GRÖBNER BASES OF IDEALS INVARIANT UNDER ENDOMORPHISMS

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Abstract. We introduce the notion of Gröbner S-basis of an ideal of the free associative algebra $K\langle X\rangle$ over a field K invariant under the action of a semigroup S of endomorphisms of the algebra. We calculate the Gröbner S-bases of the ideal corresponding to the universal enveloping algebra of the free nilpotent of class 2 Lie algebra and of the T-ideal generated by the polynomial identity $[x, y, z] = 0$, with respect to suitable semigroups S. In the latter case, if $|X| > 2$, the ordinary Gröbner basis is infinite and our Gröbner S-basis is finite. We obtain also explicit minimal Gröbner bases of these ideals.

1. Introduction

Let K be a field of any characteristic and let $X = \{x_1, x_2, \dots\}$ be a finite or countable set with more than one element. Let $K\langle X\rangle$ be the free unitary associative K-algebra generated by X. Its elements are polynomials in the noncommuting variables x_i.

In this paper we study some two-sided ideals of $K\langle X\rangle$ from computational point of view. We immediately face the problem that, even when the set X is finite, very few ideals of $K\langle X\rangle$ are finitely generated. On the other hand, quite often important ideals of $K\langle X\rangle$ have additional structure and "uniformly looking" generating sets.

For example, let $L = L(X)$ be the free Lie algebra freely generated by X and canonically embedded into $K\langle X\rangle$. It is known that the free nilpotent of class c Lie algebra $L/[\underbrace{L, \dots, L}_{c+1 \text{ times}}]$, usually denoted in the theory of varieties of Lie algebras $F(\mathfrak{N}_c)$, has a set of defining relations

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consisting of all (left normed) commutators

$$u_j = [[\dots[x_{j_1}, x_{j_2}], \dots, x_{j_c}], x_{j_{c+1}}] = 0.$$

Hence, by the Poincaré-Birkhoff-Witt theorem, its universal enveloping algebra $U(F(\mathfrak{N}_c))$ is a homomorphic image of $K\langle X\rangle$ modulo the ideal I generated by all u_j. We may define the ideal I as the minimal ideal of $K\langle X\rangle$ which contains the commutator

$$[x_1, x_2, \dots, x_c, x_{c+1}] = [[\dots[x_1, x_2], \dots, x_c], x_{c+1}]$$

and is invariant, or stable, under all endomorphisms sending X to X.

Other examples are the T-ideals of $K\langle X\rangle$. These ideals are invariant under all endomorphisms of $K\langle X\rangle$ and coincide with the ideals of polynomial identities of suitable PI-algebras. If

$$U = \{u_j(x_1, \dots, x_{n_j}) \mid j \in J\} \subset K\langle X\rangle$$

is any set, then the T-ideal generated by U is generated as a usual ideal by all $u_j(f_1, \dots, f_{n_j})$, when the n_j polynomials $f_1, \dots, f_{n_j}$ run on $K\langle X\rangle$. For infinite X nontrivial T-ideals cannot be finitely generated as ideals. If the set X is finite, then a theorem of Markov [Ma] describes the few cases when a T-ideal is finitely generated as an ideal. This happens if and only if the T-ideal contains for some c the Engel polynomial

$$[x_2, \underbrace{x_1, \dots, x_1}_{c \text{ times}}].$$

One of the classical problems in PI-theory is the Specht problem [Sp] which states whether any T-ideal is finitely generated as a T-ideal. The celebrated structure theory of T-ideals developed by Kemer, see his book [K2] for the account, allowed him [K1] in 1987 to give a positive solution to the Specht problem over a field of characteristic 0. In the case of positive characteristic there are several counterexamples. The first of them was given by Belov [B]. A good source for the state of the art of the Specht problem, as well as an improved exposition of the theory of Kemer, can be found in the recent book by Kanel-Belov and Rowen [KBR].

When the set of variables X is finite, the knowledge of a generating set of an ideal I of the polynomial algebra $K[X]$ is not always sufficient for concrete calculations with the elements of I and in the factor algebra $K[X]/I$. A similar phenomenon appears for the ideals of $K\langle X\rangle$, even if the ideal has a finite generating set. In commutative algebra the problem is solved with the technique of Gröbner bases. This is a powerful tool for computing with commutative algebras, in algebraic geometry, and in invariant theory, see e.g. the books by Adams and Loustaunau [AL], Kreuzer and Robbiano [KRo], Sturmfels [St]. In the last two decades the number of the applications of the noncommutative Gröbner bases increases, see e.g. the seminal paper by Bergman [Be] and the surveys by Mora and Ufnarovski [Mo, U]. Nevertheless, there are very few examples of ideals of the free algebras with explicitly known Gröbner bases.

In the present paper we consider ideals I of the free algebra $K\langle X\rangle$ which are invariant under the action of a subsemigroup S of the endomorphism semigroup of $K\langle X\rangle$. We introduce the notion of Gröbner S-basis of I. This is a subset B of I with the property that $S(B)$ is a Gröbner basis of I in the usual sense, with respect to some termordering of the monomials in $K\langle X\rangle$.

We handle completely two cases of Gröbner S-bases. The first is the universal enveloping algebra $U(F(\mathfrak{N}_2))$ of the free nilpotent of class 2 Lie algebra $F(\mathfrak{N}_2) = L/[L, L, L]$. The semigroup S consists of all endomorphisms which send X to X and preserve the ordering on X. The corresponding ideal I of $K\langle X\rangle$ is generated by the commutators $[x_i, x_j, x_k]$. We give a concrete finite Gröbner S-basis of I. It consists of commutators of length 3 and one more commutator of degree 4.

One may introduce Gröbner bases for ideals not only in the polynomial algebra $K[X]$ and in the free associative algebra $K\langle X\rangle$, but also for free Lie algebras and other free objects, see e.g. [BFS]. In this case one often calls the corresponding bases Gröbner-Shirshov bases instead of Gröbner bases. By a theorem of Lalonde and Ram [LR], and Bokut and Malcolmson [BM], if H is an ideal of the free Lie algebra $L(X)$ and B is its Gröbner-Shishov basis with respect to a certain ordering on a suitable basis of the vector space $L(X)$, then B is also a Gröbner basis of the ideal I of $K\langle X\rangle$ generated by H. Of course, the factorization modulo this ideal gives the universal enveloping algebra $U(L/H)$. This result easily implies that the algebra $U(F(\mathfrak{N}_2))$ does have a Gröbner basis consisting of polynomials of degree 3 and 4 only. We want to mention that our approach is direct and does not use the theorem of Lalonde-Ram [LR] and Bokut-Malcolmson [BM]. Instead, we use easy combinatorics of words and the explicit K-basis of $U(F(\mathfrak{N}_2))$.

The second example treats another algebra of importance for the theory of PI-algebras and with applications to superalgebras. This is the relatively free algebra $F(\mathrm{var}E)$ of the variety of associative algebras generated by the Grassmann (or exterior) algebra E over an infinite field of characteristic different from 2. This algebra can be considered as the generic Grassmann algebra. The structure of $F(\mathrm{var}E)$, $\mathrm{char}K = 0$, was described by Krakowski and Regev [KR], see also the paper by Di Vincenzo [DV] or the book [D] by one of the authors. It is known that the polynomial identities of E are consequences of the commutator identity $[x_1, x_2, x_3] = 0$. The defining relations of $F(\mathrm{var}E)$ in characteristic 0 were described by Latyshev [L] and consist of the polynomials

$$[x_i, x_j, x_k] = 0, \quad [x_i, x_j][x_k, x_l] + [x_i, x_k][x_j, x_l] = 0,$$

where x_i, x_j, x_k, x_l are replaced by all possible elements of X. It is well known that the same polynomials form a set of defining relations of $K\langle X\rangle/([x_1, x_2, x_3])^T$ over any field of characteristic different from 2, where $([x_1, x_2, x_3])^T$ is the T-ideal of $K\langle X\rangle$ generated by $[x_1, x_2, x_3]$. Bokut and Makar Limanov [BML] showed that, when $|X| > 2$, the ideal $([x_1, x_2, x_3])^T$ has no finite Gröbner basis. On the other hand, they introduced an extra set of generators of the algebra $F(\mathrm{var}E)$, $y_{ij} = [x_i, x_j]$, which are in its centre, and established that the corresponding Gröbner basis is finite when X is finite. In the present paper we show that, although the Gröbner basis of the T-ideal $([x_1, x_2, x_3])^T$ is infinite for $m > 2$, it is uniformly looking. We present explicitly a finite set of polynomials G and a subsemigroup S of the endomorphism semigroup of $K\langle X\rangle$ such that G is a Gröbner S-basis of the ideal. We also correct some inaccuracies in the paper by Bokut and Makar Limanov [BML]. Again, our approach is based on combinatorics of words and the explicit basis of $F(\mathrm{var}E)$.

2. S-ideals, S-bases, Gröbner S-bases

Denote by $\mathrm{End}(K\langle X\rangle)$ the semigroup of all endomorphisms of the K-algebra $K\langle X\rangle$. Let $S \subset \mathrm{End}(K\langle X\rangle)$ be a subsemigroup which includes the identity endomorphism. If I is a two-sided ideal of $K\langle X\rangle$ we say that I is an *S-invariant ideal* or simply an *S-ideal* if it is invariant under all the endomorphisms of S, i.e.

$$\varphi(I) \subset I \quad \text{for all } \varphi \in S.$$

To construct an S-ideal it is sufficient to take any subset $B \subset K\langle X\rangle$ and form the two-sided ideal I generated by $S(B)$. In this case, we say that B is an *S-basis* of I.

A natural problem is to establish if, for different choices of the semigroup S, all the S-ideals have finite S-bases. For example, the positive solution by Kemer [K1] of the Specht problem in characteristic 0 can be restated that for $S = \mathrm{End}(K\langle X\rangle)$ every S-invariant ideal is finitely S-generated.

We fix now on $K\langle X\rangle$ a *term-ordering* $<$, i.e. a linear order on the set $\langle X\rangle$ of words, or monomials, which is a multiplicatively compatible well-ordering. This means that:

(i) For every two different monomials u, v we have either $u < v$ or $v < u$;

(ii) Every subset of $\langle X \rangle$ has a minimal element;

(iii) If $u < v$ in $\langle X \rangle$, then $wu < wv$ and $uw < vw$ for every $w \in \langle X \rangle$.

If $f \in K\langle X \rangle$ is a nonzero polynomial, we denote by $\mathrm{lt}(f)$ the greatest monomial of f. We recall that a *Gröbner basis* of an ideal I of $K\langle X \rangle$ is a subset $G \subset I$ (not necessarily finite) which satisfies the following property: for any nonzero $f \in I$ there exists a $g_i \in G$ such that $\mathrm{lt}(g_i)$ is a subword of $\mathrm{lt}(f)$. By induction on the term-ordering, it is easy to prove that we can write any $f \in I$ as

$$f = \sum f_i g_i h_i,$$

where $g_i \in G$ (possibly $g_i = g_j$ for $i \neq j$) and we have $f_i, h_i \in K\langle X \rangle$, only a finite number of them different from zero, such that for all i

$$\mathrm{lt}(f) \geq \mathrm{lt}(f_i)\mathrm{lt}(g_i)\mathrm{lt}(h_i).$$

We have hence that G is also a generating set of I as a two-sided ideal of $K\langle X \rangle$. For any subset $G \subset K\langle X \rangle$ it is useful to define $\mathrm{init}(G)$ as the two-sided ideal generated by the set of monomials $\{ \mathrm{lt}(g_i) \mid g_i \in G \}$. We say that $\mathrm{init}(G)$ is the *initial ideal* generated by G. Then, we have clearly that a subset $G \subset I$ is a Gröbner basis of I if and only if $\mathrm{init}(G) = \mathrm{init}(I)$. In other words, the set of monomials of $K\langle X \rangle$

$$\{ w \in \langle X \rangle \mid \text{there exists } g_i \in G \text{ such that } \mathrm{lt}(g_i) \text{ is a subword of } w \}$$

is a K-basis of the subspace $\mathrm{init}(I) \subset K\langle X \rangle$. Then the set

$$N = \langle X \rangle \backslash \mathrm{lt}(\mathrm{init}(G))$$

of *normal words* with respect to G is a K-basis of the factor algebra $K\langle X \rangle / I$. A Gröbner basis G of an ideal I is called *reduced* if every $g_i \in G$ is a linear combination of normal words with respect to $G \backslash \{g_i\}$. Moreover, we call G *minimal* if for any $g_i \in G$ we have that $G \backslash \{g_i\}$ is not a Gröbner basis of I, i.e. $\mathrm{lt}(g_i)$ is a normal word with respect to $G \backslash \{g_i\}$. For more details about the theory of noncommutative Gröbner bases we refer to [Mo, U].

Now let S be a semigroup of endomorphisms of $K\langle X \rangle$ and let G be a subset of the S-ideal I. We say that G is a *Gröbner S-basis* of I if $S(G)$ is a Gröbner basis of I as a two-sided ideal of $K\langle X \rangle$, i.e. $\mathrm{init}(I)$ is equal to the initial ideal generated by $S(G)$.

3. Universal enevoping algebras of free nilpotent algebras

We keep K, X, and $K\langle X\rangle$ as in the previous section. We introduce the standard *deg-lex* ordering on $\langle X\rangle$. We compare the monomials first by total degree and then lexicographically, reading them from left to right and assuming that $x_1 < x_2 < \cdots$. We consider $K\langle X\rangle$ also as a multigraded vector space, counting in the monomials the number of enterings of each variable. If $f, g \in K\langle X\rangle$, *the commutator of* f, g is simply the polynomial

$$[f, g] = fg - gf.$$

We refer to the book by Bahturin [Ba] as a background on Lie algebras and their polynomial identities. Here we summarize the basic facts we need. The Lie subalgebra of $K\langle X\rangle$ generated by X with respect to the commutator operation is isomorphic to the free Lie algebra freely generated by X. We denote this algebra by $L = L(X)$. Every Lie algebra generated by a countable (or finite) set is isomorphic to L/H for some ideal H of the Lie algebra L. Then the Poincaré-Birkhoff-Witt theorem gives that the universal enveloping algebra $U(L/H)$ is isomorphic to $K\langle X\rangle/I$, where $I = K\langle X\rangle HK\langle X\rangle$ is the ideal of $K\langle X\rangle$ generated by H. If $f_1, f_2, \ldots$ is a basis of the K-vector space L/H, then $U(L/H)$ has a K-basis consisting of all "monomials" $f_1^{a_1} \cdots f_p^{a_p}$.

The algebra L has several important bases consisting of commutators. They are built on the following principle. One fixes an ordered set of associative *Lyndon-Shirshov* monomials defined in terms of some special combinatorial properties. Then, for each monomial, one arranges the Lie brackets in a certain recursive way, and obtains the basis of L. The elements of the basis are either elements of X or commutators $[[u], [v]]$, where $[u], [v]$ are also elements of the basis. The bases under consideration allow to introduce an analogue of Gröbner bases for the ideals H of L, called *Gröbner-Shirshov* bases. The subset G of H is a Gröbner-Shirshov basis of H, if for every nonzero $f \in H$ with leading commutator $[u]$ there exists a $g \in G$ with leading commutator $[v]$ such that the associative word v obtained by deleting the Lie brackets in $[v]$ is a subword of the associative word u. The theorem of [LR, BM] cited in the introduction gives that every Gröbner-Shirshov basis of the ideal H of L is a Gröbner basis of the ideal I generated in $K\langle X\rangle$ by H.

We denote by $F(\mathfrak{N}_c)$ the free nilpotent of class c Lie algebra. It is isomorphic to the factor algebra of L modulo the $(c+1)$-st member

$$\gamma_{c+1}(L) = [\underbrace{L, \ldots, L}_{c+1 \text{ times}}]$$

of the lower central series of L. It is well known that $\gamma_{c+1}(L)$ is spanned by all commutators of length $\geq c+1$ and can be generated as an ideal by commutators of length $c+1$. We apply the version of Bokut and Malcolmson of [LR, BM] to the universal enveloping algebra $U(F(\mathfrak{N}_c))$ and the ideal of $K\langle X\rangle$ generated by $\gamma_{c+1}(L)$.

Proposition 3.1. *There exists a Gröbner basis with respect to the deg-lex ordering of the ideal of $K\langle X\rangle$ generated by $\gamma_{c+1}(L)$ consisting only of commutators of length $c+1, c+2, \ldots, 2c$.*

Proof. Clearly, $\gamma_{c+1}(L)$ is a multigraded subspace of $K\langle X\rangle$ and does not contain linear combinations of commutators of length $< c+1$. Hence, if a Gröbner-Shirshov basis of $\gamma_{c+1}(L)$ consists of commutators, all commutators have to be of length $\geq c+1$. Let us fix a basis of L built on Lyndon-Shirshov words, and let G be the set of all commutators of lenght $c+1, c+2, \ldots, 2c$ from this basis. We shall show that G is a Gröbner-Shirshov basis of $\gamma_{c+1}(L)$. We apply induction on the length of the elements of the basis: If $[[u],[v]]$ is a commutator of lenght $> 2c$, then at least one of the commutators $[u],[v]$ is of length $\geq c+1$ and, by inductive arguments, at least one of the associative words u, v contains as a subword w for a suitable element $[w] \in G$. Now the proof is completed by the theorem of [BM]. $\square$

We apply Proposition 3.1 to the ideal I of $K\langle X\rangle$ generated by $\gamma_3(L)$.

Proposition 3.2. *The polynomials*

$$f'_{ij} = [[x_i, x_j], x_j], \quad f''_{ij} = [x_i, [x_i, x_j]], \quad i > j, \tag{1}$$

$$g'_{ijk} = [x_i, [x_j, x_k]], \quad g''_{ikj} = [[x_i, x_k], x_j], \quad i > j > k, \tag{2}$$

$$h_{ijk} = [[x_i, x_j], [x_i, x_k]], \quad i > j > k, \tag{3}$$

form a Gröbner basis with respect to the deg-lex ordering of the ideal $I = K\langle X\rangle\gamma_3(L)K\langle X\rangle$.

Proof. We consider the set B of all associative Lyndon-Shirshov words u defined with the property that u is bigger than all its cyclic rearrangements. The brackets on u are arranged as follows. One finds the longest right Lyndon-Shirshov subword v of u. Then $u = wv$ for some word w. It turns out that w is also a Lyndon-Shirshov word. One considers the nonassociative Lyndon-Shirshov words $[w], [v]$ corresponding to w and v. Then one defines $[u] = [[w],[v]]$.

By Proposition 3.1 we need all associative Lyndon-Shirshov words of length 3 and 4. They are

$$x_ix_jx_j, \quad x_ix_ix_j, \quad i > j, \tag{4}$$

$$x_ix_jx_k, \quad x_ix_kx_j, \quad i > j > k, \tag{5}$$

$$x_ix_jx_jx_j, \quad x_ix_ix_jx_j, \quad x_ix_ix_ix_j, \quad i > j, \tag{6}$$

$$x_ix_ix_jx_k, \quad i > j, k, \quad x_ix_jx_ix_k, \quad i > j > k, \tag{7}$$

$$x_ix_jx_kx_l, \quad i > j, k, l. \tag{8}$$

The arrangement of the brackets in the cases (4) and (5) is, respectively,

$$[[x_i, x_j], x_j], \quad [x_i, [x_i, x_j]], \quad [x_i, [x_j, x_k]], \quad [[x_i, x_k], x_j],$$

and this gives the elements f'_{ij} and f''_{ij}, $i > j$, in (1) and g'_{ijk} and g''_{ijk}, $i > j > k$, in (2). Similarly, we obtain h_{ijk}, $i > j > k$, in (3) from $x_ix_jx_ix_k$ in (7). We do not need the commutators built on the words from (6), (8), and the words $x_ix_ix_jx_k$ from (7) for the Gröbner basis of the ideal I generated by $\gamma_3(L)$ because they contain a subword of the form $u_iu_iu_j$ or $u_iu_ju_k$ with $i > j, k$. Hence the commutators (1), (2), and (3) give a Gröbner basis of I. □

Now we state Proposition 3.2 in terms of Gröbner S-bases.

Theorem 3.3. *Let X be an infinite set and let S be the semigroup consisting of all endomorphisms of $K\langle X\rangle$ which send X to X and preserve the ordering on X. Then the set of polynomials*

$$[[x_2, x_1], x_1], \quad [x_2, [x_2, x_1]], \tag{9}$$

$$[x_3, [x_2, x_1]], \quad [[x_3, x_1], x_2], \quad [[x_3, x_2], [x_3, x_1]] \tag{10}$$

is a Gröbner S-basis of the ideal of $K\langle X\rangle$ generated by $\gamma_3(L)$.

Proof. Let φ_1 be an endomorphism from S such that $\varphi_1(x_1) = x_j$ and $\varphi_1(x_2) = x_i$, $i > j$. Applying φ_1 to $[[x_2, x_1], x_1]$ and $[x_2, [x_2, x_1]]$ we obtain the elements (1). Similarly, if $i > j > k$, we start with $\varphi_2 \in S$ satisfying $\varphi_2(x_1) = x_k$, $\varphi_2(x_2) = x_j$, $\varphi_2(x_3) = x_i$. Applying it on $[x_3, [x_2, x_1]]$, $[[x_3, x_1], x_2]$, and $[[x_3, x_2], [x_3, x_1]]$, we obtain (2) and (3). In this way, acting by S on the elements from (9) and (10), we obtain the Gröbner basis of the ideal generated by $\gamma_3(L)$. □

Remark 3.4. (i) It is easy to see that applying the semigroup S from Theorem 3.3 to the Gröbner S-basis (9), (10), we obtain a minimal Gröbner basis which is not reduced. The polynomial

$$[x_3,[x_2,x_1]] = x_3x_2x_1 - x_3x_1x_2 - x_2x_1x_3 + x_1x_2x_3$$

contains as a summand the monomial $x_3x_1x_2$ which can be reduced using $[[x_3,x_1],x_2]$. The commutator $[[x_3,x_2],[x_3,x_1]]$ also needs to be reduced. These reductions can be done by easy calculations.

(ii) The restriction that X is countable is not essential. Theorem 3.3 can be restated for any infinite well-ordered set X.

(iii) When the set X is finite, the semigroup S from Theorem 3.3 consists of the identity endomorphism only. We may replace it with the semigroup generated by the endomorphisms φ_1, φ_2 of $K\langle X\rangle$ with the property $\varphi_1(X),\varphi_2(X)\subseteq X$, $\varphi_1(x_1)<\varphi_1(x_2)$, $\varphi_2(x_1)<\varphi_2(x_2)<\varphi_2(x_3)$.

We shall give another direct combinatorial description of the Gröbner basis of the ideal of $K\langle X\rangle$ generated by $\gamma_3(L)$ which we shall use later for the Gröbner basis of the T-ideal $([x_1,x_2,x_3])^T$.

Lemma 3.5. *The polynomials*

$$x_{i_1}\cdots x_{i_l}[x_{j_1},x_{k_1}]\cdots[x_{j_m},x_{k_m}], \tag{11}$$

where $i_1\leq\cdots\leq i_l$, $j_s>k_s$, $s=1,\ldots,m$, and $(j_1,k_1)\leq\cdots\leq(j_m,k_m)$ with respect to the lexicographic ordering, form a K-basis of the universal enveloping algebra $U(F(\mathfrak{N}_2))$.

Proof. The Poincaré-Birkhoff-Witt theorem gives that, if $g_1,g_2,\ldots$ is an ordered K-basis of a Lie algebra, then its universal enveloping algebra has a K-basis consisting of all $g_1^{a_1}\cdots g_n^{a_n}$. This immediately completes the proof: the free nilpotent of class 2 Lie algebra $F(\mathfrak{N}_2)$ is spanned by all commutators of length 1 and 2, i.e. by the elements x_i and $[x_i,x_j]$, and the anticommutativity allows to assume that $i>j$ in $[x_i,x_j]$. □

Lemma 3.6. *The set of normal words $N(G)$ with respect to the set G of the commutators (1), (2), and (3) consists of all monomials $w=x_{i_1}\cdots x_{i_n}$ such that*

(i) *The inequality $i_k>i_{k+1}$ implies that $i_k\leq i_{k+2}$ and if, additionally $k>1$, then $i_{k-1}<i_k$;*

(ii) *If $i_k=i_{k+2}>i_{k+1},i_{k+3}$, then $i_{k+1}\leq i_{k+3}$.*

Proof. The leading monomials of the elements of G are of three types:

(a) $x_ix_jx_j$ and $x_ix_ix_j$, where $i\geq j$;

(b) $x_ix_jx_k$, where $i>j,k$;

(c) $x_ix_jx_ix_k$, where $i>j>k$.

If the word $w = x_{i_1} \cdots x_{i_n}$ does not satisfy the condition (i), then $i_k > i_{k+1}$ for some k, but $i_{k-1} \geq i_k$ or $i_k > i_{k+2}$. In this case at least one of the subwords $x_{i_{k-1}} x_{i_k} x_{i_{k+1}}$ and $x_{i_k} x_{i_{k+1}} x_{i_{k+2}}$ is of type (a) or (b). Suppose now that w does not satisfy (ii), i.e. $i_k = i_{k+2} > i_{k+1}, i_{k+3}$ and $i_{k+1} > i_{k+3}$. Then, the subword $x_{i_k} x_{i_{k+1}} x_{i_{k+2}} x_{i_{k+3}}$ is of type (c). Moreover, the above arguments can be clearly reversed. □

Now we give an explicit bijection between the basis of $U(F(\mathfrak{N}_2))$ from Lemma 3.5 and the set of normal words from Lemma 3.6.

Proposition 3.7. *There is a one-to-one correspondence between the set B of the products (11) and the set $N(G)$ from Lemma 3.6 which preserves the multigrading.*

Proof. We consider the set of sequences of indices that parametrize the polynomials in B, say:

$$\overline{B} = \{(i_1, \ldots, i_l, (j_1, k_1), \ldots, (j_m, k_m))\}.$$

We consider also the set of sequences of indices that occur in the words of $N = N(G)$:

$$\overline{N} = \{(i_1, \ldots, i_n) \mid i_k \text{ satisfies (i),(ii)}\}$$

We define recursively a map ψ from $\overline{B}$ into the set of sequences of integers. If $u = (i_1, \ldots, i_l, (j_1, k_1), \ldots, (j_m, k_m))$ then we find the first index i_{p+1} with the property $j_1 \leq i_{p+1}$ (hence $i_p < j_1$ if $p \geq 1$) and define

$$\psi(u) = (i_1, \ldots, i_p, j_1, k_1, \psi(v))$$

where $v = (i_{p+1}, \ldots, i_l, (j_2, k_2), \ldots, (j_m, k_m))$. We shall prove that the image of ψ is contained in $\overline{N}$. Since $i_1 \leq \cdots \leq i_l$ and by the definition of ψ we have that $\psi(u)$ satisfies the condition (i). Moreover, owing to the lexicographic ordering of the pairs $(j_1, k_1), \ldots, (j_m, k_m)$ it is clear that also (ii) is verified. For example, if

$$u = (1, 2, 2, 2, 3, 4, 5, 6, (\mathbf{2}, \mathbf{1}), (\mathbf{2}, \mathbf{1}), (\mathbf{3}, \mathbf{1}), (\mathbf{3}, \mathbf{2}), (\mathbf{5}, \mathbf{2}), (\mathbf{5}, \mathbf{3}), (\mathbf{6}, \mathbf{4})),$$

(we have typesetted the pairs (j, k) in bold) then

$$\psi(u) = (1, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, 2, 2, 2, \mathbf{3}, \mathbf{1}, \mathbf{3}, \mathbf{2}, 3, 4, \mathbf{5}, \mathbf{2}, \mathbf{5}, \mathbf{3}, 5, \mathbf{6}, \mathbf{4}, 6). \tag{12}$$

We define now two maps ϑ_1, ϑ_2 from $\overline{N}$ respectively into the set of integer sequences and the set of sequences of pairs of integers. If $u = (i_1, \ldots, i_n)$ then:

$$\vartheta_1(u) = (i_1, \ldots, i_{k-1}, \vartheta_1(v)) \text{ and } \vartheta_2(u) = ((i_k, i_{k+1}), \vartheta_2(v)),$$

where $i_1 \leq \cdots \leq i_k > i_{k+1}$ and $v = (i_{k+2}, \ldots, i_n)$. Define now the map $\vartheta : v \mapsto (\vartheta_1(v), \vartheta_2(v))$. We claim that the image of ϑ is contained in $\overline{B}$.

In fact, by the condition (i) we have that $\vartheta_1(u)$ is an increasing sequence of indices. Moreover, from the definition of ϑ_2 and the condition (ii) it follows that $\vartheta_2(u)$ is a sequence of pairs (j,k) with $j>k$, which is increasing with respect to the lexicographic ordering. In the above example, if $v=\psi(u)$, then $\vartheta(v)=u$.

Finally, it is easy to check that the maps ψ and ϑ induce bijections between B and N which preserve the multigrading and are inverse of each other. $\square$

Remark 3.8. It is more convenient, compare with the example in (12), to write the normal words $N(G)$ from Lemma 3.6 in the form

$$x_1^{a_1}(x_2x_1)^{b_{21}}x_2^{a_2}(x_3x_1)^{b_{31}}(x_3x_2)^{b_{32}}x_3^{a_3}\cdots\prod_{p=1}^{m-1}(x_mx_p)^{b_{mp}}x_m^{a_m}, \tag{13}$$

where $a_i,b_{ij}\geq 0$. For example, in (12) we have the word

$$x_1(x_2x_1)^2x_2^3(x_3x_1)(x_3x_2)x_3x_4(x_5x_2)(x_5x_3)x_5(x_6x_4)x_6.$$

Let I be a multigraded ideal of $K\langle X\rangle$ and let B be a multigraded basis of $R=K\langle X\rangle/I$. If G is a subset of I and $N(G)$ is the set of normal words with respect to G, then in each multihomogeneous component of B and $N(G)$, the number of elements from B is not greater than the number of elements from $N(G)$. If the number of these elements coincides for each multihomogeneous component, we have that G is a Gröbner basis for I. Hence Proposition 3.7 implies immediately Proposition 3.2 and Theorem 3.3.

4. The polynomial identities of the Grassmann algebra

In this section we assume that the base field K is of characteristic different from 2. We consider the T-ideal $T=([x_1,x_2,x_3])^T$ of $K\langle X\rangle$ generated by the commutator $[x_1,x_2,x_3]$. We shall summarize the necessary facts, including also some proofs to make the exposition self-contained. The following proposition is well known, see Latyshev [L] for the case of characteristic 0.

Proposition 4.1. (i) *The factor algebra $K\langle X\rangle/T$ satisfies the identities*

$$[x_1,x_2]x_3=x_3[x_1,x_2], \tag{14}$$

$$[x_1,x_2][x_1,x_3]=0,\quad [x_1,x_2]x_4[x_1,x_3]=0, \tag{15}$$

$$[x_1,x_2][x_3,x_4]+[x_1,x_3][x_2,x_4]=0, \tag{16}$$

$$[x_1,x_2]x_5[x_3,x_4]+[x_1,x_3]x_5[x_2,x_4]=0. \tag{17}$$

(ii) *The products*

$$x_{i_1}\cdots x_{i_l}[x_{j_1},x_{k_1}]\cdots[x_{j_m},x_{k_m}], \tag{18}$$

$i_1\leq\cdots\leq i_l$, $k_1<j_1<\cdots<k_m<j_m$, *form a* K*-basis of* $K\langle X\rangle/T$.

Proof. (i) The identity (14) is an expanded form of $[[x_1,x_2],x_3]=0$. The first identity of (15) is obtained from $[[x_1,x_2],x_3]=0$ by the substitution x_1^2 instead of x_1 and using that the commutators are in the centre:

$$\begin{aligned}0&=[[x_1^2,x_2],x_3]\\&=x_1[x_1,x_2,x_3]+[x_1,x_2,x_3]x_1+[x_1,x_2][x_1,x_3]+[x_1,x_3][x_1,x_2]\\&=[x_1,x_2][x_1,x_3]+[x_1,x_3][x_1,x_2]=2[x_1,x_2][x_1,x_3].\end{aligned}$$

This gives $[x_1,x_2][x_1,x_3]=0$ because the characteristic is different from 2. The second identity of (15) follows from the first and (14):

$$0=x_4[x_1,x_2][x_1,x_3]=[x_1,x_2]x_4[x_1,x_3].$$

The linearization of (15) gives (16):

$$\begin{aligned}0&=[x_2+x_3,x_1][x_2+x_3,x_4]-[x_2,x_1][x_2,x_4]-[x_3,x_1][x_3,x_4]\\&=-([x_1,x_2][x_3,x_4]+[x_1,x_3][x_2,x_4]),\end{aligned}$$

and the second identity in (17) is obtained as the second identity in (15).

(ii) The algebra $K\langle X\rangle/T$ is a homomorphic image of the algebra $U(F(\mathfrak{N}_2))$. Hence it is spanned on the products from (11). The anticommutative law $[x_1,x_2]=-[x_2,x_1]$ and the identities (15), (16) allow to rearrange the indices $j_1,k_1,\ldots,j_m,k_m$ and to remove the products with two equal indices in the commutators. Hence the elements of (18) span $K\langle X\rangle/T$. In order to see that (18) are linearly independent, it is sufficient to consider only multihomogenous linear combinations of (18) with nonzero coefficients (because the ideal T is multigraded). The Grassmann algebra $E(K)$ generated by $e_1,e_2,\ldots$ over K (or over an infinite extension of K when the field K is finite) satisfies the identity $[x_1,x_2,x_3]=0$. It satisfies the relations $e_ie_j=-e_je_i$ and has a K-basis consisting of all $e_{i_1}\cdots e_{i_k}$, $i_1<\cdots<i_k$. Using a well known calculation in the theory of PI-algebras, we replace x_i with α_i+e_i, $\alpha_i\in K$, and, choosing properly the α_i's, we obtain a nonzero evaluation of the considered linear combination of (18). This guarantees that the elements (18) are linearly independent. $\square$

Theorem 4.2. *Let* $\mathrm{char}(K) \neq 2$*. The polynomials*

$$f'_{ij} = [[x_i, x_j], x_j], \quad f''_{ij} = [x_i, [x_i, x_j]], \quad i > j,$$
$$g'_{ijk} = [x_i, [x_j, x_k]], \quad g''_{ikj} = [[x_i, x_k], x_j], \quad i > j > k,$$

from (1) and (2) and the polynomials

$$t_{ij} = [x_i, x_j][x_i, x_j], \quad i > j, \tag{19}$$

$$u'_{ijk} = [x_i, x_j][x_i, x_k], \quad u''_{ijk} = [x_i, x_k][x_i, x_j], \quad i > j > k, \tag{20}$$

$$v'_{ijka} = [x_j, x_k]x_j^{a_j} \cdots x_{i-1}^{a_{i-1}}[x_i, x_k], \tag{21}$$

$$v''_{ijka} = [x_j, x_k]x_j^{a_j} \cdots x_{i-1}^{a_{i-1}}[x_i, x_j], \tag{22}$$

where $i > j > k$*,* $a_j, \ldots, a_{i-1} \geq 0$*,*

$$w'_{ijkla} = [x_j, x_k]x_j^{a_j} \cdots x_{i-1}^{a_{i-1}}[x_i, x_l] + [x_j, x_l]x_j^{a_j} \cdots x_{i-1}^{a_{i-1}}[x_i, x_k], \tag{23}$$

$$w''_{ijkla} = [x_j, x_l]x_j^{a_j} \cdots x_{i-1}^{a_{i-1}}[x_i, x_k] + [x_k, x_l]x_j^{a_j} \cdots x_{i-1}^{a_{i-1}}[x_i, x_j], \tag{24}$$

where $i > j > k > l$*,* $a_j, \ldots, a_{i-1} \geq 0$*, form a minimal Gröbner basis with respect to the deg-lex ordering of the T-ideal of* $K\langle X\rangle$ *generated by* $[x_1, x_2, x_3]$*.*

Proof. By Proposition 4.1 (i), the polynomials (1), (2), (19), (20), (21), (22), (23), (24) belong to the T-ideal T generated by $[x_1, x_2, x_3]$. Their leading terms are obtained by deleting the commutators in the corresponding elements and are, respectively,

$$x_ix_jx_j, \quad x_ix_ix_j, \quad i > j,$$
$$x_ix_jx_k, \quad x_ix_k, x_j, \quad i > j > k,$$
$$x_ix_jx_ix_j, \quad i > j,$$
$$x_ix_jx_ix_k, \quad x_ix_kx_ix_j, \quad i > j > k,$$
$$x_jx_kx_j^{a_j} \cdots x_{i-1}^{a_{i-1}}x_ix_k, \quad x_jx_kx_j^{a_j} \cdots x_{i-1}^{a_{i-1}}x_i, x_j, \quad i > j > k,$$
$$x_jx_kx_j^{a_j} \cdots x_{i-1}^{a_{i-1}}x_ix_l, \quad x_jx_lx_j^{a_j} \cdots x_{i-1}^{a_{i-1}}x_ix_k, \quad i > j > k > l,$$

and $a_j, \ldots, a_{i-1} \geq 0$. It is easy to see that these words are pairwise different. Clearly, the polynomial $u'_{ijk} = [x_i, x_j][x_i, x_k]$ from (20) has the same leading term as $h_{ijk} = [[x_i, x_j], [x_i, x_k]]$ from (3). Hence the set of normal words with respect to f'_{ij}, f''_{ij}, g'_{ijk}, g''_{ikj}, u'_{ijk} is the same as the one in Lemma 3.6 and we may assume that these normal words are in the form (13). Now we want to remove the words in (13) which contain as a subword a leading word of some t_{ij}, u''_{ijk}, v'_{ijka}, v''_{ijka}, w'_{ijkla}, w''_{ijkla}. If $b_{ij} \geq 2$ for some i, j, then we remove the word using t_{ij}. Hence we may assume that $b_{ij} \leq 1$. If $b_{ik} = b_{ij} = 1$ for some $i > j > k$, and $b_{i,k+1} = \cdots = b_{i,j-1} = 0$, then we use u''_{ijk}. Therefore, the words left in (13) are

$$x_1^{a_1}(x_2x_1)^{\varepsilon_2}x_2^{a_2}(x_3x_{k_3})^{\varepsilon_3}x_3^{a_3}\cdots(x_mx_{k_m})^{\varepsilon_m}x_m^{a_m}, \tag{25}$$

where $a_i \geq 0$, $i > k_i$, $\varepsilon_i = 0, 1$. Let us consider two consecutive nonzero ε_c and ε_d. The corresponding monomial contains a subword

$$x_cx_px_c^{a_c}\cdots x_{d-1}^{a_{d-1}}x_dx_q, \quad d > c > p, d > q. \tag{26}$$

If $p = q$ or $c = q$, then we use, respectively, v'_{dcpa} and v''_{dcp}. If c, d, p, q are pairwise different, then we have the three possibilities $p > q$, $c > q > p$, and $q > c$. The first two possibilities are excluded, respectively, using w'_{dcpqa} and w''_{dcqpa}. In this way, the only subwords (26) left are for $d > q > c > p$. Hence, we reduce the set of normal words from (25) to the words with the condition that for the nonzero $\varepsilon_{j_1}, \ldots, \varepsilon_{j_r}$ we have

$$k_{j_1} < j_1 < k_{j_2} < j_2 < \cdots < k_{j_r} < j_r.$$

Using the correspondence ϑ from Proposition 3.7, we obtain that these words are in bijection with the basis elements (18) of $K\langle X\rangle/T$ which preseves the multigrading. This implies that the polynomials f'_{ij}, f''_{ij}, g'_{ijk}, g''_{ikj}, t_{ij}, u'_{ijk}, u''_{ijk}, v'_{ijka}, v''_{ijka}, w'_{ijkla}, and w''_{ijkla} do form a minimal Gröbner basis of the T-ideal. $\square$

We can state Theorem 4.2 in the following way. We require $|X| \geq 5$ for simplification of the statement only.

Theorem 4.3. *Let* $\mathrm{char}(K) \neq 2$, $|X| \geq 5$, *and let* S *be the semigroup of* $\mathrm{End}(K\langle X\rangle)$ *generated by all endomorphisms sending* x_1, x_2, x_3, x_4 *to arbitrary elements of* X *(allowing repetitions) and* x_5 *to products of the form* $x_1^{a_1}\cdots x_m^{a_m}$, $a_i \geq 0$. *The polynomials*

$$[[x_1, x_2], x_3], \quad [x_1, x_2]x_5[x_3, x_4] + [x_1, x_3]x_5[x_2, x_4]$$

form a (nonminimal) Gröbner S*-basis with respect to the deg-lex ordering of the* T*-ideal of* $K\langle X\rangle$ *generated by* $[x_1, x_2, x_3]$.

Remark 4.4. (i) As in the previous section, the condition that X is countable can be replaced by the requirement that X is any infinite well-ordered set.

(ii) In [BML] Bokut and Makar Limanov include in the list of the Gröbner basis of the T-ideal of $K\langle x_1, x_2\rangle$ generated by $[x_1, x_2, x_3]$ the element $(x_2x_1)^2 - (x_1x_2)^2$. The evaluation of this polynomial on the Grassmann algebra $x_1 \to 1+e_1$, $x_2 \to 1+e_2$ shows that $(x_2x_1)^2-(x_1x_2)^2$

does not belong to the T-ideal. The correct Gröbner basis consists of the three polynomials

$$[[x_2, x_1], x_1], \quad [x_2, [x_2, x_1]], \quad [x_2, x_1][x_2, x_1].$$

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