Brauer groups of diagonal quartic surfaces

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Abstract

We describe explicit methods of exhibiting elements of the Brauer groups of diagonal quartic surfaces. Using these methods, we compute the algebraic Brauer–Manin obstruction in two contrasting examples. In the second example, the obstruction is found to be trivial but a computer search reveals no points of small height on the surface.

1 Introduction

We are concerned with the solubility in rational numbers of the equation

$$a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0 \qquad (1)$$

where the a_i are non-zero rational numbers. Let V denote the smooth surface defined by this equation over $\mathbb{Q}$. We will always suppose that V has points everywhere locally, that is, in the real numbers $\mathbb{R}$ and in each p-adic field $\mathbb{Q}_p$; for otherwise the equation (1) certainly has no rational solutions.

One reason that V might have no rational points is given by the Brauer–Manin obstruction, first described by Manin (1971, 1974). This obstruction is defined in terms of the Brauer group of V, which is the group of Azumaya algebras on V modulo equivalence. Our objective is to show how one part of the Brauer group, the so-called algebraic part, may be computed, and how the associated obstruction may thence be evaluated.

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The surface V is geometrically a K3 surface; for certain choices of coefficients a_i, there is a fibration $V \to \mathbb{P}^1$ defined over $\mathbb{Q}$, the fibres of which are curves of genus 1.

Colliot-Thélène et al. (1998) studied elliptic fibrations over $\mathbb{P}^1$ satisfying certain criteria: that the singular fibres are all of type I_2; and a technical condition ("Condition D") on the class of the fibration in a geometric Selmer group of its Jacobian. Conditionally on two well-known conjectures (the finiteness of the Tate–Shafarevich groups of elliptic curves, and Schinzel's hypothesis), they were able to prove that the obstruction associated to the so-called *vertical Brauer group* is the only obstruction to the Hasse principle.

Swinnerton-Dyer (2000) proved a similar result for a particular family of diagonal quartic surfaces, those where $a_0a_1a_2a_3$ is a square. In that case there is a fibration over $\mathbb{P}^1$ with six fibres each of type I_4, consisting of four straight lines arranged in a skew quadrilateral. He defined a new obstruction to the existence of rational points on these surfaces, and showed under analogous conditions to those mentioned above that this obstruction is the only obstruction to the Hasse principle. He then showed that the new obstruction is in fact a Brauer–Manin condition.

The object of this article is to show how the algebraic Brauer group and associated obstruction may be computed for any diagonal quartic surface, whether or not it has the structure of an elliptic fibration over $\mathbb{P}^1$. By automating this procedure and calculating a large number of examples, we hope to gather evidence on the Brauer–Manin obstruction on these surfaces. In addition to describing the general procedures for carrying out the calculations, we present in detail two examples which exhibit contrasting results.

The first example computed in the present article is another example of a diagonal quartic surface with a fibration. We show that there is a non-trivial Brauer–Manin obstruction and therefore that the surface has no rational points.

On the other hand, a general diagonal quartic surface has no fibration defined over $\mathbb{Q}$. In this case, the algebraic Brauer group has order 2 and it is relatively straightforward to compute the associated obstruction. Our second example has no algebraic Brauer–Manin obstruction and yet no rational points of small height; and numerical evidence suggests that this is a common occurrence.

1.1 Overview

The approach we adopt is as follows. Let $\mathrm{Br}_1 V$ denote the algebraic Brauer group, and $\bar{V}$ the extension of V to an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. It is well known that $\mathrm{Br}_1 V/\mathrm{Br}\,\mathbb{Q}$ is isomorphic to the Galois cohomology group $H^1(\mathbb{Q}, \mathrm{Pic}\,\bar{V})$. We can write down a set of generators for $\mathrm{Pic}\,\bar{V}$ and compute this cohomology group, which is finite; this process has been implemented in Magma (see Magma, 2003; Bosma et al., 1997), using Gavin Brown's algebraic geometry machinery and the group cohomology algorithms implemented by Derek Holt. Code to do this is available from the author's web page at Bright (2002). Unfortunately the isomorphism from $H^1(\mathbb{Q}, \mathrm{Pic}\,\bar{V})$ to $\mathrm{Br}_1 V/\mathrm{Br}\,\mathbb{Q}$ is rather awkward to calculate. For certain special types of algebra, though, we can write down Azumaya algebras on V and easily compute the corresponding elements of $H^1(\mathbb{Q}, \mathrm{Pic}\,\bar{V})$. In this way it is often possible to find a set of representatives for the whole algebraic Brauer group.

The first special class of algebras we consider are those which arise from a fibration of V. Let π be a surjective morphism from V to $\mathbb{P}^1$; then any element of the Brauer group of $\mathbb{Q}(\mathbb{P}^1)$ lifts to an element of the Brauer group of $\mathbb{Q}(V)$; and it is possible to write down simple criteria which determine when this resulting algebra is Azumaya. These algebras, which live in the so-called "vertical" Brauer group of V, are discussed in Section 2. We compute the algebraic Brauer–Manin obstruction for a particular diagonal quartic surface equipped with a fibration over $\mathbb{P}^1$.

The second special class of algebras are the cyclic algebras. Just as in the theory of Brauer groups of fields, it is more straightforward to compute with cyclic algebras than with general algebras. In Section 3 we describe the map from $\mathrm{Br}_1 V/\mathrm{Br}\,\mathbb{Q}$ to $H^1(\mathbb{Q}, \mathrm{Pic}\,\bar{V})$ for cyclic algebras. We compute the algebraic Brauer–Manin obstruction for a surface which has no fibration over $\mathbb{P}^1$.

Although we are primarily concerned with surfaces as above defined over the rational numbers, the techniques described apply more generally. Throughout, k will denote a number field over which V is defined.

1.2 The Brauer group

For background information on Brauer groups of schemes, we refer the reader to Grothendieck (1968).

Let V be any smooth surface over k. We will write $\mathrm{Br}_0 V$ for the image of $\mathrm{Br}\,k$ in $\mathrm{Br}\,V$; when V has points everywhere locally, this map is injective. Let $\mathrm{Br}_1 V$ denote the kernel of $\mathrm{Br}\,V \to \mathrm{Br}\,\bar{V}$, the "algebraic" part of the Brauer group.

As described by Skorobogatov (2001, pp. 116–117), there is an isomorphism between $\mathrm{Br}_1 V/\mathrm{Br}\,k$ and $H^1(k, \mathrm{Pic}\,\bar{V})$. We now discuss computational aspects of this map.

Proposition 1 *Let* $\operatorname{Prin} V$ *denote the group of principal divisors on the variety* V*. There are isomorphisms as follows:*

$$\begin{array}{c}
\operatorname{Br}_1 V / \operatorname{Br} k \\
\text{easy} \upharpoonleft\downharpoonleft \text{easy} \\
\ker(H^2(k, \bar{k}(V)^\times) \to H^2(k, \operatorname{Div} \bar{V})) / \operatorname{Br} k \\
\text{hard} \upharpoonleft\downharpoonleft \text{easy} \\
\ker(H^2(k, \operatorname{Prin} \bar{V}) \to H^2(k, \operatorname{Div} \bar{V})) \\
\text{easy} \upharpoonleft\downharpoonleft \text{easy} \\
H^1(k, \operatorname{Pic} \bar{V})
\end{array}$$

where the arrows are labelled to show ease of computation.

PROOF. The top isomorphism is the correspondence between an Azumaya algebra on V (specified as a vector space over $k(V)$ together with a multiplication law) and the corresponding 2-cocycle (or factor set, in old terminology). This is well known and described explicitly in, say, Deuring (1935).

For the middle isomorphism, note that, as $H^3(k, \bar{k}^\times) = 0$ (see Tate, 1967, p. 199), there is an exact sequence

$$H^2(k, \bar{k}^\times) \to H^2(k, \bar{k}(V)^\times) \to H^2(k, \bar{k}(V)^\times / \bar{k}^\times) \to 0$$

and so the map from $H^2(k, \bar{k}(V)^\times)$ to $H^2(k, \operatorname{Div} \bar{V})$, being trivial on $\operatorname{Br} k$, factors through that quotient. Now the group $\bar{k}(V)^\times / \bar{k}^\times$ is isomorphic to the group $\operatorname{Prin} \bar{V}$ of principal divisors on $\bar{V}$. There are two problems with computing this map: firstly, we must make effective the triviality of $H^3(k, \bar{k}^\times)$; and secondly, we must have some procedure which, given a divisor known to be principal, finds a function having that divisor. Hence the upward arrow is labelled "hard".

The lower isomorphism is the coboundary map from the long exact sequence in cohomology associated to

$$0 \to \operatorname{Prin} \bar{V} \to \operatorname{Div} \bar{V} \to \operatorname{Pic} \bar{V} \to 0$$

where $H^1(k, \operatorname{Div} \bar{V}) = 0$ because it is a permutation module. Computing the upward arrow is easy: given a 1-cocycle with values in $\operatorname{Pic} \bar{V}$, we lift to $\operatorname{Div} \bar{V}$ and take the coboundary. For the downward arrow, we must compute the inverse of the coboundary map; splitting $\operatorname{Div} \bar{V}$ into its irreducible components makes this a simple problem in linear algebra. $\square$

For more details on how these isomorphisms may be computed, see Bright and Swinnerton-Dyer (2004).

There are two main reasons why the Picard group of $\bar{V}$ is particularly amenable to computational methods. Firstly, the surface $\bar{V}$ is a K3 surface, which means that it has trivial canonical sheaf and

$$H^1(\bar{V}, \mathcal{O}_{\bar{V}}) = 0$$

and therefore $\operatorname{Pic} \bar{V}$ is finitely generated. Moreover, this means that divisors are linearly equivalent if and only if they are numerically equivalent, that is, their difference has intersection number zero with all divisors. The geometry of K3 surfaces is thoroughly covered by Barth et al. (1984).

Secondly, there is a very convenient set of generators. Let ϵ denote a fixed primitive eighth root of unity and set

$$\alpha_{ij} = \sqrt[4]{a_i/a_j}$$

where the a_i are the coefficients from (1) and the fourth roots are chosen such that

$$\alpha_{ij}\alpha_{jk} = \alpha_{ik}$$

for all i, j, k. Then we can write down the equations of 48 straight lines contained in $\bar{V}$:

$$\begin{array}{rll} L^{123}_{mn}: & X_0 = \epsilon^m \alpha_{10} X_1, & X_2 = \epsilon^n \alpha_{32} X_3; \\ L^{231}_{mn}: & X_0 = \epsilon^m \alpha_{20} X_2, & X_3 = \epsilon^n \alpha_{13} X_1; \\ L^{312}_{mn}: & X_0 = \epsilon^m \alpha_{30} X_3, & X_1 = \epsilon^n \alpha_{21} X_2. \end{array}$$

Here in each case m and n take the values $1, 3, 5, 7$.

Let Λ denote the (free) subgroup of $\operatorname{Div} \bar{V}$ generated by the 48 lines. The intersection numbers of straight lines are easy to compute; let Λ^0 denote the subgroup of Λ consisting of those linear combinations of lines which are numerically equivalent to 0, in other words, principal.

The following is well known; a proof may be found in Pinch and Swinnerton-Dyer (1991, Lemma 1).

Proposition 2 *The Picard group* $\operatorname{Pic} \bar{V}$ *is free of rank 20, and is generated by the classes of the 48 lines* L^{pqr}_{mn}. $\square$

Let K denote the field

$$K = k(\epsilon, \alpha_{10}, \alpha_{20}, \alpha_{30}).$$

Then all 48 lines are clearly defined over K, and in fact K is the smallest common field of definition of the 48 lines. The Galois action on the lines therefore factors through

$$G = \mathrm{Gal}(K/k)$$

and the inflation-restriction sequence gives us $H^1(k, \operatorname{Pic} \bar{V}) \cong H^1(G, \Lambda/\Lambda^0)$.

Let m be an integer known to annihilate $H^1(G, \operatorname{Pic} \bar{V})$, such as $|G|$. By considering the long exact sequence in cohomology associated to the exact sequence

$$0 \to \Lambda/\Lambda^0 \xrightarrow{\times m} \Lambda/\Lambda^0 \to \Lambda/\Lambda^0 \otimes \mathbb{Z}/m\mathbb{Z} \to 0$$

(remember that Λ/Λ^0 is torsion-free), we see that there is an isomorphism

$$\frac{H^0(G, \Lambda/\Lambda^0 \otimes \mathbb{Z}/m\mathbb{Z})}{H^0(G, \Lambda/\Lambda^0) \otimes \mathbb{Z}/m\mathbb{Z}} \xrightarrow{\inf \circ d} H^1(k, \operatorname{Pic} \bar{V}) \tag{2}$$

where the map is the coboundary map followed by inflation.

The Galois action on the 48 lines is straightforward. Using the above isomorphism it is easy to compute $H^1(k, \operatorname{Pic} \bar{V})$ for any diagonal quartic surface. In fact the author has computed this group for all diagonal quartic surfaces. These results, together with some Magma and Pari scripts for producing them, are available electronically (Bright, 2002).

2 The vertical Brauer group of a fibration

2.1 The vertical Brauer group

In this section we suppose that the surface V has a surjective morphism $\pi : V \to \mathbb{P}^1$, whose fibres must be curves of genus 1. This gives a useful way of producing Azumaya algebras on V. We choose once and for all a coordinate t on $\mathbb{P}^1$, such that the fibre of π over $t = \infty$ is non-degenerate. Whenever we write $\mathbb{A}^1$, we mean that open subscheme of $\mathbb{P}^1$ where $t \neq \infty$.

Definition 3 *We define the following groups associated to the fibration π:*

$$B := \{\mathcal{A} \in \operatorname{Br} k(\mathbb{P}^1) \mid \pi^*\mathcal{A} \in \operatorname{Br} V\} / \operatorname{Br} k$$
$$\mathrm{Br}_{\mathrm{vert}} := \pi^*(\operatorname{Br} k(\mathbb{P}^1)) \cap \operatorname{Br} V$$
$$\mathrm{Pic}_{\mathrm{vert}} := \ker(\operatorname{Pic} \bar{V} \to \operatorname{Pic} \bar{V}_\eta)$$

where η is the generic point of $\mathbb{P}^1$, and V_η the generic fibre of π.

Tsen's theorem (Milne, 1998, 13.5(b)) shows that $\mathrm{Br}_{\mathrm{vert}}$ is contained in $\mathrm{Br}_1 V$. A simple piece of diagram-chasing gives the following result.

Proposition 4 *There is an isomorphism $B \cong H^1(k, \mathrm{Pic}_{\mathrm{vert}})$, such that the diagram*

$$\begin{array}{ccc} H^1(k, \mathrm{Pic}_{\mathrm{vert}}) & \xrightarrow{\sim} & B \\ \downarrow & & \downarrow \\ H^1(k, \mathrm{Pic}\,\bar{V}) & \xrightarrow{\sim} & \mathrm{Br}_1 V / \mathrm{Br}\,k \end{array}$$

commutes. $\square$

If we can compute $\mathrm{Pic}_{\mathrm{vert}}$, then Proposition 4 allows us to find out how much of $\mathrm{Br}_1 V/\mathrm{Br}\,k$ comes from the $\mathrm{Br}_{\mathrm{vert}}$ without having to compute with elements of the Brauer group. In general, finding a generating set for $\mathrm{Pic}_{\mathrm{vert}}$ is not totally straightforward. Such a generating set is given by the irreducible components of the singular fibres of π, as described by Shioda (1972). The procedure of Section 1.3 gives us $H^1(k, \mathrm{Pic}\,\bar{V})$ in terms of the straight lines L^{pqr}_{mn}. In order to compute $H^1(k, \mathrm{Pic}_{\mathrm{vert}})$ compatibly with this, we need to know a sum of classes of straight lines which is equal to the class in $\mathrm{Pic}\,\bar{V}$ of each irreducible component of each singular fibre. However, it is possible to find these without ever knowing explicitly what the singular fibres are, as follows.

- Each component of a singular fibre is a rational curve, of self-intersection -2; and the degree (that is, intersection number with a plane section) of a component is bounded by the degree of a general fibre of π, which is easily found from any explicit formula for π.
- There exist only finitely many divisor classes of a given self-intersection and bounded degree, and when the degree is reasonably small they may easily be listed.
- Any component of a fibre has intersection number 0 with the class of a fibre; this condition rules out any rational curves which are transverse to the fibration.
- Each fibre is connected; this means that, by considering the intersection numbers of the rational curves we find, we can partition them into possible fibres.

In this way it is possible to find a generating set for $\mathrm{Pic}_{\mathrm{vert}}$.

2.2 Structure of the vertical Brauer group

To compute the Brauer–Manin obstruction coming from $\mathrm{Br}_{\mathrm{vert}}$, we want to write down elements of B.

Lemma 5 *For any closed point P of $\mathbb{A}^1$, identify the residue field $k(P)$ with $k[t]/F(t)$, where $F = N_{k(P)/k}(t - t_P)$ is the monic irreducible polynomial generating the maximal ideal associated to P. When $c \in H^1(\ell, \mathbb{Q}/\mathbb{Z})$ and $f \in \ell(t)$*

for some field ℓ, let (c, f) denote the algebra in $\operatorname{Br} \ell(t)$ associated to the 2-cocycle $f \cup dc$.

Any element $\mathcal{A}$ of $\operatorname{Br} k(\mathbb{P}^1)$ may be written uniquely as

$$\mathcal{A} = \sum_P \operatorname{cores}_{k(P)/k}(c_P, t - t_P) + \alpha \tag{3}$$

where the sum is over finitely many distinct closed points P of $\mathbb{P}^1$; c_P lies in $H^1(k(P), \mathbb{Q}/\mathbb{Z})$; and α lies in $\operatorname{Br} k$.

PROOF. Colliot-Thélène and Swinnerton-Dyer (1994, Proposition 1.2.1). □

The following description of which of these algebras lift to Azumaya algebras on V is stated by Swinnerton-Dyer (2000).

Proposition 6 *The algebra $\mathcal{A}$ described in* (3) *lies in B if and only if the following two conditions hold:*

(1) $\sum_P \operatorname{cores}_{k(P)/k}(c_P) = 0$ in $H^1(k, \mathbb{Q}/\mathbb{Z})$;
(2) for each P, $\operatorname{res}_{\ell/k(P)}(c_P) = 0$ whenever ℓ is the least field of definition of an irreducible component of the fibre $\pi^{-1}P$. □

Note that condition 2 of the proposition shows that non-trivial Azumaya algebras are only obtained by taking the points P to lie below geometrically reducible fibres of π. Therefore there are, up to elements of $\operatorname{Br} k$, only finitely many classes of Azumaya algebras arising by this method: at each of the finitely many geometrically reducible fibres of π, the group

$$\ker\Big(H^1(k(P), \mathbb{Q}/\mathbb{Z}) \to H^1(\ell, \mathbb{Q}/\mathbb{Z})\Big) = H^1(\ell/k(P), \mathbb{Q}/\mathbb{Z})$$

is finite.

2.3 Example

2.3.1 A surface and a fibration

We consider the diagonal quartic surface V defined by

$$X_0^4 + X_1^4 = 6X_2^4 + 12X_3^4. \tag{4}$$

It is easily checked with Magma, using routines by Nils Bruin, that the surface has points everywhere locally. A search using an algorithm described by Bernstein (2001) finds that there are no global solutions with the $|X_i|$ less than

10^4. The 48 straight lines on this surface are defined over the field

$$L = \mathbb{Q}(\epsilon, \sqrt[4]{2}, \sqrt[4]{3})$$

of degree 32 over $\mathbb{Q}$. (Here ϵ denotes, as always, a primitive eighth root of unity.) Looking at the Galois action on the lines shows that $\operatorname{Pic} V$ is free of rank 2, generated by the classes of a plane section and of the divisor

$$F = L_{11}^{123} + L_{15}^{123} + L_{31}^{123} + L_{35}^{123}.$$

While the class of F is defined over $\mathbb{Q}$ and contains a divisor defined over $\mathbb{Q}$, the divisor F itself is defined only over $K = \mathbb{Q}(\sqrt{-2})$. We will write σ for the non-trivial automorphism of $K/\mathbb{Q}$. The conjugate over $\mathbb{Q}$ of F is another divisor

$$F^\sigma = L_{53}^{123} + L_{57}^{123} + L_{73}^{123} + L_{77}^{123}$$

linearly equivalent to F. The individual straight lines involved are defined over $\mathbb{Q}(i, \sqrt[4]{2})$.

The divisor F has self-intersection 0 and hence arithmetic genus 1; therefore the linear system $|F|$ defines a fibration $\pi : V \to \mathbb{P}^1$ whose generic fibre is a curve of genus 1. Although it would be possible to compute an equation, defined over $\mathbb{Q}$, for this fibration, it will not be necessary. The morphism π is defined only up to an automorphism over $\mathbb{Q}$ of $\mathbb{P}^1$; we will later find a fibre of π defined over $\mathbb{Q}$ and choose this to be the fibre at infinity.

As described in Section 1.3, we can compute $H^1(\mathbb{Q}, \operatorname{Pic} \bar{V})$. Using Magma and the routines available at Bright (2002), we find that it is of order 2:

```
Magma V2.10-8      Tue Oct 14 2003 15:29:43 on kiwi
Type ? for help.  Type <Ctrl>-D to quit.
> load "quartic";
Loading "quartic"
> V := DiagonalQuarticSurface([1,1,-6,-12]);
> V;
Scheme over Rational Field defined by
X0^4 + X1^4 - 6*X2^4 - 12*X3^4
> H1Pic(V);
Full Quotient RSpace of degree 1 over Integer Ring
Column moduli:
[ 2 ]
```

Thus $\operatorname{Br}_1 V / \operatorname{Br} k$ is also of order 2. It would be possible to work out the singular fibres of the fibration and hence the associated $\operatorname{Pic}_{\text{vert}}$, and then to show that the non-trivial element of $\operatorname{Br}_1 V / \operatorname{Br} k$ lies in the image of $\operatorname{Br}_{\text{vert}}$; but this is unnecessary, as any non-trivial Azumaya algebra which we produce with the help of Proposition 6 must lie in the unique non-trivial class of Azumaya algebras.

2.3.2 An Azumaya algebra

We will now use Proposition 6 to write down an Azumaya algebra on V arising from the fibration described above. As we already know that F is one geometrically reducible fibre of the fibration, we will try to find an Azumaya algebra without having to compute all the singular fibres.

Lemma 7 *Let g be a function on V, defined over K, with divisor $(g) = F - F_\infty$, where F_∞ is a geometrically irreducible closed curve defined over $\mathbb{Q}$. Then the quaternion algebra $\mathcal{A} = (-1, N_{K/\mathbb{Q}} g)$ is Azumaya on V.*

PROOF. Choose the fibration π so that F_∞ lies above the point at infinity, that is, $\pi^*\infty = F_\infty$. Let P be the closed point of $\mathbb{A}^1_{\mathbb{Q}}$ over which the two fibres F and F^σ lie; so $\pi^* P = F + F^\sigma$ and $k(P) = K$. Then g is, up to a constant multiple, equal to $\pi^*(t - t_P)$. We have

$$\mathrm{cores}_{k(P)/\mathbb{Q}}(-1, g) = (-1, N_{K/\mathbb{Q}} g) = \mathcal{A}$$

as the quadratic character on $K^\times$ associated to -1 comes by restriction from $\mathbb{Q}^\times$. All we must check are the conditions of Proposition 6. Condition 1 is satisfied because $N_{K/\mathbb{Q}}(-1)$ is a square. Condition 2 is satisfied because the least field of definition of each irreducible component of F or F^σ is $\mathbb{Q}(i, \sqrt[4]{-2})$, in which -1 is a square. $\square$

The function

$$f(X_0, X_1, X_2, X_3) = \frac{X_0^2 - \sqrt{-2} X_0 X_1 - X_1^2}{X_2^2 + \sqrt{-2} X_3^2}$$

is found to have divisor $F - F^\sigma$. Therefore the map from V to $\mathbb{P}^1$, defined over K by f, differs from π by an automorphism of $\mathbb{P}^1$ over K. Note also the following property of f: that f^σ has a zero at F^σ and a pole at F; and therefore $N_{K/\mathbb{Q}} f = f f^\sigma$ is a constant function. In fact,

$$N_{K/\mathbb{Q}} f = \frac{X_0^4 + X_1^4}{X_2^4 + 2X_3^4} = 6. \tag{5}$$

Suppose we can find a $c \in K$ such that the fibre $\{f = c\}$ is defined over $\mathbb{Q}$ and geometrically irreducible; then we can take

$$g = \frac{f}{f - c}$$

to satisfy the requirements of Lemma 7.

Lemma 8 *The fibre* $\{f = c\}$ *is defined over* $\mathbb{Q}$ *if and only if* $N_{K/\mathbb{Q}}(c) = 6$.

PROOF. The conjugate fibre to $\{f = c\}$ is defined by the equation $f^\sigma = c^\sigma$; by (5) this is equivalent to $f = 6/c^\sigma$. The fibre is defined over $\mathbb{Q}$ if and only if the two equations define the same fibre, that is, $c = 6/c^\sigma$. □

We will take
$$c = 2 - \sqrt{-2}.$$
The fibre over c is defined over $\mathbb{Q}$, and is easily checked to be non-singular. Following Lemma 7, we define

$$\begin{aligned} G &= N_{K/\mathbb{Q}}\left(\frac{f}{f-c}\right) \\ &= \frac{X_0^4 + X_1^4}{2(X_0^4 + X_1^4) - 4(X_2^2 + X_3^2)(X_0^2 - X_1^2) - 4(X_2^2 - 2X_3^2)X_0X_1} \end{aligned}$$

and then $\mathcal{A} = (-1, G)$ is an Azumaya algebra on V.

2.3.3 Computing the obstruction

To compute the Brauer–Manin obstruction on V, we must calculate the value of
$$\sum_v \operatorname{inv}_v \mathcal{A}(x_v)$$
for each adèlic point (x_v) in $V(\mathbb{A}_\mathbb{Q})$. This task is made easier by several facts:

- The function $\operatorname{inv}_v \mathcal{A}(x_v)$ is continuous, hence locally constant, on $V(\mathbb{Q}_v)$ for each v.
- The support of (G) (and, indeed, of any divisor) is closed in each $V(\mathbb{Q}_v)$, under the appropriate complete (real or p-adic) topology (see Colliot-Thélène et al., 1980, Lemma 3.1.2). It is therefore unnecessary to evaluate the local invariants along zeros or poles of G.
- At finite primes v where both the variety V and the algebra $\mathcal{A}$ extend to $\mathbb{Z}_v$, the function $\operatorname{inv}_v \mathcal{A}(x_v)$ is zero (see Skorobogatov, 2001, p. 101).

We must therefore evaluate $\operatorname{inv}_v \mathcal{A}(x_v)$ at $v = 2$, 3 and at the infinite prime.

On $\mathbb{R}$, the algebra $\mathcal{A} = (-1, G)$ is trivial where G is positive, and non-trivial where G is negative. Now G has no zeros in $V(\mathbb{R})$, and a double pole along a curve of genus 1; therefore G is either everywhere positive or everywhere negative. Evaluating G at any one point of $V(\mathbb{R})$ gives a positive value; hence the local invariant is everywhere zero on $V(\mathbb{R})$.

Over $\mathbb{Q}_3$, we ask MAGMA to compute the kernel of the local restriction map $H^1(\mathbb{Q}, \operatorname{Pic} \bar{V}) \to H^1(\mathbb{Q}_3, \operatorname{Pic} \bar{V})$:

```
> RestrictionKernel(V,3);
Full Quotient RSpace of degree 1 over Integer Ring
Column moduli:
[ 2 ]
```

This computation involves finding the decomposition group at 3, and then using Derek Holt's group cohomology functions to realise the restriction map. The necessary code can be found at Bright (2002). It turns out that the restriction map is the zero map; so any Azumaya algebra on V over $\mathbb{Q}_3$ is equivalent to a constant algebra over $\mathbb{Q}_3$. We therefore only need to evaluate $(-1, G)$ at one point of $V(\mathbb{Q}_3)$ to discover that the invariant is everywhere equal to 0.

On $\mathbb{Q}_2$, we must work a little harder. Looking modulo 16 shows that every primitive solution of (4) in $\mathbb{Z}_2$ must have all the X_i units. Hensel's Lemma then shows that any such solution is equivalent, modulo 2^5, to an integer point with

$$X_0^4 + X_1^4 \equiv 6X_2^4 + 12X_3^4 \pmod{2^7}. \tag{6}$$

Modulo squares, G is equal to

$$(X_0^4 + X_1^4)\Big(2(X_0^4 + X_1^4) - 4(X_2^2 + X_3^2)(X_0^2 - X_1^2) - 4(X_2^2 - 2X_3^2)X_0X_1\Big) \tag{7}$$

and knowing the X_i modulo 2^5 gives the value of (7) modulo 2^8. We list the odd (X_i) modulo 2^5 satisfying (6) and evaluate (7) at each such point; it turns out that the value of G is always either 96 or 176 modulo 2^8. In either case, the formula for the Hilbert symbol given in Serre (1973, Chapter III, 1.2, Theorem 1) shows that the local invariant is $\frac{1}{2}$.

Combining these calculations shows that

$$\sum_v \operatorname{inv}_v \mathcal{A}(x_v) = \frac{1}{2}$$

for all points $(x_v) \in V(\mathbb{A}_\mathbb{Q})$. Therefore there is a Brauer–Manin obstruction to the existence of rational points on V, and so there are no rational solutions to (4).

3 Cyclic algebras

In this section we look at another class of Azumaya algebras on a surface V which are often straightforward to find, that is, cyclic Azumaya algebras.

3.1 Background

The following well-known proposition is a straightforward generalisation of the Lemma 11 stated by Swinnerton-Dyer (2000).

Proposition 9 *Let $\mathcal{A} = (K/k, f)$ be a cyclic algebra on V, where K is a cyclic extension of the base field k and f is some rational function in $k(V)$. Then $\mathcal{A}$ is Azumaya if and only if the divisor (f) is the norm of some divisor D defined over K. If moreover we assume that V has points in every completion of k, then $\mathcal{A}$ is equivalent to a constant algebra if and only if we can take D to be principal.* □

In the examples which follow, we will never need to compute the element of $H^1(k, \operatorname{Pic} \bar{V})$ corresponding to a cyclic algebra: the mere fact of being non-trivial gives it away. In more complex examples, though, this may be useful and we include it here for completeness.

Proposition 10 *Let $\mathcal{A} = (K/k, f)$ be a cyclic Azumaya algebra on V, with $(f) = N_{K/k} D$ for some divisor D defined over K. Let α be the 1-cocycle with values in $\operatorname{Pic} V_K$ defined by*

$$\alpha(1) = 0$$
$$\alpha(\sigma^r) = \sum_{i=0}^{r-1} \sigma^i D$$

where σ is an (implicitly fixed) generator of $\operatorname{Gal}(K/k)$. Then the class $\inf_{\bar{k}/K} \alpha$ in $H^1(k, \operatorname{Pic} \bar{V})$ corresponds to the class of $\mathcal{A}$ in $\operatorname{Br}_1 V / \operatorname{Br} k$, under the isomorphism descibed in Proposition 1.

PROOF. Using the convention that an empty sum is 0, we work through the isomorphisms described in Proposition 1. Considering α as a 1-cocycle with values in $\operatorname{Div} V_K$, its coboundary is given by

$$\begin{aligned} d\alpha(\sigma^r, \sigma^s) &= \sum_{i=0}^{r+s-1} \sigma^i D - \sum_{i=0}^{((r+s) \bmod n)-1} \sigma^i D \\ &= \begin{cases} 0 & r+s<n \\ N_{K/k} D = (f) & r+s \geq n. \end{cases} \end{aligned}$$

This is indeed the image of $(K/k, f)$ in $H^2(k, \operatorname{Prin} V_K)$. $\square$

3.2 Example

The example of Section 2.3 describes a cyclic algebra: there, the algebra produced was split by the extension $\mathbb{Q}(i)$. The function G had divisor $F + F^\sigma - 2F_\infty$, which is easily shown to be the norm from $\mathbb{Q}(i)$ to $\mathbb{Q}$ of the divisor

$$L_{11}^{123} + L_{15}^{123} + L_{53}^{123} + L_{57}^{123} - F_\infty$$

and so $(-1, G)$ is Azumaya.

3.3 Example

In this section we will describe another example, where the surface has no fibrations in curves of genus 1 defined over $\mathbb{Q}$. The approach used is a general one which extends to other similar cases. In this example, we will show that there is no algebraic Brauer–Manin obstruction on the chosen surface. The surface V this time is given by

$$7X_0^4 + 15X_1^4 = 2X_2^4 + 6X_3^4. \tag{8}$$

This diagonal quartic surface shows the "most general" possible Galois module structure on the 48 straight lines: that is, the least common field of definition of the lines,

$$\mathbb{Q}(\epsilon, \sqrt[4]{15 \times 7}, \sqrt[4]{2 \times 7}, \sqrt[4]{6 \times 7}),$$

is as large as it can be, of degree 256 over $\mathbb{Q}$. The cohomology computations described earlier show that $\operatorname{Pic} V = H^0(\mathbb{Q}, \operatorname{Pic} \bar{V})$ is of rank 1, generated by the class Π of a plane section; and $\operatorname{Br}_1 V / \operatorname{Br} k$ is again of order 2. A computer search reveals that there are no integer solutions with the $|X_i|$ less than 10^4.

We will use a fibration of V in curves of genus 1, defined over a quadratic extension K of $\mathbb{Q}$, to find a divisor F whose norm is linearly equivalent to twice a plane section. This will allow us to construct a function f satisfying the conditions described in Proposition 9, and hence an Azumaya algebra.

3.3.1 A useful fibration

There are no fibrations from V to $\mathbb{P}^1$ defined over $\mathbb{Q}$, for there are no divisors over $\mathbb{Q}$ which could be fibres of such a fibration. However, Swinnerton-Dyer (2000) describes two conjugate fibrations which are defined over

$$K = \mathbb{Q}(\sqrt{7 \times 15 \times 2 \times 6}) = \mathbb{Q}(\sqrt{35}).$$

We will not reproduce the details of the geometry of these fibrations here, but will describe as much as is necessary for this example. The recipe given for finding such fibrations is as follows. Let

$$(a_0, a_1, a_2, a_3) = (7, 15, -2, -6)$$

be the coefficients in the equation of the surface V.

Lemma 11 *Let θ, r_1, r_2, r_3 be elements of K such that*

$$a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2 = 0, \qquad \theta^2 = a_0 a_1 a_2 a_3.$$

Define polynomials A, B, C, D by

$$\begin{aligned} A &= \theta r_2 X_0^2 + a_1 a_3 (r_3 X_1^2 - r_1 X_3^2) \\ B &= \theta r_3 X_0^2 - a_1 a_2 (r_2 X_1^2 + r_1 X_2^2) \\ C &= a_3 \theta r_3 X_0^2 - a_1 a_2 a_3 (r_2 X_1^2 - r_1 X_2^2) \\ D &= -a_2 \theta r_2 X_0^2 - a_1 a_2 a_3 (r_3 X_1^2 + r_1 X_3^2); \end{aligned}$$

then the original equation (8) *factorises over K as*

$$\begin{aligned} &A(X_0, X_1, X_2, X_3) D(X_0, X_1, X_2, X_3) \\ &\qquad\qquad = B(X_0, X_1, X_2, X_3) C(X_0, X_1, X_2, X_3). \end{aligned}$$

The two morphisms $\pi_i : V \to \mathbb{P}^1$ given by

$$\begin{aligned} \pi_1 &= (A : B) = (C : D) \\ \pi_2 &= (A : C) = (B : D) \end{aligned}$$

are fibrations whose fibres are curves of genus 1. *The classes in* $\operatorname{Pic} V_K$ *of the fibres of π_1 and π_2 respectively are conjugate over $\mathbb{Q}$, and the sum of these two classes is equal to twice the class of a plane section.*

PROOF. See Swinnerton-Dyer (2000). □

If we write (A) for the divisor of zeros of the polynomial A, and so on, then

$$\begin{aligned} (A) &= F_1 + F_2 \qquad & (B) &= F_1' + F_2 \\ (C) &= F_1 + F_2' \qquad & (D) &= F_1' + F_2' \end{aligned} \tag{9}$$

where F_1 and F_1' are two fibres of π_1, each defined over K, and similarly F_2 and F_2' are two fibres of π_2.

Now consider the divisor F_1^σ, where σ is the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$. This is a smooth curve lying in the class in $\operatorname{Pic} V_K$ of the fibres of π_2, and therefore is a fibre of π_2, defined by an equation $B/D = c$ for some $c \in K$. To find c, it suffices to evaluate B/D at any point of F_1^σ. We first find a $\bar{\mathbb{Q}}$-valued point on F_1: such a point is a common zero of A and C. Rearranging the equation $A = 0$ gives us X_3^2 in terms of X_0 and X_1; similarly $C = 0$ yields an expression for X_2^2. Setting $X_0 = 0$ and $X_1 = 1$ gives

$$X_2^2 = \frac{r_2}{r_1} \qquad X_3^2 = \frac{r_3}{r_1}$$

and, taking conjugates and substituting into B/D, we obtain

$$c = \frac{B}{D}(F_1^\sigma) = \frac{r_1 r_2^\sigma + r_2 r_1^\sigma}{a_3(r_1 r_3^\sigma + r_3 r_1^\sigma)}. \tag{10}$$

Lemma 12 *The function*

$$f = \frac{C}{X_0^2}\left(\frac{B}{D} - c\right) = \frac{A - cC}{X_0^2}$$

has divisor $N_{K/\mathbb{Q}}(F_1 + D_0)$, where D_0 is the plane section divisor defined by $\{X_0 = 0\}$. The algebra

$$\mathcal{A} = (K/\mathbb{Q}, f)$$

is a non-trivial Azumaya algebra on V.

PROOF. The expression for (f) comes straight from (9) and (10). Proposition 9 then shows that $\mathcal{A}$ is Azumaya. Proposition 10 and the isomorphism (2) show that it is non-trivial. □

3.3.2 Computing the obstruction

To compute the algebra described in Lemma 12, we follow Lemma 11 and find r_1, r_2 and r_3 in $\mathbb{Q}(\sqrt{35})$ satisfying

$$15r_1^2 - 2r_2^2 - 6r_3^2 = 0. \tag{11}$$

This is not entirely straightforward. Such conic equations over the rational numbers have been well studied: Legendre famously described an algorithm for solving quadratic forms in three variables, and more recent work on efficient solution is described in Cremona and Rusin (2003). However, these methods run into problems when applied over a number field, as described in Simon's thesis (Simon, 1998). In the present case, several steps of Legendre's descent technique reduce the equation (11) to

$$x^2 - \omega y^2 = 3z^2$$

where $\omega = 6+\sqrt{35}$ is a fundamental unit in K. Fortunately, this new equation succumbs to a computer search, and we deduce the solution

$$(r_1, r_2, r_3) = (20 + 2\sqrt{35}, 15, 20 + 5\sqrt{35})$$

to (11). Note the similarity to the example in Section 2.3, where finding the rational fibres of a fibration also involved solving a norm equation. Substituting these values into the expression in (10) gives $c = -1$. Lemma 12 produces, after removing a constant factor,

$$f = \frac{7X_0^2 + 5X_1^2 - 4X_2^2 - 2X_3^2}{X_0^2} \tag{12}$$

such that $(35, f)$ is an Azumaya algebra on V.

We now show that there is no Brauer–Manin obstruction on V associated to the algebra $\mathcal{A}$.

Lemma 13 *Let $\mathcal{A}$ be an element of the 2-torsion subgroup of* $\operatorname{Br} V$. *Suppose that, for some place w of $\mathbb{Q}$, the local invariant function*

$$\operatorname{inv}_w \mathcal{A}(x_w)$$

takes both values 0 and $\frac{1}{2}$ for x_w in $V(\mathbb{Q}_w)$. Then there is no Brauer–Manin obstruction associated to $\mathcal{A}$.

PROOF. Let (x_v) be a point of $V(\mathbb{A}_\mathbb{Q})$. Then $\sum_v \operatorname{inv}_v \mathcal{A}(x_v)$ is equal to either 0 or $\frac{1}{2}$. In the first case, this shows that there is no Brauer–Manin obstruction associated to $\mathcal{A}$. In the second case, we replace x_w by x'_w so as to change the value of $\operatorname{inv}_w \mathcal{A}(x_w)$ and make the sum equal 0. □

In our example, looking at $\mathbb{Q}_5$ shows that there is no Brauer–Manin obstruction. Using the formula in Serre (1973, Chapter III, 1.2, Theorem 1), we have

$$(35, b)_5 = \left(\frac{7}{5}\right)^\beta \left(\frac{u}{5}\right) = (-1)^\beta \left(\frac{u}{5}\right)$$

where $b = 5^\beta u$ with u a 5-adic unit. Now let

$$g = 7X_0^2 + 5X_1^2 - 4X_2^2 - 2X_3^2;$$

then $(35, f) = (35, g)$. It is enough to show that g takes both square and non-square values in $\mathbb{Z}_5^\times$ when evaluated at points of $V(\mathbb{Q}_5)$. For example, the point $(2 : 1 : 1 : 5)$ satisfies

$$7X_0^4 + 15X_1^4 - 2X_2^4 - 6X_3^4 \equiv 0 \pmod{5^3}$$

and so by Hensel's Lemma is equivalent modulo 5^3 to a point of $V(\mathbb{Q}_5)$. But putting these values into g gives

$$g(2, 1, 1, 5) \equiv 4 \pmod{5^2}$$

which is a square in $\mathbb{Q}_5$. On the other hand, the point $(2 : 2 : 7 : 5)$ also lifts to a point of $V(\mathbb{Q}_5)$, this time modulo 5^2, but we have

$$g(2, 2, 7, 5) \equiv 2 \pmod{5^2}$$

which is not a square in $\mathbb{Q}_5$. Therefore $(35, f)$ takes values both $+1$ and -1 on $V(\mathbb{Q}_5)$, and so $\mathrm{inv}_5\, \mathcal{A}(x)$ takes values 0 and $\frac{1}{2}$. By Lemma 13, there is no Brauer–Manin obstruction associated to $\mathcal{A}$, and hence no algebraic Brauer–Manin obstruction on V.

That the quadratic form g should take both square and non-square values in $\mathbb{Z}_5^\times$ on the surface V seems unsurprising; indeed, one might be surprised if the opposite were true. It seems likely that the absence of an algebraic Brauer–Manin obstruction on this particular surface is one instance of a more general trend. This is supported by the evidence described below.

On the other hand, it is possible to list surfaces which fall into this "most general" class, and to search for rational points of bounded height on them.

There are 424 everywhere locally soluble diagonal quartic surfaces of the form

$$aX_0^4 + bX_1^4 = cX_2^4 + dX_3^4$$

with $|a|, |b|, |c|, |d|$ less than 16, and "sufficiently general" in the sense described at the beginning of this section. Checking these conditions is simple with Magma: the only one not yet described is local solubility, for which the bounds given by the Weil conjectures show that we only need to check solubility at 2, 5 and the primes dividing the coefficients.

It is reasonably straightforward (see Bernstein, 2001) to search for solutions with the $|X_i|$ less than 10^4. Let $N(H)$ denote the number of such surfaces with no point satisfying $|X_i| < H$ for all i; then we obtain the following data.

H	100	1000	10000
$N(H)$	81	69	61

There appears to be a trend for surfaces in this family to have a point of reasonably small height if at all. Should this pattern continue, there would be some surfaces either with points of extremely large height, or with no points at all. Moreover, the same construction as used above allows us to compute the algebraic Brauer–Manin obstruction, which turns out to be trivial on all of these surfaces: in each case, there is some prime at which the local invariant takes both values 0 and $\frac{1}{2}$. Although very vague, this evidence does suggest that some diagonal quartic surfaces may have trivial algebraic Brauer–Manin obstruction but no rational points.

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