THE DEGREE COMPLEXITY OF SMOOTH SURFACES OF CODIMENSION 2

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Abstract. For a given term order, the degree complexity of a projective scheme is defined by the maximal degree of the reduced Gröbner basis of its defining saturated ideal in generic coordinates [2]. It is well-known that the degree complexity with respect to the graded reverse lexicographic order is equal to the Castelnuovo-Mumford regularity [3]. However, much less is known if one uses the graded lexicographic order [1], [5].

In this paper, we study the degree complexity of a smooth irreducible surface in $\mathbb{P}^4$ with respect to the graded lexicographic order and its geometric meaning. Interestingly, this complexity is closely related to the invariants of the double curve of a surface under a generic projection. As results, we prove that except a few cases, the degree complexity of a smooth surface S of degree d with $h^0(\mathcal{I}_S(2)) \neq 0$ in $\mathbb{P}^4$ is given by $2+\binom{\deg Y_1(S)-1}{2} - g(Y_1(S))$, where $Y_1(S)$ is a double curve of degree $\binom{d-1}{2} - g(S\cap H)$ under a generic projection of S. In particular, this complexity is actually obtained at the monomial

$$x_0x_1x_3^{\binom{\deg Y_1-1}{2}-g(Y_1(S))}$$

where $k[x_0,x_1,x_2,x_3,x_4]$ is a polynomial ring defining $\mathbb{P}^4$. Exceptional cases are a rational normal scroll, a complete intersection surface of $(2,2)$-type, or a Castelnuovo surface of degree 5 in $\mathbb{P}^4$ whose degree complexities are in fact equal to their degrees. This complexity can also be expressed in terms of degrees of defining equations of I_S in the same manner as the result of A. Conca and J. Sidman [5]. We also provide some illuminating examples of our results via calculations done with *Macaulay 2* [10].

Contents

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1. Introduction

D. Bayer and D. Mumford in [2] have introduced the degree complexity of a homogeneous ideal I with respect to a given term order τ as the maximal degree of the reduced Gröbner basis of I, and this is exactly the highest degree of minimal generators of the initial ideal of I. Even though degree complexity depends on the choice of coordinates, it is constant in generic coordinates since the initial ideal of I is invariant under a generic change of coordinates, which is the so-called the generic initial ideal of I [7].

For the graded lexicographic order (resp. the graded reverse lexicographic order), we denote by $M(I)$ (resp. $m(I)$) the degree complexity of I *in generic coordinates*. For a projective scheme X, the degree complexity of X can also be defined as $M(I_X)$ (resp. $m(I_X)$) for the graded lexicographic order (resp. the graded reverse lexicographic order) where I_X is the defining saturated ideal of X.

D. Bayer and M. Stillman have shown in [3] that $m(I)$ is exactly equal to the Castelnuovo-Mumford regularity $\operatorname{reg}(I)$. Then what can we say about $M(I)$? A. Conca and J. Sidman proved in [5] that if I_C is the defining ideal of a smooth irreducible complete intersection curve C of type (a, b) in $\mathbb{P}^3$ then $M(I_C)$ is $1 + \frac{ab(a-1)(b-1)}{2}$ with the exception of the case $a = b = 2$, where $M(I_C)$ is 4. Recently, J. Ahn has shown in [1] that if I_C is the defining ideal of a non-degenerate smooth integral curve of degree d and genus $g(C)$ in $\mathbb{P}^r$ (for $r \geq 3$), then $M(I_C) = 1+\binom{d-1}{2} - g(C)$ with two exceptional cases.

In this paper, we would like to compute the degree complexity of a smooth surface S in $\mathbb{P}^4$ with respect to the graded lexicographic order. Interestingly, this complexity is closely related to the invariants of the double curve of S under the generic projection. Our main results are: if $S \subset \mathbb{P}^4$ is a smooth irreducible surface of degree d with $h^0(\mathcal{I}_S(2)) \neq 0$, then the degree complexity $M(I_S)$ of S is given by $2+\binom{\deg Y_1(S)-1}{2} - g(Y_1(S))$ with three exceptional cases, where $Y_1(S)$ is a smooth double curve of S in $\mathbb{P}^3$ under a generic projection and $\deg Y_1(S) = \binom{d-1}{2} - g(S \cap H)$. Moreover, this complexity is actually obtained at the monomial

$$x_0 x_1 x_3^{\binom{\deg Y_1-1}{2}-g(Y_1(S))}$$

where $k[x_0, x_1, x_2, x_3, x_4]$ is a polynomial ring defining $\mathbb{P}^4$.

On the other hand, $M(I_S)$ can also be expressed in terms of degrees of defining equations of I_S in the same manner as the result of A. Conca and J. Sidman [5] (see Theorem 4.9). Note that if S is a locally Cohen-Macaulay surface with $h^0(\mathcal{I}_S(2)) \neq 0$ then there are two types of S. One is a complete intersection of $(2, \alpha)$-type and the other is projectively Cohen-Macaulay of degree $2\alpha - 1$. For those cases, $\deg Y_1(S)$, $g(Y_1(S))$ and $g(S \cap H)$ can be obtained in terms of α.

Consequently, if S is a complete intersection of $(2, \alpha)$-type for some $\alpha \geq 3$ then $M(I_S) = \frac{1}{2}(\alpha^4 - 4\alpha^3 + 5\alpha^2 - 2\alpha + 4)$. If S is projectively Cohen-Macaulay of degree $2\alpha - 1$, $\alpha \geq 4$, then $M(I_S) = \frac{1}{2}(\alpha^4 - 6\alpha^3 + 13\alpha^2 - 12\alpha + 8)$ (see

Theorem 4.9). Exceptional cases are a rational normal scroll, a complete intersection surface of $(2,2)$-type, or a Castelnuovo surface of degree 5 in $\mathbb{P}^4$. In these cases, $M(I_S) = \deg(S)$ (see Theorem 4.5).

The main ideas are divided into two parts: one is to show that the degree complexity $M(I_S)$ is given by the maximum of $\mathrm{reg}(\mathrm{Gin}_{\mathrm{GLex}}(K_i(I_S))) + i$ for $i = 0, 1$ and the other part is to compare the schemes of multiple loci defined by partial elimination ideals and their classical scheme structures defined by the Fitting ideals of an $\mathcal{O}_{\mathbb{P}^3}$-module $\pi_*\mathcal{O}_S$ where π is a generic projection of S to $\mathbb{P}^3$.

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2. Notations and basic facts

- We work over an algebraically closed field k of characteristic zero.
- Let $R = k[x_0, \ldots, x_r]$ be a polynomial ring over k. For a closed subscheme X in $\mathbb{P}^r$, we denote the defining saturated ideal of X by
$$I_X = \bigoplus_{m=0}^{\infty} H^0(\mathcal{I}_X(m))$$
- For a homogeneous ideal I, the Hilbert function of R/I is defined by $H(R/I, m) := \dim_k(R/I)_m$ for any non-negative integer m. We denote its corresponding Hilbert polynomial by $P_{R/I}(z) \in \mathbb{Q}[z]$. If $I = I_X$ then we simply write $P_X(z)$ instead of $P_{R/I_X}(z)$.
- We write $\rho_a(X) = (-1)^{\dim(X)}(P_X(0) - 1)$ for the arithmetic genus of X.
- For a homogeneous ideal $I \subset R$, consider a minimal free resolution
$$\cdots \to \bigoplus_j R(-i-j)^{\beta_{i,j}(I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(I)} \to I \to 0$$
of I as a graded R-modules. We say that I is m-regular if $\beta_{i,j}(I) = 0$ for all $i \geq 0$ and $j \geq m$. The Castelnuovo-Mumford regularity of I is defined by
$$\mathrm{reg}(I) := \min\{\, m \mid I \text{ is } m\text{-regular}\}.$$
- Given a term order τ, we define the initial term $\mathrm{in}_\tau(f)$ of a homogeneous polynomial $f \in R$ to be the greatest monomial of f with respect to τ. If $I \subset R$ is a homogeneous ideal, we also define the initial ideal $\mathrm{in}_\tau(I)$ to be the ideal generated by $\{\mathrm{in}_\tau(f) \mid f \in I\}$. A set $G = \{g_1, \ldots, g_n\} \subset I$ is said to be a Gröbner basis if
$$(\mathrm{in}_\tau(g_1), \ldots, \mathrm{in}_\tau(g_n)) = \mathrm{in}_\tau(I).$$

- For an element $\alpha = (\alpha_0, \ldots, \alpha_r) \in \mathbb{N}^r$ we define the notation $x^\alpha = x_0^{\alpha_0} \cdots x_r^{\alpha_r}$ for monomials. Its degree is $\mid \alpha \mid = \sum_{i=0}^r \alpha_i$.

  For two monomial terms x^α and x^β, the *graded lexicographic order* is defined by $x^\alpha \geq_{\mathrm{GLex}} x^\beta$ if and only if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and if the left most nonzero entry of $\alpha - \beta$ is positive. The *graded reverse lexicographic order* is defined by $x^\alpha \geq_{\mathrm{GRLex}} x^\beta$ if and only if we have $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and if the right most nonzero entry of $\alpha - \beta$ is negative.
- In characteristic 0, we say that a monomial ideal I has the Borel-fixed property if, for some monomial m, we have $x_i m \in I$, then $x_j m \in I$ for all $j \leq i$.
- Given a homogeneous ideal $I \subset R$ and a term order τ, there is a Zariski open subset $U \subset GL_{r+1}(k)$ such that $\mathrm{in}_\tau(g(I))$ is constant. We will call $\mathrm{in}_\tau(g(I))$ for $g \in U$ the generic initial ideal of I and denote it by $\mathrm{Gin}_\tau(I)$. Generic initial ideals have the Borel-fixed property (see [7],[8]).
- For a homogeneous ideal $I \subset R$, let $m(I)$ and $M(I)$ denote the maximum of the degrees of minimal generators of $\mathrm{Gin}_{\mathrm{GRLex}}(I)$ and $\mathrm{Gin}_{\mathrm{GLex}}(I)$ respectively.
- If I is a Borel fixed monomial ideal then $\mathrm{reg}(I)$ is exactly the maximal degree of minimal generators of I (see [3],[8]). This implies that

$$m(I) = \mathrm{reg}(\mathrm{Gin}_{\mathrm{GRLex}}(I)) \text{ and } M(I) = \mathrm{reg}(\mathrm{Gin}_{\mathrm{GLex}}(I)).$$

3. Gröbner bases of partial elimination ideals

Definition 3.1. Let I be a homogeneous ideal in R. If $f \in I_d$ has leading term $\mathrm{in}(f) = x_0^{d_0} \cdots x_r^{d_r}$, we will set $d_0(f) = d_0$, the leading power of x_0 in f. We let

$$\widetilde{K}_i(I) = \bigoplus_{d \geq 0} \{f \in I_d \mid d_0(f) \leq i\}.$$

If $f \in \widetilde{K}_i(I)$, we may write uniquely

$$f = x_0^i \overline{f} + g,$$

where $d_0(g) < i$. Now we define $K_i(I)$ as the image of $\widetilde{K}_i(I)$ in $\bar{R} = k[x_1 \ldots x_r]$ under the map $f \to \overline{f}$ and we call $K_i(I)$ the i-th partial elimination ideal of I.

Remark 3.1. We have an inclusion of the partial elimination ideals of I:

$$I \cap \bar{R} = K_0(I) \subset K_1(I) \subset \cdots \subset K_i(I) \subset K_{i+1}(I) \subset \cdots \subset \bar{R}.$$

Note that if I is in generic coordinates and $i_0 = \min\{i \mid I_i \neq 0\}$ then $K_i(I) = \bar{R}$ for all $i \geq i_0$.

The following result gives the precise relationship between partial elimination ideals and the geometry of the projection map from $\mathbb{P}^r$ to $\mathbb{P}^{r-1}$. For a proof of this proposition, see [8, Propostion 6.2].

Proposition 3.2. Let $X \subset \mathbb{P}^r$ be a reduced closed subscheme and let I_X be the defining ideal of X. Suppose $p = [1, 0, \dots, 0] \in \mathbb{P}^r \setminus X$ and that $\pi : X \to \mathbb{P}^{r-1}$ is the projection from the point $p \in \mathbb{P}^r$ to $x_0 = 0$. Then, set-theoretically, $K_i(I_X)$ is the ideal of $\{q \in \pi(X) \mid \mathrm{mult}_q(\pi(X)) > i\}$.

For each $i \geq 0$, note that we can give a scheme structure on the set

$$Y_i(X) := \{q \in \pi(X) \mid \mathrm{mult}_q(\pi(X)) > i\}$$

from the i-th partial elimination ideal $K_i(I)$. Let

$$Z_i(X) := \mathrm{Proj}(\bar{R}/K_i(I_X)),$$

where $\bar{R} = k[x_1 \dots x_r]$. Then it follows from Proposition 3.2 that

$$Z_i(X)_{\mathrm{red}} = Y_i(X).$$

Remark 3.3. Let $X \subset \mathbb{P}^r$ be a smooth variety of codimension two and let $\pi : X \to \mathbb{P}^{r-1}$ be a generic projection of X. A classical scheme structure on the set $Y_i(X)$ is given by i-th Fitting ideal of the $\mathcal{O}_{\mathbb{P}^{r-1}}$-module $\pi_*\mathcal{O}_X$ (see [12],[14]). Throughout this paper, we use the notation $Y_i(X)$ in the sense that it is a closed subscheme defined by Fitting ideal of $\pi_*\mathcal{O}_X$, as distinguished from the notation $Z_i(X)$. We show that if $S \subset \mathbb{P}^4$ is a smooth surface lying in a quadric surface then $Y_1(S)$ and $Z_1(S)$ have the same reduced scheme structure (see Theorem 4.2), which will be used in the proof of Proposition 4.5.

It is natural to ask: what is a Gröbner basis of $K_i(I)$? Recall that any non-zero polyomial f in R can be uniquely written as $f = x^t \bar{f} + g$ where $d_0(g) < t$. A. Conca and J. Sidman [5] show that if G is a Gröbner basis for an ideal I then the set

$$G_i = \{\bar{f} \mid f \in G \text{ with } d_0(f) \leq i\}$$

is a Gröbner basis for $K_i(I)$. However if I is in generic coordinates then there is a more refined Gröbner basis for $K_i(I)$, which plays an important role in this paper.

Proposition 3.4. Let I be a homogeneous ideal *in generic coordinates* and G be a Gröbner basis for I with respect to the graded lexicographic order. Then, for each $i \geq 0$,

(a) the i-th partial elimination ideal $K_i(I)$ is in generic coordinates;
(b) $G_i = \{\bar{f} \mid f \in G \text{ with } d_0(f) = i\}$ is a Gröbner basis for $K_i(I)$.

Proof. (a) is in fact proved in Proposition 3.3 in [5]. For a proof of (b), it suffices to show that $\langle \mathrm{in}(G_i)\rangle = \mathrm{in}(K_i(I))$ by the definition of Gröbner bases. Since $G_i \subset K_i(I)$, we only need to show that $\langle \mathrm{in}(G_i)\rangle \supset \mathrm{in}(K_i(I))$. Now, we denote $\mathcal{G}(I)$ by the set of minimal generators of I. Let $m \in \mathrm{in}(K_i(I))$ be a monomial. Then there is a monomial generator $M \in \mathcal{G}(\text{in}(K_i(I)))$ such that M divide m.

We claim that $x_0^i M \in \mathcal{G}(\text{in}(I))$ if and only if $M \in \mathcal{G}(\text{in}(K_i(I)))$.

If the claim is proved then we will be done. Indeed, for $M \in \mathcal{G}(\text{in}(K_i(I)))$, we see that $x_0^i M \in \mathcal{G}(\text{in}(I))$. This implies that there exists a polynomial $f = x_0^i \bar{f} + g \in G$ with $d_0(g) < i$ such that

$$\text{in}(f) = x_0^i \text{in}(\bar{f}) = x_0^i M.$$

This means that $M = \text{in}(\bar{f}) \in \langle \text{in}(G_i) \rangle$. Thus we have $m \in \langle \text{in}(G_i) \rangle$.
Here is a proof of the claim: suppose that $x_0^i M \in \mathcal{G}(\text{in}(I))$ then we can say that $x_0^i M \in \text{in}(I)$. Thus there is a polynomial $f = x_0^i \bar{f} + g \in I$ such that $d_0(g) < i$ and $\text{in}(f) = x_0^i \text{in}(\bar{f}) = x_0^i M$. By the definition of partial elimination ideals, we have that $\bar{f} \in K_i(I)$, which means $M \in \text{in}(K_i(I))$. Assume that $M \notin \mathcal{G}(\text{in}(K_i(I)))$. Then for some monomial $N \in \mathcal{G}(\text{in}(K_i(I)))$ such that N divide M. This implies that

$$x_0^i N \in \text{in}(I) \text{ and } x_0^i N \mid x_0^i M,$$

which contradicts the fact that $x_0^i M$ is a minimal generator of $\text{in}(I)$. Thus M is contained in $\mathcal{G}(\text{in}(K_i(I)))$.

Conversely, suppose that there is $M \in \mathcal{G}(\text{in}(K_i(I)))$ such that $x_0^i M \notin \mathcal{G}(\text{in}(I))$. Then we may choose a monomial $x_0^j N \in \mathcal{G}(\text{in}(I))$ satisfying

$$x_0 \nmid N \text{ and } x_0^j N \mid x_0^i M. \tag{1}$$

Note that (1) implies that $i \geq j \geq 0$. Since $N \in \text{in}(K_j(I))$ and $K_0(I) \subset K_1(I) \subset \cdots$, it is obvious that $N \in \text{in}(K_i(I))$ and N divides M. Now, we claim that N can be chosen to be different from M. If $N = M$ then j must be less than i. Denote N by $x_1^{j_1} \cdots x_r^{j_r}$ and choose $j_t \neq 0$. By (a), note that $K_i(I)$ is in generic coordinates and so we may assume that $\text{in}(K_i(I))$ has the Borel-fixed property. Therefore, if we set $N' = N/x_{j_t}$ then $x_0^{j+1} N' \in \text{in}(I)$. Replace $x_0^j N$ by $N'' = x_0^{j+1} N'$. Then $N' \in \text{in}(K_{j+1}(I))$. Since $j+1 \leq i$, we can say that $N' \in \text{in}(K_i(I))$ and N' divides M with $N' \neq M$. This contradicts the assumption that $M \in \mathcal{G}(\text{in}(K_i(I)))$. □

Remark 3.5. The condition "in generic coordinates" is crucial in Proposition 3.4 (b) as the following example shows. Let $I = (x_0^2, x_0x_1, x_0x_2, x_3)$ be a monomial ideal. Then $G = \{x_0^2, x_0x_1, x_0x_2, x_3\}$ is a Gröbner basis for I. Then we can easily check that

$$G_1 = \{\bar{f} \mid f \in G \text{ with } d_0(f) \leq 1\} = (x_1, x_2, x_3),$$
$$G_1' = \{\bar{f} \mid f \in G \text{ with } d_0(f) = 1\} = (x_1, x_2).$$

This shows that G_1' is not a Gröbner basis for $K_1(I)$.

We have the following corollary from Proposition 3.4.

Corollary 3.6. For a homogeneous ideal $I \subset R = k[x_0, \dots, x_r]$ *in generic coordinates*, we have

$$M(I) = \max\{M(K_i(I)) + i \mid 0 \le i \le \beta\},$$

where $\beta = \min\{j \mid I_j \neq 0\}$.

Proof. Note that $K_\beta(I) = \bar{R}$ for $\beta = \min\{j \mid I_j \neq 0\}$ by definition. We know that $M(I)$ can be obtained from the maximal degree of generators in $\mathrm{Gin}(I)$. Remember that $\mathcal{G}(I)$ is the set of minimal generators of I. Then by Proposition 3.4, every generator of $\mathrm{Gin}(I)$ is of the form $x_0^i M$ where $M \in \mathcal{G}(\mathrm{Gin}(K_i(I)))$ for some i. This means that $M(I) \le M(\mathrm{Gin}(K_i(I))) + i$ for some i. On the other hand, if for each i, we choose $M \in \mathcal{G}(K_i(I))$, then by Proposition 3.4, $x_0^i M$ is contained in $\mathcal{G}(\mathrm{Gin}(I))$. Hence we conclude that

$$M(I) = \max\{M(K_i(I)) + i \mid 0 \le i \le \beta\}.$$

□

Corollary 3.6 with the following theorem can be used to obtain the degree-complexities of the smooth surface lying in a quadric hypersurface in $\mathbb{P}^4$. For a proof of this theorem, see [1, Theorem 4.4].

Theorem 3.7. Let C be a non-degenerate smooth curve of degree d and genus $g(C)$ in $\mathbb{P}^r$ for some $r \ge 3$. Then,

$$M(I_C) = \max\{d, 1 + \binom{d-1}{2} - g(C)\}.$$

4. Degree complexity of smooth irreducible surfaces in $\mathbb{P}^4$

Let S be a non-degenerate smooth irreducible surface of degree d and arithmetic genus $\rho_a(S)$ in $\mathbb{P}^4$ and let I_S be the defining ideal of S in $R = k[x_0, \dots, x_4]$. In this section, we study the scheme structure of

$$Z_i(S) := \mathrm{Proj}(\bar{R}/K_i(I_S)), \ \text{ where } \bar{R} = k[x_1, x_2, x_3, x_4]$$

arising from a generic projection in order to get a geometric interpretation of the degree-complexity $M(I_S)$ of S in $\mathbb{P}^4$ with respect to the degree lexicographic order.

We recall without proof the standard facts concerning generic projections of surfaces in $\mathbb{P}^4$ to $\mathbb{P}^3$.

Let $S \subset \mathbb{P}^4$ be a non-degenerate smooth irreducible surface of degree d and arithmetic genus $\rho_a(S)$ and $\pi : S \to \pi(S) \subset \mathbb{P}^3$ be a generic projection.

(a) The singular locus of $\pi(S)$ is a curve $Y_1(S)$ with only singularities a number t of ordinary triple points with transverse tangent directions. The inverse image $\pi^{-1}(Y_1(S))$ is a curve with only singularities $3t$ nodes, 3 nodes above each triple point of $Y_1(S)$ (see [15]). This implies (using Proposition 3.2) that the ideals $K_j(I_S)$ have finite colength if $j > 2$. This fact is used in the proofs of Propostion 4.6 and Theorem 4.3.

(b) If a smooth surface $S \subset \mathbb{P}^4$ is contained in a quadric hypersurface then there are no ordinary triple points in $Y_1(S)$. This implies that the double curve $Y_1(S)$ is smooth by (a).

(c) The double curve $Y_1(S)$ is irreducible unless S is a projected Veronese surface in $\mathbb{P}^4$ (see [14]).

(d) The reduced induced scheme structure on $Y_1(S)$ is defined by the first Fitting ideal of the $\mathcal{O}_{\mathbb{P}^3}$-module $\pi_*\mathcal{O}_S$ (see [14]).

(e) The degree of $Y_1(S)$ is $\binom{d-1}{2} - g(S \cap H)$ where $S \cap H$ is a general hyperplane section and the number of apparent triple points t is given in [13] by

$$t = \binom{d-1}{3} - g(S \cap H)(d-3) + 2\chi(\mathcal{O}_S) - 2.$$

The following lemma shows that the Hilbert function of I_S can be obtained from those of partial elimination ideals $K_i(I_S)$.

Lemma 4.1. Let $S \subset \mathbb{P}^4$ be a smooth surface with the defining ideal I_S in $R = k[x_0, x_1, \ldots, x_4]$. Consider a projection $\pi_q : S \longrightarrow \mathbb{P}^3$ from a general point $q = [1, 0, 0, 0, 0] \notin S$. Then,

$$H(R/I_S, m) = \sum_{i \geq 0} H(\bar{R}/K_i(I_S), m - i).$$

In particular,

$$P_S(z) = P_{Z_0(S)}(z) + P_{Z_1(S)}(z-1) + P_{Z_2(S)}(z-2).$$

Proof. The equality on Hilbert functions basically comes from the following combinatorial identity

$$\binom{m+d}{d} = \sum_{i=0}^{d} \binom{m-1+d-i}{d-i}.$$

For a smooth surface $S \subset \mathbb{P}^4$, $Z_i(S) = \emptyset$ for $i \geq 3$ by the (dimension +2)-secant lemma (see [16]) and so $\bar{R}/K_i(I_S)$ is Artinian. Thus $P_{Z_i(S)}(z) = 0$ for $i \geq 3$ (see [1, Lemma 3.4] for details). □

The following theorem says that the first partial elimination ideal $K_1(I_S)$ gives the reduced induced scheme structure on the double curve $Y_1(S)$ in $\mathbb{P}^3$ (i.e., $I_{Z_1(S)} = I_{Y_1(S)}$).

Theorem 4.2. Suppose that S is a reduced irreducible surface in $\mathbb{P}^4$. Then,

(a) the first partial elimination ideal $K_1(I_S)$ is a saturated ideal, so we have $K_1(I_S) = I_{Z_1(S)}$;

(b) if S is a smooth surface contained in a quadric hypersurface, then $K_1(I_S) = I_{Y_1(S)}$, which implies that $K_1(I_S)$ is a reduced ideal.

Proof. (a) Assume that S is a reduced irreducible surface in $\mathbb{P}^4$ of degree d. Take a general point $q \in \mathbb{P}^4$; we may assume $q = [1, 0, \ldots, 0]$. Then the generic projection of S into $\mathbb{P}^3$ from the point q is defined by a single polynomial $F \in k[x_1, x_2, x_3, x_4]$ of degree d and $K_0(I_S) = (F)$, which is a reduced ideal.

Let $\bar{\mathcal{M}} = (x_1, x_2, x_3, x_4)$ be the irrelevant maximal ideal of $\bar{R} = k[x_1, x_2, x_3, x_4]$. By the definition of saturated ideal, $K_1(I_S)$ is saturated if and only if

$$(K_1(I_S) : \bar{\mathcal{M}}) = K_1(I_S).$$

Hence it is enough to show that

$$(K_1(I_S) : \bar{\mathcal{M}})/K_1(I_S) = 0.$$

For the proof, consider the Koszul complex

$$\cdots \to \mathcal{K}_m^{-p-1} \to \mathcal{K}_m^{-p} \to \mathcal{K}_m^{-p+1} \to \cdots,$$

where $\mathcal{K}_m^{-p} = \bigwedge^p \bar{\mathcal{M}} \bigotimes K_0(I_S)_{m-p}$. From Corollary 6.7 in [8], the $\bar{R}$-module $(K_1(I_S) : \bar{\mathcal{M}})_d/K_1(I_S)_d$ injects into $H^{-1}(\mathcal{K}^\bullet_{d+3})$ for each d. Note that

$$H^{-1}(\mathcal{K}^\bullet_{d+3}) = H(\bigwedge^1 \bar{\mathcal{M}} \bigotimes K_0(I_S)_{d+2}) = \mathrm{Tor}_1^{\bar{R}}(\bar{R}/\bar{\mathcal{M}}, K_0(I_S))_{d+3}.$$

Since the ideal $K_0(I_S)$ is generated by a single polynomial F, we have that

$$\mathrm{Tor}_1^{\bar{R}}(\bar{R}/\bar{\mathcal{M}}, K_0(I_S)) = 0.$$

This proves that $(K_1(I_S) : \bar{\mathcal{M}})/K_1(I_S) = 0$, as we wished.

(b) Since S is contained in a quadric hypersurface and the center of projection is outside a quadric, we have a surjection $\varphi : \bar{R}(-1) \oplus \bar{R} \to R/I_S$ as a $\bar{R}$-module homomorphism with the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0(I_S) & \longrightarrow & \bar{R} & \longrightarrow & \bar{R}/K_0(I_S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{K}_1(I_S) & \longrightarrow & \bar{R} \oplus \bar{R}(-1) & \xrightarrow{\varphi} & R/I_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1(I_S)(-1) & \longrightarrow & \bar{R}(-1) & \longrightarrow & \bar{R}/K_1(I_S)(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\widetilde{K}_1(I_S) = \{f \in I_S \mid d_0(f) \leq 1\}$ is an $\bar{R}$-module. Let $\mathcal{O}_{Z_1(S)}$ be the sheafification of $\bar{R}/K_1(I_S)$. By sheafifying the rightmost vertical sequence, we have

$$0 \longrightarrow \mathcal{O}_{\pi(S)} \longrightarrow \pi_* \mathcal{O}_S \longrightarrow \mathcal{O}_{Z_1(S)}(-1) \longrightarrow 0. \tag{2}$$

Let $\mathcal{I}_{Z_1(S)} = \mathcal{K}_1(I_S)$ be the sheafification of the ideal $K_1(I_S)$. In [12, (3.4.1), p. 302], S. Kleiman, J. Lipman and B. Ulrich proved that

$$\mathcal{I}_{Y_1(S)} = \mathrm{Fitt}_1^{\mathbb{P}^3}(\pi_* \mathcal{O}_S) = \mathrm{Fitt}_0^{\mathbb{P}^3}(\pi_* \mathcal{O}_S/\mathcal{O}_{\pi(S)}) = \mathrm{Ann}_{\mathbb{P}^3}(\mathcal{O}_{Z_1(S)}(-1)),$$

and this defines *the reduced scheme structure* on $Y_1(S)$ (see [14, p. 3]).

On the other hand, from the sequence (2), we have

$$\mathcal{I}_{Y_1(S)} = \mathrm{Ann}_{\mathbb{P}^3}(\mathcal{O}_{Z_1(S)}(-1)) = \mathcal{K}_1(I_S) = \mathcal{I}_{Z_1(S)}.$$

Then it follows from (a) that

$$I_{Z_1(S)} = K_1(I_S)^{\mathrm{sat}} = K_1(I_S) = I_{Y_1(S)}.$$

Since $I_{Y_1(S)}$ is a reduced ideal, we conclude that $I_{Z_1(S)} = K_1(I_S)$ is also a reduced ideal. □

If $S \subset \mathbb{P}^4$ is contained in a quadric hypersurface, then by Theorem 4.2, $K_1(I_S)$ is saturated and reduced. So, it defines the reduced scheme structure on $Y_1(S)$. Note also that the double curve $Y_1(S)$ is smooth (see the standard fact (b) in the beginning of this section). We use this fact to prove the following theorem.

Theorem 4.3. Let S be a smooth irreducible surface of degree d lying on a quadric hypersurface in $\mathbb{P}^4$. Let $Y_1(S)$ be the double curve of genus $g(Y_1(S))$ defined by a generic projection π of S to $\mathbb{P}^3$. Then, we have the following;

(a) $M(I_S) = \max\{d, 1 + \deg Y_1(S), 2 + \binom{\deg Y_1(S)-1}{2} - g(Y_1(S))\}$;

(b) $M(I_S)$ can be obtained at one of monomials

$$x_1^d,\ x_0x_2^{\deg Y_1(S)},\ x_0x_1x_3^{\binom{\deg Y_1(S)-1}{2}-g(Y_1(S))}.$$

Proof. Note that by Corollary 3.6,

$$M(I_S) = \max_{0\le i\le \beta}\{\mathrm{reg}(\mathrm{Gin}(K_i(I_S))) + i\},$$

where $\beta = \min\{j \mid K_j(I_S) = \bar{R}\}$. Since S is contained in a quadric hypersurface, $\mathrm{Gin}(I_S)$ contains the monomial x_0^2. This means that $\mathrm{Gin}(K_2(I_S)) = \bar{R}$. On the other hand, $\mathrm{Gin}(K_0(I_S)) = (x_1^d)$ by the Borel fixed property because $\pi(S)$ is a hypersurface of degree d in $\mathbb{P}^3$ and $I_{\pi(S)} = K_0(I_S)$. Thus $\mathrm{Gin}(I_S)$ is of the form

$$(x_0^2, x_0g_1, x_0g_2, \dots, x_0g_m, x_1^d).$$

Note that $g_1, \dots g_m$ are monomial generators of $\mathrm{Gin}(K_1(I_S)) = \mathrm{Gin}(I_{Y_1(S)})$ by Proposition 3.4.

Therefore, by Theorem 3.7,

$$\mathrm{reg}(\mathrm{Gin}(K_1(I_S))) = \max\{\deg Y_1(S), 1 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S))\}$$

and consequently,

$$M(I_S) = \max\{d, 1 + \deg Y_1(S), 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S))\}.$$

For a proof of (b), consider $\mathrm{Gin}(K_1(I_S)) = \langle g_1, g_2, \dots, g_m\rangle$ in (a). Note that the double curve $Y_1(S)$ is smooth in $\mathbb{P}^3$. By the similar argument used in (a), $\mathrm{Gin}(K_1(I_S))$ contains $x_2^{\deg(Y_1(S))}$ because the image of $Y_1(S)$ under a generic projection to $\mathbb{P}^2$ is a plane curve of degree $\deg(Y_1(S))$. Finally, consider all monomial generators of the form $x_1 \cdot h_j(x_2, x_3, x_4)$ in $\{g_1, g_2, \ldots, g_m\}$. Then, $\{h_j(x_2, x_3, x_4) \mid 1 \le j \le m\}$ is a minimal generating set of $\mathrm{Gin}(K_1(I_{Y_1(S)}))$ by Proposition 3.4. Recall that $K_1(I_{Y_1(S)})$ defines $\binom{\deg Y_1(S)-1}{2} - g(Y_1(S))$ distinct nodes in $\mathbb{P}^2$. So, $\mathrm{Gin}(K_1(I_{Y_1(S)}))$ should contain the monomial $x_3^{\binom{\deg Y_1(S)-1}{2}-g(Y_1(S))}$ (see also [5, Corollary 5.3]). Therefore, $\mathrm{Gin}(I_S)$ contains monomials $x_1^d, x_0 x_2^{\deg(Y_1(S))}$ and $x_0x_1x_3^{\binom{\deg Y_1(S)-1}{2}-g(Y_1(S))}$. $\square$

Remark 4.4. In the proof of Theorem 4.3, we showed that if a smooth irreducible surface S is contained in a quadric hypersurface then $M(I_S)$ is determined by two partial elimination ideals $K_0(I_S)$ and $K_1(I_S)$ since $K_i(I_S) = \bar{R}$ for all $i \ge 2$.

The following theorem shows that if $d \ge 6$ then $M(I_S)$ is determined by the degree complexity of the first partial elimination ideal $K_1(I_S)$.

Proposition 4.5. Let S be a smooth irreducible surface of degree d in $\mathbb{P}^4$. Suppose that S is contained in a quadric hypersurface. Then

$$M(I_S) = \begin{cases} 3 & \text{if } S \text{ is a rational normal scroll with } d = 3 \\ 4 & \text{if } S \text{ is a complete intersection of (2,2)-type} \\ 5 & \text{if } S \text{ is a Castelnuovo surface with } d = 5 \\ 2 + \binom{\deg Y_1(S)-1}{2} - g(Y_1(S)) & \text{for } d \ge 6 \end{cases}$$

where $Y_1(S) \subset \mathbb{P}^3$ is a double curve of degree $\binom{d-1}{2} - g(S \cap H)$ under a generic projection of S to $\mathbb{P}^3$.

Proof. Since $K_2(I_S) = \bar{R}$, Theorem 4.3 implies that

$$M(I_S) = \max\{d, 1 + \deg Y_1(S), 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S))\}.$$

If $\deg Y_1(S) \ge 5$ then by the genus bound,

$$1 + \deg Y_1(S) \le 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)).$$

We claim that if $d \ge 6$, then $d \le 1 + \deg Y_1(S)$. Notice that from our claim, we have the degree complexity of a surface lying on a quadric hypersurface in $\mathbb{P}^4$ for $d \ge 6$ as follows;

$$M(I_S) = 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)).$$

Note again that

$$g(S \cap H) \le \pi(d, 3) = \begin{cases} (\frac{d}{2} - 1)^2 & \text{if } d \text{ is even;} \\ (\frac{d-1}{2})(\frac{d-3}{2}) & \text{if } d \text{ is odd.} \end{cases}$$

Then we can show that $\pi(d,3) \leq \binom{d-1}{2} - d + 1$ if $d = \deg(S \cap H) \geq 6$. Thus, if $d \geq 6$ then

$$d \leq 1 + \binom{d-1}{2} - g(S \cap H) = 1 + \deg Y_1(S).$$

So, our claim is proved and only three cases of $d = 3, 4, 5$ are remained.

Case 1: If $\deg S = 3$ then S is a rational normal scroll with $g(S \cap H) = 0$ and the double curve $Y_1(S)$ is a line. So, by simple computation, $M(I_S) = 3$.

Case 2: If $\deg S = 4$ then S is a complete intersection of (2,2)-type with $g(S \cap H) = 1$ and the double curve $Y_1(S)$ is a plane conic of $\deg Y_1(S) = 2$. So, by simple computation, $M(I_S) = 4$.

Case 3: If $\deg S = 5$ then S is a Castelnuovo surface with $g(S \cap H) = 2$ and the double curve $Y_1(S) \subset \mathbb{P}^3$ is a smooth elliptic curve of degree 4. In this case, we can also compute

$$M(I_S) = 5 = \deg S > 2 + \binom{\deg Y_1(S) - 1}{2} - g(Y_1(S)) = 4.$$

□

Proposition 4.6. Let S be a smooth irreducible surface of degree d and arithmetic genus $\rho_a(S)$ in $\mathbb{P}^4$. Let $Y_i(S)$ be the multiple locus defined by a generic projection of S to $\mathbb{P}^3$ for $i \geq 0$. Assume that S is contained in a quadric hypersurface. Then, the following identity holds;

$$g(Y_1(S)) = \binom{d-1}{3} - \binom{d-1}{2} + g(S \cap H) - \rho_a(S) + 1.$$

Proof. Let $P_S(z)$ be the Hilbert polynomial of a smooth irreducible surface of degree d and arithmetic genus $\rho_a(S)$. Since $Y_2(S) = \emptyset$, $P_{Y_2(S)}(z) = 0$ and, by Lemma 4.1,

$$P_S(z) = P_{Y_0(S)}(z) + P_{Y_1(S)}(z-1).$$

Plugging $z = 0$, $P_S(0) = \rho_a(S) + 1, P_{Y_0(S)}(0) = \binom{d-1}{3} + 1$, and

$$P_{Y_1(S)}(-1) = -\deg Y_1(S) + 1 - g(Y_1(S)) = -\binom{d-1}{2} + g(S \cap H) + 1 - g(Y_1(S)).$$

Therefore, we have the following identity:

$$g(Y_1(S)) = \binom{d-1}{3} - \binom{d-1}{2} + g(S \cap H) - \rho_a(S) + 1.$$

□

Remark 4.7. By Proposition 4.6, when $d \geq 6$, $M(I_S)$ can be expressed with only three invariants of S: its degree, sectional genus, and arithmetic genus, as follows:

$$M(I_S) = \binom{\binom{d-1}{2} - g(S \cap H) - 1}{2} - \binom{d-1}{3} + \binom{d-1}{2} - g(S \cap H) + \rho_a(S) + 1.$$

□

In order to compute $M(I_S)$ in terms of degrees of defining equations as A. Conca and J. Sidman did in [5], we need the following remark. This shows that a smooth surface in $\mathbb{P}^4$ has a nice algebraic structure when it is contained in a quadric hypersurface.

Remark 4.8. Let S be a locally Cohen-Macaulay surface lying on a quadric hypersurface Q in $\mathbb{P}^4$. Then S satisfies one of following conditions (see [11, Theorem 2.1]);

- (a) S is a complete intersection of $(2, \alpha)$-type.
  - (i) $I_S = (Q, F)$, where F is a polynomial of degree α.
  - (ii) $\operatorname{reg}(S) = \alpha + 1$.
- (b) S is projectively Cohen-Macaualy of degree $2\alpha - 1$.
  - (i) $I_S = (Q, F_1, F_2)$, where F_1 and F_2 are polynomials of degree α.
  - (ii) $\operatorname{reg}(S) = \alpha$.

From the above Remark 4.8, we can compute $g(S \cap H)$ and $\rho_a(S)$ in terms of the degree of defining equations of S by finding the Hilbert polynomial of S in two ways. Therefore, we have the following Theorem.

Theorem 4.9. Let $S \subset \mathbb{P}^4$ be a smooth irreducible surface of degree d and arithmetic genus $\rho_a(S)$, which is contained in a quadric hypersurface.

- (a) Suppose S is of degree $2\alpha, \alpha \geq 3$. Then,
$$M(I_S) = \frac{1}{2}(\alpha^4 - 4\alpha^3 + 5\alpha^2 - 2\alpha + 4).$$
- (b) Suppose S is of degree $2\alpha - 1, \alpha \geq 4$. Then
$$M(I_S) = \frac{1}{2}(\alpha^4 - 6\alpha^3 + 13\alpha^2 - 12\alpha + 8).$$

Proof. For a proof of (a), by Koszul complex we have the minimal free resolution of the defining ideal I_S as follows:
$$0 \longrightarrow R(-\alpha - 2) \longrightarrow R(-2) \oplus R(-\alpha) \longrightarrow I_S \longrightarrow 0,$$
Hence the Hilbert function of R/I_S is given by
$$\begin{aligned} H(R/I_S, m) =& \alpha m^2 + (-\alpha^2 + 3\alpha)m + \frac{1}{6}\alpha(2\alpha^2 - 9\alpha + 13) \\ =& \frac{2\alpha}{2}m^2 + (\alpha + 1 - g(S \cap H))\, m + \rho_a(S) + 1. \end{aligned}$$
Hence $g(S \cap H) = (\alpha - 1)^2$ and $\rho_a(S) = \frac{1}{6}\alpha(2\alpha^2 - 9\alpha + 13) - 1$.
If $Y_1(S)$ is the double curve of S then
$$\deg Y_1(S) = \binom{2\alpha - 1}{2} - g(S \cap H) = \alpha(\alpha - 1).$$
By Remark 4.7,
$$g(Y_1(S)) = \binom{2\alpha - 1}{3} - \binom{2\alpha - 1}{2} + g(S \cap H) - \rho_a(S) + 1.$$

Thus we conclude that

$$\begin{aligned}M(I_S) =& 2+\binom{\alpha(\alpha-1)-1}{2}-g(Y_1(S))\\ =&\binom{\alpha(\alpha-1)-1}{2}-\binom{2\alpha-1}{3}+\binom{2\alpha-1}{2}-(\alpha-1)^2+\rho_a(S)+1\\ =&\frac{1}{2}(\alpha^4-4\alpha^3+5\alpha^2-2\alpha+4).\end{aligned}$$

For a proof of (b), let S be a smooth surface of degree $2\alpha-1$ lying on a quadric hypersurface in $\mathbb{P}^4$. Note that S is arithmetically Cohen-Macaulay of codimension 2. By the Hilbert-Burch Theorem [6] we have the minimal free resolution of the defining ideal I_S as follows:

$$0 \longrightarrow R(-\alpha-1)^2 \xrightarrow{\begin{pmatrix} L_1 & L_2\\ L_3 & L_4\\ F_5 & F_6\end{pmatrix}} R(-2)\oplus R(-\alpha)^2 \longrightarrow I_S \longrightarrow 0,$$

where L_1, L_2, L_3, L_4 are linear forms and F_5, F_6 are forms of degree $\alpha-1$. Hence the Hilbert function of R/I_S is given by

$$\begin{aligned}H(R/I_S,m) =& \frac{1}{2}(2\alpha-1)m^2+\left(4\alpha-\alpha^2-\frac{3}{2}\right)m+\frac{1}{3}\alpha^3-2\alpha+\frac{11}{3}\alpha-1\\ =& \frac{(2\alpha-1)}{2}m^2+\left(\frac{2\alpha-1}{2}+1-g(S\cap H)\right)m+\rho_a(S)+1.\end{aligned}$$

Hence we have that $g(S\cap H)=2\binom{\alpha-1}{2}$ and $\rho_a(S)=2\binom{\alpha-1}{3}$.

If $Y_1(S)$ be the double curve of S then

$$\deg Y_1(S)=\binom{2\alpha-2}{2}-g(S\cap H)=\binom{2\alpha-2}{2}-2\binom{\alpha-1}{2}.$$

On the other hand, we have

$$\begin{aligned}g(Y_1(S)) &= \binom{2\alpha-2}{3}-\binom{2\alpha-2}{2}+g(S\cap H)-\rho_a(S)+1\\ &=(\alpha-2)(\alpha^2-3\alpha+1)\end{aligned}$$

and thus we conclude that

$$\begin{aligned}M(I_S) &= 2+\binom{\deg Y_1(S)-1}{2}-g(Y_1(S))\\ &=\frac{1}{2}(\alpha^4-6\alpha^3+13\alpha^2-12\alpha+8).\end{aligned}$$

□

Example 4.10 (Macaulay 2)**.** We give some examples of $\mathrm{Gin}(I_S)$ and $M(I_S)$ computed by using *Macaulay 2*.

(a) Let S be a rational normal scroll in $\mathbb{P}^4$ whose defining ideal is

$$I_S = (x_0x_3 - x_1x_2, x_0x_1 - x_3x_4, x_0^2 - x_2x_4).$$

Using Macaulay 2, we can compute the generic initial ideal of I_S with respect to GLex:

$$\mathrm{Gin}(I_S) = (x_0^2, x_0x_1, x_0x_2, \mathbf{x_1^3}).$$

Thus $\mathrm{reg}(\mathrm{Gin}_{\mathrm{GLex}}(K_0)) = 3$ and $\mathrm{reg}(\mathrm{Gin}_{\mathrm{GLex}}(K_1)) = 1$. Therefore,

$$M(I_S) = \deg S = 3.$$

(b) Let S be a complete intersection of $(2,2)$-type in $\mathbb{P}^4$. Then,

$$\mathrm{Gin}(I_S) = (x_0^2, x_0x_1, \mathbf{x_1^4}, x_0x_2^2).$$

Hence, we see $M(I_S) = \deg S = 4$.

(c) Let S be a Castelnuovo surface of degree 5 in $\mathbb{P}^4$. Then, we can compute

$$\mathrm{Gin}(I_S) = (x_0^2, x_0x_1^2, \mathbf{x_1^5}, x_0x_1x_2, x_0x_2^4, x_0x_1x_3^2).$$

Hence, we see $M(I_S) = \deg S = 5$.

(d) Let S be a complete intersection of $(2,3)$-type in $\mathbb{P}^4$. Then, we see that $M(I_S) = 8$ from Theorem 4.9. On the other hand, we can compute the generic initial ideal:

$$\mathrm{Gin}(I_S) = (x_0^2, x_0x_1^2, x_1^6, x_0x_1x_2^2, x_0x_2^6, x_0x_1x_2x_3^2, \mathbf{x_0x_1x_3^6}, x_0x_1x_2x_3x_4^2, x_0x_1x_2x_4^4).$$

This also shows $M(I_S) = 8$.

(e) Let S be a smooth surface of degree 7 lying on a quadric which is not a complete intersection in $\mathbb{P}^4$. Then, the minimal resolution of I_S is given by Hilbert-Burch Theorem and thus we have

$$I_S = (L_1L_4 - L_2L_3, L_1F_5 - L_2F_6, L_3F_5 - L_4F_6),$$

where L_i is a linear form and F_5, F_6 are forms of degree 3. This is the case of $\alpha = 4$ in Theorem 4.9 and we see $M(I_S) = 20$. This can also be obtained by the computation of generic initial ideal of I_S using *Macaulay 2*:

$$\begin{aligned}\mathrm{Gin}(I_S) =&(x_0^2,\ x_0x_1^3,\ x_1^7,\ x_0x_1^2x_2,\ x_0x_1x_2^4,\ x_0x_2^9,\ x_0x_1^2x_3^2,\ x_0x_1x_2^3x_3^2,\ x_0x_1x_2^2x_3^5,\\ &x_0x_1x_2x_3^8,\ \mathbf{x_0x_1x_3^{18}},\ x_0x_1x_2^2x_3^4x_4,\ x_0x_1^2x_3x_4^2,\ x_0x_1x_2^3x_3x_4^2,\ x_0x_1x_2^2x_3^3x_4^2,\\ &x_0x_1x_2x_3^7x_4^2,\ x_0x_1x_2^3x_4^3,\ x_0x_1^2x_4^4,\ x_0x_1x_2^2x_3^2x_4^4,\ x_0x_1x_2x_3^6x_4^4,\ x_0x_1x_2^2x_3x_4^5,\\ &x_0x_1x_2x_3^5x_4^6,\ x_0x_1x_2^2x_4^7,\ x_0x_1x_2x_3^4x_4^8,\ x_0x_1x_2x_3^3x_4^{10},\ x_0x_1x_2x_3^2x_4^{12},\\ &x_0x_1x_2x_3x_4^{14},\ x_0x_1x_2x_4^{16})\end{aligned}$$

(f) Let S be a complete intersection of $(2,4)$-type in $\mathbb{P}^4$. Then, we see that $M(I_S) = 38$ from Theorem 4.9. This can be given by the computation of generic initial ideal of I_S:

$$\begin{aligned}
\mathrm{Gin}(I_S) = (&x_0^2,\ x_0x_1^3,\ x_1^8,\ x_0x_1^2x_2^2,\ x_0x_1x_2^6,\ x_0x_2^{12},\ x_0x_1^2x_2x_3^2,\ x_0x_1x_2^5x_3^2,\\
&x_0x_1^2x_3^5,\ x_0x_1x_2^4x_3^5,\ x_0x_1x_2^3x_3^7,\ x_0x_1x_2^2x_3^{11},\ x_0x_1x_2x_3^{17},\ \mathbf{x_0x_1x_3^{36}},\\
&x_0x_1^2x_3^4x_4,\ x_0x_1x_2^4x_3^4x_4,\ x_0x_1x_2^3x_3^6x_4,\ x_0x_1x_2^2x_3^{10}x_4,\ x_0x_1^2x_2x_3x_4^2,\\
&x_0x_1x_2^5x_3x_4^2,\ x_0x_1^2x_3^3x_4^2,\ x_0x_1x_2^4x_3^3x_4^2,\ x_0x_1x_2^2x_3^9x_4^2,\ x_0x_1x_2x_3^{16}x_4^2,\\
&x_0x_1^2x_2x_4^3,\ x_0x_1x_2^5x_4^3,\ x_0x_1x_2^4x_3^2x_4^3,\ x_0x_1x_2^3x_3^5x_4^3,\ x_0x_1^2x_3^2x_4^4,\\
&x_0x_1x_2^3x_3^4x_4^4,\ x_0x_1x_2^2x_3^8x_4^4,\ x_0x_1x_2x_3^{15}x_4^4,\ x_0x_1^2x_3x_4^5,\ x_0x_1x_2^4x_3x_4^5,\\
&x_0x_1x_2^3x_3^3x_4^5,\ x_0x_1x_2^2x_3^7x_4^5,\ x_0x_1x_2^4x_4^6,\ x_0x_1x_2x_3^{14}x_4^6,\ x_0x_1^2x_4^7,\\
&x_0x_1x_2^3x_3^2x_4^7,\ x_0x_1x_2^2x_3^6x_4^7,\ x_0x_1x_2^3x_3x_4^8,\ x_0x_1x_2^2x_3^5x_4^8,\ x_0x_1x_2x_3^{13}x_4^8,\\
&x_0x_1x_2^3x_4^9,\ x_0x_1x_2^2x_3^4x_4^{10},\ x_0x_1x_2x_3^{12}x_4^{10},\ x_0x_1x_2^2x_3^3x_4^{11},\ x_0x_1x_2x_3^{11}x_4^{12},\\
&x_0x_1x_2^2x_3^2x_4^{13},\ x_0x_1x_2^2x_3x_4^{14},\ x_0x_1x_2x_3^{10}x_4^{14},\ x_0x_1x_2^2x_4^{16},\ x_0x_1x_2x_3^9x_4^{16},\\
&x_0x_1x_2x_3^8x_4^{18},\ x_0x_1x_2x_3^7x_4^{20},\ x_0x_1x_2x_3^6x_4^{22},\ x_0x_1x_2x_3^5x_4^{24},\ x_0x_1x_2x_3^4x_4^{26},\\
&x_0x_1x_2x_3^3x_4^{28},\ x_0x_1x_2x_3^2x_4^{30},\ x_0x_1x_2x_3x_4^{32},\ x_0x_1x_2x_4^{34}).
\end{aligned}$$

Even though we cannot compute the generic initial ideals for the cases $\alpha \geq 5$ by using computer algebra systems, we know the degree-complexity of smooth surfaces lying on a quadric by theoretical computations. We give the following tables:

Table 1 The complete intersection S of $(2,\alpha)$-type in $\mathbb{P}^4$

α	5	6	7	8	9	10	20	50	100
$M(I_S)$	122	302	632	1178	2018	3242	64982	2881202	48024902
$m(I_S)$	6	7	8	9	10	11	21	51	101

Table 2 The smooth surface $S \subset \mathbb{P}^4$ of degree $(2\alpha - 1)$ lying on a quadric.

α	5	6	7	8	9	10	20	50	100
$M(I_S)$	74	202	452	884	1570	2594	58484	2765954	47064404
$m(I_S)$	5	6	7	8	9	10	20	50	100

Remark and Question 4.11. Let S be a non-degenerate smooth surface of degree d and arithmetic genus $\rho_a(S)$, not necessarily contained in a quadric hypersurface in $\mathbb{P}^4$. Our question is: What can be the degree complexity

$M(I_S)$ of S? It is expected that $K_1(I_S)$ and $K_2(I_S)$ are reduced ideals and the degree-complexity $M(I_S)$ is given by

$$\begin{aligned} M(I_S) &= \max \begin{cases} \deg(S) \\ \operatorname{reg}(\operatorname{Gin}_{\mathrm{GLex}}(K_1(I_S))) + 1 \\ \operatorname{reg}(\operatorname{Gin}_{\mathrm{GLex}}(K_2(I_S))) + 2 \end{cases} \\ &= \max \begin{cases} d \\ M(I_{Y_1(S)}) + 1 \\ t + 2. \end{cases} \end{aligned}$$

Note that t is the number of apparent triple points of $S \subset \mathbb{P}^4$ and $Y_1(S)$ is the double curve (possibly singular with ordinary double points) under a generic projection.

$\square$

References

[1] Ahn, Jeaman *The degree-complexity of the defining ideal of a smooth integral curve.* J. Symbolic Comput. 43 (2008), no. 6-7, 422–441.

[2] Bayer, Dave; Mumford, David *What can be computed in algebraic geometry?* Computational algebraic geometry and commutative algebra (Cortona, 1991), 1–48, Sympos. Math., XXXIV, Cambridge Univ. Press, Cambridge, 1993.

[3] Bayer, David; Stillman, Michael *A criterion for detecting m-regularity.* Invent. Math. 87 (1987), no. 1, 1–11.

[4] Conca, Aldo *Koszul homology and extremal properties of Gin and Lex.* Trans. Amer. Math. Soc. 356 (2004), no. 7, 2945–2961.

[5] Conca, Aldo; Sidman, Jessica *Generic initial ideals of points and curves.* J. Symbolic Comput. 40 (2005), no. 3, 1023–1038.

[6] Eisenbud, David *The geometry of syzygies. A second course in commutative algebra and algebraic geometry.* Graduate Texts in Mathematics, 229. Springer-Verlag, New York, 2005.

[7] Eisenbud, David *Commutative Algebra with a View toward Algebraic Geometry.* Graduate Texts in Mathematics, **150**, Springer-Verlag, New York, 1995.

[8] Green, Mark *Generic Initial Ideals*, in Six lectures on Commutative Algebra, (Elias J., Giral J.M., Miró-Roig, R.M., Zarzuela S., eds.), Progress in Mathematics **166**, Birkhäuser, 1998, 119–186.

[9] Galligo, André, *A propos du théorème de préparation de Weierstrrass*, in: Fonctions de Plusieurs Variables Complexes, Lecture Note in Mathematics, Springer, Berlin, 1974, pp. 543–579.

[10] Grayson, Daniel R.; Stillman, Michael E. *Macaulay 2: a software system for algebraic geometry and commutative algebra*, available over the web at http://www.math.uiuc.edu/Macaulay2.

[11] Kwak, Sijong *Castelnuovo-Mumford regularity bound for smooth threefolds in $\mathbb{P}^5$ and extremal examples.* J. Reine Angew. Math. 509 (1999), 21–34.

[12] Kleiman, Steven; Lipman, Joseph; Ulrich, Bernd *The multiple-point schemes of a finite curvilinear map of codimension one.* Ark. Mat. 34 (1996), no. 2, 285–326.

[13] Le Barz, Patrick *Formules pour les multisecantes des surfaces. (French) [Formulas for the multisecants of surfaces]* C. R. Acad. Sci. Paris Ser. I Math. 292 (1981), no. 17, 797–800.
[14] Mezzetti, Emilia; Portelli, Dario *A tour through some classical theorems on algebraic surfaces.* An. Stiint. Univ. Ovidius Constanta Ser. Mat. 5 (1997), no. 2, 51–78.
[15] Pinkham, Henry C. *A Castelnuovo bound for smooth surfaces.* Invent. Math. 83 (1986), no. 2, 321–332.
[16] Ran, Ziv. *The (dimension+2)-secant lemma.* Invent. Math. 106(1991), no. 1, 65-71.

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