# Rational Two-Parameter Families of Spheres and Rational Offset Surfaces

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**Abstract**

The present paper investigates two-parameter families of spheres in $\mathbb{R}^3$ and their corresponding two-dimensional surfaces $\Phi$ in $\mathbb{R}^4$. Considering a rational surface $\Phi$ in $\mathbb{R}^4$, the envelope surface $\Psi$ of the corresponding family of spheres in $\mathbb{R}^3$ is typically non-rational. Using a classical sphere-geometric approach we prove that the envelope surface $\Psi$ and its offset surfaces admit rational parameterizations if and only if $\Phi$ is a sub-variety of a so-called isotropic hyper-surface in $\mathbb{R}^4$. The close relation between the envelope surfaces $\Psi$ and rational offset surfaces in $\mathbb{R}^3$ is elaborated in detail. This connection leads to explicit rational parameterizations for all rational surfaces $\Phi$ in $\mathbb{R}^4$ whose corresponding two-parameter families of spheres possess envelope surfaces admitting rational parameterizations. Finally we discuss several classes of surfaces sharing this property.

*Key words:* space of spheres, envelope surface, Minkowski space, rational offset surface

## 1. Introduction

The rationality of the envelope surfaces of one- and two-parameter families of spheres has attracted the interest of several researchers, see [10, 11, 12, 13, 14, 15, 16, 24]. Considering $\mathbb{R}^4$ as a model space of the oriented spheres in $\mathbb{R}^3$, these families of spheres appear as curves and surfaces. A recent paper [9] investigated the problem which two-dimensional rational surfaces $\Phi$ in $\mathbb{R}^4$ correspond to two-parameter families of spheres whose envelope surfaces in $\mathbb{R}^3$ admit rational parameterizations. It has been shown that these surfaces are characterized by the fact that they possess a rational parameterization $\mathbf{p}(u, v) \in \mathbb{R}^4$ satisfying a condition involving the partial derivatives $\mathbf{p}_u, \mathbf{p}_v$ of $\mathbf{p}$, see Eqn. (9). The authors have introduced the notion *MOS-surfaces* for these rational surfaces in $\mathbb{R}^4$.

The problem is not only theoretically interesting but has relations to the medial axis representation of objects in $\mathbb{R}^3$. Given a compact set $A$ in $\mathbb{R}^3$ with piecewise smooth boundary $\partial A$, the medial axis $m(A)$ of $A$ is the set of centers of maximal balls $S_a \subset A$. The set of centers together with their radii is called *medial axis transform* $M(A)$ of $A$. It can be considered as set in $\mathbb{R}^4$ and is a unique description of $A$. Since rational or piecewise rational representations are the industrial standard for describing geometric objects, the question of rationality of envelopes of families of spheres has some practical background.

The article [9] gives the analytic condition (9) for MOS-surfaces and discusses special cases, but the particular geometric properties and characteristics of these surfaces are not studied in detail. Here we want to show that a sphere-geometric approach helps to understand more about the relations between surfaces in $\mathbb{R}^4$ and the envelopes of their corresponding families of spheres. The main contributions of the paper are the following:

- Geometric characterization of surfaces $\Phi$ in $\mathbb{R}^4$ whose corresponding two-parameter families of spheres have envelopes admitting rational parameterizations. We prove that MOS-surfaces are exactly the two-dimensional sub-varieties of so-called isotropic hyper-surfaces in Sect. 4.
- Relation to rational offset surfaces in $\mathbb{R}^3$: It is shown that the envelope surface of a two-parameter family of spheres corresponding to a MOS-surface is a rational offset surface in Sect. 4.
- Explicit parametric representations for MOS-surfaces: Based on universal parameterizations of the unit sphere in $\mathbb{R}^3$ and the relations between MOS-surfaces and rational offset surfaces we provide explicit rational parameterizations of all MOS-surfaces in Sect. 5.
- Known and new examples of classes of MOS-surfaces are presented in Sect. 6.

## 2. Geometric background and notations

Points in $\mathbb{R}^n$ are represented by their coordinate vectors $\mathbf{x} = (x_1, \dots, x_n)$. The canonical scalar product of two vectors $\mathbf{p}$ and $\mathbf{q}$ is denoted by $\mathbf{p} \cdot \mathbf{q}$. The projective closure of $\mathbb{R}^n$ is denoted by $\mathbb{P}^n$ and points in $\mathbb{P}^n$ are identified with their *homogeneous coordinate vectors* $\mathbf{y}\mathbb{R} = (y_0, y_1, \dots, y_n)\mathbb{R} = (y_0 : y_1 : \dots : y_n)$, with $\mathbf{y} \neq \mathbf{o}$. Choosing the plane at infinity $\omega$ as $y_0 = 0$, the interchange between homogeneous and Cartesian coordinates for points in $\mathbb{R}^n$ is realized by

$$x_1 = \frac{y_1}{y_0},\ x_2 = \frac{y_2}{y_0}, \dots,\ x_n = \frac{y_n}{y_0}.$$

Later on we use the notion of Plücker coordinates for lines in $\mathbb{P}^3$. In order to introduce them, let a line $G$ be spanned by two different points $P = \mathbf{p}\mathbb{R}$ and $Q = \mathbf{q}\mathbb{R}$. The *Plücker coordinates* $(g_1, \dots, g_6)$ of $G$ are defined by

$$(p_0q_1 - p_1q_0, p_0q_2 - p_2q_0, p_0q_3 - p_3q_0, p_2q_3 - p_3q_2, p_3q_1 - p_1q_3, p_1q_2 - p_2q_1). \tag{1}$$

These coordinates are homogeneous and independent of the choice of the points $P$ and $Q$ on $L$. Additionally, the $g_i$'s satisfy the *Plücker relation*

$$g_1 g_4 + g_2 g_5 + g_3 g_6 = 0. \tag{2}$$

For more details on the geometry of lines we refer to the textbook [25].

### *2.1. The Minkowski-space $\mathbb{R}^4$ as space of spheres*

An oriented sphere $S : (\mathbf{x}-\mathbf{m})^2 = r^2$ in Euclidean 3-space $\mathbb{R}^3$ is uniquely determined by its center $\mathbf{m}$ and its signed radius $r$. The family of oriented spheres in $\mathbb{R}^3$ will be denoted by $\mathcal{S}$. The coordinate vector $\mathbf{m}$ together with the signed radius $r$ represent a point $\mathbf{s} = (\mathbf{m}, r)$ in $\mathbb{R}^4$.

On the other hand any point $\mathbf{p} \in \mathbb{R}^4$ corresponds to an oriented sphere in $\mathbb{R}^3$ with center $(p_1, p_2, p_3)$ and radius $p_4$. Points $\mathbf{p} = (p_1, p_2, p_3, 0)$ in $\mathbb{R}^3 : x_4 = 0$ are identified with spheres of radius zero. Thus, there exists a bijective correspondence between points in $\mathbb{R}^4$ and oriented spheres (including points) in $\mathbb{R}^3$ which is called *cyclographic mapping*,

$$\gamma : \mathbb{R}^4 \to \mathcal{S}, \quad \mathbf{s} = (\mathbf{m}, r) \mapsto S : (\mathbf{x}-\mathbf{m})^2 = r^2. \tag{3}$$

Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be the *orthogonal projection* (top view) of points $\mathbf{s} \in \mathbb{R}^4$ to points $\pi(\mathbf{s}) = (s_1, s_2, s_3)$ in $\mathbb{R}^3 : x_4 = 0$. The sphere $S = \gamma(\mathbf{s})$ has center $\mathbf{m} = \pi(\mathbf{s})$ and is given by the equation $S = \gamma(\mathbf{s}) : (\mathbf{x}-\mathbf{m})^2 = s_4^2$. Figure 1(a) provides a lower dimensional illustration of $\gamma$.

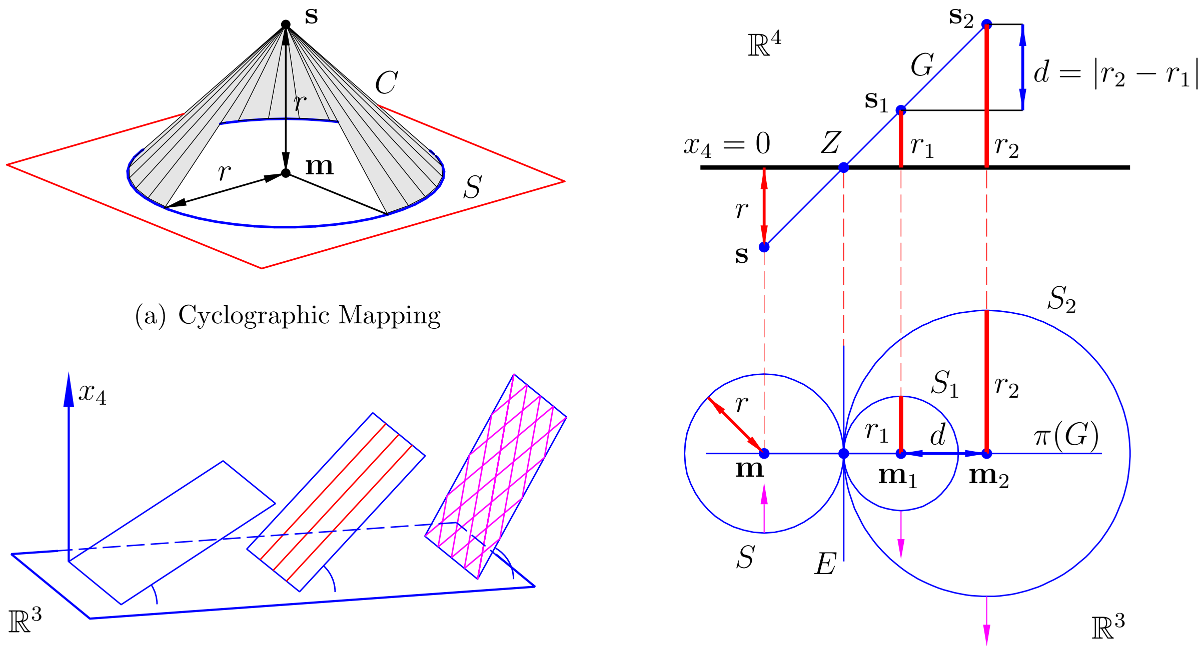
A one-parameter family of spheres $S(t) : (\mathbf{x} - \mathbf{m}(t))^2 = r(t)^2$ corresponds to a curve $\mathbf{s}(t) = (\mathbf{m}, r)(t)$ in $\mathbb{R}^4$. Likewise any two-parameter family of spheres corresponds to a surface $\mathbf{s}(u, v)$ in $\mathbb{R}^4$. The envelope computation for a one-parameter family of spheres $S(t)$ shows that $S(t)$ has a real envelope exactly if $\dot{\mathbf{m}}(t)^2 - \dot{r}(t)^2 \geq 0$, where $\dot{x}(t)$ is the derivative of $x(t)$ with respect to $t$. This motivates to introduce the scalar product

$$\langle \mathbf{x}, \mathbf{x} \rangle := \mathbf{x}^T \cdot D \cdot \mathbf{x}, \text{ with } D = \operatorname{diag}(1, 1, 1, -1), \tag{4}$$

which lets $\mathbb{R}^4$ be a *Minkowski space*. Some authors use the notion $\mathbb{R}^{3,1}$ to emphasize the signature of $D$.

Let $G : \mathbf{p} + t\mathbf{g}$ be a line in $\mathbb{R}^4$. A line $G$ is called *Euclidean*, *isotropic* or *pseudo-Euclidean* depending on whether the direction vector $\mathbf{g}$ satisfies $\langle \mathbf{g}, \mathbf{g} \rangle > 0$, $\langle \mathbf{g}, \mathbf{g} \rangle = 0$, or $\langle \mathbf{g}, \mathbf{g} \rangle < 0$. The respective notations *space-like*, *light-like* or *time-like* are also commonly used.

Let $E : \mathbf{p} + u\mathbf{g} + v\mathbf{h}$ be a plane in $\mathbb{R}^4$ spanned by $\mathbf{p}$ and the vectors $\mathbf{g}$ and $\mathbf{h}$. The plane $E$ is called *Euclidean* if all direction vectors $\mathbf{r}(u, v) = u\mathbf{g} + v\mathbf{h}$ are space-like. Thus $\langle \mathbf{r}, \mathbf{r} \rangle = 0$ has no real zeros. A plane $E$ is called *isotropic* if $\langle \mathbf{r}, \mathbf{r} \rangle = 0$ has a double zero. The corresponding direction is the only isotropic direction in $E$. A plane $E$ is called *pseudo-Euclidean* if $\langle \mathbf{r}, \mathbf{r} \rangle = 0$ has two real zeros corresponding to two isotropic directions in $E$. Figure 1(b) illustrates these types of planes together with their respective isotropic lines.




(a) Cyclographic Mapping

(b) Three Types of Planes

(c) Touching spheres

Figure 1: Minkowski Space and Cyclographic Mapping.

Applying the cyclographic mapping (3) to lines $G$ in $\mathbb{R}^4$, space-like lines $G$ map to spheres which envelope cones of revolution $\gamma(G)$. Light-like lines do not correspond to real envelope surfaces. Of particular interest are isotropic lines $G$. Let $G : \mathbf{x}(t) = \mathbf{p} + t\mathbf{g}$, $t \in \mathbb{R}$ be an isotropic line in $\mathbb{R}^4$, thus $\langle \mathbf{g}, \mathbf{g} \rangle = 0$. Let $\pi(G)$ be $G$'s orthogonal projection in $\mathbb{R}^3$, and let $E \in \mathbb{R}^3$ be a plane orthogonal to $\pi(G)$ passing through the intersection point $Z = G \cap \mathbb{R}^3$. The oriented spheres $S(t) = \gamma(\mathbf{x}(t))$ corresponding to $G$ touch each other at $Z$ with common tangent plane $E$, see Figure 1(c).

Conversely, if two spheres $S_1$ and $S_2$ in $\mathbb{R}^3$ are in *oriented contact*, then the line $G$ connecting the corresponding points $\mathbf{s}_1 = \gamma^{-1}(S_1)$ and $\mathbf{s}_2 = \gamma^{-1}(S_2)$ is isotropic. The distance $d = \|\mathbf{m}_2 - \mathbf{m}_1\|$ between the centers of two spheres $S_1$ and $S_2$ equals the absolute value of the difference of their radii $|r_2 - r_1|$. Consequently we obtain

**Lemma 1.** *Two oriented spheres $S_1$ and $S_2$ are in oriented contact if and only if the corresponding points $\mathbf{s}_1$ and $\mathbf{s}_2$ in $\mathbb{R}^4$ satisfy $\langle \mathbf{s}_2 - \mathbf{s}_1, \mathbf{s}_2 - \mathbf{s}_1 \rangle = 0$.*

### *2.2. Lorentz and Laguerre transformations*

We consider transformations in $\mathbb{R}^3$ which map

- oriented spheres to oriented spheres including points,

- oriented planes to oriented planes, and which
- preserve the oriented contact of spheres.

According to Lemma 1 this implies that the corresponding transformations in $\mathbb{R}^4$ shall preserve isotropic lines. A mapping $T : \mathbb{R}^4 \to \mathbb{R}^4$ is called a *Lorentz transformation* if it is given by

$$T(\mathbf{x}) = A \cdot \mathbf{x} + \mathbf{a}, \text{ with } A^T \cdot D \cdot A = D \text{ and } D = \mathrm{diag}(1, 1, 1, -1). \tag{5}$$

Since $\mathbf{g}^T \cdot D \cdot \mathbf{g} = 0$ implies $\mathbf{g}^T \cdot A^T \cdot D \cdot A \cdot \mathbf{g} = 0$ and vice versa, the linear transformation $\mathbf{x} \mapsto A \cdot \mathbf{x}$ with $A^T \cdot D \cdot A = D$ maps isotropic vectors $\mathbf{g}$ to isotropic vectors $A \cdot \mathbf{g}$. Applying the cyclographic projection (3), the Lorentz transformations induce *contact transformations* in the family of oriented spheres in $\mathbb{R}^3$, called *Laguerre transformations* [24].

Combining (5) with a uniform scaling $\mathbf{x} \mapsto \lambda \mathbf{x}$ preserves the required properties and we obtain more generally $T(\mathbf{x}) = \lambda A \cdot \mathbf{x} + \mathbf{a}$. Moreover it can be shown that these are the only transformations in $\mathbb{R}^4$ which induce contact transformations in $\mathbb{R}^3$ mapping oriented spheres to spheres (including points), oriented planes to planes and preserve oriented contact. Note that Möbius transformations in $\mathbb{R}^3$ are contact transformations but operate on a different set, namely on the set of non-oriented spheres including planes.

### *2.3. Envelope construction*

Let $\Phi$ be a two-dimensional surface in $\mathbb{R}^4$, parameterized by $\mathbf{p}(u, v)$. Further, let $\mathbf{m}(u, v) = \pi(\mathbf{p})$ be its top view in $\mathbb{R}^3$ and $r(u, v) = p_4(u, v)$. An implicit equation of the envelope $\Psi = \gamma(\Phi)$ of the two-parameter family of spheres $S(u, v) : (\mathbf{x} - \mathbf{m})^2 = r^2$ can be obtained eliminating the parameters $u$ and $v$ from the equations

$$S : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0,\ S_u : (\mathbf{x} - \mathbf{m}) \cdot \mathbf{m}_u + r r_u = 0,\ S_v : (\mathbf{x} - \mathbf{m}) \cdot \mathbf{m}_v + r r_v = 0. \tag{6}$$

Since the partial derivatives $S_u := \partial S/\partial u$, $S_v := \partial S/\partial v$ are planes, the envelope $\Psi$ contains the intersection points of the lines $S_u \cap S_v$ and the spheres $S(u, v)$.

Consider a point $\mathbf{p}$ in $\mathbb{R}^4$. The isotropic lines passing through $\mathbf{p}$ form the *light cone* $C : \langle \mathbf{x} - \mathbf{p}, \mathbf{x} - \mathbf{p} \rangle = 0$, see Figure 1(a). The intersection of $C$ with $x_4 = 0$ is the non-oriented sphere $\gamma(\mathbf{p})$. Now consider a parameterization $\mathbf{p}(u, v)$ of a surface $\Phi$ in $\mathbb{R}^4$ and let $C(u, v)$ be the corresponding two-parameter family of light-cones. The envelope of $C(u, v)$ is computed as solution of

$$C : \langle \mathbf{x} - \mathbf{p}, \mathbf{x} - \mathbf{p} \rangle = 0,\ C_u : \langle \mathbf{x} - \mathbf{p}, \mathbf{p}_u \rangle = 0,\ C_v : \langle \mathbf{x} - \mathbf{p}, \mathbf{p}_v \rangle = 0, \tag{7}$$

The hyperplanes $C_u$ and $C_v$ have normal vectors $\mathbf{p}_u$ and $\mathbf{p}_v$, and their intersection $N = C_u \cap C_v$ is the normal plane of $\Phi$ at $\mathbf{p}$, where orthogonality is always considered with respect to (4). The solution of (7) consists of all isotropic lines orthogonal to $\Phi$ and therefore it is called *isotropic hypersurface* $\Gamma(\Phi)$, passing through $\Phi$. Comparing the systems (6) and (7) one obtains the following result:

**Lemma 2.** *Let $\Phi$ be a two-dimensional surface in $\mathbb{R}^4$ and let $S(u,v)$ be the corresponding two-parameter family of oriented spheres in $\mathbb{R}^3$. The envelope $\Psi = \gamma(\Phi)$ of the spheres $S(u,v)$, called the cyclographic image of $\Phi$, is the intersection of the isotropic hyper-surface $\Gamma(\Phi)$ with $x_4 = 0$. The offset surfaces of the envelope surface $\Psi$ at oriented distance $d$ are obtained by intersecting $\Gamma(\Phi)$ with $x_4 = d$.*

## 3. MOS-Surfaces in $\mathbb{R}^4$

**Definition 1.** *A two-parameter family of spheres $S(u,v)$ is called rational if the corresponding surface $\gamma^{-1}(S(u,v))$ admits a rational parameterization.*

Let $\Phi \subset \mathbb{R}^4$ be a rational two-dimensional surface, parameterized by $\mathbf{p}(u,v)$, and let $S(u,v)$ be the corresponding rational two-parameter family of spheres in $\mathbb{R}^3$. At a regular point $\mathbf{p}$, the surface $\Phi$ has a tangent plane $T$ spanned by the partial derivative vectors $\mathbf{p}_u$ and $\mathbf{p}_v$, and $\Phi$ possesses a normal plane $N$ which is orthogonal to $\mathbf{p}_u$ and $\mathbf{p}_v$ in the sense of (4). If $T$ is a Euclidean tangent plane of $\Phi$ at $\mathbf{p}$, its normal plane $N$ is pseudo-Euclidean, and is spanned by $\mathbf{p}$ and two *isotropic normal vectors* $\mathbf{n}_1$ and $\mathbf{n}_2$, see Figure 2(a).

What follows is an investigation of conditions and geometric properties which are necessary and sufficient for the fact that the envelope surface $\Psi = \gamma(\Phi)$ of the spheres $S(u,v)$ admits rational parameterizations. One important issue is that $\Phi \in \mathbb{R}^4$ possesses a *rational* isotropic normal vector field $\mathbf{n}(u,v)$ which is characterized by

$$\langle \mathbf{n}, \mathbf{p}_u \rangle = \langle \mathbf{n}, \mathbf{p}_v \rangle = 0, \text{ and } \langle \mathbf{n}, \mathbf{n} \rangle = 0. \tag{8}$$

### *3.1. Rational isotropic normal vector fields*

For a given rational surface $\Phi$ in $\mathbb{R}^4$, it is proved in [9] that the existence of rational parameterizations of the envelope surface $\Psi = \gamma(\Phi)$ in $\mathbb{R}^3$ is equivalent to the existence of a rational parameterization $\mathbf{p}(u,v)$ of $\Phi$ in $\mathbb{R}^4$ satisfying the condition

$$A(\mathbf{p}) := \langle \mathbf{p}_u, \mathbf{p}_u \rangle \langle \mathbf{p}_v, \mathbf{p}_v \rangle - \langle \mathbf{p}_u, \mathbf{p}_v \rangle^2 = \sigma(u,v)^2. \tag{9}$$

This exresses that $A(\mathbf{p})$ is required to be the perfect square of a rational function $\sigma$. According to [9], the surfaces $\Phi$ in $\mathbb{R}^4$ which possess a parameterization $\mathbf{p}(u,v)$ satisfying (9) are called *MOS-surfaces*. The connection between the condition (9) and rational isotropic normal vector fields $\mathbf{n}(u,v)$ of $\mathbf{p}(u,v)$ is described by

**Corollary 3.** *A rationally parameterized surface $\mathbf{p}(u,v)$ in $\mathbb{R}^4$ possesses a rational isotropic normal vector field $\mathbf{n}(u,v)$ if and only if the condition $A(\mathbf{p}) = \sigma(u,v)^2$ holds, with a rational function $\sigma(u,v)$.*

*Proof:* Given $\mathbf{p}(u, v) \in \mathbb{R}^4$, we construct an isotropic normal vector field $\mathbf{n}(u, v)$ of $\mathbf{p}(u, v)$. At a regular surface point $\mathbf{p}$, its tangent plane $T$ is spanned by the partial derivative vectors $\mathbf{p}_u = (a, b, c, d)$, and $\mathbf{p}_v = (\alpha, \beta, \gamma, \delta)$. We introduce the vector $\mathbf{G} = \mathbf{p}_u \wedge \mathbf{p}_v \in \mathbb{R}^6$,

$$\mathbf{G} = (g_1, \ldots, g_6) = (a\beta - b\alpha, a\gamma - c\alpha, a\delta - d\alpha, c\delta - d\gamma, d\beta - b\delta, b\gamma - c\beta). \tag{10}$$

The normal plane $N$ of $\Phi$ at $\mathbf{p}$ is spanned by vectors $\mathbf{v} = (0, g_4, g_5, -g_6)$ and $\mathbf{w} = (g_6, -g_2, g_1, 0)$, satisfying the relations $\langle \mathbf{p}_u, \mathbf{v} \rangle = \langle \mathbf{p}_v, \mathbf{v} \rangle = 0$ and $\langle \mathbf{p}_u, \mathbf{w} \rangle = \langle \mathbf{p}_v, \mathbf{w} \rangle = 0$. An isotropic normal vector field $\mathbf{n} = \mathbf{v} + t\mathbf{w}$ of $\mathbf{p}(u, v)$ is characterized by the condition $\langle \mathbf{n}, \mathbf{n} \rangle = 0$ and thus determined by the zeros of the polynomial

$$\langle \mathbf{v} + t\mathbf{w}, \mathbf{v} + t\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle + 2t\langle \mathbf{v}, \mathbf{w} \rangle + t^2 \langle \mathbf{w}, \mathbf{w} \rangle. \tag{11}$$

Note that the coordinates $g_i$ of $\mathbf{v}$ and $\mathbf{w}$ are rational functions in $u$ and $v$. If the tangent plane $T$ of $\Phi$ at $\mathbf{p}$ is Euclidean, (11) has two real zeros $t_1$ and $t_2$. The corresponding isotropic normal vectors $\mathbf{n}_1 = \mathbf{v} + t_1\mathbf{w}$ and $\mathbf{n}_2 = \mathbf{v} + t_2\mathbf{w}$ are rational, if and only if the discriminant

$$d = \langle \mathbf{v}, \mathbf{w} \rangle^2 - \langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{w} \rangle = \alpha(u, v)^2, \tag{12}$$

is the perfect square of a rational function $\alpha(u, v)$. To show the equivalence of the conditions (9) and (12) we rewrite the discriminant $d$ in matrix notation

$$d = (\mathbf{v}^T \cdot D \cdot \mathbf{w})(\mathbf{v}^T \cdot D \cdot \mathbf{w}) - (\mathbf{w}^T \cdot D \cdot \mathbf{w})(\mathbf{v}^T \cdot D \cdot \mathbf{v}), \text{ with } D = \mathrm{diag}(1, 1, 1, -1).$$

Letting $K = D \cdot \mathbf{w} \cdot \mathbf{v}^T \cdot D$ we finally get $d = \mathbf{v}^T \cdot (K - K^T) \cdot \mathbf{w} = g_6^2(g_1^2 + g_2^2 - g_3^2 - g_4^2 - g_5^2 + g_6^2)$. Elaborating $A(\mathbf{p})$ in terms of the Plücker coordinates $g_i$ gives

$$A(\mathbf{p}) = \mathbf{p}_u^T \cdot (M - M^T) \cdot \mathbf{p}_v = g_1^2 + g_2^2 - g_3^2 - g_4^2 - g_5^2 + g_6^2 = \frac{d}{g_6^2} \tag{13}$$

where $M = \mathbf{p}_u \cdot \mathbf{p}_v^T$. Thus we conclude that if $A(\mathbf{p}) = \sigma(u, v)^2$ is a perfect square, there exist rational isotropic normal vector fields $\mathbf{n}_1(u, v)$ and $\mathbf{n}_2(u, v)$ of $\mathbf{p}(u, v)$. Given one of these vector fields, the other one is computed linearly, see Section 5.1.

The converse direction is obvious: If the surface $\mathbf{p}(u, v)$ possesses a rational isotropic normal vector field $\mathbf{n}$, then the zeros of (11) are rational functions in $u$ and $v$ and (13) is a perfect square of a rational function $\sigma$. $\square$

**Remark:** Let $\mathbb{P}^4$ be the projective extension of $\mathbb{R}^4$ and let $\omega = \mathbb{P}^4 \setminus \mathbb{R}^4$ be the ideal hyperplane of $\mathbb{P}^4$ with respect to $\mathbb{R}^4$. Ideal points $X \in \omega$ are described by homogeneous coordinate vectors $(0, \mathbf{x})\mathbb{R}$ with $\mathbf{x} \in \mathbb{R}^4$. Those ideal points $X$ which also satisfy $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ form the quadric $\Omega$.

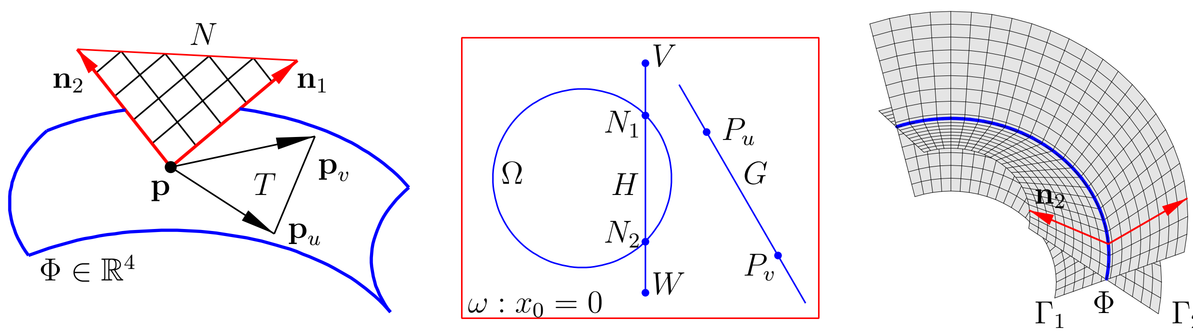
The ideal line $G$ of $\Phi$'s tangent plane $T$ connects the ideal points $P_u = (0, \mathbf{p}_u)\mathbb{R}$ and $P_v = (0, \mathbf{p}_v)\mathbb{R}$ and is represented by the Plücker coordinates $\mathbf{G} = (g_1, \ldots, g_6)$ from (10).

Consider the ideal line $H$ of $\Phi$'s normal plane $N$ at $\mathbf{p}$. It is spanned by the points $V = (0, \mathbf{v})\mathbb{R}$ and $W = (0, \mathbf{w})\mathbb{R}$ with $\mathbf{v} = (0, g_4, g_5, -g_6)$ and $\mathbf{w} = (g_6, -g_2, g_1, 0)$. The intersection points of $H$ with $\Omega$ are the ideal points $N_1 = (0, \mathbf{n}_1)\mathbb{R}$ and $N_2 = (0, \mathbf{n}_2)\mathbb{R}$ of the isotropic normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ at $\mathbf{p}$. See Fig. 2(b) for a schematic illustration.

**Remark:** The matrices $K - K^T$ and $M - M^T$ occurring in the proof of Corollary 3 are both skew symmetric,

$$K - K^T = \begin{pmatrix} 0 & g_4 & g_5 & g_6 \\ -g_4 & 0 & -g_3 & g_2 \\ -g_5 & g_3 & 0 & -g_1 \\ -g_6 & -g_2 & g_1 & 0 \end{pmatrix}, \quad M - M^T = \begin{pmatrix} 0 & g_1 & g_2 & -g_3 \\ -g_1 & 0 & g_6 & g_5 \\ -g_2 & -g_6 & 0 & -g_4 \\ g_3 & -g_5 & g_4 & 0 \end{pmatrix},$$

Their determinants equal $\det K = \det M = (g_1 g_4 + g_2 g_5 + g_3 g_6)^2$ and vanish according to the Plücker relation (2). Further $\operatorname{rk} K = \operatorname{rk} M = 2$ holds.




(a) Isotropic Normal Vectors (b) Sketch of ideal elements (c) Isotropic Hypersurface

Figure 2: Minkowski Space and Cyclographic Mapping.

*3.2. Isotropic hyper-surfaces*

Consider a surface $\Phi \in \mathbb{R}^4$ parameterized by $\mathbf{p}(u, v)$, and let $T : \mathbf{p} + s\mathbf{p}_u + t\mathbf{p}_v$ be a Euclidean tangent plane. There exist two different real isotropic normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ at the considered point $\mathbf{p}$. In an open neighborhood of $\mathbf{p}$ there exist two different sheets of isotropic hyper-surfaces $\Gamma_1$ nd $\Gamma_2$ passing through $\Phi$; see Figure 2(c) for a schematic illustration. Parameterizations $\mathbf{g}_i$ of $\Gamma_i$ are obtained by

$$\mathbf{g}_i(u, v, w) = \mathbf{p}(u, v) + w\mathbf{n}_i(u, v), \ i = 1, 2. \tag{14}$$

If $\Phi$ possesses rational isotropic normal vector fields $\mathbf{n}_1$ and $\mathbf{n}_2$, the isotropic hyper-surfaces $\Gamma_1$ and $\Gamma_2$ admit rational parameterizations. If we are considering only one sheet $\Gamma_1$ or $\Gamma_2$, we simply write $\Gamma$ and $\mathbf{n}$ for the isotropic hyper-surface and the normal vector field.

Consider a rational isotropic hyper-surface $\Gamma$. Since $\Gamma$ carries straight lines, any hyper-planar intersection is rational. Moreover, any rational function $w(u,v)$ defines a rational two-dimensional sub-variety $\mathbf{q}(u,v) = \mathbf{p}(u,v) + w(u,v)\mathbf{n}(u,v)$ of $\Gamma$. The following corollary shows that the property that $\mathbf{n}$ is a rational isotropic normal vector field is independent of the subvariety $\mathbf{p}(u,v)$.

**Corollary 4.** *Let $\mathbf{q}(u,v) = \mathbf{p}(u,v) + w(u,v)\mathbf{n}(u,v)$ be a rational two-dimensional sub-variety of the rational isotropic hyper-surface $\Gamma$, where $w(u,v)$ denotes some rational function in $u$ and $v$. If $\mathbf{n}(u,v)$ is a rational isotropic normal vector field of $\mathbf{p}(u,v)$, then this property is true for any rational two-dimensional sub-variety $\mathbf{q}(u,v)$ of $\Gamma$.*

*Proof:* The identity $\langle \mathbf{n}, \mathbf{n} \rangle = 0$ implies the identities $\langle \mathbf{n}, \mathbf{n}_u \rangle = 0$ and $\langle \mathbf{n}, \mathbf{n}_v \rangle = 0$. Computing the partial derivatives $\mathbf{q}_u = \mathbf{p}_u + w_u \mathbf{n} + w \mathbf{n}_u$ and likewise for $\mathbf{q}_v$ and using the relations (8) one immediately obtains

$$\langle \mathbf{q}_u, \mathbf{n} \rangle = \langle \mathbf{q}_v, \mathbf{n} \rangle = 0.$$

$\square$

This implies that for a given MOS-surface $\mathbf{p}(u,v)$, any two-dimensional rational sub-variety $\mathbf{q}(u,v) = \mathbf{p}(u,v) + w(u,v)\mathbf{n}(u,v)$ of the isotropic hyper-surface $\Gamma$ through $\mathbf{p}(u,v)$ is again a MOS-surface. From Corollary 3 it follows that the parameterization $\mathbf{q}(u,v)$ satisfies condition (9). Thus there exists a rational function $\tau(u,v)$ such that $A(\mathbf{q}) = \tau(u,v)^2$. This property can also be verified by a direct but lengthy calculation using the fact that $\mathbf{n}$'s partial derivatives $\mathbf{n}_u$ and $\mathbf{n}_v$ can be expressed by linear combinations of $\mathbf{n}$ and $\mathbf{p}_u$ and $\mathbf{p}_v$. Summarizing we obtain the following geometric characterization of condition (9) by

**Theorem 5.** *MOS-Surfaces are exactly the rational two-dimensional sub-varieties of rational isotropic hyper-surfaces in $\mathbb{R}^4$.*

In the next section we study further geometric properties of MOS-surfaces, in particular in connection with two-parameter families of spheres.

## 4. Rational envelope surfaces in $\mathbb{R}^3$

Let $\Phi$ be a rational two-dimensional surface in $\mathbb{R}^4$ parameterized by $\mathbf{p}(u,v)$ and let $S(u,v)$ be the corresponding rational two-parameter family of spheres in $\mathbb{R}^3$. According to Corollaries 3 and 4 and Theorem 5 the following statement holds.

**Corollary 6.** *The envelope surface $\Psi \subset \mathbb{R}^3$ of the rational two-parameter family of spheres $S(u,v)$ corresponding to the surface $\Phi$ in $\mathbb{R}^4$ admits a rational parameterization $\mathbf{q}(u,v)$ if and only if $\Phi$ is a rational sub-variety of a rational isotropic hyper-surface $\Gamma$.*

*Proof:* According to Lemma 2 the envelope surface $\Psi$ is the intersection of $\Gamma(\Phi)$ with $\mathbb{R}^3$. Since $\Gamma(\Phi)$ carries a two-parameter family of straight lines (isotropic normals), rationality of $\Gamma(\Phi)$ implies rationality of $\Psi$.

On the other hand, assume that $\mathbf{p}(u,v) \in \mathbb{R}^4$ and $\mathbf{q}(u,v) \in \mathbb{R}^3$ are corresponding rational parameterizations of $\Phi$ and $\Psi$. Since spheres $S = \gamma(\mathbf{p})$ envelope the surface $\Psi$, the correspondence guarantees that the sphere $\gamma(\mathbf{p})$ is touching $\Psi$ at the point $\mathbf{q}$. By embedding $\Psi$ into $\mathbb{R}^4$ by $\mathbf{q} = (q_1, q_2, q_3, 0)$ we form the hyper-surface

$$\mathbf{g}(u,v,w) = (1-w)\mathbf{q} + w\mathbf{p}$$

According to Lemma 1, the rational hyper-surface parameterized by $\mathbf{g}(u,v,w)$ consists of isotropic lines and is thus the isotropic hyper-surface passing through $\Phi$ and $\Psi$. $\square$

For an arbitrary rational surface $\Phi$ in $\mathbb{R}^4$ the envelope surface $\Psi$ of the spheres corresponding to $\Phi$ is typically non-rational. We note that even if $\Psi$ admits rational parameterizations, not any rational parameterization $\mathbf{p}(u,v)$ of $\Phi$ satisfies condition (9) and leads to a rational parameterization $\mathbf{q}(u,v)$ of $\Psi$. In practice one usually has to find a suitable reparameterization to construct a rational parameterization $\mathbf{q}(u,v)$ of $\Psi$. Examples illustrating this fact are found in Section 6.6.

### *4.1. Relations to rational offset surfaces*

Rational envelope surfaces of rational two-parameter families of spheres in $\mathbb{R}^3$ are in close relation to rational surfaces with rational offsets. The latter ones have been studied at first in detail in [21].

**Definition 2.** *A surface $F$ in $\mathbb{R}^3$ is called Pythagorean normal surface or PN surface if it possesses a rational parameterization $\mathbf{f}(u,v)$ and a rational unit normal vector field $\mathbf{n}(u,v)$ corresponding to $\mathbf{f}(u,v)$.*

The representation $\mathbf{f}(u,v)$ is denoted as *PN-parameterization.* The rationality of the unit normal vector field $\mathbf{n}(u,v)$ implies that the offset surfaces $F_d$ of $F$ at oriented distance $d$ admit rational parameterizations $\mathbf{f}_d(u,v) = \mathbf{f}(u,v) + d\mathbf{n}(u,v)$. The following corollary enlightens the close relation between PN-surfaces in $\mathbb{R}^3$ and isotropic hyper-surfaces in $\mathbb{R}^4$. For this reason let $\mathbb{R}^3$ be embedded into $\mathbb{R}^4$ as hyperplane $x_4 = 0$. Vectors in $\mathbb{R}^4$ are denoted as usual whereas vectors in $\mathbb{R}^3$ are indicated by $\tilde{\mathbf{x}}$.

**Corollary 7.** *A rational surface $\Psi \in \mathbb{R}^3$ is a PN-surface if and only if it is the intersection of a rational isotropic hypersurface $\Gamma$ with $x_4 = 0$.*

*Proof:* If $\Psi$ is a PN-surface with PN-parameterization $\tilde{\mathbf{q}}(u,v) = (q_1, q_2, q_3)(u,v)$, it possesses a rational unit normal vector field $\tilde{\mathbf{n}}(u,v) = (n_1, n_2, n_3)(u,v)$. This vector field

$\tilde{\mathbf{n}}$ induces two rational isotropic normal vector fields $\mathbf{n}_1 = (\tilde{\mathbf{n}}, 1)$ and $\mathbf{n}_2 = (\tilde{\mathbf{n}}, -1)$ of $\Psi \in \mathbb{R}^4$. Consequently the isotropic hyper-surfaces $\Gamma_i$ through $\Psi \in \mathbb{R}^4$ are rational and parameterized by

$$\mathbf{g}_i(u, v, w) = \mathbf{q}(u, v) + w\mathbf{n}_i(u, v)), \text{ with } \mathbf{q} = (q_1, q_2, q_3, 0), \text{ and } , i = 1, 2.$$

Conversely, let $\Gamma$ be a rational isotropic hypersurface in $\mathbb{R}^4$ and $\Psi = \Gamma \cap \mathbb{R}^3$. We may assume that $\Gamma$ is represented by $\mathbf{g}(u, v, w) = \mathbf{p}(u, v) + w\mathbf{n}(u, v)$, where $\mathbf{p}(u, v)$ and $\mathbf{n}(u, v)$ are rational and $\langle \mathbf{n}, \mathbf{n} \rangle = 0$ holds. Without loss of generality we can assume $n_4 = 1$. Intersecting $\Gamma$ with $x_4 = 0$, a rational parameterization of $\Psi$ is obtained by $\mathbf{q}(u, v) = \mathbf{p} - p_4\mathbf{n}$. According to Corollary 4, $\mathbf{n}(u, v)$ is a rational isotropic normal vector field of any two-dimensional sub-variety of $\Gamma$ and the relations $\langle \mathbf{n}, \mathbf{q}_u \rangle = 0$, and $\langle \mathbf{n}, \mathbf{q}_v \rangle = 0$, and $\langle \mathbf{n}, \mathbf{n} \rangle = 0$ hold.

Further we have to show that $\tilde{\mathbf{n}} = (n_1, n_2, n_3)$ is a rational unit normal vector field of $\tilde{\mathbf{q}} = (q_1, q_2, q_3)$ in $\mathbb{R}^3$. Since $q_4 = 0$ we have $\tilde{\mathbf{n}} \cdot \tilde{\mathbf{q}}_u = 0$ and $\tilde{\mathbf{n}} \cdot \tilde{\mathbf{q}}_v = 0$. The normalization $\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} = 1$ follows from $\langle \mathbf{n}, \mathbf{n} \rangle = 0$ and $n_4 = 1$, which completes the proof. $\square$

Finally let us consider a rational two-parameter family of spheres $S(u, v)$ whose corresponding rational surface $\Phi$ in $\mathbb{R}^4$ satisfies condition (9). We prove that the envelope surface $\Psi$ of $S(u, v)$ is a PN-surface. According to Corollaries 3 and 6, $\Phi$ is contained in a rational isotropic hyper-surface $\Gamma$. According to Lemma 2, the envelope surface $\Psi$ is obtained as intersection of $\Gamma$ with $\mathbb{R}^3$. Finally Corollary 7 proves that $\Psi$ is a PN-surface. Summarizing this construction we obtain

**Theorem 8.** *Let $S(u, v)$ be a rational two-parameter family of spheres in $\mathbb{R}^3$ and let $\Phi$ be the corresponding rational surface in $\mathbb{R}^4$. The envelope surface $\Psi$ of $S(u, v)$ admits rational parameterizations if and only if $\Psi$ is a PN-surface.*

In general there are two sheets of isotropic hyper-surfaces $\Gamma_1$ and $\Gamma_2$ through a surface $\Phi$ in $\mathbb{R}^4$. Likewise the envelope surface $\Psi$ of a two-parameter family of spheres consists of two sheets $\Psi_1$ and $\Psi_2$ which are the intersections of $\Gamma_1$ and $\Gamma_2$ with $\mathbb{R}^3$. Considering algebraic objects $\Phi$ in $\mathbb{R}^4$, these two sheets $\Gamma_1$ and $\Gamma_2$ belong to one algebraic hyper-surface $\Gamma$. The same holds for the envelope surface $\Psi$ in $\mathbb{R}^3$.

## 5. Rational parameterizations of MOS-surfaces

As defined in [9], MOS-surfaces $\Phi$ in $\mathbb{R}^4$ can be characterized by the fact that they admit rational parameterizations $\mathbf{p}(u, v)$ which satisfy the condition (9). Of course it is desirable to find explicit representations for these surfaces. The approach in [9] tries first to find six polynomials $g_1(u, v), \ldots, g_6(u, v)$ satisfying $g_1^2 + g_2^2 - g_3^2 - g_4^2 - g_5^2 + g_6^2 = \sigma(u, v)^2$ and the

Plücker condition (2). Comparing with (13) shows that this represents condition (9) in terms of the Plücker coordinates $g_i$ of the ideal lines of $\Phi$'s tangent planes. At second the parameterization $\mathbf{p}(u,v)$ shall be derived from these polynomials $g_i(u,v)$. This idea has been successful unfortunately only in some special cases, like representing MOS-surfaces in hyperplanes. We present another approach which uses the relations between MOS-surfaces and PN-surfaces and leads to explicit rational representations of all MOS-surfaces.

### *5.1. From PN-surfaces to MOS-surfaces*

We are given a rational PN-surface $\Psi_1$ with PN-parameterization $\tilde{\mathbf{q}}_1(u,v) = (q_1, q_2, q_3)(u,v)$ and a rational unit normal vector field $\tilde{\mathbf{n}}_1(u,v)$ of $\tilde{\mathbf{q}}_1$. By prescribing a rational radius function $r(u,v)$ we may define a rational two-parameter family of spheres $S(u,v)$ of radius $r(u,v)$ centered at $\tilde{\mathbf{p}} = \tilde{\mathbf{q}}_1 + r\tilde{\mathbf{n}}_1$. These spheres envelope the PN-surface $\Psi_1$ and define a rational surface $\Phi$ in $\mathbb{R}^4$, represented by

$$\mathbf{p}(u,v) = (\tilde{\mathbf{p}}, r) = (q_1 + rn_1, q_2 + rn_2, q_3 + rn_3, r). \tag{15}$$

One part $\Gamma_1$ of the isotropic hyper-surface $\Gamma$ through $\Phi$ is already known. Embedding $\Psi$ in to $\mathbb{R}^4$ by $\mathbf{q}_1(u,v) = (\tilde{\mathbf{q}}_1, 0)(u,v)$, a rational parameterization of $\Gamma_1$ is given by $\mathbf{g}_1(u,v,w) = \mathbf{q}(u,v) + w\mathbf{n}_1(u,v)$, where $\mathbf{n}_1 = (\tilde{\mathbf{n}}_1, 1)$ is a rational isotropic normal vector field of $\Phi$. We want to construct the second rational isotropic normal vector field $\mathbf{n}_2$ of $\Phi$ and the second part $\Gamma_2$ of the the isotropic hyper-surface $\Gamma$ through $\Phi$.

The normal vectors of $\mathbf{p}(u,v)$ can be parameterized by $t\mathbf{n}_1 + \mathbf{w}$ where $\mathbf{w}$ satisfies $\langle \mathbf{p}_u, \mathbf{w}\rangle = \langle \mathbf{p}_v, \mathbf{w}\rangle = 0$, compare also (11) in Section 3.1. From $\langle t\mathbf{n}_1 + \mathbf{w}, t\mathbf{n}_1 + \mathbf{w}\rangle = 0$ and because of $\langle \mathbf{n}_1, \mathbf{n}_1\rangle = 0$ the second isotropic normal vector field is linearly computed by

$$\mathbf{n}_2(u,v) = -2\langle \mathbf{w}, \mathbf{w}\rangle \mathbf{n}_1 + \langle \mathbf{n}_1, \mathbf{w}\rangle \mathbf{w}. \tag{16}$$

The isotropic hyper-surface $\Gamma_2$ is thus parameterized by $\mathbf{g}_2(u,v,w) = \mathbf{p}(u,v) + \mathbf{n}_2(u,v)$ and the second part $\Psi_2$ of the envelope surface of the spheres $S(u,v)$ is obtained by intersecting $\Gamma_2$ with the hyperplane $\mathbb{R}^3 : x_4 = 0$. This construction is summarized in the following

**Theorem 9.** *A MOS-surface $\Phi \in \mathbb{R}^4$ can be constructed as follows: We are given a PN-surface in $\mathbb{R}^3$ parameterized by $\tilde{\mathbf{q}}(u,v)$ with rational unit normal vectors $\tilde{\mathbf{n}}$. Prescribing a rational radius function $r(u,v)$, the rational surface $\mathbf{p}(u,v) = (\tilde{\mathbf{p}}, r)$ according to* (15) *is a MOS-surface in $\mathbb{R}^4$.*

### *5.2. Explicit representations*

We describe a method to provide rational parameterizations for all MOS-surfaces by using the relation to rational offset surfaces. The technique is based on the fact that rational parameterizations of PN-surfaces in $\mathbb{R}^3$ are derived from rational parameterizations of the unit sphere, see [21]. Applying Theorem 9 it is possible to specify explicit rational parameterizations of all MOS-surfaces $\Phi$ in $\mathbb{R}^4$.

The parameterization of PN-surfaces used in [21] is derived from a stereographic projection of the unit sphere $S^2$. Choosing $(a/c, b/c)$ as Cartesian coordinates in $\mathbb{R}^2$ and $(0,0,1)$ as center of projection, a parameterization $\mathbf{n}$ of $S^2$ reads $\mathbf{n} = (2ac/f, 2bc/f, (a^2+b^2-c^2)/f)$, with $f = (a^2+b^2+c^2)$. Choosing polynomials $a(u,v)$, $b(u,v)$ and $c(u,v)$ and a rational function $h(u,v)$, the tangent planes $T(u,v)$ of a PN-surface $\Psi$ are given by $T(u,v)$: $2acx + 2bcy + (a^2+b^2-c^2)z = h$. A parameterization of $\Psi$ is obtained by intersecting $T \cap T_u \cap T_v$.

Here we construct rational parameterizations of $S^2$ in a different way, following the approach in [2]. Choosing four polynomials $a(u,v)$, $b(u,v)$, $c(u,v)$ and $d(u,v)$, and letting

$$A = 2(ac+bd),\ B = 2(bc-ad),\ C = a^2+b^2-c^2-d^2,\ D = a^2+b^2+c^2+d^2,$$

the unit sphere $S^2$ admits the *universal parameterization* $\tilde{\mathbf{n}} = (A/D, B/D, C/D)$. The property $\tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} = 1$ implies that $\mathbf{n}(u,v) = (A,B,C,D)(u,v)$ satisfies $\langle \mathbf{n}, \mathbf{n} \rangle = 0$. Thus $\mathbf{n}$ is a rational isotropic vector field in $\mathbb{R}^4$, and $\lambda \mathbf{n}(u,v)$ is a parameterization of the light cone with vertex at the origin.

The tangent planes of a PN-surface $\Psi$ in $\mathbb{R}^3$ can be represented by $T(u,v) : Ax+By+Cz = h$, with a rational function $h(u,v)$. A point representation $\tilde{\mathbf{q}}(u,v)$ of $\Psi$ follows by $T \cap T_u \cap T_v$ and reads

$$\tilde{\mathbf{q}}(u,v) = \frac{1}{F} \begin{pmatrix} B(C_u h_v - C_v h_u) + C(h_u B_v - h_v B_u) + h(B_u C_v - B_v C_u) \\ C(A_u h_v - A_v h_u) + A(h_u C_v - h_v C_u) + h(C_u A_v - C_v A_u) \\ A(B_u h_v - B_v h_u) + B(h_u A_v - h_v A_u) + h(A_u B_v - A_v B_u) \end{pmatrix}, \tag{17}$$

with $F = A(B_u C_v - B_v C_u) + B(C_u A_v - A_u C_v) + C(A_u B_v - A_v B_u)$. Embedding $\Psi$ into $\mathbb{R}^4$ by $\mathbf{q} = (q_1, q_2, q_3, 0)$, the rational isotropic hyper-surface $\Gamma$ through $\Psi$ admits the rational parameterization $\mathbf{g} = \mathbf{q}(u,v) + w\mathbf{n}(u,v)$, with $\mathbf{n}(u,v) = (A,B,C,D)(u,v)$. Prescribing a rational function $w(u,v)$, an explicit representation $\mathbf{p}(u,v)$ of all MOS-surfaces $\Phi$ in $\mathbb{R}^4$ is given by $\mathbf{p}(u,v) = \mathbf{q}(u,v) + w(u,v)\mathbf{n}(u,v)$. Substituting the parameterizations for $\mathbf{q}$ and $\mathbf{n}$ we obtain $\mathbf{p}(u,v)$ in terms of the polynomials $A, B, C, D$ and the rational functions $h$ and $w$,

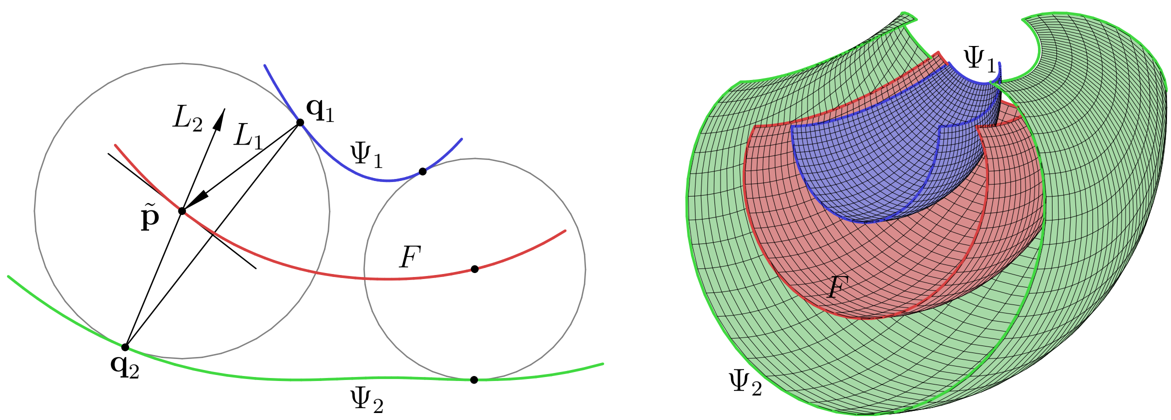
$$\mathbf{p}(u,v) = \frac{1}{F} \begin{pmatrix} B(C_u h_v - C_v h_u) + C(h_u B_v - h_v B_u) + h(B_u C_v - B_v C_u) + wFA \\ C(A_u h_v - A_v h_u) + A(h_u C_v - h_v C_u) + h(C_u A_v - C_v A_u) + wFB \\ A(B_u h_v - B_v h_u) + B(h_u A_v - h_v A_u) + h(A_u B_v - A_v B_u) + wFC \\ wFD \end{pmatrix}. \tag{18}$$

The corresponding two-parameter family of spheres $S(u,v)$ corresponding to the MOS-surface $\Phi$ has centers at $\tilde{\mathbf{p}}(u,v) = (p_1, p_2, p_3)(u,v)$ and its radius function is given by $w(u,v)D(u,v)$.

**Theorem 10.** *A MOS-surface $\Phi$ in $\mathbb{R}^4$ admits a rational parameterization of the form* (18) *which is derived from a universal parameterization of the unit sphere $S^2$ in $\mathbb{R}^3$.*

### *5.3. Geometrical Optics*

The construction of Section 5.1 admits the following interpretation in the sense of geometrical optics. We are given a two-parameter family of light rays $L_1(u, v)$ orthogonal to a surface $\Psi_1$ and a mirror surface $F = \pi(\Phi)$. The mirror surface is considered as orthogonal projection of a surface $\Phi \in \mathbb{R}^4$ onto $\mathbb{R}^3$. Assuming a parameterization $\tilde{\mathbf{q}}_1(u, v)$ and unit normal vectors $\tilde{\mathbf{n}}_1(u, v)$ of $\Psi_1$, the light rays $L_1(u, v)$ are represented by $\tilde{\mathbf{q}}_1 + w\tilde{\mathbf{n}}_1$. Reflecting the light rays $L_1$ at $F$ yields a two-parameter system of light rays $L_2(u, v)$ which are again orthogonal to a surface $\Psi_2$ according to a Theorem by Malus-Dupin, see for instance [1]. The surface $\Psi_2$ is called an anticaustic of reflection with respect to illumination orthogonal to $\Psi_1$ and reflection at $F$. A parameterization $\tilde{\mathbf{q}}_2(u, v)$ of $\Psi_2$ is constructed by reflecting the points $\tilde{\mathbf{q}}_1(u, v)$ at the tangent planes at $\tilde{\mathbf{p}}$ of $F$, see Fig. 3(a). Equivalently $\Psi_2$ is the intersection of the isotropic hypersurface $\Gamma_2$ through $\Phi$ with $\mathbb{R}^3 : x_4 = 0$.




(a) Principle of the anticaustic construction.

(b) Envelope surfaces $\Psi_1$ and $\Psi_2$ and mirror surface $F$ (center surface).

Figure 3: Construction of an Anticaustic $\Psi_2$ with respect to illumination orthogonal to $\Psi_1$ and reflection at $F$.

**Example:** We are given the surface $\Psi_1$ (parabolic Dupin cyclide) by the parameterization

$$\tilde{\mathbf{q}}_1(u, v) = \left( \frac{u(1 + bv^2(b - a))}{1 + a^2u^2 + b^2v^2}, \frac{v(1 + au^2(a - b))}{1 + a^2u^2 + b^2v^2}, \frac{au^2 + bv^2}{1 + a^2u^2 + b^2v^2} \right),$$

with rational unit normal vector field

$$\tilde{\mathbf{n}}_1(u, v) = \left( \frac{2au}{1 + a^2u^2 + b^2v^2}, \frac{2bv}{1 + a^2u^2 + b^2v^2}, \frac{a^2u^2 + b^2v^2 - 1}{1 + a^2u^2 + b^2v^2}, \right).$$

Prescribing a rational radius function $r(u,v)$ one obtains a two-parameter family of spheres $S(u,v)$. Its corresponding surface $\Phi \in \mathbb{R}^4$ is represented by

$$\mathbf{p}(u,v) = (\tilde{\mathbf{q}}_1 + r\tilde{\mathbf{n}}_1, r).$$

Since $\tilde{\mathbf{n}}_1$ is a rational unit normal vector field of $\Psi_1$, a rational isotropic normal vector field of $\Phi$ is given by $\mathbf{n}_1 = (2au, 2bv, a^2u^2 + b^2v^2 - 1, a^2u^2 + b^2v^2 + 1)$. The second isotropic normal vector field is constructed linearly following (16). Fig. 3(b) illustrates the envelope surfaces $\Psi_1$ and $\Psi_2$ of the spheres centered at $F$ with radius $r(u,v) = 1/2 - 1/4u^2$.

## 6. Examples of MOS-surfaces in $\mathbb{R}^4$

Although explicit parameterizations of MOS-surfaces in $\mathbb{R}^4$ are given by (18), it is difficult to decide which surfaces belong to this class. In the following we provide an overview of classes of rational surfaces satisfying the required properties. Besides the class of surfaces discussed in Sect. 6.6, the rationality of the corresponding envelope surfaces of the given examples has already been proved in [16].

### *6.1. Rational ruled surfaces*

A rational ruled surface $\Phi$ in $\mathbb{R}^4$ corresponds to a special two-parameter family of spheres. Since Euclidean rulings $G$ of $\Phi$ are mapped to cones of revolution $D = \gamma(G)$, the envelope $\gamma(\Phi)$ is also obtained as envelope of a one-parameter family of cones of revolution $D(t)$ in $\mathbb{R}^3$.

Let $\Phi$ be rationally parameterized by $\mathbf{p}(u,v) = \mathbf{a}(u) + v\mathbf{g}(u)$. We have to find a rational isotropic normal vector field $\mathbf{n}(u,v)$, satisfying $\langle \mathbf{n}, \mathbf{p}_u \rangle = 0$, $\langle \mathbf{n}, \mathbf{p}_v \rangle = 0$ and $\langle \mathbf{n}, \mathbf{n} \rangle = 0$, see (8). Setting up these equations in a straight forward way results in a quadratic equation whose solution involves square roots and does not lead to rational expressions. We demonstrate another approach proving the existence of rational parameterizations of the envelope surface $\gamma(\Phi)$ for non-developable (skew) ruled surfaces $\Phi$, see also [17].

#### *6.1.1. Non-developable ruled surfaces*

The ruled surface $\Phi$ parameterized by $\mathbf{p}(u,v) = \mathbf{a}(u) + v\mathbf{g}(u)$ is called non-developable if the vectors $\dot{\mathbf{a}}(u)$, $\mathbf{g}(u)$ and $\dot{\mathbf{g}}(u)$ are linearly independent except for finitely many $u_i$'s. Linearly dependence characterizes *torsal rulings* which possess a fixed tangent plane. The tangent planes $T(u^\star)$ along a non-torsal ruling $G(u^\star) : \mathbf{p}(u^\star, v)$ turn around this line. There is a projective correspondence between the tangent planes $T(u^\star)$ and the points of contact $\mathbf{p}(u^\star, v)$. The normal plane $N(u)$ at $\mathbf{p}(u,v)$ can be spanned by two vectors $\mathbf{n}_1(u,v)$ and $\mathbf{n}_2(u) = \mathbf{m}(u)$ where the latter one is orthogonal to the 3-space spanned by $\dot{\mathbf{a}}(u)$, $\mathbf{g}(u)$ and

$\dot{\mathbf{g}}(u)$. The first one $\mathbf{n}_1$ is linear in $v$, thus $\mathbf{n}_1(u,v) = \mathbf{k}(u) + v\mathbf{l}(u)$. This implies that the normal vectors $\mathbf{n}(u,v)$ of $\mathbf{p}(u,v)$ can be represented as linear combination

$$\mathbf{n}(u,v) = \mathbf{k}(u) + v\mathbf{l}(u) + w\mathbf{m}(u).$$

In order to determine a rational isotropic normal vector field, we require $\langle \mathbf{n}, \mathbf{n} \rangle = 0$. Inserting the linear combination for $\mathbf{n}$ leads to the quadratic equation

$$(1, v, w)^T \cdot \begin{pmatrix} \langle \mathbf{k}, \mathbf{k} \rangle & \langle \mathbf{k}, \mathbf{l} \rangle & \langle \mathbf{k}, \mathbf{m} \rangle \\ \langle \mathbf{k}, \mathbf{l} \rangle & \langle \mathbf{l}, \mathbf{l} \rangle & \langle \mathbf{l}, \mathbf{m} \rangle \\ \langle \mathbf{k}, \mathbf{m} \rangle & \langle \mathbf{l}, \mathbf{m} \rangle & \langle \mathbf{m}, \mathbf{m} \rangle \end{pmatrix} \cdot \begin{pmatrix} 1 \\ v \\ w \end{pmatrix} = 0. \tag{19}$$

Replacing $v$ and $w$ by the homogeneous coordinates $\mathbf{y} = (y_0, y_1, y_2)$ with $v = y_1/y_0$ and $w = y_2/y_0$ and denoting the symmetric coefficient matrix by $A(u)$, whose entries are rational in $u$, we rewrite (19) by

$$K(u) : \mathbf{y}^T \cdot A(u) \cdot \mathbf{y} = 0, \tag{20}$$

which defines a one-parameter family of curves $K(u)$ of degree two, typically conics. Assume that $K(u)$ contains more than one real point for all $u \in \mathbb{R}$. If this reality condition holds only for an interval $[a, b] \subset \mathbb{R}$, it is possible to re-parameterize by $s = (a+u^2)/(b+s^2)$ such that $K(s)$ contains real points for all $s \in \mathbb{R}$. For one-parameter families of real conics $K(u)$ it is proved in [10] and [17] that there exists a rational vector $\mathbf{y}(u)$ such that (20) is satisfied identically. The solution $\mathbf{y}(u)$ can be computed explicitly and defines rational $v(u) = y_1(u)/y_0(u)$ and $w(u) = y_2(u)/y_0(u)$ such that

$$\mathbf{n}'(u) = \mathbf{k}(u) + v(u)\mathbf{l}(u) + w(u)\mathbf{m}(u)$$

is a rational isotropic normal vector field of $\mathbf{p}(u,v)$. This vector field is orthogonal to $\Phi$ at points of a curve $C \in \Phi$, parameterized by

$$\mathbf{c}(u) = \mathbf{a}(u) + v(u)\mathbf{g}(u).$$

We note that $\mathbf{n}'(u)$ is in particular orthogonal to $\mathbf{g}(u)$. In the next step we compute an isotropic normal vector field $\mathbf{n}(u,v)$ of $\Phi$. Therefore we let $\mathbf{n}$ be the linear combination $\mathbf{n} = \mathbf{n}' + \lambda(\mathbf{k} + t\mathbf{l})$ and determine $\lambda$ such that $\mathbf{n}$ is isotropic. This results in

$$\mathbf{n}(u,t) = \langle \mathbf{k} + t\mathbf{l}, \mathbf{k} + t\mathbf{l} \rangle \mathbf{n}' - 2\langle \mathbf{n}', \mathbf{k} + t\mathbf{l} \rangle (\mathbf{k} + t\mathbf{l}).$$

Let $u = u^\star$ be fixed. Then $\mathbf{n}(u^\star, t)$ are the isotropic normal vectors along the ruling $\mathbf{a}(u^\star) + v\mathbf{g}(u^\star)$ of $\Phi$. The reparameterization along the rulings has to be performed in a way that $\mathbf{n}$ is an isotropic normal vector at $\mathbf{a} + v\mathbf{g}$. Since $\langle \mathbf{n}, \dot{\mathbf{a}} + v\dot{\mathbf{g}} \rangle = 0$ shall hold, one obtains

$$v(u,t) = -\frac{\langle \mathbf{n}(u,t), \dot{\mathbf{a}}(u) \rangle}{\langle \mathbf{n}(u,t), \dot{\mathbf{g}}(u) \rangle}.$$

Since $\Phi$ has a rational isotropic normal vector field $\mathbf{n}(u,t)$, the surface is a rational two-dimensional sub-variety of a rational isotropic hyper-surface $\Gamma$. Applying Theorem 5 we obtain

**Corollary 11.** *A rational non-developable ruled surface in $\mathbb{R}^4$ is a MOS-surface.*

Real rational parameterizations $\mathbf{p}(u, v)$ for rational ruled surfaces $\Phi$ which satisfy condition (9) are only possible for those patches of $\Phi$ consisting only of Euclidean generating lines, since only for these parts the isotropic hyper-surface $\Gamma(\Phi)$ is real.

#### *6.1.2. Developable ruled surfaces*

The result obtained for rational skew ruled surfaces does not hold in general for rational developable ruled surfaces. Along each generating line there exists a fixed tangent plane and thus a fixed normal plane. Usually there exist two isotropic normal vector fields $\mathbf{n}_1(u)$ and $\mathbf{n}_2(u)$ which are typically non-rational.

A developable surface $\Phi$ is closely related to its curve of regression $C$, since $\Phi$ is formed by the tangent lines of $C$. If $\Phi$ is rational, then $C$ is rational too, and vice versa. As we will see in Section 6.2, a rational curve $C$ possesses a field of rational isotropic normal vectors.

### *6.2. Rational curves*

A rational curve $C$ in $\mathbb{R}^4$ corresponds to a rational canal surface $\gamma(C)$ in $\mathbb{R}^3$. The isotropic hypersurface $\Gamma$ through $C$ consists of a one-parameter family of isotropic (light) cones $D(u)$ with vertices on $C$. The carrier 3-spaces of these cones $D(u)$ are orthogonal to the tangent lines of $C$ in the Minkowski sense. Considering the parameterization $\mathbf{c}(u)$ of $C$, the isotropic cones $D(u)$ are determined by the equations

$$\langle \mathbf{x} - \mathbf{c}, \mathbf{x} - \mathbf{c} \rangle = 0, \ \langle \mathbf{x} - \mathbf{c}, \dot{\mathbf{c}} \rangle = 0.$$

The construction of a rational parameterization of $\Gamma$, works as follows: We are looking for a vector field $\mathbf{r}(u)$ satisfying $\langle \mathbf{r}, \dot{\mathbf{c}} \rangle = 0$ and $\langle \mathbf{r}, \mathbf{r} \rangle = 0$. Let $\mathbf{k}(u)$, $\mathbf{l}(u)$ and $\mathbf{m}(u)$ be linearly independent vectors satisfying the first equation $\langle \mathbf{r}, \dot{\mathbf{c}} \rangle = 0$. Evaluating the second equation for the linear combination $y_0\mathbf{k} + y_1\mathbf{l} + y_2\mathbf{m}$ results in a quadratic equation analogously to (19) and (20) defining a family of conics typically. As discussed in Section 6.1.1 there exist rational functions $(y_0, y_1, y_2)(u)$ such that $\mathbf{n}'(u) = (y_0\mathbf{k} + y_1\mathbf{l} + y_2\mathbf{m})(u)$ is a rational isotropic normal vector field of the rational curve $\mathbf{c}(u)$.

Once having found a rational solution $\mathbf{n}'(u)$, stereographic projection or similar methods as proposed in Section 6.1.1 can be applied to construct a rational isotropic normal vector field $\mathbf{n}(u, v)$ of $\mathbf{c}(u)$. For lowest possible degree solutions of this problem see [10]. Only those parts of the isotropic hyper-surface $\Gamma$ through $C$ are real which correspond to parts of $C$ with only Euclidean tangent lines. This construction proves that

**Corollary 12.** *A rational curve in $\mathbb{R}^4$ possesses a rational isotropic hyper-surface $\Gamma$. Any two-dimensional rational sub-variety of $\Gamma$ is a MOS-surface.*

This result is a reformulation of the already known result that canal surfaces defined by rational curves and rational radius functions admit rational parameterizations.

In particular, if $C$ in $\mathbb{R}^4$ is a circle in the Minkowski sense then there exists a totally orthogonal circle $D$. The isoptropic hyper-surface $\Gamma$ is formed by all lines joining points on $C$ and $D$. The cyclographic image $\gamma(C) = \gamma(D)$ is a Dupin cyclide.

*6.3. Two-dimensional regular quadrics*

It is proved in [17] that regular quadrics and their Laguerre transforms $\Psi$ are PN-surfaces in $\mathbb{R}^3$. Thus any rational two-dimensional surface $\Phi$ in the isotropic hyper-surface $\Gamma$ through $\Psi$ is a MOS-surface.

The fundamental idea to obtain this result is as follows: Consider a two-dimensional quadric $\Phi$ in $\mathbb{R}^4$. It is obviously contained in a hyperplane $H$. If $H$ is an isotropic hyper-plane it is proved in Section 6.4 that $\Phi$ satisfies condition (9) and the envelope surface $\gamma(\Phi)$ of the corresponding family of spheres is rational. If $\Phi$ is a regular ruled quadric (non developable), Section 6.1.1 proves the existence and shows how to construct rational parameterizations of $\Phi$ satisfying the condition (9).

Otherwise let $\Phi$ be a regular quadric in a space-like or time-like hyperplane. The isotropic hypersurface $\Gamma$ through $\Phi$ contains also three other quadrics $F_1, F_2, F_3$ and $\Gamma$ intersects the ideal hyperplane $\omega$ in the quadric $\Omega$. For all these quadrics $F_i$ the envelope surfaces $\gamma(F_i)$ in $\mathbb{R}^3$ of their corresponding two-parameter families of spheres coincide with $\Psi = \Gamma \cap \mathbb{R}^3$. It is shown in [17] that one of these quadrics $F_i$ is a real ruled quadric. Corollary 11 shows that $\Gamma$ is a rational isotropic hypersurface and therefore there exist rational parameterizations of $\Phi$ satisfying condition (9).

**Corollary 13.** *Regular two-dimensionial quadrics $\Phi$ in $\mathbb{R}^4$ are MOS-surfaces.*

*6.4. Rational surfaces in hyperplanes*

There exist three types of hyperplanes $H : \langle \mathbf{e}, \mathbf{x} \rangle = c$ in $\mathbb{R}^4$ depending on whether their normal vector $\mathbf{e} \in \mathbb{R}^4$ is time-like, light-like or space-like. The respective hyperplanes are denoted as space-like, light-like and time-like. The types of MOS-surfaces in hyperplanes are discussed in [9] and it is shown that MOS-surfaces in space-like hyperplanes are Lorentz-transforms of PN-surfaces, combined with a uniform scaling. MOS-surfaces in time-like hyperplanes can be constructed in a similar way, but the condition for a rational unit normal vector field has to be adjusted according to the scalar product (4).

Finally, let $\Phi$ be a rational two-dimensional surface in an isotropic hyperplane $H$. Since $H$ is a rational isotropic hyper-surface through $\Phi$ in a trivial way, Theorem 5 implies

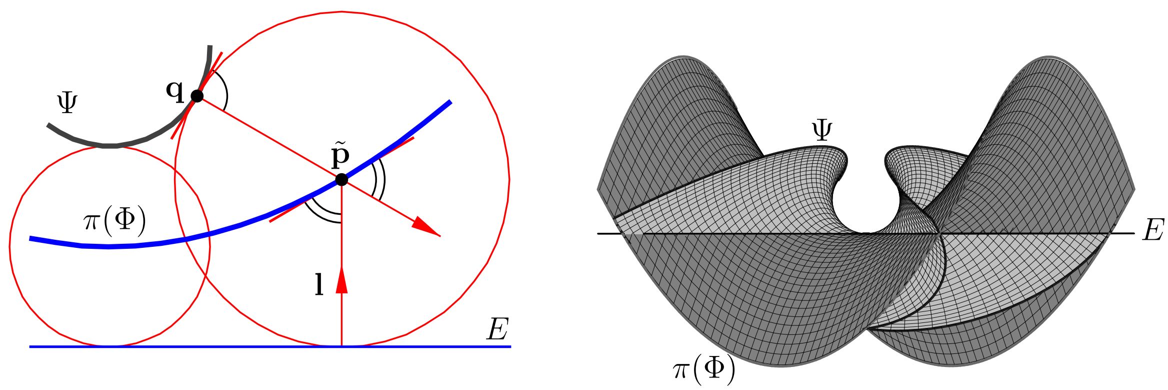
**Corollary 14.** *A rational two-dimensional sub-variety $\Phi$ of an isotropic hyperplane in $\mathbb{R}^4$ is a MOS-surface.*

Let $H$ be given by $\langle \mathbf{e}, \mathbf{x} \rangle = 0$. Since $\mathbf{e} = (e_1, e_2, e_3, e_4)$ is a constant isotropic normal vector of $\Phi$, the additional rational isotropic normal vector $\mathbf{n}$ can be computed linearly. The envelope surfaces $\Psi = \gamma(\Phi)$ in $\mathbb{R}^3$ of two-parameter family of spheres corresponding to $\Phi$ appear also in [24] as anticaustics of a rational surface $\pi(\Phi)$ with respect to an illumination parallel to $(e_1, e_2, e_3)$ and reflection at $\pi(\Phi)$, see Figure 4(a).

**Example:** Let $\Phi$ be a paraboloid in the isotropic 3-space $x_3 = x_4$, parameterized by $\mathbf{p} = (u, v, \frac{a}{2}u^2 + \frac{b}{2}v^2, \frac{a}{2}u^2 + \frac{b}{2}v^2)$. The envelope surface of the family of spheres centered at $\pi(\mathbf{p}) = \tilde{\mathbf{p}} = (p_1, p_2, p_3)$ with radii $p_4$ consists of the plane $E : x_3 = 0$ and the parabolic Dupin cyclide $\Psi$, parameterized by

$$\mathbf{q}(u, v) = \left( u\frac{1 + bv^2(b-a)}{1 + a^2u^2 + b^2v^2}, v\frac{1 + au^2(a-b)}{1 + a^2u^2 + b^2v^2}, \frac{au^2 + bv^2}{1 + a^2u^2 + b^2v^2} \right)$$

It is an anticaustic of reflection of the paraboloid $\tilde{\mathbf{p}}$ with respect to illumination parallel to the $x_3$-axis, see Fig. 4(b).




(a) Anticaustic of reflection w.r.t parallel illumination.

(b) Parabolic Dupin Cyclide as anticaustic of reflection of a paraboloid.

Figure 4: Anticaustic of reflection with respect to parallel illumination

*6.5. Rational surfaces in light-cones*

Besides the isotropic hyperplanes $H$ the light cones $C$ in $\mathbb{R}^4$ are also simple examples of rational isotropic hyper-surfaces. Applying Theorem 5 we obtain

**Corollary 15.** *A rational two-dimensional sub-variety of a light-cone in $\mathbb{R}^4$ is a MOS-surface.*

Consider the light cone $C$ with vertex at the origin, given by $\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$. Let further $\mathbf{f}(u, v)$ be a rational parameterization of a two-dimensional surface $\Phi$ in $C$. Because of $\langle \mathbf{f}, \mathbf{f} \rangle$=0 and $\langle \mathbf{f}, \mathbf{f}_u \rangle$=$\langle \mathbf{f}, \mathbf{f}_v \rangle$=0, one rational isotropic normal vector field of $\mathbf{f}$ is

$\mathbf{f}$ itself. To compute the second one, we consider the Plücker coordinates $G = (g_1, \ldots, g_6)$ of the line connecting the ideal points $(0, \mathbf{f}_u)\mathbb{R}$ and $(0, \mathbf{f}_v)\mathbb{R}$. The normal plane $N$ of $\mathbf{f}$ is spanned by $\mathbf{v} = (0, g_4, g_5, -g_6)$ and $\mathbf{w} = (g_6, -g_2, g_1, 0)$. Determining the second isotropic normal vector field $\mathbf{c}$, we set $\mathbf{c} = \mathbf{f} - \mu\mathbf{w}$. Requiring $\langle \mathbf{c}, \mathbf{c} \rangle = 0$ leads to

$$\mathbf{c}(u, v) = \mathbf{f} - 2\frac{\langle \mathbf{f}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w}.$$

The envelope surface $\Psi$ is obtained by intersecting the isotropic lines $\mathbf{f} + \lambda\mathbf{c}$ with the 3-space $\mathbb{R}^3 : x_4 = 0$. Since $w_4 = 0$, the envelope surface $\Psi$ admits the rational parameterization

$$\mathbf{q}(u, v) = \mathbf{f}(u, v) - \mathbf{c}(u, v) = 2\frac{\langle \mathbf{f}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w}. \tag{21}$$

Consequently, $\Psi$'s offsets at distance $d$ are the intersections of $\mathbf{f} + \lambda\mathbf{c}$ with the 3-spaces $x_4 = d$. The orthogonal projection $\mathbf{n} = (c_1, c_2, c_3)/c_4$ onto $x_4 = 0$ is a rational unit normal vector field of $\tilde{\mathbf{q}}(u, v) = (q_1, q_2, q_3)(u, v)$.

These envelope surfaces in $\mathbb{R}^3$ are also obtained as anticaustics with respect to central illumination with the origin as light source and reflection at the orthogonal projection $\pi(\Phi)$ of $\Phi$, parameterized by $\tilde{\mathbf{p}}(u, v) = (f_1, f_2, f_3)(u, v)$ in $\mathbb{R}^3$. Let $\tilde{\mathbf{m}}(u, v) = \tilde{\mathbf{p}}_u \times \tilde{\mathbf{p}}_v$ be a normal vector field of $\tilde{\mathbf{p}}(u, v)$. In terms of the Plücker coordinates $g_1, \ldots, g_6$ this reads $\tilde{\mathbf{m}} = (g_6, -g_2, g_1)$. Because of $w_4 = 0$ we have $\pi(\mathbf{w}) = \tilde{\mathbf{m}}$. Reflecting the origin at the tangent planes of $\tilde{\mathbf{p}}(u, v)$ leads to

$$\tilde{\mathbf{q}} = 2\frac{\tilde{\mathbf{p}} \cdot \tilde{\mathbf{m}}}{\tilde{\mathbf{m}} \cdot \tilde{\mathbf{m}}}\tilde{\mathbf{m}}, \tag{22}$$

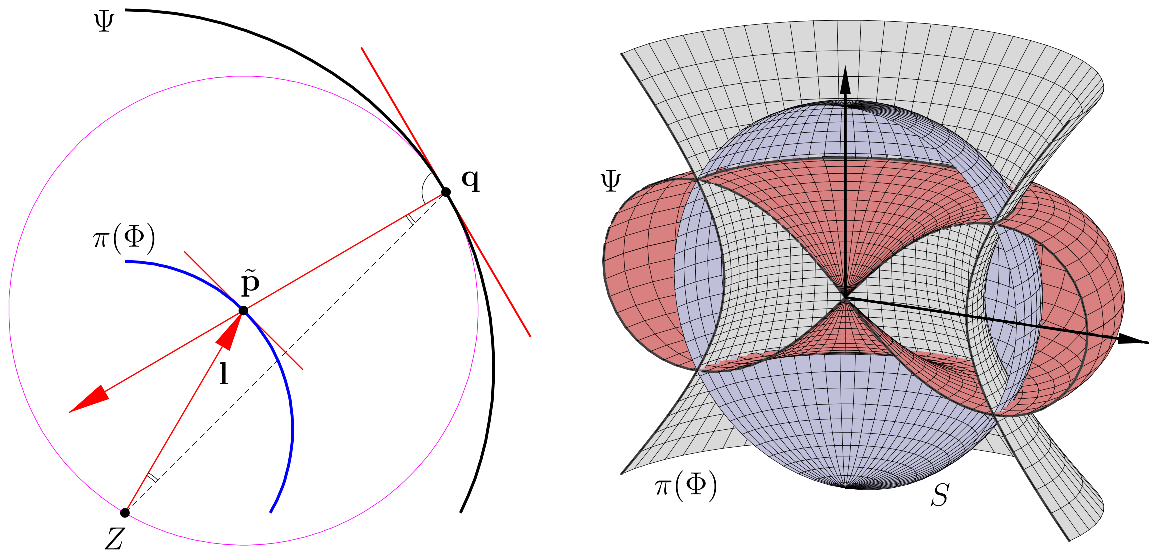
which agrees with (21). This construction is illustrated in Fig. 5(a).

Figure 5(b) illustrates the envelope $\Psi$ of a two-parameter family of spheres whose centers are located at the one-sheet hyperboloid $\pi(\Phi) : x_1^2 + x_2^2 - x_3^2 = 1$. The spheres pass through the origin, the center of $\pi(\Phi)$. The envelope $\Psi$ is a surface of revolution and the zero set of the polynomial $(x_1^2 + x_2^2 + x_3^2)^2 + 4(x_3^2 - x_1^2 - x_2^2)$. It is generated by a Bernoulli lemniscate as meridian curve. This surface can also be obtained by applying an inversion to $\pi(\Phi)$ at the sphere $S$ of radius $\sqrt{2}$, centered at the origin. The displayed parameter lines (Fig. 5(b)) do not represent a rational parameterization of $\Psi$.

**Remark:**

According to the construction and equation (22), *anticaustics of reflection* $\Psi$ *of rational surfaces* $F = \pi(\Phi)$ *with respect to central illumination are always rational*, see Fig. 5(a). This is even true for general rational surfaces $F$. Usually $\Psi$'s offsets are non-rational.

The surface $\Phi \in \mathbb{R}^4$ with top view $F$ is a two-dimensional surface in the light cone with vertex $Z$. Also for rational $F$, the surface $\Phi$ is typically non-rational. This implies that the isotropic hyper-surface through $\Phi$ is non-rational and decomposes into the rational light-cone $C$ and a non-rational isotropic hyper-surface $\Gamma$. Thus $\Phi$ is not a MOS-surface.




(a) Anticaustic of reflection with respect to central illumination, with center $Z$.

(b) Anticaustic of reflection $\Psi$ of a one-sheet hyperboloid $\pi(\Phi)$, and sphere of inversion $S$.

Figure 5: Construction of an Anticaustic and Example.

We conclude that there exist rational envelope surfaces $\Psi = \gamma(\Phi)$ of two-dimensional families of spheres determined by non-rational two-dimensional surfaces $\Phi \in \mathbb{R}^4$. The offset surfaces of $\Psi$ are typically non-rational.

### *6.6. Generalized LN-surfaces*

We study a special class of surfaces in $\mathbb{R}^4$ belonging to the class of MOS-surfaces, which generalize *LN-surfaces* of $\mathbb{R}^3$, see [6, 26]. Latter ones are rational surfaces in $\mathbb{R}^3$ admitting parameterizations $\mathbf{p}(u,v)$ such that their normal vectors are linear in $u$ and $v$. Considering only non-developable LN-surfaces $F$, we may assume that the normal vector field of $\mathbf{p}(u,v)$ is $\mathbf{n}(u,v) = (u,v,1)$. Consequently the tangent planes $T$ of $F$ are given by $xu + yv + z = f(u,v)$, with some rational function $f(u,v)$. A parameterization of $F$ is obtained by $T \cap T_u \cap T_v$ and reads $\mathbf{p}(u,v) = (f_u, f_v, f - uf_u - vf_v)$. Since the family of tangent planes of LN-surfaces are graphs $(u,v,f(u,v))$ of rational functions, their generalizations to $\mathbb{R}^4$ can be defined as follows:

**Definition 3.** *A rational surface $\Phi$ in $\mathbb{R}^4$ is called generalized LN-surface if it admits a parameterization $\mathbf{p}(u,v)$ in a way that $\Phi$'s tangent planes are spanned by vectors $\mathbf{s}(u,v)$ and $\mathbf{t}(u,v)$ which are linear in the surface parameters $u$ and $v$.*

As discussed in [19], generalized LN-surfaces are obtained as solutions of a system of linear equations

$$\begin{array}{llll} E: & \mathbf{e}\cdot\mathbf{x} = a, & F: & \mathbf{f}\cdot\mathbf{x} = b, \\ E_u: & \mathbf{e}_u\cdot\mathbf{x} = a_u, & F_u: & \mathbf{f}_u\cdot\mathbf{x} = b_u, \\ E_v: & \mathbf{e}_v\cdot\mathbf{x} = a_v, & F_v: & \mathbf{f}_v\cdot\mathbf{x} = b_v, \end{array} \tag{23}$$

for particular choices of vectors $\mathbf{e}(u,v)$ and $\mathbf{f}(u,v)$ and rational functions $a(u,v)$ and $b(u,v)$. Before some examples will be presented, we prove that generalized LN-surfaces are MOS-surfaces.

Let $\Phi \in \mathbb{R}^4$ be a generalized LN-surface, parameterized by $\mathbf{p}(u,v)$. According to Definition 3 the partial derivative vectors $\mathbf{p}_u$ and $\mathbf{p}_v$ can be represented as linear combinations $\mathbf{p}_u = \alpha\mathbf{s} + \beta\mathbf{t}$ and $\mathbf{p}_u = \gamma\mathbf{s} + \delta\mathbf{t}$ of the vectors $\mathbf{s}$ and $\mathbf{t}$, which are linear in $u$ and $v$, with certain rational functions $\alpha$, $\beta$, $\gamma$ and $\delta$. The linearity of $\mathbf{s}(u,v)$ and $\mathbf{t}(u,v)$ in $u$ and $v$ implies that for any given vector $\mathbf{z} \in \mathbb{R}^4$, the equations

$$\langle \mathbf{z}, \mathbf{s}(u,v)\rangle = 0,\ \langle \mathbf{z}, \mathbf{t}(u,v)\rangle = 0 \tag{24}$$

have rational solutions $u = u(z_1,\ldots,z_4)$ and $v = v(z_1,\ldots,z_4)$.

To prove that the isotropic hyper-surface $\Gamma(\Phi)$ through a generalized LN-surface $\Phi$ in $\mathbb{R}^4$ is rational, let $\mathbf{z}(s,t)$ be a rational parameterization of an isotropic cone. We may choose $\mathbf{z}(s,t) = (2s, 2t, 1-s^2-t^2, 1+s^2+t^2)$. Setting up the equations (24), the parameters $u$ and $v$ can be expressed by rational functions $u(s,t)$ and $v(s,t)$. Substituting these functions into the initial parameterization $\mathbf{p}(u,v)$ of $\Phi$ yields the reparameterized representation $\mathbf{p}(s,t)$. Since equations (24) hold, $\mathbf{z}(s,t)$ is a rational isotropic normal vector field of $\mathbf{p}(s,t)$. Consequently the isotropic hyper-surface $\Gamma(\Phi)$ of $\Phi$ is rationally parameterized by $\mathbf{p}(s,t) + w\mathbf{z}(s,t)$.

**Theorem 16.** *Generalized LN-surfaces $\Phi$ in $\mathbb{R}^4$ are MOS-surfaces.*

To illustrate generalized LN-surfaces in $\mathbb{R}^4$, we study three examples. One necessary ingredient for the construction are *LN-curves* in $\mathbb{R}^2$. A rational curve $C$ is said to be an LN-curve, if it admits a parameterization $\mathbf{c}(t)$ such that its normal vectors are linear in $t$. This implies that its tangent lines are given by $tx_1 + x_2 = f(t)$. Computing the envelope of the tangent lines one obtains the parameterization $\mathbf{c}(t) = (f_t, f - tf_t)(t)$.

*6.6.1. Surfaces of type 1*

The generalized LN-surfaces $\Phi$ of type 1 are *translational surfaces* with planar LN-curves as profile curves. The vectors $\mathbf{e}$ and $\mathbf{f}$ can be chosen as $\mathbf{e}(u) = (1,0,u,0)$ and $\mathbf{f}(v) = (0,1,0,v)$, and $a(u)$ and $b(v)$ are arbitrary univariate rational functions. A rational parameterization of $\Phi$ is given by

$$\begin{aligned} &\mathbf{p}(u,v) = \mathbf{c}(u) + \mathbf{d}(v), \qquad \text{with} \\ &\mathbf{c}(u) = (a - ua_u, 0, a_u, 0)\,, \quad \mathbf{d}(v) = (0, b - vb_v, 0, b_v)\,. \end{aligned} \tag{25}$$

The tangent planes $T$ of $\Phi$ are spanned by $\mathbf{p}$ and the vectors $\mathbf{s} = (-u, 0, 1, 0)$ and $\mathbf{t} = (0, -v, 0, 1)$. The reparameterization according to (24) reads

$$u = \frac{z_3}{z_1},\ v = -\frac{z_4}{z_2}. \tag{26}$$

*Example:* Choosing quadratic polynomials $a(u) = 1/2u^2$ and $b(v) = 1/2v^2$, the surface of type 1 is $\mathbf{p}(u,v) = (-1/2u^2, -1/2v^2, u, v)$. It is generated by translating two parabolas along each other.

### 6.6.2. Surfaces of type 2

The generalized LN-surfaces $\Phi$ of type 2 are *translational surfaces* with conjugate complex planar LN-curves as profile curves. It can be shown that these surfaces are Euclidean minimal surfaces in $\mathbb{R}^4$. The vectors $\mathbf{e}$ and $\mathbf{f}$ can be chosen as $\mathbf{e}(u,v) = (1, 0, -u, v)$ and $\mathbf{f}(u,v) = (0, -1, v, u)$, and the rational functions $a(u,v)$ and $b(u,v)$ have to satisfy the conditions $a_u = -b_v$ and $a_v = b_u$. A rational parameterization of $\Phi$ with respect to the real parameters $u$ and $v$ is given by

$$\mathbf{p}(u,v) = (a - ua_u - va_v, -b - va_u + ua_v, -a_u, a_v)\,. \tag{27}$$

The tangent planes $T$ of $\Phi$ are spanned by $\mathbf{p}$ and the vectors $\mathbf{s} = (u, v, 1, 0)$ and $\mathbf{t} = (-v, u, 0, 1)$. The reparameterization according to (24) is a solution of the system of linear equations

$$\begin{pmatrix} z_1 & z_2 \\ z_2 & -z_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -z_3 \\ z_4 \end{pmatrix}. \tag{28}$$

*Example:* For the choice $a = 1/2(v^2 - u^2)$ and $b = uv$ one obtains the parameterization $\mathbf{p}(u,v) = (1/2(u^2 - v^2), uv, u, v)$ of a generalized LN-surface of type 2. A projection of $\Phi$ onto $x_3 = 0$ together with the envelope surface $\Psi$ of the two-parameter family of spheres corresponding to $\Phi$ is displayed in Fig. 6.

### 6.6.3. Surfaces of type 3

To obtain generalized LN-surfaces $\Phi$ of type 3, we may choose vectors $\mathbf{e}$ and $\mathbf{f}$ by $\mathbf{e} = (u, 0, -1, -u^2 + v)$ and $\mathbf{f} = (-v, 1, 0, uv)$. The rational functions $a$ and $b$ have to satisfy the conditions $b_u = va_v$ and $b_v = -a_u - ua_v$. This implies that $a$ is a solution of

$$a_{uu} = -ua_{uv} - 2a_v - va_{vv}. \tag{29}$$

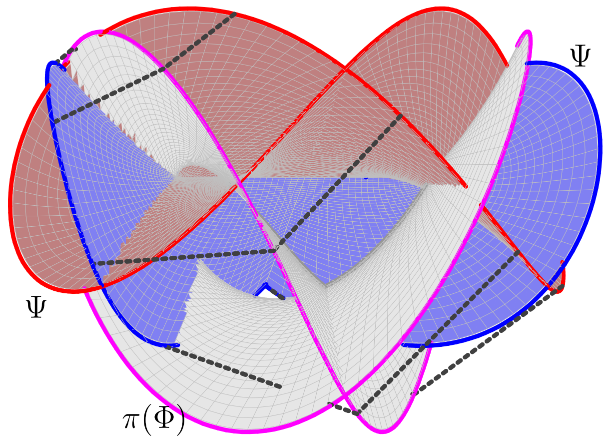



Figure 6: Projection $\pi(\Phi)$ of a surfaces of type 2 and envelope surface $\Psi$.

and $b(u,v)$ is obtained by

$$b = \int v a_v du - \int (\int (v a_{vv} + a_v) du + a_u + u a_v) dv + C. \tag{30}$$

A rational parameterization of a surface $\Phi$ in $\mathbb{R}^4$ of type 3 is

$$\mathbf{p}(u,v) = (u a_v - b_v, b - v b_v, -a + b_u - u b_v, a_v)\,. \tag{31}$$

The tangent planes $T$ of $\Phi$ are spanned by $\mathbf{p}$ and the vectors $\mathbf{s} = (u, 0, v, 1)$ and $\mathbf{t} = (1, v, u, 0)$. The reparameterization according to (24) is a solution of the system of linear equations

$$\begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} z_4 \\ -z_1 \end{pmatrix}. \tag{32}$$

*Example:* Choosing polynomials $a(u,v) = -1/2u^3 + uv$ and $b(u,v) = 1/2u^2v - 1/2v^2$ we obtain a quadratically parameterized surface, represented by $\mathbf{p}(u,v) = (1/2u^2 + v, 1/2v^2, uv, u)$.

## Conclusion

Representing two-parameter families of spheres $S(u,v)$ in $\mathbb{R}^3$ by surfaces $\Phi$ in $\mathbb{R}^4$, we have studied those rational surfaces $\Phi$ whose envelope surfaces $\Psi$ of their corresponding two-parameter families of spheres $S(u,v)$ admit rational parameterizations. It is proved that these so-called MOS-surfaces $\Phi$ possess rational isotropic normal vector fields and occur as two-dimensional sub-varieties of isotropic hyper-surfaces in $\mathbb{R}^4$. We have studied various characterizations as well as their relations to rational offset surfaces. Finally it is shown that rational ruled surfaces, quadrics and generalized LN-surfaces are MOS-surfaces.

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