of the Curveball Markov Chain.

Proof. Any Markov Chain that is irreducible, aperiodic and symmetric converges to a uniform distribution [47, Th. 7.10]. We show that G-ES-MC has these three properties.

For **irreducibility** we observe that whenever there exists an edge switch from state A to B , there also exists a global switch from A to B (e.g. if $\ell = 1$). The state graph of ES-MC is therefore a subgraph of the state graph of G-ES-MC. In addition, both Markov Chains share the same set of states, i.e., the set of all simple graphs with the given degree sequence. Then, since the state graph of ES-MC is already strongly connected [48], so is the state graph of G-ES-MC, and thus both Markov Chains are irreducible.

For **aperiodicity** we note that a global switch $\Gamma = (\pi, \ell)$ may not alter the graph (e.g., if $\ell = 0$). Thus each state in the Markov Chain has a self-loop with strictly positive probability mass. This guarantees aperiodicity.

It remains to show the **symmetry** of transition probabilities. Let \mathcal{S}_{AB} be the set of global switches Γ that transform graph A into graph B . Then the transition probability P_{AB} from A to B equals the probability of drawing a global switch from \mathcal{S}_{AB} ,

$$P_{AB} = \sum_{\Gamma \in \mathcal{S}_{AB}} P(\Gamma),$$

where $P(\Gamma)$ is the probability of selecting Γ . A global switch $\Gamma = (\pi, \ell)$ is selected by drawing its permutation π and executing ℓ edge switches. In particular, we have

$$P(\Gamma) = \underbrace{\frac{1}{m!}}_{\text{uniform } \pi} \cdot \underbrace{\binom{\lfloor m/2 \rfloor}{\ell} (1 - P_L)^\ell P_L^{\lfloor m/2 \rfloor - \ell}}_{\text{binomially distributed } \ell}.$$

Observe that $P(\Gamma = (\pi, \ell))$ depends on ℓ , but neither on a specific choice of π nor the states A and B . Thus the symmetry $P_{AB} = P_{BA}$ follows by establishing a bijection μ_{AB} between any forward global switch $\Gamma = (\pi, \ell) \in \mathcal{S}_{AB}$ and an inverse global switch $\tilde{\Gamma} = (\tilde{\pi}, \ell) \in \mathcal{S}_{BA}$ with matching ℓ .

We construct the bijection μ_{AB} as follows. For a global switch $\Gamma = (\pi, \ell)$ that executes the edge switches $\sigma_1, \dots, \sigma_\ell$ with $\sigma_k = (\pi(2k-1), \pi(2k), g_k)$ in sequence, define the inverse global switch $\tilde{\Gamma} = (\tilde{\pi}, \ell)$. The global switch $\tilde{\Gamma}$ executes the inverse edge switches in reverse order, i.e., $\tilde{\sigma}_{\ell-k+1}$ recovers the effect of the forward edge switch σ_k .

Recall that (i) the inverse of an accepted edge switch is given by a direction flag $g = 0$, and that (ii) the forward direction bit is defined as $g_k = \mathbb{1}_{\pi(2k-1) < \pi(2k)}$. Thus, if σ_k is legal and $g_k = 1$, we need to switch the order of the edge indices in the inverse switch $\tilde{\sigma}_{\ell-k+1}$. This implies $\tilde{\pi}$ on positions $[2\ell]$. In particular, we have for $k \in [\ell]$:

$$\begin{aligned} & \left(\tilde{\pi}(2[\ell-k+1] - 1), \tilde{\pi}(2[\ell - k + 1]) \right) = \\ & \begin{cases} (\pi(2k-1), \pi(2k)) & \sigma_k \text{ is illegal or } g_k = 0 \\ (\pi(2k), \pi(2k-1)) & \sigma_k \text{ is legal and } g_k = 1 \end{cases} \end{aligned}$$

For the unused entries $i \in [2\ell+1..m]$ choose $\tilde{\pi}(i) = \pi(i)$. ◀

3.1 Mixing Time of G-ES-MC in relation to ES-MC

For simple graphs, the mixing time of variants of ES-MC have been studied for many families of degree sequences (c.f. [23, 32–36]). Recently, Erdős et al. [37] provide a survey that unifies many proofs on rapidly mixing Switching Markov chains on different types of degree sequences. Furthermore, for bipartite degree sequences a theoretical comparison

between a variant of ES-MC and the more recent Curveball Markov Chain [30] has been established [49].² This proved to be the first result regarding the mixing time for Curveball Markov Chains.

In this subsection we present the comparison framework that was used in [49] and highlight why it is difficult to apply for a comparison of G-ES-MC with ES-MC. The framework considers a Markov Chain \mathcal{M} and its *heat-bath variant* $\mathcal{M}_{\text{heat}}$ and relates the second largest eigenvalues of both Markov Chains to then compare the mixing times. For the sake of uniformity we use the same terminology: for an ergodic Markov Chain $\mathcal{M} = (\Omega, P)$ with stationary distribution π where $\pi(x) > 0$ for all $x \in \Omega$, that can be decomposed as

$$P = \sum_{a \in \mathcal{L}} \rho(a) \sum_{R \in \mathcal{R}_a} P_R$$

where

- i) \mathcal{L} is a finite index set,
- ii) ρ a probability distribution over \mathcal{L} ,
- iii) $\mathcal{R}_a = \bigcup R_{\ell,a}$ a partition of Ω for $a \in \mathcal{L}$

and where the restriction of a matrix P_R to the rows and columns of $R = R_{\ell,a}$ defines the transition matrix of an ergodic, time-reversible Markov Chain on R (and is zero elsewhere), with stationary distribution $\tilde{\pi}_R(x) = \pi(x)/\pi(R)$ for $x \in R$. As highlighted, a state transition of \mathcal{M} can be thought of as drawing an index a from \mathcal{L} and then performing a transition on R .

Then, the *heat-bath variant* $\mathcal{M}_{\text{heat}}$ of the Markov chain \mathcal{M} is given by the transition matrix

$$P_{\text{heat}} = \sum_{a \in \mathcal{L}} \rho(a) \sum_{R \in \mathcal{R}_a} \mathbf{1} \cdot \sigma_R$$

where σ_R is a row-vector given by $\sigma_R(x) = \tilde{\pi}_R(x)$ if $x \in R$ and zero otherwise, and $\mathbf{1}$ the all-ones column vector. Similar to before, $\mathcal{M}_{\text{heat}}$ can be thought of first drawing an index a from \mathcal{L} but then simply drawing a state x in R with probability $\tilde{\pi}_R(x)$.

For our purposes, in order to apply this framework we require a suitable index set \mathcal{L} , probability distribution ρ and state space decompositions \mathcal{R}_a for $a \in \mathcal{L}$, such that G-ES-MC is the heat-bath variant $\mathcal{M}_{\text{heat}}$ of ES-MC. However, in this setting G-ES-MC must by construction already provide a transition matrix $\mathbf{1} \cdot \sigma_R$ that reflects the stationary distribution π up to scaling on all restrictions R for any a . Naturally and analogously to the original proof in [49], the choice for \mathcal{L} is a set of entries in the adjacency matrix of size m that need to be switched in some order. In this case, any possible order of execution must be reflected in \mathcal{R}_a given the chosen entries provided by the initial choice a . Additionally, a transition to any such possible graph must be uniformly given by σ_R which in general does not hold.

4 Parallel Algorithms for ES-MC and G-ES-MC

In this section, we describe parallel algorithms for ES-MC and G-ES-MC. Both algorithms rely on PARALLELSUPERSTEP, a parallel algorithm that performs a superstep of edge switches $\sigma_1, \dots, \sigma_\ell$ without source dependencies while preserving the observable outcome, i.e. the graph produced is the same as if the switches were performed sequentially.

² For a general survey on random graph generation we refer the interested reader to the recent survey by Greenhill [50].

To this end, we detect all edge switches in $\sigma_1, \dots, \sigma_\ell$ that have target dependencies on other switches, and ensure that if a switch σ depends on another switch σ' , then σ is decided³ only after σ' is. For our purposes, it is convenient to think in terms of the following two types of target dependencies:

- An edge switch σ_k has an *erase dependency* via edge e on switch σ_p if σ_k has e as target edge, σ_p has e as source edge and $k > p$. In this case, σ_k will attempt to insert an edge that σ_p erases, but since $k > p$, it may well be the case that σ_k is legal, if σ_p is legal, and thus we must ensure that σ_p is decided before σ_k .
- An edge switch σ_k has an *insert dependency* via edge e on switch σ_q if both σ_k and σ_q have e as target edge and $k > q$. In this case, σ_k will attempt to insert an edge that σ_q inserts, but since $k > q$, it is the case that σ_k is illegal, if σ_q is legal, and thus we must ensure that σ_q is decided before σ_k .

Observation 2. Given a graph $G = (V, E)$ a superstep of edge switches without source dependencies attempts to remove only edges $e \in E$ and does so only once. As a direct consequence all erase dependencies for some edge $e \in E$ originate in the same switch σ . Also, for any number of insert dependencies for some edge e at most one switch σ can be successful. \triangle

Algorithm 1 shows an implementation of PARALLELSUPERSTEP in pseudocode. Before performing a superstep, we store the dependencies of the edge switches $\sigma_1, \dots, \sigma_\ell$ in a concurrent hash table T . Then, while attempting to decide a switch, this allows us to lookup the dependencies of the switch and check if the switch is ready to be decided or has unresolved dependencies. A superstep is then performed incrementally during multiple rounds. In each round, we only decide switches that have no dependencies on undecided switches. This in turn resolves the dependencies of all switches that only depend on the decided switches, and as the dependencies cannot be circular, we eventually decide all switches in this way. Then, finally, once all switches have been decided, the superstep has been performed.

We store the dependencies as tuples in a concurrent hash table, and index them by the source or target edge. For each switch σ_k , we store four tuples, one for each source and each target edge, containing the edge e , the index k of the switch, the type of operation the switch attempts to perform on the edge $t_{e,k} \in \{\text{ERASE}, \text{INSERT}\}$ and a status flag $s_k \in \{\text{UNDECIDED}, \text{LEGAL}, \text{ILLEGAL}\}$, that is initially set to UNDECIDED. We also use the same data structure to lookup the existence of edges. To this end, we assume that T implicitly stores a tuple $(e, \infty, \text{ERASE}, \text{ILLEGAL})$ for each edge that is in the graph, but not a source edge of any switch in the batch, causing a switch that attempts to insert such an edge to be decided as illegal.

Now, when attempting to decide a switch σ_k , we lookup all tuples where the edge is one of its target edges. By Observation 2, for each target edge e , there is at most one tuple stored by a switch σ_p where $t_{e,p} = \text{ERASE}$, i.e. that erases the edge. Similarly, for each target edge e , there is at most one tuple stored by a switch σ_q , that is legal, and inserts the target edge. Specifically, at any point, the only tuple that needs to be considered is the tuple with the smallest index q where $t_{e,q} = \text{INSERT}$ and $s_q \neq \text{ILLEGAL}$.

We then use this information to decide if σ_k is legal, illegal, or has to be delayed. If $k < p$ or $s_p = \text{ILLEGAL}$, then a target edge of the switch σ_k is only erased by a later switch σ_p , or not erased at all, and thus the switch is illegal. Similarly, if $k > q$ and $s_q = \text{LEGAL}$,

³ We say that a single edge switch is *decided* after we rewire the edges, if it is legal, or reject the switch, if it is illegal.

■ **Algorithm 1** PARALLELSUPERSTEP

```

Data: edge list  $E$ , superstep of switches  $S$ 
1  $T \leftarrow \emptyset$  // Initialize dependency table  $T$ 
2 for  $\sigma_k \in S$  in parallel
3    $(i, j, g) \leftarrow \sigma_k, e_1 \leftarrow E[i], e_2 \leftarrow E[j]$ 
4   Compute target edges  $(\vec{e}_3, \vec{e}_4) \leftarrow \tau(\vec{e}_1, \vec{e}_2, g)$ 
5    $\forall e_a \in \{e_1, e_2\}: T.\text{store}(e_a, k, \text{ERASE}, \text{UNDECIDED})$ 
6    $\forall e_b \in \{e_3, e_4\}: T.\text{store}(e_b, k, \text{INSERT}, \text{UNDECIDED})$ 
7  $U \leftarrow S$  // Initialize array  $U$  for undecided switches
8 while  $U$  not empty do // Perform superstep  $S$ 
9    $D \leftarrow \emptyset$  // Initialize array  $D$  for delayed switches
10  for  $\sigma_k \in S$  in parallel
11     $(i, j, g) \leftarrow \sigma_k, e_1 \leftarrow E[i], e_2 \leftarrow E[j]$ 
12    Compute target edges  $(\vec{e}_3, \vec{e}_4) \leftarrow \tau(\vec{e}_1, \vec{e}_2, g)$ 
13     $s_k \leftarrow \text{LEGAL}$  // Initialize status  $s_k$  of  $\sigma_k$ 
14    for  $e \in \{e_3, e_4\}$  do // Lookup dependencies
15      if  $e$  is self-loop then
16         $s_k \leftarrow \text{ILLEGAL}$ 
17       $p, s_p \leftarrow T.\text{lookup}(e, \text{ERASE})$ 
18       $q, s_q \leftarrow T.\text{lookup\_min}(e, \text{INSERT})$ 
19      if  $k < p \vee s_p = \text{ILLEGAL}$  then
20         $s_k \leftarrow \text{ILLEGAL}$ 
21      if  $k > q \wedge s_q = \text{LEGAL}$  then
22         $s_k \leftarrow \text{ILLEGAL}$ 
23      if  $s_k \neq \text{ILLEGAL}$  then
24        if  $k > p \wedge s_p = \text{UNDECIDED}$  then
25           $s_k \leftarrow \text{UNDECIDED}$ 
26        if  $k > q \wedge s_q = \text{UNDECIDED}$  then
27           $s_k \leftarrow \text{UNDECIDED}$ 
28      if  $s_k = \text{LEGAL}$  then
29         $E[i] \leftarrow e_3, E[j] \leftarrow e_4$  // Success, rewire the edges
30      else if  $s_k = \text{UNDECIDED}$  then
31         $D.\text{append}(\sigma_k)$  // Delay the switch until the next round
32       $\forall e_a \in \{e_1, e_2\}: T.\text{update}(e_a, k, \text{ERASE}, s_k)$ 
33       $\forall e_b \in \{e_3, e_4\}: T.\text{update}(e_b, k, \text{INSERT}, s_k)$ 
34  barrier: wait until all switches completed
35   $U \leftarrow D$ 

```

then a target edge of σ_k is already inserted by the earlier switch σ_q and thus the switch is illegal. Otherwise, if σ_k has an erase or insert dependency, but the status of the other switch is UNDECIDED, the switch cannot be decided yet, and must be delayed. In any other case the switch is legal.

If the switch is legal, the edges are rewired and the status of the tuples is set to LEGAL. Otherwise, if the switch is illegal, the status of the tuples is set to ILLEGAL. Finally, if the switch has unresolved dependencies, we delay it until the next round.

4.1 ParES

We first describe how PARES, a parallelization of ES-MC, can be implemented by using PARALLELSUPERSTEP (see Algorithm 2).

The algorithm first populates an array R with the requested number of switches r by sampling two edge indices and a direction bit for each switch. The switches in R are then performed during multiple iterations, where each iteration performs a superstep. In each iteration, we let s denote the number of switches that were already performed. We then identify the next superstep by finding the smallest index $t > s$ of a switch that has a source collision with another switch in the sequence $\sigma_s, \dots, \sigma_r$. To this end, we insert for each switch σ_k two tuples (i, k) and (j, k) into a concurrent hash set. If for any edge index i , a different

■ **Algorithm 2** PARES

Data: edge list E , requested number of switches r
// Initialize array of requested switches R

```

1  $R \leftarrow \emptyset$ 
2 for  $k$  from 1 to  $r$  in parallel
3   Sample edge indices  $i, j \sim [m]$  with  $i \neq j$ 
4   Sample direction bit  $g \sim \{0, 1\}$ 
5    $R[k] \leftarrow (i, j, g)$ 
6  $s \leftarrow 1$ 
7 while  $s \leq r$  do
8   // Initialize hash set  $H$ 
9    $H \leftarrow \emptyset$ 
10  // Find superstep without source dependencies
11   $t \leftarrow r + 1$ 
12  for  $k$  from  $s$  to  $(t - 1)$  in parallel
13    // Check for collision and update lower bound  $t$ 
14     $(i, j, g) \leftarrow R[k]$ 
15     $k' \leftarrow H.\text{insert\_if\_min}(i, k)$ 
16     $k'' \leftarrow H.\text{insert\_if\_min}(j, k)$ 
17     $t' \leftarrow \max\{k, k'\}, t'' \leftarrow \max\{k, k''\}$ 
18     $t \leftarrow \min\{t, t', t''\}$ 
19  // Perform superstep  $\sigma_s, \dots, \sigma_{t-1}$ 
20   $S[k - s + 1] \leftarrow R[k] \quad \forall s \leq k \leq t - 1$ 
21   $\text{PARALLELSUPERSTEP}(E, S)$ 
22   $s \leftarrow t$ 

```

■ **Algorithm 3** PARGLOBALES

Data: edge list E , requested number of global switches r

```

1 for  $i$  from 1 to  $r$  do
2   // Select random global switch  $\Gamma = (\pi, \ell)$ 
3   Sample random permutation  $\pi$  of  $[m]$ 
4   Sample  $\ell \sim \text{Binom}(\lfloor m/2 \rfloor, 1 - P_L)$ 
5   // Perform global switch
6    $S[k] \leftarrow (\pi(2k - 1), \pi(2k), \mathbb{1}_{\pi(2k-1) < \pi(2k)}) \quad \forall 1 \leq k \leq \ell$ 
7    $\text{PARALLELSUPERSTEP}(E, S)$ 

```

switch $\sigma_{k'}$ exists which has inserted the same index i , we know that the first collision occurs at most at index $t' = \max\{k, k'\}$ so we set the lower bound to $t \leftarrow \min\{t, t'\}$. In addition, if $k < k'$, we replace the tuple (i, k') by (i, k) to detect further switches $k < k'' < k'$ which may update the lower bound. Once all switches up to the lower bound t have been inserted, we know that $\sigma_s, \dots, \sigma_{t-1}$ is a sequence of edge switches without source dependencies. These edge switches are then performed by using PARALLELSUPERSTEP.

4.2 ParGlobales

Implementing the parallelization PARGLOBALES of G-ES-MC is much simpler (see Algorithm 3). As a global switch Γ contains no source dependencies by definition, the full algorithm only consists of one loop, which in each iteration selects a random global switch $\Gamma = (\pi, \ell)$ and then calls on PARALLELSUPERSTEP to perform this switch.

4.3 Analysis of ParGlobales

A necessary condition for good scaling of PARGLOBALES is that the number of rounds to perform a global switch is small. Note that in practice, the number of rounds depends on the scheduling and assignment of switches to processors. This is because we store the

dependencies in a concurrent hash table: if a switch σ depends on a switch σ' , but the processor assigned to σ' finishes writing the result before the processor assigned to σ attempts to decide its switch, then both switches can be decided in the same round, and the dependency will not affect the number of rounds. Similarly, if σ and σ' are assigned to the same processor, the dependency will not affect the number of rounds. Still, even for a worst-case scheduler, that always moves dependent switches to the next round, we can show that the expected number of rounds is small for bounded degree graphs, regular graphs, and power-law graphs with sufficiently high degree exponent.

Theorem 2. Let R denote the number of rounds needed to perform a global switch $\Gamma = (\pi, \ell)$ on a graph $G = (V, E)$. If G has $m = |E|$ edges and each node has at most degree $d \leq \Delta$, we have $R \leq 4\Delta^2/m$ in expectation over π .

Corollary 1. If each node in G has at most degree $d \leq \sqrt{m}$, we have $R \leq 4$ in expectation over π .

Corollary 2. If G is a d -regular graph, we have $R \leq 4$ in expectation over π .

Proof. Observe that $R < k$ unless there exists a chain of switches $\sigma^{(1)}, \dots, \sigma^{(k)} \in \Gamma$, so that each switch $\sigma^{(r+1)}$ with $1 \leq r < k$ depends on switch $\sigma^{(r)}$ and cannot be decided before round $r + 1$. There are four ways that $\sigma^{(r+1)}$ can depend on $\sigma^{(r)}$, either (1) $\sigma^{(r)}$ erases edge e_a , and $\sigma^{(r+1)}$ inserts e_a , or (2) $\sigma^{(r)}$ inserts edge e_b , and $\sigma^{(r+1)}$ inserts e_b , or (3) and (4), the symmetric cases for the other two edges inserted or erased by $\sigma^{(r)}$. Recall that a switch $\sigma_i \in \Gamma$ inserts edge $e = \{u, v\}$ if the permutation π contains an edge incident with u and an edge incident with v in positions $2i - 1$ and $2i$. There are d_u and d_v such edges, respectively, and as only one assignment of the edges to the positions will give the necessary direction bit g_i , each edge must be assigned to one specific position. Thus, the probability that switch σ_i inserts edge e is at most

$$\frac{\overbrace{\# \text{ permutations such that } \sigma_i \text{ inserts } e}^{d_u d_v (m-2)!}}{m!} = \frac{d_u d_v}{m(m-1)} \leq \frac{\Delta^2}{m(m-1)} \leq \frac{2\Delta^2}{m^2}. \quad (1)$$

total permutations

Hence, for each switch $\sigma_i \in \Gamma$, the probability that the switch with the next index σ_{i+1} inserts one of the four edges, and depends on σ_i , is at most $8\Delta^2/m^2$. Then, in expectation, if switch $\sigma^{(r)}$ has index i , $\sigma^{(r+1)}$ has index at least $i + m^2/8\Delta^2$, and since Γ contains only $m/2$ switches, each chain has expected length $k \leq 4\Delta^2/m$. \blacktriangleleft

The following theorem gives a sharper bound for graphs with a skewed degree sequence such as power-law graphs.

Theorem 3. Let R denote the number of rounds needed to perform a global switch $\Gamma = (\pi, \ell)$ on a graph $G = (V, E)$, let $S = \{\{u, v\} : u, v \in V, u \neq v\}$ denote the set of possible edges, and let $P_2 = \sum_{e=\{u,v\} \in S} (d_u d_v / m(m-1))^2$. Then, we have $R = \mathcal{O}(P_2 m)$ in expectation over π .

Proof. For each switch $\sigma_i \in \Gamma$, the probability that the switch with the next index σ_{i+1} inserts a specific edge e that σ_i erases is

$$\sum_{e=\{u,v\} \in S} \frac{\mathbb{1}_{e \in E}}{m} \frac{d_u d_v}{\underbrace{m(m-1)}_{\substack{\sigma_i \text{ erases } e \\ \sigma_{i+1} \text{ inserts } e}}} \leq \frac{1}{m} \sum_{e=\{u,v\} \in S} \frac{d_u d_v}{m(m-1)} \leq \frac{1}{m}. \quad (2)$$

For an insert dependency to arise, σ_{i+1} must insert a specific edge e that σ_i inserts, and the probability for this event is given by

$$\sum_{e=\{u,v\} \in S} \underbrace{\frac{d_u d_v}{m(m-1)}}_{\sigma_i \text{ inserts } e} \underbrace{\frac{(d_u-1)(d_v-1)}{(m-2)(m-3)}}_{\sigma_{i+1} \text{ inserts } e} = \mathcal{O}(P_2). \quad (3)$$

Hence, for each switch $\sigma_i \in \Gamma$, the probability that the switch with the next index σ_{i+1} depends on σ_i is at most $2/m + \mathcal{O}(P_2)$, and as Γ contains $m/2$ switches, each dependency chain has expected length $\mathcal{O}(P_2 m)$. ◀

Lemma 1. If G is a power-law graph with degree exponent $\gamma > 2$, we have $P_2 = \mathcal{O}\left(1/n + n^{(2+\gamma-\gamma^2)/(\gamma-1)}\right)$.

Corollary 3. If G is a power-law graph with degree exponent $\gamma > 1 + \sqrt{2}$, we have $R = \mathcal{O}(1)$ in expectation over π .

Proof. We will use known properties of power-law graphs to find a suitable upper bound on P_2 . Recall that if G is a power-law graph with degree exponent $\gamma > 2$, then G has $N_d = \mathcal{O}(nd^{-\gamma})$ nodes with degree d and maximum degree $\Delta = \mathcal{O}(n^{1/(\gamma-1)})$. Rewriting the expression for P_2 as a sum over degree groups yields

$$P_2 = \sum_{i=1}^{\Delta} N_i \sum_{j=1}^{\Delta} N_j \left(\frac{ij}{m(m-1)} \right)^2 \quad (4)$$

$$\leq \sum_{i=1}^{\Delta} N_i \sum_{j=1}^{\Delta} N_j \frac{i^2 j^2}{n^4} \quad (5)$$

$$= \mathcal{O} \left(\sum_{i=1}^{\Delta} \frac{1}{i^\gamma} \sum_{j=1}^{\Delta} \frac{1}{j^\gamma} \frac{i^2 j^2}{n^2} \right). \quad (6)$$

Observe that the terms in P_2 where $ij < n$ sum to at most $\mathcal{O}(1/n)$. For each of the remaining terms, we have $ij \geq n$, so we may write their sum as

$$\sum_{i=1}^{\Delta} \frac{1}{i^\gamma} \sum_{j=\lfloor \frac{n}{i} \rfloor}^{\Delta} \frac{1}{j^\gamma} \frac{i^2 j^2}{n^2}. \quad (7)$$

Now, using $j^\gamma > j^2$, and substituting $\lfloor \frac{n}{i} \rfloor$ for j , we see that i^γ/n^γ is an upper bound on each term of the inner sum over j , and obtain

$$\sum_{i=1}^{\Delta} \frac{1}{i^\gamma} \sum_{j=\lfloor \frac{n}{i} \rfloor}^{\Delta} \frac{1}{j^\gamma} \frac{i^2 j^2}{n^2} \leq \sum_{i=1}^{\Delta} \frac{1}{i^\gamma} \sum_{j=\lfloor \frac{n}{i} \rfloor}^{\Delta} \frac{i^\gamma}{n^\gamma} \leq \frac{\Delta^2}{n^\gamma} = \mathcal{O}\left(n^{(2+\gamma-\gamma^2)/(\gamma-1)}\right). \quad (8)$$

◀

5 Implementation

In this section, we describe the implementation of our sequential and parallel algorithms and the data structures used therein. In addition to the PARGLOBALES algorithm described in Section 4, we implement the following sequential and parallel algorithms.

- SEQES: a fast sequential implementation of ES-MC.
- SEQGLOBALES: a sequential implementation of G-ES-MC.
- NAIVEPARES: a simplistic parallelization of ES-MC.

All algorithms (including previously existing ones) are implemented in C++.

5.1 NaiveParES

To establish a performance baseline for parallel algorithms, we implement NAIVEPARES, a simplistic parallelization of ES-MC.

Each PU (processing unit) performs switches independently while synchronizing implicitly only by preventing concurrent updates of individual edges. To ensure that no edge is erased or inserted twice, we store the edges in a concurrent hash-set using the following semantics: to remove an edge from the set, a ticket has to be acquired first; this can be done by locking an existing edge or by inserting-and-locking a new edge. These operations are implemented using a compare-and-exchange primitive. Concurrent updates to the same edge are sequenced by the hardware.

Note that this implementation performs all edge switches that are legal after synchronization but ignores dependencies between edge switches. In contrast to the exact parallelizations PARES and PARGLOBALES, it can thereby deviate from the intended Markov Chain.

5.2 Graph and dependency representation

Most ES-MC implementations use an adjacency list to store the graph and manipulate it with each switching [40, 51–54]⁴. This design choice often leads to an easy integration with other algorithms. However, ES-MC requires a graph representation that efficiently supports edge insertion, deletion, and existence queries. Unfortunately, an adjacency list cannot support updates and search both in constant time (cf. the hybrid data structure of [54]).

In contrast, hash-sets support all required operations in expected constant time. To this end we first identify each possible edge with a unique integer. For instance, if nodes $u, v \in V$ are stored as 32-bit integers, then each possible edge $\{u, v\} \in E$ can be identified by the 64-bit integer where the first 32-bits are set equal to the smaller node $u < v$ and the remaining 32-bits are set equal to the larger node $v > u$ (recall that we disallow loops). To store a graph in a hash set we insert for each edge its unique identifier. Checking if an edge is contained in the graph is possible by checking if its identifier is contained in the hash-set. When performing a legal edge switch, we delete both identifiers of the source edges and then insert both identifiers of the target edges.

From a practical point of view, we require a hash-set implementation that can handle a roughly balanced mix of insertions, deletions, and search queries. After preliminary experiments on various graphs, machines, and hash-set implementations⁵, we find that in most cases ROBINMAP with a maximum load-factor of 1/2 is the fastest sequential solution. Observe that for performance reasons, all our implementations use hash-tables where the number of buckets is a power-of-two; hence the actual load-factor can be lower. Our hash

⁴ [40] use a *reduced* adjacency matrix that only stores one directed edge. [53] use a different MC with the same switchings. [54] interleave ES-MC with connectivity checks after each edge switching. They use an adjacency list where high-degree nodes store their neighborhoods in individual hash tables.

⁵ We considered the following hash-sets: <https://gcc.gnu.org/>, <https://github.com/{Tessil/robin-map, Tessil/hopscotch-map, sparsehash/sparsehash}>.

function uses the 64bit variant of the `crc32` instruction available on x64 processes with SSE 4.2 [55].

Our parallel algorithms require concurrent hash-tables with stable iterators (i.e., once an element is placed into a bucket, it is not moved until it gets erased). It is folklore that such a data structure can be efficiently implemented with open addressing and lock-free compare-and-swap instructions (cf. [56]).

We implement the locking of edges as follows: each edge is kept in an 64bit-wide bucket, where 56 bits are used to store the edge and 8 bit are reserved for locking. To acquire a lock, a PU tries to compare-and-swap its thread id into the lock bits and succeeds only if the bucket previously kept the edge in an unlocked state. This implementations allows us to process graphs with up-to $n \leq 2^{28}$ nodes on $\mathcal{P} < 256$ threads. Observe that these restrictions can be lifted quite easily, as virtually all relevant processors support 128bit compare-and-swap instructions with only moderate performance penalties.

5.3 Sampling edges

Pseudo-random bits are generated using the MT19937-64 variant of the Mersenne Twister [57] implemented by `libstdc++` and translated into unbiased random integers using [58]. Random permutations are sampled in parallel with an optimized implementation inspired by [58, 59].

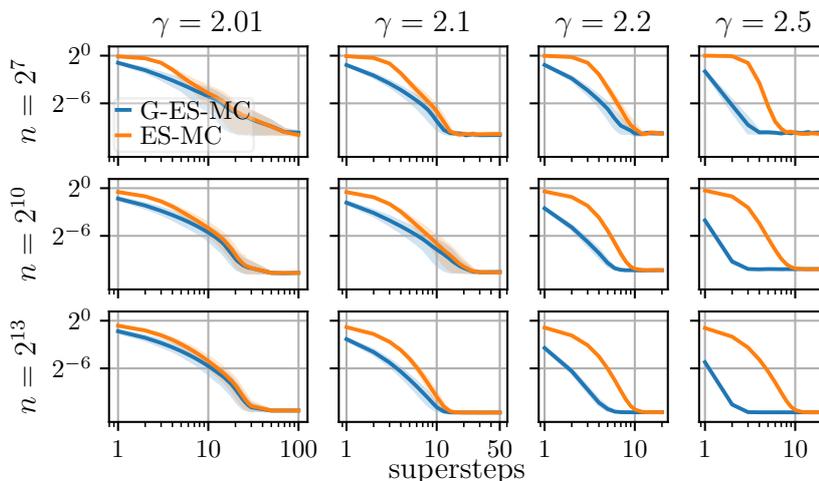
To sample edges uniformly at random we consider two options: Firstly, we maintain an auxiliary array of edges. In order to sample an edge uniformly at random, we read from a random index — this closely resembles the way we introduced edge switching in the previous chapters. Secondly, the use of open-addressing hash-tables allows us to directly sample from the hash-set by repeatedly drawing random buckets until we hit the first non-empty one.

While the second option avoids additional memory, it leads to a time trade-off: while all queries discussed in Section 5.2 benefit from a low load-factor L , the sampling time is geometrically distributed with a success probability of L . It, thus, favors a high load-factor. In preliminary experiments, we found that decreasing the load factor to allow faster queries and sampling using an additional array yields up-to 30 % faster overall performance compared to balancing the load factor for both queries and sampling.

5.4 Prefetching

The rewiring of random edges inherently leads to unstructured accesses to main memory, especially if the graph is represented in a hash set. While [44] propose an I/O-efficient edge switching implementation for graphs exceeding main memory, their solution requires to repeatedly sort the edge list (and other data structures). In the context of a parallel algorithm, this sorting step alone is more expensive than a global switch using unstructured accesses.

We therefore accept the random I/Os and accelerate them using prefetching instructions. To this end, we split all insertion, deletion, and search queries to our hash-sets into two: in a first step, we hash the key and identify the bucket in which the item is placed if there is no collision. We then prefetch this bucket as well as its direct successor, and return precomputed values that are required when we carry out the actual operation in a second step. Since we use linear probing hash-sets with a low load factor and a prefetch in advance, we effectively eliminate almost all cache misses if there is sufficient time between both steps. To increase this time window, we use a pipeline of four edge switches in different progress stages.



■ **Figure 2** Fraction of non-independent edges as a function of the thinning value k as a multiple of the supersteps for the SYNPLD dataset where $(n, \gamma) \in \{2^7, 2^{10}, 2^{13}\} \times \{2.01, 2.1, 2.2, 2.5\}$. Each data point is represented by its mean value μ (line) and its 2σ error (shade).

6 Experiments

In the following, we empirically investigate the mixing times of the Markov Chains and the runtime performances of their derived algorithms. The performance benchmarks are built with GNU g++-9.3 and executed on a machine equipped with an AMD EPYC 7702P 64-core processor running Ubuntu 20.04.

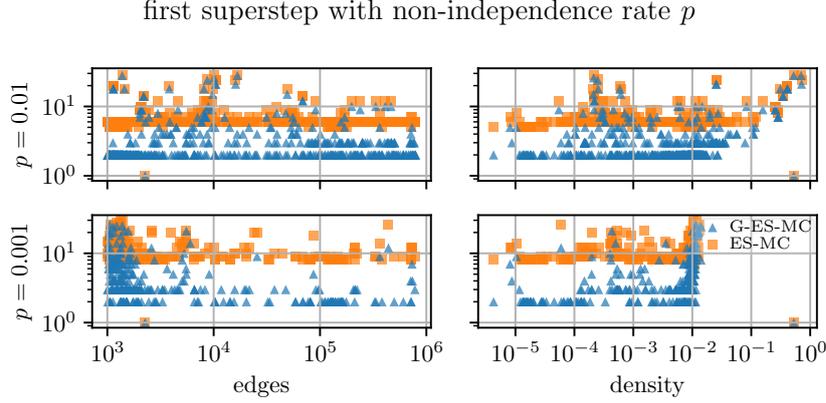
Our experiments are performed on the following datasets:

- (SYNGNP) — We generate $G(n, p)$ graphs [60] (each edge exists independently with probability p) for varying node counts n and p .
- (SYNPLD) — For varying node counts n and degree exponents γ , we generate power-law degree sequences according to the degree distribution $\text{PLD}([1.. \Delta], \gamma)$ where the maximum degree is set to $\Delta = n^{1/(\gamma-1)}$ matching the analytic bound [61]. Thereafter, the generated sequences are materialized by the Havel-Hakimi algorithm [8,9]. Both steps are performed using NetworkKit [51].
- (NETREP) — We consider graphs from the network repository [62] where we exclude the unsuitable categories *dimacs*, *dimacs10*, *graph500* (benchmark graphs), *dynamic*, *misc* (unclassified graphs), *rand* (synthetic graphs) and *tsc* (temporal graphs). To ensure that the graphs are simple and undirected, we perform the following modifications: all directed edges (u, v) are replaced by the undirected edges $\{u, v\}$, and self-loops and multi-edges are removed.

6.1 Empirical Mixing Time of ES-MC and G-ES-MC

Here, we compare the mixing times of ES-MC and the novel G-ES-MC. While we argue in Section 3 that the lack of source dependencies of G-ES-MC improves parallelizability over ES-MC, it is a priori unclear whether this restriction affects the randomization quality of the Markov Chain.

In practice, the mixing time is approximated by empirical proxies or estimated by data driven methods [63,64]. The former measures the convergence to the stationary distribution



■ **Figure 3** Scatterplots the NETREP dataset where the x -coordinate either denotes the number of edges m (left) or the density $m/\binom{n}{2}$ (right) of a graph with n nodes and m edges. The y -coordinate represents the first superstep k at which the mean fraction of non-independent edges drops below either 1×10^{-2} (top) or 1×10^{-3} (bottom). We do not use large primes and numbers with many divisors as thinning values. This yields an uneven (but inconsequential) quantization of the y -axis. The outlier that merely requires $k = 1$ supersteps for both Markov Chains possesses only two unique node degrees.

by convergence of its proxy or by some aggregated value. In doing so, the Markov Chain is reflected by a projection which may converge faster [65]. The result depends on the proxy and might be insufficient for other more sensitive proxies. Additionally, it has been observed that common measures, e.g. assortativity coefficients, clustering coefficients, diameter, maximum eigenvalue and triangle count, are less sensitive than data-driven methods [44, 64]. Thus, we consider the *autocorrelation analysis*, an approximate non-parametric method [64].

The autocorrelation analysis proceeds in two steps. First, execute the Markov Chain for a large number of steps K , and for each possible edge e track in a binary time-series $\{Z_t\}$ whether edge e exists at time t . By its own, $\{Z_t\}$ and its transitions will be correlated [63] which naturally indicates that a single Markov Chain step is insufficient.

Consider now the k -thinned chain $\{Z_t^k\}$ which retains every k -th entry of $\{Z_t\}$. The k -thinned chain will have smaller autocorrelation and begin to resemble independent draws from a distribution for sufficiently large k . At some point, the thinned time-series should resemble an independent process more than a first-order Markov process. To determine which of the models is a better fit, the Bayesian Information Criterion (BIC) is computed using the G^2 -statistic [66] (see [64] for details). Thus, in a second step, the time-series $\{Z_t\}$ is progressively thinned to determine independent edges for the thinned time-series $\{Z_t^k\}$ for increasing values of k .

For our purposes, instead of first computing the whole time-series $\{Z_t\}$ and then considering increasing thinning values in a post-processing step, we define a fixed set of thinning values T and aggregate relevant entries of $\{Z_t\}$ on-the-fly for each $k \in T$. While this approach is far less memory-consuming, we cannot recover for each edge the earliest point of time it would have been deemed independent. Instead, for a thinning value k , we report the fraction of edges that would be deemed independent irrespective of a smaller thinning value $k' < k$.

In this context, we compare ES-MC and G-ES-MC. In order to visually align the results we define a *superstep* for both Markov Chains. To this end, let $m/2$ uniform random edge switches and one uniform random global switch be a designated superstep for ES-MC and

G-ES-MC, respectively. This accounts for the fact that one global switch potentially executes $m/2$ (non-uniform) edge switches.

We first consider SYNPLD and generate for each $(n, \gamma) \in \{2^7, 2^{10}, 2^{13}\} \times \{2.01, 2.1, 2.2, 2.5\}$ forty power-law graphs (we limit the largest node count to $n = 2^{13}$ since the longest individual run already took 18 hours using an Intel Skylake Gold 6148 processor). In Figure 2 we report the mean fraction of non-independent edges depending on the number of supersteps for a subset of the node counts and degree exponents. For highly skewed degree sequences, e.g. $\gamma = 2.01$, the G-ES-MC performs slightly better than the ES-MC for small supersteps. Increasing the number of supersteps results in matching performances for both. For larger degree exponents $\gamma \geq 2.2$ G-ES-MC consistently outperforms ES-MC where the advantage increases with γ . We observe both features for up to two orders of magnitude, and we expect this to hold for even larger values of n .

Next we investigate real-world graphs of the NETREP dataset. Due to the high computational cost, we restrict ourselves to graphs with $1000 \leq m \leq 800\,000$ edges. To further reduce the cost, we perform the autocorrelation analysis only for the edges of the initial graph, reducing the memory footprint of each thinning to $\Theta(m)$ where m is the number of edges. In Figure 3 we present for each graph the first reported superstep at which the mean fraction of non-independent edges of at least 15 runs is below a threshold τ . For $\tau = 1 \times 10^{-2}$, G-ES-MC seems to consistently outperform ES-MC except for very dense graphs where the performance is similar. The $\tau = 1 \times 10^{-3}$ is reached by only 46% of the 594 instances within 30 supersteps. Here, G-ES-MC still outperforms ES-MC on most instances except for moderately dense graphs on which both chains converge significantly slower.

6.2 Performance Benchmarks

In this section, we benchmark our ES-MC and G-ES-MC implementations. In each experiment, we run a subset of the implementations on the same initial graph and measure the average time required to initialize the data structures and perform 20 supersteps (e.g. 10 switches per edge). In practice, common choices [24, 38, 39] are 10 to 30 switches per edge. As G-ES-MC typically requires fewer supersteps (compare Section 6.1), this gives a slight advantage to ES-MC over the G-ES-MC implementations.

6.2.1 Runtime

We compare existing sequential implementations to our solutions and report absolute runtimes. To this end, we benchmark all implementations on a sample of graphs from NETREP and report their runtimes in Table 4. We select the graphs in this sample to cover a variety of sizes, average degrees and maximum degrees. As some of the networks are quite large, we set a timeout of 1000 seconds.

We first compare SEQES and SEQGLOBALES, our sequential ES-MC and G-ES-MC solutions, with existing implementations from NetworKit [51] and Gengraph [31]. Our solutions run 15-50 times faster than NetworKit and 5-10 times faster than Gengraph. We also observe that SEQGLOBALES is faster than SEQES on large graphs, where shuffling is more efficient than sampling the edges, whereas SEQES runs faster on small graphs. In conclusion, our sequential implementations provide a meaningful baseline to measure further speed-ups.

Next, we report the runtimes of the parallel algorithms. For $\mathcal{P} = 32$ PUs, all parallel implementations run much faster than the sequential implementations. On the largest graph, only the parallel implementations were able to perform 20 supersteps before the

■ **Figure 4** Runtimes in seconds on a sample of graphs from NETREP sorted by network size. The left columns lists the graph, number of nodes n , number of edges m and maximum degree d_{\max} . The center columns list the sequential and parallel implementations for $\mathcal{P} = 1$ PU. The right columns list the parallel implementations for $\mathcal{P} = 32$ PUs. The best time in each group is indicated by the bold font. A dash (—) indicates a runtime of more than 1000 sec. for 20 supersteps.

Graph	n	m	d_{\max}	NetworkKit	Gengraph	SEQES	SEQGLOBALES	NAIVEPARES	PARGLOBALES	NAIVEPARES	PARGLOBALES
				$\mathcal{P} = 1$				$\mathcal{P} = 32$			
soc-twitter-mpi-sws	41 M	1.2 B	2.9 M	—	—	—	—	—	—	251	397
bn-human-Jung2015	1.8 M	146 M	8.7 K	—	—	517	460	448	784	20.0	36.7
tech-p2p	5.7 M	140 M	675 K	—	—	530	464	477	788	21.3	37.2
socfb-konect	59 M	92 M	4.9 K	—	—	287	253	228	459	11.9	21.7
ca-hollywood2009	1 M	56 M	11 K	—	686	140	112	116	244	8.1	11.4
inf-road-usa	23 M	28 M	9	619	186	49.2	41.4	53.6	97.0	5.2	5.1
bio-human-gene1	220 K	12 M	7.9 K	512	109	12.3	12.5	18.1	32.0	1.3	2.0
web-wikipedia2009	1.8 M	4.5 M	2.6 K	65.4	36.5	4.7	4.9	6.6	9.7	0.58	0.95
cit-HepTh	22 K	2.4 M	8.7 K	45.0	20.4	2.2	2.3	3.4	5.3	0.25	0.47
email-enron-large	33 K	180 K	1.3 K	0.92	0.44	0.12	0.14	0.21	0.37	0.06	0.07
rec-amazon	91 K	120 K	5	0.57	0.16	0.10	0.11	0.18	0.17	0.06	0.05

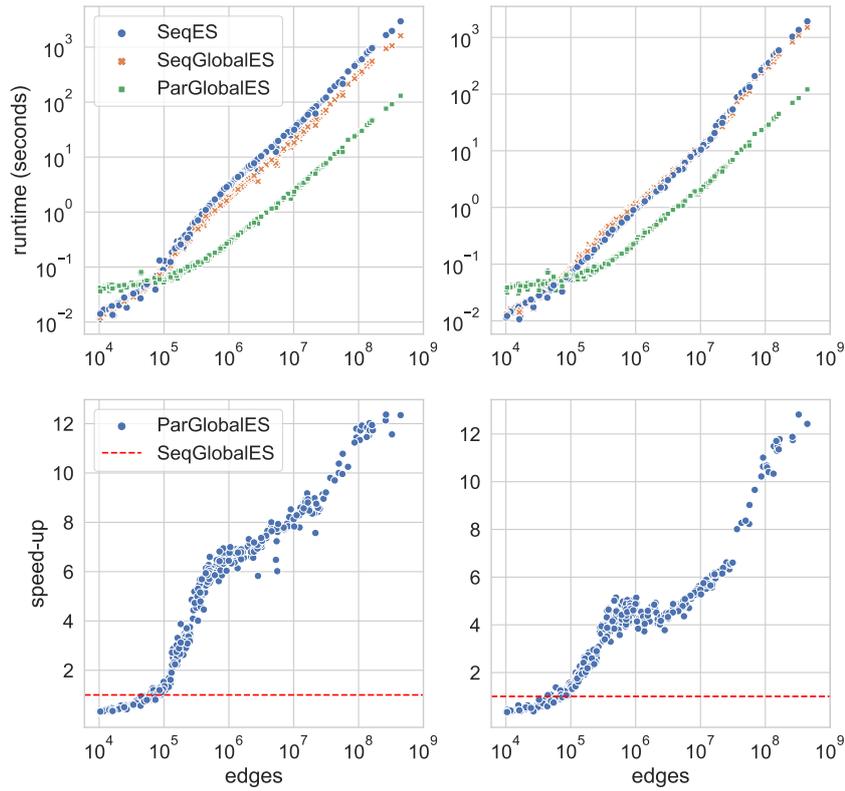
timeout. Here, PARGLOBALES is up to 12 times faster than SEQGLOBALES. On all graphs, PARGLOBALES only shows a slowdown of at most 2 compared to the baseline algorithm NAIVEPARES. In this context, it is important to recall that NAIVEPARES is not an exact parallelization since it ignores dependencies between edge switches.

In Figure 5, we evaluate SEQES, SEQGLOBALES and PARGLOBALES on all graphs from NETREP with at least $m = 10^4$ edges. For each graph, we run SEQES and SEQGLOBALES on one PU and PARGLOBALES on $\mathcal{P} = 32$ PUs and report the absolute runtimes and the speed-up of PARGLOBALES over SEQGLOBALES. On all graphs with $m > 10^5$, the parallel algorithm is faster than the sequential algorithms and the observed speed-up increases with the size of the graph.

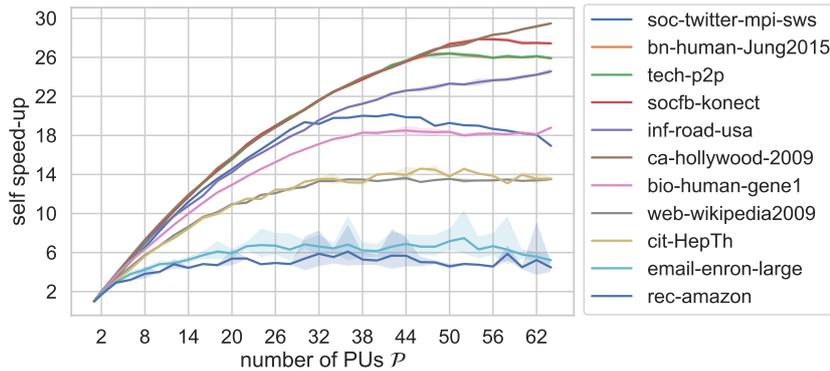
6.2.2 Scaling

We first report the self speed-up of PARGLOBALES on the sample of graphs from NETREP in dependence of \mathcal{P} (Figure 6). For larger graphs, the maximum speed-up ranges between 20 and 30 using 32 to 64 PUs. On the two smallest graphs, the work is likely too small to be efficiently parallelized (e.g. compare Figure 4). An outlier is the largest graph (soc-twitter-mpi-sws); a likely explanation for the slightly low speed-up on this graph is that the highly skewed degree sequence increases the number of target dependencies and the synchronization overhead.

To measure the influence of the graph properties on the runtime, we consider synthetic graphs. We first consider graphs from SYNGNP for various n and p , and plot the runtime as a function of the average degree $\bar{d} = 2m/n$ in Figure 7. The edge probability p seems to have no significant effect on the runtime, even in the case where the average degree approaches the possible maximum $n - 1$ (bottom-right in the plot). This is a consequence of Theorem 2. As



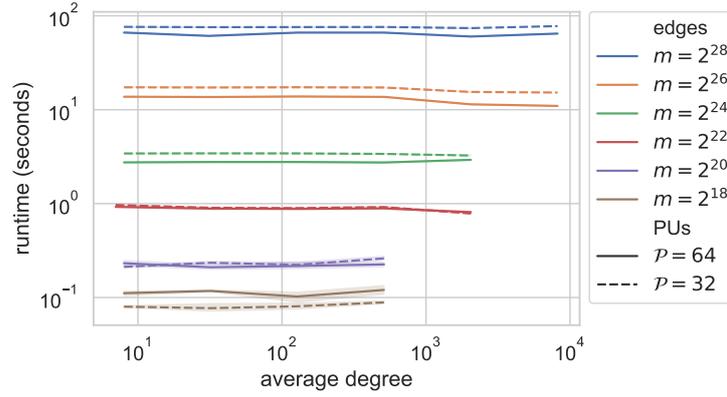
■ **Figure 5** Scatterplot showing runtimes and speed-ups on NETREP. Top row: runtimes of SEQES and SEQGLOBALES on $\mathcal{P} = 1$ PU and PARGLOBALES on $\mathcal{P} = 32$ PUs. The color and symbol indicates the algorithm. Bottom row: speed-ups of PARGLOBALES over SEQGLOBALES. The height of the red dashed line corresponds to a speed-up of 1. Left column: without prefetching. Right column: with prefetching.



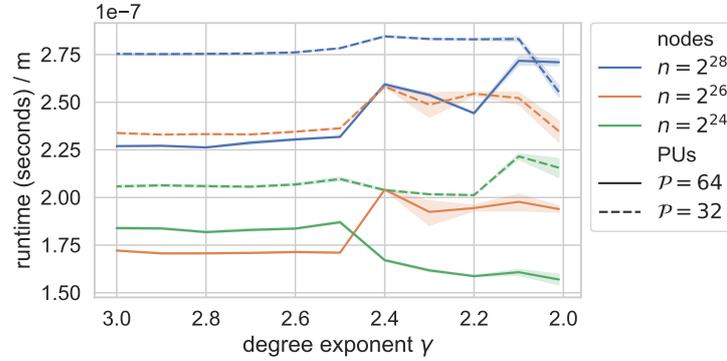
■ **Figure 6** Strong scaling of PARGLOBALES on a sample of graphs from NETREP for $1 \leq \mathcal{P} \leq 64$. The line colors indicate the graph and are sorted by graph size.

$G(n, p)$ graphs are sufficiently close to regular with high probability, the number of rounds required to perform a global switch is constant regardless of p .

Next, we consider power-law graphs from SYNPLD with degree exponent $3 \geq \gamma \geq 2.01$ to evaluate the influence of the degree distribution’s skewness. Note that increasing γ increases



■ **Figure 7** Runtime of PARGLBALES on graphs from SYNGNP where $m \in \{2^{18}, 2^{20}, 2^{22}, 2^{24}, 2^{26}\}$ in dependence of the average degree $\bar{d} = 2m/n$. The color indicates the number of edges and the line style the number of PUs.

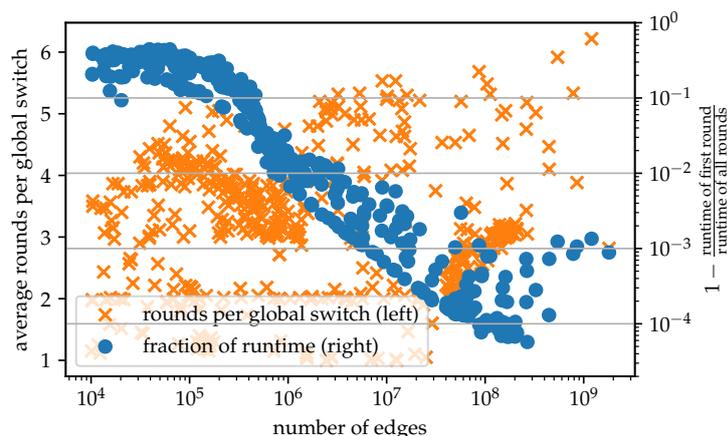


■ **Figure 8** Runtime per edge of PARGLBALES on graphs from SYNPLD where $n \in \{2^{24}, 2^{26}, 2^{28}\}$ in dependence of the degree exponent γ . The color indicates the number of nodes and the line style the number of PUs.

the number of edges even when fixing n , therefore we normalize the runtime by dividing by the number of edges. We report the runtime per edge as a function of γ in Figure 8. We observe an effect both in n and γ . For $n \geq 2^{26}$, the runtime on 64 PUs increases slightly as γ approaches 2. This matches the analysis given for Theorem 3: for graphs with a highly skewed degree sequence, most edge switches will attempt to create the same few edges, causing many target dependencies and more synchronization overhead.

6.2.3 Rounds per Global Switch

Recall that PARGLBALES may delay edge switches to resolve target dependencies. It does so by executing several rounds. To study the performance impact, we execute 20 global switches per graph of NETREP using $\mathcal{P} = 32$ PUs and record the number of rounds per global switch, and the time accumulated on all rounds excluding the first one. As reported in Figure 9, the average number of rounds is low with a mean of 2.2, and the maximum number of rounds we observed was 8. For all networks with more than 4M edges the first round also accounts for more than 99% of the runtime. This suggests that even in a case where more rounds are required, the performance impact of the following rounds is negligible



■ **Figure 9** Rounds per global switch performed by PARGLIBALES on NETREP. Each orange cross corresponds the average number of rounds recorded for a graph; the blue dots indicate the fractional runtime of all rounds excluding the first one.

for sufficiently large graphs.

7 Conclusions

We propose G-ES-MC as an ES-MC variant that exhibits more parallelism due to the absence of source dependencies. In addition, we propose PARES and PARGLIBALES as exact parallelizations of ES-MC and G-ES-MC. These algorithms avoid possible deviations from the intended random process by accounting for the complex dependencies between edge switches. To the best of our knowledge, they are the first exact parallelizations of switch Markov Chains for the uniform sampling of simple undirected graphs.

An autocorrelation analysis suggests that G-ES-MC typically requires fewer steps than standard ES-MC to randomize a graph. On $\mathcal{P} = 32$ PUs, our parallel algorithm executes 10 – 12 times faster than our sequential G-ES-MC implementation, and 50 – 100 faster than existing ES-MC implementations. We investigate the number of rounds needed for PARGLIBALES to perform a global switch and find that very few rounds are required in practice. The experiments on the influence of graph properties match the theoretical predictions. For regular graphs, the performance is not affected by the density of the graph. For power-law graphs with very small degree exponents $\gamma < 2.2$, there is a slight slowdown, due to the increased number of target dependencies. However, in our experiment on over 600 real graphs, this occurs for only very few outliers.

We expect that dedicated base cases for small graphs can further reduce the overhead due to the synchronization and concurrent data structures and thereby improve the scaling on such graphs. We are also still interested in analyzing if G-ES-MC is rapidly mixing for any class of undirected graphs. While we expect lower speed-ups than for PARGLIBALES, an alternative could be to investigate the scalability of PARES since this algorithm inherits the rapid mixing property of ES-MC.

Acknowledgments

Extensive calculations on the Goethe-HLR high-performance computer of the Goethe University Frankfurt were conducted for this research. The authors would like to acknowledge the CSC team for their support.

References

- 1 Paolo Boldi and Sebastiano Vigna. Axioms for centrality. *Internet Math.*, 2014.
- 2 Albert-László Barabási and Márton Pósfai. *Network science*. 2016.
- 3 G. W. Cobb and Y.-P. Chen. An application of Markov Chain Monte Carlo to community ecology. *American Math. Monthly*, 110, 2003.
- 4 S. Itzkovitz, R. Milo, N. Kashtan, G. Ziv, and U. Alon. Subgraphs in random networks. *Phys. Rev. E*, 68, 2003.
- 5 Wolfgang Eugen Schlauch and Katharina Anna Zweig. Influence of the null-model on motif detection. In *ASONAM*, 2015.
- 6 C. J. Carstens. *Topology of Complex Networks: Models and Analysis*. PhD thesis, RMIT University, 2016.
- 7 Manuel Penschuck, Ulrik Brandes, Michael Hamann, Sebastian Lamm, Ulrich Meyer, Ilya Safro, Peter Sanders, and Christian Schulz. Recent advances in scalable network generation. *CoRR*, abs/2003.00736, 2020.
- 8 Václav Havel. Poznámka o existenci konečných grafů. *Časopis pro pěstování matematiky*, 080, 1955.
- 9 Seifollah L. Hakimi. On realizability of a set of integers as degrees of the vertices of a linear graph. i. *J. SIAM*, 10, 1962.
- 10 Joseph K. Blitzstein and Persi Diaconis. A sequential importance sampling algorithm for generating random graphs with prescribed degrees. *Internet Math.*, 6, 2011.
- 11 Md Hasanuzzaman Bhuiyan, Maleq Khan, and Madhav Marathe. A parallel algorithm for generating a random graph with a prescribed degree sequence. In *IEEE BigData*, 2017.
- 12 F. Chung and L. Lu. Connected components in random graphs with given expected degree sequences. *A. of Comb.*, 6, 2002.
- 13 Sebastián Moreno, Joseph J. Pfeiffer III, and Jennifer Neville. Scalable and exact sampling method for probabilistic generative graph models. *Data Min. Knowl. Discov.*, 32, 2018.
- 14 E. A. Bender and E. R. Canfield. The asymptotic number of labeled graphs with given degree sequences. *J. of Comb. Theo. A*, 1978.
- 15 A. Békéssy, P. Békéssy, and J. Komlós. Asymptotic enumeration of regular matrices. *Stud. Sci. Math. Hungar.*, 7, 1972.
- 16 M. E. J. Newman. *Networks: An Introduction*. 2010.
- 17 B. Bollobás. *Random graphs*. 1985.
- 18 B. Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *Eur. J. Comb.*, 1, 1980.
- 19 B. D. McKay and N. C. Wormald. Uniform generation of random regular graphs of moderate degree. *J. Algorithms*, 11, 1990.
- 20 P. Gao and N. C. Wormald. Uniform generation of random regular graphs. In *FOCS*, 2015.
- 21 A. Arman, P. Gao, and N. C. Wormald. Fast uniform generation of random graphs with given degree sequences. In *FOCS*, 2019.
- 22 M. Jerrum and A. Sinclair. Fast uniform generation of regular graphs. *TCS*, 73, 1990.
- 23 C. Cooper, M. E. Dyer, and C. S. Greenhill. Sampling regular graphs and a peer-to-peer network. *Comb. Probab. Comput.*, 2007.
- 24 Christos Gkantsidis, Milena Mihail, and Ellen W. Zegura. The Markov Chain simulation method for generating connected power law random graphs. In *ALENEX*, 2003.
- 25 C. S. Greenhill. The switch Markov Chain for sampling irregular graphs. In *SODA*, 2015.

- 26 R. Kannan, P. Tetali, and S. S. Vempala. Simple Markov Chain algorithms for generating bipartite graphs and tournaments. *RSA*, 1999.
- 27 P. Mahadevan, D. V. Krioukov, K. R. Fall, and V. Vahdat. Systematic topology analysis and generation using degree correlations. In *SIGCOMM*, 2006.
- 28 I. Stanton and A. Pinar. Sampling graphs with a prescribed joint degree distribution using Markov Chains. In *ALLENEX*, 2011.
- 29 G. Strona, D. Nappo, F. Boccacci, S. Fattorini, and J. San-Miguel-Ayanz. A fast and unbiased procedure to randomize ecological binary matrices with fixed row and column totals. *Nature comm.*, 2014.
- 30 N. D. Verhelst. An efficient MCMC algorithm to sample binary matrices with fixed marginals. *Psychometrika*, 73, 2008.
- 31 F. Viger and M. Latapy. Efficient and simple generation of random simple connected graphs with prescribed degree sequence. *J. Complex Networks*, 4, 2016.
- 32 Péter L. Erdős, Sándor Z. Kiss, István Miklós, and Lajos Soukup. Approximate counting of graphical realizations. *PLOS ONE*, 10, 2015.
- 33 Catherine S. Greenhill and Matteo Sfragara. The switch Markov Chain for sampling irregular graphs and digraphs. *TCS*, 719, 2018.
- 34 Péter L. Erdős, Catherine Greenhill, Tamás Róbert Mezei, István Miklós, Dániel Soltész, and Lajos Soukup. The mixing time of switch markov chains: A unified approach. *Eur. J. Comb.*, 99, 2022.
- 35 Georgios Amanatidis and Pieter Kleer. Rapid mixing of the switch Markov Chain for strongly stable degree sequences. *RSA*, 57, 2020.
- 36 Pu Gao and Catherine S. Greenhill. Mixing time of the switch Markov Chain and stable degree sequences. *Discret. Appl. Math.*, 291, 2021.
- 37 Péter L. Erdős, Catherine S. Greenhill, Tamás Róbert Mezei, István Miklós, Daniel Soltész, and Lajos Soukup. The mixing time of switch markov chains: A unified approach. *Eur. J. Comb.*, 99:103421, 2022.
- 38 R. Milo, N. Kashtan, S. Itzkovitz, M. E. J. Newman, and U. Alon. On the uniform generation of random graphs with prescribed degree sequences. *CoRR*, abs/cond-mat/0312028, 2003.
- 39 Jaideep Ray, Ali Pinar, and C. Seshadhri. Are we there yet? when to stop a Markov Chain while generating random graphs. In *WAW*, volume 7323 of *LNCS*, 2012.
- 40 Md Hasanuzzaman Bhuiyan, Maleq Khan, Jiangzhuo Chen, and Madhav V. Marathe. Parallel algorithms for switching edges in heterogeneous graphs. *J. Parallel Distributed Comput.*, 104, 2017.
- 41 D. Allendorf, U. Meyer, M. Penschuck, and H. Tran. Parallel global edge switching for the uniform sampling of simple graphs with prescribed degrees. In *2022 IEEE International Parallel and Distributed Processing Symposium (IPDPS)*, pages 269–279, Los Alamitos, CA, USA, jun 2022. IEEE Computer Society.
- 42 C. J. Carstens, M. Hamann, U. Meyer, M. Penschuck, H. Tran, and D. Wagner. Parallel and I/O-efficient randomisation of massive networks using Global Curveball trades. *ESA*, 2018.
- 43 Santosh Pandey, Lingda Li, Adolfo Hoisie, Xiaoye S Li, and Hang Liu. C-saw: A framework for graph sampling and random walk on gpus. In *SC20: International Conference for High Performance Computing, Networking, Storage and Analysis*, pages 1–15. IEEE, 2020.
- 44 Michael Hamann, Ulrich Meyer, Manuel Penschuck, Hung Tran, and Dorothea Wagner. I/O-efficient generation of massive graphs following the *LFR* benchmark. *ACM J. Exp. Algorithmics*, 23, 2018.
- 45 Artur Czumaj and Andrzej Lingas. On truly parallel time in population protocols. *CoRR*, abs/2108.11613, 2021.
- 46 Corrie Jacobien Carstens, Annabell Berger, and Giovanni Strona. Curveball: a new generation of sampling algorithms for graphs with fixed degree sequence. *CoRR*, abs/1609.05137, 2016.
- 47 Michael Mitzenmacher and Eli Upfal. *Probability and Computing: Randomized Algorithms and Probabilistic Analysis*. 2005.

- 48 A. Ramachandra Rao, Rabindranath Jana, and Suraj Bandyopadhyay. A Markov Chain Monte Carlo method for generating random $(0, 1)$ -matrices with given marginals. *Sankhyā: The Indian J. of Statistics A*, 58, 1996.
- 49 Corrie Jacobien Carstens and Pieter Kleer. Speeding up switch Markov Chains for sampling bipartite graphs with given degree sequence. In *APPROX/RANDOM*, volume 116 of *LIPIcs*, 2018.
- 50 Catherine S. Greenhill. Generating graphs randomly. In Konrad K. Dabrowski, Maximilien Gadouleau, Nicholas Georgiou, Matthew Johnson, George B. Mertzios, and Daniël Paulusma, editors, *Surveys in Combinatorics, 2021: Invited lectures from the 28th British Combinatorial Conference, Durham, UK, July 5-9, 2021*, pages 133–186. Cambridge University Press, 2021.
- 51 C. L. Staudt, A. Sazonovs, and H. Meyerhenke. Networkkit: A tool suite for large-scale complex network analysis. *Netw. Sci.*, 4, 2016.
- 52 Aric A. Hagberg, Daniel A. Schult, and Pieter J. Swart. Exploring network structure, dynamics, and function using NetworkX. In *SciPy*, 2008.
- 53 A. Lancichinetti and S. Fortunato. Benchmarks for testing community detection algorithms on directed and weighted graphs with overlapping communities. *Phys. Rev. E*, 80, 2009.
- 54 Fabien Viger and Matthieu Latapy. Fast generation of random connected graphs with prescribed degrees. *CoRR*, abs/cs/0502085, 2005.
- 55 Intel Corporation. *Intel®64 and IA-32 Architectures — Software Developer’s Manual — Vol. 2*, 2019.
- 56 Tobias Maier, Peter Sanders, and Roman Dementiev. Concurrent hash tables: Fast and general(!)? *ACM Trans. Parallel Comput.*, 5(4), 2019.
- 57 Makoto Matsumoto and Takuji Nishimura. Mersenne twister: A 623-dimensionally equidistributed uniform pseudo-random number generator. *ACM Trans. Model. Comput. Simul.*, 8(1), 1998.
- 58 D. Lemire. Fast random integer generation in an interval. *ACM Trans. Model. Comput. Simul.*, 29, 2019.
- 59 P. Sanders. Random permutations on distributed, external and hierarchical memory. *Inf. Process. Lett.*, 67, 1998.
- 60 E. N. Gilbert. Random Graphs. *Ann. Math. Stat.*, 30(4), 1959.
- 61 P. Gao and N. C. Wormald. Uniform generation of random graphs with power-law degree sequences. In *SODA*, 2018.
- 62 R. A. Rossi and N. K. Ahmed. The network data repository with interactive graph analytics and visualization. In *AAAI*, 2015.
- 63 Isabelle Stanton and Ali Pinar. Constructing and sampling graphs with a prescribed joint degree distribution. *ACM J. Exp. Algorithmics*, 17.
- 64 Jaideep Ray, Ali Pinar, and C. Seshadhri. A stopping criterion for Markov Chains when generating independent random graphs. *J. Complex Networks*, 3, 2015.
- 65 Annabell Berger and Corrie Jacobien Carstens. Smaller universes for uniform sampling of $0,1$ -matrices with fixed row and column sums. *CoRR*, abs/1803.02624, 2018.
- 66 Yvonne M Bishop, Stephen E Fienberg, and Paul W Holland. *Discrete multivar. analysis: theo. and practice*. 2007.